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Hodge type of projective varieties of low degree

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Let S be a complex projective variety embedded in $\mathbf{P}^n$ with complement U, such that S is defined by r equations of homogeneous degrees $d_1 \geqq d_2 \geqq \cdots \geqq d_r$. One defines the *Hodge type* of S to be the largest integer a for which the Hodge-Deligne filtration $F^\bullet$ on the de Rham cohomology of U with compact supports satisfies

$$F^a H^i_c(U) = H^i_c(U) \quad \forall i .$$

If our equations are defined over the finite field $\mathbf{F}_q$, then a theorem of Ax and Katz yields the congruence

$$\# U(\mathbf{F}_q) \equiv 0 \,(\mathrm{mod}\ q^\kappa)$$

where

$$\kappa = \left[\frac{n - \sum_{i=2}^{r} d_i}{d_1} \right]$$

(here $[a]$ denotes the integral part of a). This suggests that the motive associated to the cohomology of U should be the product of the Tate motive $\mathbb{Q}(-\kappa)$ with an effective motive (see [DD, Introduction]). If so, then

$$\text{(D):} \quad \text{the Hodge type of } S \text{ should be} \geqq \kappa ,$$

as precisely formulated in [DD, Sect. 1].

This conjecture on the Hodge type is therefore directly inspired by the philosophy of motives.

When S is a smooth complete intersection, Deligne proved (D) in SGA 7 [D]. In [DD], the case $r = 1$ is proven by comparing the Hodge filtration on the de Rham cohomology $H^i(U)$ to the filtration defined by the order of the poles along the divisor S. In higher codimension, the first author reduced the general problem of computing the Hodge type of a variety to some vanishing theorem, true for complete intersections [E].

In this note, we explain two different geometric constructions, leading to situations where one may apply a variant of the reduction mentioned above, to prove (D) in general. The first one was used in the second author's work on cycles [N], and consists of considering S as a special fibre of the 'deformation' of S given by the universal family of varieties in $\mathbf{P}^n$ defined by equations of the given degrees. The second one was inspired by Terasoma's work [T] and consists of considering the 'universal pencil' spanned by the equations of S and having S as the base locus.

Both ways allow one to prove:

Theorem. *Let S be a complex projective variety defined in $\mathbf{P}^n$ by r equations of homogeneous degrees $d_1 \geqq d_2 \geqq \cdots \geqq d_r$. Then the Hodge type of S is at least*

$$\kappa = \left[\frac{n - \sum_{i=2}^{r} d_i}{d_1}\right]$$

1 Deformation

1.1

We consider the following diagram

$$\begin{array}{c} X \to Y = \mathbf{P}^n \times V \\ p \searrow \downarrow p_2 \\ V \end{array}$$

where

$$V = \operatorname{Spec} \operatorname{Sym}(H^0(\mathbf{P}^n, \mathscr{E})^\vee) \quad \text{where } \mathscr{E} = \oplus_{i=1}^r \mathcal{O}_{\mathbf{P}^n}(d_i)\,,$$

$$X = \text{universal subvariety defined by equations of degrees } d_1, \ldots, d_r\,.$$

In other words,

$$X = \operatorname{Spec} \operatorname{Sym}(\mathscr{R}^\vee)$$

where $\mathscr{R}$ is the locally free sheaf defined by the exact sequence

$$0 \to \mathscr{R} \to H^0(\mathbf{P}^n, \mathscr{E}) \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}^n} \to \mathscr{E} \to 0\,,$$

and the superscript $^\vee$ denotes the dual of a locally free sheaf. For $f \in V$, one has $f = (f_1, \ldots, f_r)$, $f_i \in H^0(\mathbf{P}^n, \mathcal{O}(d_i))$, and $X_f = p^{-1}(f) = \{x \in \mathbf{P}^n \mid f_i(x) = 0 \,\forall\, 1 \leqq i \leqq r\}$. We now mimic [E, (1.1) and (1.2)] in this relative situation. We denote by $|_A$ the topological restriction of sheaves to a subspace A.

Proposition 1.1.1. *Let $\Omega^{<\kappa}_{Y,X} \xrightarrow{i} \Omega^{<\kappa}_Y$ be a map from a complex supported in degrees $< \kappa$ to the truncated holomorphic de Rham complex such that for some $f \in V$ one has*

(i) $i|_{\mathbf{P}^n\times\{f\}}$ *is an isomorphism on* $U_f = \mathbf{P}^n \times \{f\} - X_f$

(ii) *let* $\sigma\colon Z \to \mathbf{P}^n \times \{f\}$ *be a desingularization of* X_f *such that* $\tilde{X}_f := \sigma^{-1} X_f$ *is a normal crossing divisor. Then there is a factorization* σ^{-1}:

$$\begin{array}{ccc} \Omega^j_{Y,X}|_{\mathbf{P}^n\times\{f\}} & \xrightarrow{\sigma^{-1}\circ i|_{\mathbf{P}^n\times\{f\}}} & \sigma^{-1}\Omega^j_Y|_{\mathbf{P}^n\times\{f\}} \\ & & \downarrow \\ \sigma^{-1}\downarrow & & \sigma^{-1}\Omega^j_{\mathbf{P}^n\times\{f\}} \\ & & \downarrow \\ \Omega^j_Z(\log \tilde{X}_f)(-\tilde{X}_f) & \to & \Omega^j_Z \end{array}$$

for any $0 \leqq j < \kappa$.

Then σ^{-1} *defines a surjection (independent of the desingularization)*

$$(\mathbf{R}^i(p_2)_*\Omega^{<\kappa}_{Y,X})|_{\{f\}} \xrightarrow{\sigma^{-1}} H^i_c(U_f)/F^\kappa H^i_c(U_f)$$

for any i.

Proof. Let $\mathscr{I}$ be the ideal sheaf of X in Y. For $l \gg 0$, define

$$\Omega'^{\bullet}_{Y,X} := 0 \to \mathscr{I}^l \to \mathscr{I}^{l-1} \otimes \Omega^1_{Y,X} \to \cdots \to \mathscr{I}^{l-\kappa+1} \otimes \Omega^{\kappa-1}_{Y,X}$$
$$\to \mathscr{I}^{l-\kappa} \otimes \Omega^\kappa_Y \to \cdots \to \mathscr{I}^{l-\dim Y} \otimes \Omega^{\dim Y}_Y \to 0\,.$$

Then the natural map $\Omega'^{\bullet}_{Y,X} \to \Omega^\bullet_Y$ fulfills (i) and (ii) for $\kappa = \dim Y + 1$ and for the same f as before. Let $U = Y - X \overset{j}{\hookrightarrow} Y$ and $j_f := j|_{\mathbf{P}^n\times\{f\}}\colon U_f \to \mathbf{P}^n \times \{f\}$. One has a natural map $j_!\mathbf{C} \to \Omega'^{\bullet}_{Y,X}$, and hence maps

$$(j_f)_!\mathbf{C} \to \Omega'^{\bullet}_{Y,X}|_{\mathbf{P}^n\times\{f\}} \xrightarrow{\sigma^{-1}} \mathbf{R}\sigma_*\Omega^\bullet_Z(\log \tilde{X}_f)(-\tilde{X}_f)$$

which, as in [E, (1.1)] defines a surjection

$$(\mathbf{R}^i(p_2)_*\Omega'^{\bullet}_{Y,X})|_{\{f\}} \xrightarrow{\sigma^{-1}} H^i_c(U_f)\,.$$

Then, as in [E, (1.2)], one has a commutative diagram

$$\begin{array}{ccc} (\mathbf{R}^i(p_2)_*\Omega'^{\bullet}_{Y,X})|_{\{f\}} & \twoheadrightarrow & H^i_c(U_f) \\ \downarrow & & \downarrow \\ (\mathbf{R}^i(p_2)_*\Omega^{<\kappa}_{Y,X})|_{\{f\}} & \to & H^i_c(U_f)/F^\kappa H^i_c(U_f)\,. \end{array}$$

□

Corollary 1.1.2. *To prove the theorem, it suffices to find* $\Omega^{<\kappa}_{Y,X}$ *as in Proposition* 1.1.1 *such that* $\mathbf{R}^i(p_2)_*\Omega^{<\kappa}_{Y,X} = 0$ *for all i. In fact it suffices to find such a complex with* $R^i(p_2)_*\Omega^j_{Y,X} = 0$ *for all* $0 \leqq j < \kappa$ *and all i.*

1.2

Choose as in [E, (2.2)]

$$\Omega^j_{Y,X} := \mathscr{I}^{\kappa-j} \otimes \Omega^j_Y,\quad 0 \leqq j < \kappa\,.$$

Then Ω^j_Y has a resolution by direct sums of sheaves $p_1^*\mathcal{O}_{\mathbf{P}^n}(-m)$, $0 \leqq m \leqq j$, and as X is a smooth complete intersection in Y defined by a section of $\mathscr{E}$, $\mathscr{I}^{\kappa-j}$ has a resolution by direct sums of sheaves $p_1^*\mathcal{O}_{\mathbf{P}^n}(-l)$, $0 \leqq l \leqq (\kappa - j)d_1 + d_2 + \cdots + d_r$. Therefore $\Omega^j_{Y,X}$ has a resolution by sheaves $p_1^*\mathcal{O}_{\mathbf{P}^n}(-s)$, $0 \leqq s \leqq \kappa d_1 + d_2 + \cdots + d_r - jd_1 + j$. Since

$$H^i(Y, \Omega^j_{Y,X}) = H^0(V, R^i(p_2)_* \Omega^j_{Y,X}) = 0 \iff R^i(p_2)_* \Omega^j_{Y,X} = 0 \,,$$

the desired vanishing statement holds if

$$0 \leqq \kappa d_1 + d_2 + \cdots + d_r - jd_1 + j \leqq n \quad \text{for } 0 \leqq j < \kappa \,,$$

that is for

$$\kappa = \left[\frac{n - \sum_{i=2}^{r} d_i}{d_1}\right]$$

1.3

Another way of obtaining the theorem from the above geometric construction is motivated by the arguments in [N]. This is the way in which the theorem was initially proved.

Define

$$\Omega^\bullet_{(Y,X)} := \ker(\Omega^\bullet_Y \to \Omega^\bullet_X) \,.$$

As X is smooth, $\Omega^\bullet_{Y,X}$ is quasi-isomorphic to $j_!\mathbf{C}$.

Proposition 1.3.1. *Suppose* $\mathbf{R}^i(p_2)_*\Omega^{<\kappa}_{(Y,X)} = 0$ *for all i. Then the theorem holds.*

Proof. For any embedded desingularization $\sigma: Z \to \mathbf{P}^n \times \{f\}$ of X_f such that $\tilde{X}_f = f^{-1}(X_f)$ is a normal crossing divisor, we have a commutative diagram

$$\begin{array}{ccccccc} j_!\mathbf{C} & \simeq & \Omega^\bullet_{(Y,X)} & \to & \Omega^\bullet_{(Y,X)}|_{\mathbf{P}^n\times\{f\}} \simeq (j_f)_!\mathbf{C} & \simeq & \mathbf{R}\sigma_*(\Omega^\bullet_Z(\log \tilde{X}_f)(-\tilde{X}_f)) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \Omega^{<\kappa}_{(Y,X)} & \to & \Omega^{<\kappa}_{(Y,X)}|_{\mathbf{P}^n\times\{f\}} & \to & \mathbf{R}\sigma_*(\Omega^{<\kappa}_Z(\log \tilde{X}_f)(-\tilde{X}_f)) \,. \end{array}$$

Hence there is a commutative diagram

$$\begin{array}{ccc} (\mathbf{R}^i(p_2)_*\Omega^\bullet_{(Y,X)})|_{\{f\}} & \overset{\cong}{\to} & H^i_c(U_f) \\ \downarrow & & \downarrow \\ (\mathbf{R}^i(p_2)_*\Omega^{<\kappa}_{(Y,X)})|_{\{f\}} & \to & H^i_c(U_f)/F^\kappa H^i_c(U_f) \,. \end{array}$$

□

Now in order to prove the theorem, we want to show that $R^i(p_2)_*\Omega^j_{(Y,X)} = 0$ for all $i \geqq 0$, $0 \leqq j < \kappa$. $\Omega^i_{(Y,X)}$ is filtered by $\Omega^j_{(Y,X)/\mathbf{P}^n} \otimes p_1^*\Omega^{i-j}_{\mathbf{P}^n}$. Since $X = \operatorname{Spec} \operatorname{Sym}(\mathscr{R})$, we have an exact sequence

$$0 \to \Omega^s_{(Y,X)/\mathbf{P}^n} \to \bigwedge^s H^0(\mathbf{P}^n, \mathscr{E})^\vee \otimes \mathcal{O}_Y \to p_1^*\mathscr{R}^\vee \otimes \mathcal{O}_X \to 0 \,.$$

Using the Koszul resolution for $\mathscr{I}$, the ideal sheaf of X, and the exact sequence

$$0 \to \mathscr{R} \to H^0(\mathbf{P}^n, \mathscr{E}) \otimes \mathcal{O}_{\mathbf{P}^n} \to \mathscr{E} \to 0,$$

one again reduces (as in [N, Sect. 3]) the desired vanishing statement to that for suitable direct sums of suitable $\mathcal{O}_{\mathbf{P}^n}(k)$.

2 Pencil

2.1

We consider the following commutative diagram, inspired by [T]:

$$\begin{array}{ccccccc} \mathbf{P}_S(\mathscr{E}|_S) & \hookrightarrow & Y & \hookrightarrow & Q & \hookleftarrow & V \\ & \searrow & & \searrow & \downarrow \pi & & \downarrow \\ & & S & \hookrightarrow & \mathbf{P}^n & \hookleftarrow & U \end{array}$$

where Q is the projective bundle $\mathbf{P}_{\mathbf{P}^n}(\mathscr{E})$, $\mathscr{E} = \bigoplus_{i=1}^r \mathcal{O}_{\mathbf{P}^n}(d_i)$, S is the subvariety of $\mathbf{P}^n$ defined by the $f_i \in H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d_i))$, $U = \mathbf{P}^n - S$, Y is the hypersurface in Q defined by the section $f = (f_1, \ldots, f_r) \in H^0(\mathbf{P}^n, \mathscr{E}) = H^0(\mathbf{P}^n, \pi_* \mathcal{O}_Q(1)) = H^0(Q, \mathcal{O}_Q(1))$, and finally $V = Q - Y$. Clearly $\pi^{-1}(S) = \mathbf{P}_S(\mathscr{E}|_S) \subset Y$.

Since $\mathcal{O}_Q(Y) = \mathcal{O}_Q(1)$, $\pi \colon V \to U$ is an affine bundle with fibres $\mathbf{A}^{r-1}$ and one has

$$F^a H^i_c(U) = F^{r-1+a} H_c^{2(r-1)+i}(V) \; \forall i, a .$$

So one has to prove that the Hodge type of Y is $\geqq \kappa' = \kappa + r - 1$.

2.2

According to [E, (1.2) and (1.3)], it is enough to compute the following.

Lemma 2.2.1. $H^i(Q, \mathcal{O}_Q(-(\kappa' - j)) \otimes \Omega^j_Q) = 0 \; \forall\, i, \forall\, 0 \leqq j \leqq \kappa' - 1.$

Proof. Ω^j_Q is resolved by sheaves $\pi^* \Omega^k_{\mathbf{P}^n} \otimes \Omega^{j-k}_{Q/\mathbf{P}^n}$, $0 \leqq k \leqq j$, and $\pi^* \Omega^k_{\mathbf{P}^n}$ is resolved by direct sums of $\pi^* \mathcal{O}_{\mathbf{P}^n}(-l)$, $0 \leqq l \leqq k$. Finally $\Omega^{j-k}_{Q/\mathbf{P}^n}$ is resolved by sheaves $(\pi^* \bigwedge^m \mathscr{E}) \otimes \mathcal{O}_Q(-m)$, $0 \leqq m \leqq j - k$. Hence one has to compute

$$H^i(\mathbf{P}^n, (\bigwedge^m \mathscr{E}(-l)) \otimes R\pi_* \mathcal{O}_Q(-(\kappa' - j + m))$$

for

$$0 \leqq l \leqq k \leqq \kappa' - 1, \quad 0 \leqq m \leqq j - k .$$

One has *a priori* contributions only for $\kappa' - j + m \geqq r$, and in this case, duality reduces the computation to the vanishing of

$$H^i(\mathbf{P}^n, (\bigwedge^m \mathscr{E}(-l)) \otimes (S^{m+\kappa'-j-r} \mathscr{E})^\vee \otimes (\det \mathscr{E})^{-1}),$$

which is true if for any

$$j_1 > j_2 > \cdots > j_l, \quad i_1 \geqq i_2 \geqq \cdots \geqq i_{m+\kappa'-j-r}$$

one has

$$1 \leqq l - (d_{j_1} + \cdots + d_{j_m}) + (d_{i_1} + \cdots + d_{i_{m+\kappa'-j-r}}) + (d_1 + \cdots + d_r) \leqq n .$$

The left inequality is obviously valid, whereas the right one reads

$$l + (d_1 + \cdots + d_{r-m}) + (m - \kappa - 1 - j)d_1 \leqq n$$

or

$$\kappa d_1 + (m - j)d - 1 + d_2 + \cdots + d_{r-m} + l \leqq n$$

which, from the extreme case $m = j = l = 0$ gives

$$\kappa = \left[\frac{n - \sum_{i=2}^{r} d_i}{d_1}\right].$$

□

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