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Characteristic divisors on complex manifolds

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Introduction

If M is a 4-dimensional, C^∞, oriented, closed manifold, and $W \subset M$ an oriented 2-submanifold, then Rochlin in [10] defined W to be *characteristic* if its integral homology class in M represents the Poincaré dual of a cohomology class whose reduction mod 2 is the 2^{nd} Stiefel-Whitney class of M. In this situation, Rochlin proved a remarkable theorem:

If $\sigma(M)$ is the signature of M (see [4]*) and W^2 is the self-intersection number of W in M, then*

$$\sigma(M) - W^2 \equiv 8\,\mathrm{Arf}(W) \mod 16$$

where $\mathrm{Arf}(W) \in \{0, 1\}$ *is a (topologically defined)* mod 2*-invariant associated to W and M.*

When both manifolds M and W are complex analytic, all terms involved in Rochlin's theorem have geometric interpretations. In fact, Hirzebruch's signature theorem [4], together with the Hirzebruch-Riemann-Roch formula [2] imply

$$\sigma(M) - W^2 = -8\chi(M, \mathscr{D})$$

where $\mathscr{D}$ is a holomorphic line bundle over M such that W is a divisor of $\mathscr{L} = \omega_M \otimes \mathscr{D}^{-2}$, ω_M is the canonical bundle of M and $\chi(M, \mathscr{D})$ is the Euler-Poincaré characteristic of $\mathscr{D}$. As we note in § 1 below, the restriction to W of the bundle $\mathscr{D}$ is a holomorphic square root of the canonical (or dualizing) sheaf of W. In this situation, Atiyah and Rees [3] (also [1], [7]) introduced a mod 2-invariant of W and $\mathscr{D}$, the mod 2-index of the $\bar{\partial}$-operator acting on differential forms on W with coefficients in $\mathscr{D}|_W$:

$$\mathfrak{h}(W, \mathscr{D}) = \dim H^0(W, \mathscr{D}|_W) \mod 2.$$

By [5], or [12], Prop. 1.6, $\mathfrak{h}(W, \mathscr{D})$ is the invariant $\mathrm{Arf}(W)$ in Rochlin's theorem. Hence, Rochlin's theorem for a complex surface M with a characteristic complex submanifold W reads

$$\mathfrak{h}(W, \mathscr{D}) = \chi(M, \mathscr{D}) \mod 2.$$

In this work, following Rochlin's definition of characteristic submanifolds, we introduce the notion of a *characteristic divisor on a complex manifold* (cf. [9], [12]). This means an effective divisor W in a complex (possibly non-compact) manifold M, such that $|W|$ is compact, and W is of the form $W = 2D - K$, where K is a divisor of ω_M, the canonical sheaf, D is some divisor in M and $|K|$, $|D|$ are both compact. When $n = \dim(M)$ is of the form $n = 4k + 2$, we can follow [3] and define *the* mod 2-*index* of the characteristic divisor W:

$$\mathfrak{h}(W, \mathscr{D}) = \sum_{i=0}^{2k} h^{2i}(W, \mathscr{D}|_W) \mod 2.$$

We prove in § 2 below that if M is compact of $\mathbb{C}$-dimension $4k + 2$, then (Theorem 2.1),

$$\mathfrak{h}(W, \mathscr{D}) = \chi(M, \mathscr{D}) \mod 2$$

generalizing Rochlin's theorem to higher dimensions and to possibly singular and non-reduced, reducible divisors (see [9]). When W is non-singular, this theorem is in fact a direct application of Theorem 7.4 in [3]. The proof is based on some interpretation of Serre's duality.

In § 3 we apply the method of § 2 to desingularizations of isolated Gorenstein singularities, to obtain formulae relating the mod 2-index of a characteristic divisor, to the genus of the singularity, the difference between these two being a "Riemann-Roch difference" (see 3.5). Finally in § 4 we apply the results of § 3 to the surface case. If X is a Gorenstein surface with an isolated singular point P, π: $Y \to X$ is a desingularization, D and W are divisors supported in $\pi^{-1}(P)$, and $W \geqq 0$ is characteristic with $\mathcal{O}_Y(W) \otimes \omega_Y = \mathcal{O}_Y(2D)$, then

$$h^0(W, \mathcal{O}_W(D|_W)) + \frac{W^2 - K^2}{8} \equiv p_g \mod 2$$

where p_g is the genus and K is a divisor of ω_Y with support in $\pi^{-1}(P)$. In particular, if K is even and if we take $W = 0$, we obtain

$$\frac{K^2}{8} \equiv p_g \mod 2$$

thus, if K is divisible by 4, p_g is even. Moreover, if $K \leqq 0$, for example when the desingularization Y is minimal, then these congruences become equalities.

Previous proofs of the congruences above for surfaces were given in [6], [11], [12], but those proofs assumed (X, P) to be smoothable and W non-singular. It was hoped [6] that the non-validity of those congruences could be an obstruction to smoothability. Alas, since we show them to hold in general, this is not the case (see 4.2. b) and c)). B. Steer has obtained 4.1 for certain homogeneous singularities.

This note grew out of a workshop at CIMAT, in Guanajuato, where the second named author explained the methods contained in § 1 and § 2, and it was written up at Instituto de Matemáticas, UNAM, in Mexico City. The first and last authors would like to thank the mexican mathematicians for their hospitality. The second author is grateful to N. J. Hitchin for fruitful discussions while beginning this work, and he thanks the ICTP, at Trieste, Italy, for hospitality and support while working on it.

§ 1. Characteristic divisors

Let M be a complex space of pure complex dimension n, with a canonical (dualizing) sheaf ω_M.

1.1. Definition. A *theta characteristic* on M is (the isomorphism class of) a holomorphic line bundle $\mathscr{L}$ on M with $\mathscr{L}^2 \cong \omega_M$, i.e. a holomorphic square root of ω_M. When M admits a theta characteristic we say that M is *spin* (see [1]).

It is clear that if $K = \sum n_i E_i$ is a divisor of ω_M, then M admits a theta characteristic if all n_i's are even.

1.2. Definition. Let M be a (non-singular) complex n-manifold, and let $W \geqq 0$ be a divisor on M, with $|W|$ compact (or empty). W is a *characteristic divisor* if it can be expressed as $W = 2D - K$, where D is some divisor on M and K is a divisor of ω_M.

1.3. Lemma. *If W is a characteristic divisor of M then*:

i) *The line bundle $\mathscr{L}$ associated to W is of the form*

$$\mathscr{L} \cong \omega_M^{-1} \otimes \mathscr{D}^2 ,$$

where $\mathscr{D}$ is the bundle of D.

ii) *If $W \neq 0$, then the restriction of $\mathscr{D}$ to W is a theta characteristic on W. That is,*

$$\mathscr{D}^2|_W \cong \omega_W$$

where ω_W is the canonical sheaf of W.

Proof. Statement i) is immediate from the definitin of characteristic divisors. To prove ii) we recall that the dualizing sheaf always exists for (non-reduced, reducible) effective divisors on complex manifolds. Adjunction formula says:

$$\omega_W \cong \omega_M|_W \otimes \mathcal{O}_W(W)$$

where $\mathcal{O}_W(W) \cong \mathcal{L}|_W$ is the normal sheaf. Hence,

$$\omega_W \cong (\omega_M \otimes \mathcal{L})|_W \cong \mathcal{D}^2|_W$$

as stated.

From now on, we assume that the dimension n of M is of the form $n = 4k+2$. Following [3], [7], let us define a mod 2 invariant as follows:

1.4. Definition. The mod 2-*index* of the characteristic divisor W is

$$\mathfrak{h}(W, \mathcal{D}) = \sum_{i=0}^{2k} \dim H^i(W, \mathcal{D}) \quad \mathrm{mod}\, 2$$

if $W \neq 0$. If $W = 0$, then $\mathfrak{h}(0, \mathcal{D}) = 0$.

1.5. Remark (cf. [1], [5]). If the divisor W is non-singular, then the theta-characteristics on W are classified by $H^1(W; \mathbb{Z}/2)$. Thus, for example, if $k = 0$, so that W is a compact Riemann surface of genus g in M, then W has 2^{2g} non-equivalent theta characteristics and, generally speaking, there is not a preferred one. However, 1.3. ii) tells us that whenever $\mathcal{O}_M(W) = \mathcal{D}^2 \otimes \omega_M^{-1}$, then $\mathcal{D}$ gives a fixed theta characteristic on W.

§ 2. The compact case

In this section we prove the following theorem:

2.1. Theorem. *Let M be a compact, complex manifold of $\mathbb{C}$-dimension $n = 4k+2$, for some $k \geqq 0$. If W is a characteristic divisor of M, and if $\mathcal{D}$ is associated to W as in* § 1, *then,*

$$\mathfrak{h}(W, \mathcal{D}) \equiv \chi(M, \mathcal{D}) \quad \mathrm{mod}\, 2$$

where $\chi(M, \mathcal{D}) = \sum_{i=0}^{4k+2} (-1)^i h^i(M, \mathcal{D})$ is the analytic Euler-Poincaré characteristic of M with coefficients in $\mathcal{D}$.

In particular, if M is spin and one takes $W = 0$, one has $\chi(M, \mathcal{D}) = 0 \bmod 2$. This is Atiyah and Hirzebruch's theorem [4].

Proof. We assume first that $W \neq 0$. Consider the exact sequence of sheaves over M,

$$0 \to \omega_M \otimes \mathcal{D}^{-1} \xrightarrow{s^*} \mathcal{D} \xrightarrow{r} \mathcal{D}|_W \to 0$$

where s^* is multiplication by the section s of $\mathcal{L}$ that defines W and r is the restriction to W, with associated long exact cohomology sequence

$$\dots \to H^{2k}(W; \mathcal{D}|_W) \xrightarrow{\alpha} H^{2k+1}(M; \omega_M \otimes \mathcal{D}^{-1}) \xrightarrow{\beta} H^{2k+1}(M; \mathcal{D}) \to \dots .$$

By exactness we get

$$(1)\qquad \sum_{i=0}^{2k}\left[h^i(M;\omega_M\otimes\mathscr{D}^{-1})+h^i(M;\mathscr{D})+h^i(W;\mathscr{D}|_W)\right]\equiv \dim\operatorname{Im}(\alpha)\mod 2.$$

By Serre's duality, one has

$$H^i(M;\omega_M\otimes\mathscr{D}^{-1})\cong H^{4k+2-i}(M;\mathscr{D})\quad\text{for } i=0,\dots,2k$$

and

$$H^{2i+1}(W;\mathscr{D}|_W)\cong H^{4k-2i}(W;\mathscr{D}|_W)\quad\text{for } i=0,\dots,2k-1.$$

Thus, from (1) we obtain

$$\mathfrak{h}(W,\mathscr{D})+\chi(M;\mathscr{D})\equiv h^{2k+1}(M;\mathscr{D})+\dim\operatorname{Im}(\alpha)\mod 2$$

and the theorem will be proved if we show

$$\dim\ker(\beta)\equiv h^{2k+1}(M;\mathscr{D})\mod 2.$$

By Serre's duality, β is a bilinear form: it is defined by the cup product

$$H^{2k+1}(M;\omega_M\otimes\mathscr{D}^{-1})\times H^{2k+1}(M;\omega_M\otimes\mathscr{D}^{-1})\to H^{4k+2}(M;\omega_M^2\otimes\mathscr{D}^{-2})$$

followed by the maps

$$H^{4k+2}(M;\omega_M^2\otimes\mathscr{D}^{-2})\xrightarrow{s^*}H^{4k+2}(M;\omega_M)\xrightarrow{\operatorname{tr}}\mathbb{C}.$$

As $(2k+1)$ is odd, the cup product is skew, hence β can be written as a matrix of the form

$$\begin{pmatrix} 0 & b_1 & & & \\ -b_1 & 0 & & & \\ & & 0 & b_2 & \\ & & -b_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

and the result follows.

For completeness, let us give the proof for $W=0$ as well. By Serre's duality,

$$h^i(M,\mathscr{D})=h^{4k+2-i}(M,\omega_M\otimes\mathscr{D}^{-1}).$$

Since $\mathscr{D}=\omega_M\otimes\mathscr{D}^{-1}$, one has

$$\chi(M,\mathscr{D})-h^{2k+1}(M,\mathscr{D})\equiv 0\mod 2.$$

The cup product on $H^{2k+1}(M, \mathscr{D})$ is a non-degenerate skew-symmetric form, and hence $h^{2k+1}(M, \mathscr{D})$ is even.

2.2. Corollary. *The invariant $\mathfrak{h}(W, \mathscr{D})$ is stable under deformations. To be precise, if (M_t, W_t) is a continuous family of compact, complex surfaces M_t with characteristic divisors W_t, then $\mathfrak{h}(W_t, \mathscr{D}_t)$ is constant.*

Proof. By 2.1, for each t,

$$\mathfrak{h}(W_t, \mathscr{D}_t|_W) \equiv \chi(M_t, \mathscr{D}_t) \mod 2 .$$

Hence, by the formula of Hirzebruch-Riemann-Roch [2], $\mathfrak{h}(W_t, \mathscr{D}_t)$ is determined by the Chern class of $\mathscr{D}_t$ and the Chern classes of M_t, and all of these classes are invariant under deformations.

If the deformation in 2.2 is flat algebraic, 2.2 follows from the invariance of the Euler-Poincaré characteristic.

When $k = 0$, Riemann-Roch for surfaces says

$$\begin{aligned}\chi(M, \mathscr{D}) &= \chi(M, \mathcal{O}) + \frac{1}{2}(D^2 - D \cdot K)\\ &= \chi(M, \mathcal{O}) + \frac{1}{8}(W^2 - K^2) .\end{aligned}$$

Therefore,

2.3. Corollary. *Let M be a compact, complex surface and let W be a characteristic divisor on M associated to a bundle $\mathscr{L}$ of the form $\mathscr{L} = \omega_M^{-1} \otimes \mathscr{D}^2$. Then,*

$$\mathfrak{h}(W, \mathscr{D}) \equiv \chi(M, \mathcal{O}) + \frac{1}{8}(W^2 - K^2) \mod 2$$

where K is a canonical divisor.

2.4. Remarks. a) W being given, $\mathscr{D}$ as an element in Pic M is well defined modulo the 2-torsion in Pic M. However, by Hirzebruch-Riemann-Roch, $\chi(M, \mathscr{D})$ depends only on the numerical equivalence class of $\mathscr{D}$. Therefore the invariant $\mathfrak{h}(W, \mathscr{D})$ does not depend on the choice of $\mathscr{D}$ in its class modulo 2-torsion. So we write it $\mathfrak{h}(\mathrm{W})$.

b) A complex manifold M is canonically spin^c, and different spin^c structures on M correspond to line bundles of the form $\mathscr{L} = -\omega_M \otimes \mathscr{D}^2$. Spin and spin^c manifolds have an associated Dirac operator. From this point of view, Theorem 2.1 says that if the $\mathbb{C}$-dimension of M is $n = 4k + 2$, and if W is non-singular, then the index of the Dirac operator on M (with respect to the spin^c structure determined by W) modulo 2 equals the mod 2-index of the Dirac operator on W for the spin structure determined by $\mathscr{D}|_W$, where $\mathscr{D} \otimes \mathscr{D} = \mathcal{O}_M(W) \otimes \omega_M$. It would be interesting to know whether this theorem is true in general for spin^c manifolds of real dimension $8k+4$. This would have some applications. For instance when M is a complex surface and W is singular, Rochlin's theorem and 2.1 allow us to define

$$\mathfrak{h}(W) = \mathrm{Arf}(W_t) \mod 2$$

where W_t is any C^∞ smoothing of W. A similar definition of $\mathfrak{h}(W)$ in higher dimensions would follow from the C^∞ version of 2.1.

c) Given a compact, complex manifold X of dimension $4k+1$ with a theta characteristic $\mathscr{D}$, i.e. $\mathscr{D}^2 = \omega_X$, Atiyah and Rees [3] (also [7]) showed that the mod 2-invariant

$$\mathfrak{h}(X, \mathscr{D}) = \sum_{i=0}^{2k} h^{2i}(X, \mathscr{D}) \mod 2$$

is stable under deformations (see [3], 4.1). The invariant $\mathfrak{h}(W, \mathscr{D})$ of 2.1 above provides a similar invariant for manifolds of dimension $4k+2$, and Corollary 2.2 says that this is also stable under deformations.

d) Let us mention some special cases of 2.3 for surfaces.

i) If $K < 0$, and $W = -K$, so that $D = 0$, we obtain

$$h^0(-K, \mathcal{O}) \equiv \chi(M, \mathcal{O}) \mod 2.$$

ii) If $K > 0$, and $W = K = D$, then

$$h^0(K, \omega_M|_K) \equiv \chi(M, \mathcal{O}) \mod 2$$

because $\chi(M, \omega_M) = \chi(M, \mathcal{O})$.

iii) If W is non-singular of genus 0, then W has a unique theta characteristic and $\mathfrak{h}(W) = 0$. Hence

$$\chi(M, \mathcal{O}) \equiv \frac{1}{8}(\mathrm{W}^2 - K^2) \mod 2.$$

iv) If M is spin, then $\frac{1}{8}K^2$ is an integer congruent to $\chi(M, \mathcal{O})$ mod 2.

§ 3. Normal isolated Gorenstein singularities

In this section we apply 2.1 to isolated normal Gorenstein singularities in order to obtain 3.5.

Even though the necessary evaluation of the Hirzebruch-Riemann-Roch theorem does not give a formula as nice as in the surface case, we will allow higher dimensions in this section.

Let $\pi\colon Y \to X$ be a desingularization of an isolated complex, normal Gorenstein singularity (X, P). Let D be any divisor on Y with $\pi(|D|) = P$, or, as we will say in the sequel, a *vertical* divisor. If $\bar{Y}$ is a smooth compactification of Y we write, for $a \in \mathbb{Z}$,

$$g(a \cdot D) = \chi(\mathcal{O}_{\bar{Y}}(a \cdot D)) - \chi(\mathcal{O}_{\bar{Y}}),$$

the difference of the Euler-Poincaré characteristics of $\mathcal{O}_{\bar{Y}}(a \cdot D)$ and $\mathcal{O}_{\bar{Y}}$.

3.1. Properties of $g(a \cdot D)$. a) *If $D = D_1 - D_2$ for effective divisors D_1 and D_2, then*

$$g(a \cdot D) = \chi(\mathcal{O}_{a \cdot D_1}(a \cdot D)) - \chi(\mathcal{O}_{a \cdot D_2}).$$

b) *$g(a \cdot D)$ is independent of the compactification $\bar{Y}$ chosen.*

c) *$g(a \cdot D)$ is a polynomial of degree at most* $\dim X$ *in a with rational coefficients.*

d) *If Y is a surface and K_Y is the canonical divisor of Y, then* $g(a \cdot D) = \frac{1}{2} a \cdot D \cdot (a \cdot D - K_Y)$.

Proof. a) is just the additivity of the Euler-Poincaré characteristic for short exact sequences. b) follows from a), and c) and d) follow from the Hirzebruch-Riemann-Roch theorem.

3.2. Remark. We will use several well known relative vanishing theorems, which easily follow from the classical ones. Let ω_Y be the canonical sheaf of Y, and let $D \geqq 0$ be a vertical divisor. Assume that either $-D$ is relatively ample for $\pi\colon Y \to X$ or that $D = 0$. Then

a) $$R^i \pi_* \omega_Y(-D) = 0 \quad \text{for } i > 0.$$

b) $$R^i \pi_* \mathcal{O}_Y(D) = 0 \quad \text{for } 0 < i < \dim X - 1.$$

Let us sketch the proof of b): We choose non-singular projective compactifications $\bar{X}$ and $\bar{Y}$ such that $\bar{\pi}\colon \bar{Y} \to \bar{X}$ is an isomorphism outside of $\bar{\pi}^{-1}(P)$. Let $\mathcal{A}$ be an ample sheaf on $\bar{X}$. By Serre's vanishing theorem and Serre-Grothendieck's duality we can find some v_0 such that for $v \gg v_0$ and $i < n = \dim X$ we have

(*) $$H^i(\bar{X}, \mathcal{A}^{-v}) = H^{n-i}(\bar{X}, \mathcal{A}^v \otimes \omega_{\bar{X}}) = 0.$$

Moreover, if $D \neq 0$, we can choose v such that $\pi * \mathcal{A}^v \otimes \mathcal{O}_{\bar{Y}}(-D)$ is ample on $\bar{Y}$. By Kodaira (or Grauert-Riemenschneider if $D = 0$) vanishing one has

(**) $$H^k(\bar{Y}, \bar{\pi} * \mathcal{A}^{-v} \otimes \mathcal{O}_{\bar{Y}}(D)) = 0 \quad \text{for } k < n.$$

One has the Leray spectral sequence

$$E_2^{p,q} = H^p(\bar{X}, R^q \bar{\pi}_* \mathcal{O}_{\bar{Y}}(D) \otimes \mathcal{A}^{-\nu}) \Rightarrow H^{p+q}(\bar{Y}, \bar{\pi}_* \mathcal{A}^{-\nu} \otimes \mathcal{O}_{\bar{Y}}(D))$$

and (**) implies that $E_\infty^{0,q} = 0$ for $q < n$. $E_{s+1}^{0,q}$ is the kernel of $E_s^{0,q} \to E_s^{s,q-s+1}$ for $s \geqq 2$. However, $E_s^{s,q-s+1}$ as a subquotient of $E_2^{s,q-s+1}$ can only be non-zero for $s = n$ and $q - s + 1 = 0$ by (*). Hence, as long as $q < n - 1$ we obtain that $E_{s+1}^{0,q} = E_s^{0,q}$ for any $s \geqq 2$, and therefore that $E_\infty^{0,q} = E_2^{0,q}$.

3.3. Assumptions. We assume that $\dim X = n = 4k + 2$ for some $k \in \mathbb{N}$. As (X, P) is Gorenstein, ω_Y may be written as $\omega_Y = \mathcal{O}_Y(K_Y)$ for some vertical divisor K_Y, at least if we replace X by some affine neighbourhood of P. Let D be a vertical divisor such that $W = 2D - K_Y \geqq 0$. Let us consider the invariant

$$\mathfrak{h}(W) = \sum_{i=0}^{2k} (-1)^i h^i(W, \mathcal{O}_Y(D)|_W), \quad \text{if } W > 0,$$

and $\mathfrak{h}(W) = 0$ if $W = 0$. This invariant reduced modulo 2 is the invariant $\mathfrak{h}(W, \mathcal{O}_Y(D))$ introduced in § 1.

Finally recall that

$$p_g = \dim R^{n-1} \pi_* \mathcal{O}_Y$$

is the *genus* of the singularity (X, P).

3.4. Lemma. *For any effective vertical divisor Δ we have*

$$\mathfrak{h}(W) + g(D) \equiv \mathfrak{h}(W + 2\Delta) + g(D + \Delta) \mod 2.$$

Proof. It is clear that $W' = W + 2\Delta$ and $D' = D + \Delta$ satisfy the assumptions made in 3.3. Let $\bar{Y}$ be a non-singular compactification of Y and $\bar{\mathcal{D}}$ be an extension of $\mathcal{O}_Y(D)$ to an invertible sheaf on $\bar{Y}$. For some divisor Γ_∞, supported in $\bar{Y} - Y$, we have

$$\omega_{\bar{Y}} \otimes \mathcal{O}_{\bar{Y}}(W + \Gamma_\infty) = \bar{\mathcal{D}}^2.$$

Choosing $\bar{\mathcal{D}}$ large enough, we may assume Γ_∞ to be effective. The pairs $(W + \Gamma_\infty, \bar{\mathcal{D}})$ and $(W' + \Gamma_\infty, \bar{\mathcal{D}}' = \bar{\mathcal{D}} \otimes \mathcal{O}_{\bar{Y}}(\Delta))$ satisfy the assumptions of (2.1) and hence

$$\mathfrak{h}(W + \Gamma_\infty) - \mathfrak{h}(W' + \Gamma_\infty) \equiv \chi(\bar{Y}, \bar{\mathcal{D}}) - \chi(\bar{Y}, \bar{\mathcal{D}}') \mod 2.$$

As Γ_∞ does not meet the support of W or W', one has

$$\bar{\mathcal{D}}|_{W + \Gamma_\infty} = \mathcal{O}_Y(D)|_W \oplus \bar{\mathcal{D}}|_{\Gamma_\infty}$$

and

$$\bar{\mathscr{D}}'|_{W'+\Gamma_\infty} = \mathcal{O}_Y(D')|_{W'} \oplus \bar{\mathscr{D}}'|_{\Gamma_\infty}$$
$$= \mathcal{O}_Y(D')|_{W'} \oplus \bar{\mathscr{D}}|_{\Gamma_\infty}$$

and therefore

$$\mathfrak{h}(W) - \mathfrak{h}(W') = \mathfrak{h}(W + \Gamma_\infty) - \mathfrak{h}(W' + \Gamma_\infty).$$

On the other hand, we have exact sequences

$$0 \to \bar{\mathscr{D}} \xrightarrow{\Delta} \bar{\mathscr{D}}' \to \mathcal{O}_{\bar{Y}}(D')|_\Delta \to 0$$

and

$$0 \to \mathcal{O}_{\bar{Y}}(D) \to \mathcal{O}_{\bar{Y}}(D') \to \mathcal{O}_{\bar{Y}}(D')|_\Delta \to 0.$$

Therefore

$$\chi(\bar{Y}, \bar{\mathscr{D}}) - \chi(\bar{Y}, \bar{\mathscr{D}}') = \chi(\bar{Y}, \mathcal{O}_{\bar{Y}}(D)) - \chi(\bar{Y}, \mathcal{O}_{\bar{Y}}(D'))$$
$$= g(D) - g(D').$$

3.5. Theorem. *Keeping the assumptions made in* 3.3 *we have*:

a) $$\mathfrak{h}(W) + g(D) \equiv p_g \mod 2.$$

b) *If* $D = 0$, *or if* $D > 0$ *and* $-D$ *is relatively ample for* π, *then*

$$\mathfrak{h}(W) + g(D) = p_g.$$

Proof. Since we can always choose some effective vertical divisor Δ such that $D + \Delta > 0$ and $-(D + \Delta)$ is relatively ample for π, part a) of 3.5 follows from part b) and 3.4.

To prove b), let assume first that $W > 0$. As

$$\mathcal{O}_Y(D)|_W = \omega_W \otimes \mathcal{O}_Y(-D)|_W,$$

Serre duality says

$$\mathfrak{h}(W) = \sum_{i=2k+1}^{4k+1} (-1)^{i+1} h^i(W, \mathcal{O}_Y(D)|_W).$$

Under the assumptions made in b), we can apply 3.2. a) and b) to obtain

$$R^i\pi_* \omega_Y(-D) = 0 \quad \text{for } 0 < i \leqq 4k+1$$

and

$$R^i \pi_* \mathcal{O}_Y(D) = 0 \quad \text{for } 0 < i < 4k+1 .$$

Moreover, from the exact sequence

$$0 \to \omega_Y(-D) \to \mathcal{O}_Y(D) \to \mathcal{O}_Y(D)|_W \to 0$$

we deduce $H^i(W, \mathcal{O}_Y(D)|_W) = 0$ for $0 < i < 4k+1$ and

$$R^{4k+1} \pi_* \mathcal{O}_Y(D) = H^{4k+1}(W, \mathcal{O}_Y(D)|_W) .$$

Hence $\mathfrak{h}(W) = h^{4k+1}(W, \mathcal{O}_Y(D)|_W)$ and therefore

$$\mathfrak{h}(W) = \dim R^{4k+1} \pi_* \mathcal{O}_Y(D) .$$

The last equality holds true as well for $W = 0$. In fact, the left hand side is zero by definition and since $\mathcal{O}_Y(D) = \omega_Y(-D)$ in this case, the right hand side is zero by 3.2. a).

If $D = 0$, $g(D) = 0$ by definition and we are done. If $D > 0$, the exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(D) \to \mathcal{O}_D(D) \to 0$$

and 3.2. b) tell us that

$$H^0(D, \mathcal{O}_D(D)) = \operatorname{coker}(\pi_* \mathcal{O}_Y \to \pi_* \mathcal{O}_Y(D)) = 0 ,$$

that

$$H^i(D, \mathcal{O}_D(D)) = 0 \quad \text{for } 0 < i < 4k ,$$

and that

$$0 \to H^{4k}(D, \mathcal{O}_D(D)) \to R^{4k+1} \pi_* \mathcal{O}_Y \to R^{4k+1} \pi_* \mathcal{O}_Y(D) \to H^{4k+1}(D, \mathcal{O}_D(D)) \to 0$$

is exact. All together one obtains

$$\begin{aligned} \mathfrak{h}(W) &= \dim R^{4k+1} \pi_* \mathcal{O}_Y(D) \\ &= p_g - h^{4k}(D, \mathcal{O}_D(D)) + h^{4k+1}(D, \mathcal{O}_D(D)) \\ &= p_g - \chi(D, \mathcal{O}_D(D)) \end{aligned}$$

and by 3.1. a) $\chi(D, \mathcal{O}_D(D)) = g(D)$, proving Theorem 3.5.

3.6. Remarks. a) If K_Y is divisible by 2 (see 1.1), one can choose $W = 0$ in 3.5 and one obtains for $D = \frac{1}{2} K_Y$

$$g(D) \equiv p_g \mod 2.$$

b) If $K_Y \leqq 0$ one gets $\mathfrak{h}(-K_Y) = p_g$.

§ 4. Application to surface singularities

Let us assume from now on that (X, P) is a two-dimensional normal Gorenstein singularity and let us choose D and W as in 3.3. Then, since

$$\frac{1}{2} D(D - K_Y) = \frac{1}{8}(W + K_Y)(W - K_Y)$$

we obtain from 3.5 and 3.6, using 3.1. d):

4.1. Theorem. *Under the above assumptions one has:*

a) $\mathfrak{h}(W) + \frac{1}{8}(W^2 - K_Y^2)$ *is an integer congruent to* p_g mod 2.

b) *If* $D = 0$ *or* $D > 0$ *and* $-D$ *relatively ample for* π, *then*

$$\mathfrak{h}(W) + \frac{1}{8}(W^2 - K_Y^2) = p_g.$$

c) *If* $K_Y \leqq 0$, *for example when* Y *is a minimal resolution of* (X, P), *then for all* $D \geqq 0$, $W = 2D - K_Y$,

$$\mathfrak{h}(W) + \frac{1}{8}(W^2 - K_Y^2) = p_g.$$

d) *In particular, if* $K_Y \leqq 0$, *then*

$$\mathfrak{h}(-K_Y) = p_g.$$

e) *If* K_Y *is divisible by* 2, *then* $\frac{1}{8} K_Y^2$ *is an integer congruent to* p_g mod 2.

Proof. The only part which is more than a translation of 3.5 or 3.6 is part c). Since $K_Y \leqq 0$, $D \geqq 0$ imply that $\omega_Y(-D) \to \mathcal{O}_Y(D)$ factorizes through

$$\omega_Y(D) \to \omega_Y \to \omega_Y(-K_Y + D) = \mathcal{O}_Y(D).$$

Hence, by 3.2. a) for $D = 0$,

$$R^1 \pi_* \omega_Y(-D) \to R^1 \pi_* \mathcal{O}_Y(D)$$

is the zero map. This is enough to deduce

$$R^1 \pi_* \mathcal{O}_Y(D) = H^1(W, \mathcal{O}_Y(D)|_W)$$

and without the assumption "$-D$ relatively ample" one can deduce c) from this and from the arguments in the second half of the proof of 3.5. b).

4.2. Remarks. If K_Y is not divisible by 2, the most natural choice for W is $W_0 = \sum E_i$ where the sum is taken over all components of $\pi^{-1}(P)$ which have odd multiplicity in K_Y. With this choice, assuming that $E = (\pi^{-1}(P))_{\text{red}}$ is a normal crossings divisor and $b^1(E) = \dim H^1(E; \mathbb{Z}) = 0$, one gets

$$\frac{1}{8}(W_0^2 - K_Y^2) \equiv p_g \mod 2.$$

In fact, $b^1(E) = 0$ implies that $H^0(E, \omega_E) = 0$ and hence that $H^0(W_0, \omega_{W_0}) = 0$. Since $\mathcal{O}_{W_0}(D)^2 = \omega_{W_0}$, one gets $H^0(W_0, \mathcal{O}_{W_0}(D)) = 0$.

This remark, together with the simple combinatorial argument contained in [6] imply theorem 2 in that paper without the assumption that the singularity is smoothable.

b) If K_Y is divisible by 4, then $p_g \equiv 0 \mod 2$.

c) In [6] the authors present four examples of singularities with K_Y even, for which the computer gave $p_g \not\equiv \frac{1}{8} K_Y^2 \mod 2$, in contradiction with 4.1. e).

4.3. A Gorenstein surface singularity (X, P) has an associated e-invariant $e(X, P) \in (1/24) \cdot \mathbb{Z}$, mod $\mathbb{Z}$, see [11]. If

$$\mathcal{F} : \mathcal{X} \to \Delta \subset \mathbb{C}$$

is a flat morphism from an analytic space $\mathcal{X}$ onto the open unit disc Δ with $(X, P) \cong (X_0, P_0) = \mathcal{F}^{-1}(0)$, and $(X_t, P_t) = \mathcal{F}^{-1}(t)$ an isolated (Gorenstein) surface singularity for all t, then each fibre (X_t, P_t) has an e-invariant, and this is constant under the deformation, because it is a framed cobordism invariant of the link. If $\pi_t \colon Y_t \to X_t$ is a desingularization of (X_t, P_t), for some t, then ([12], Th. 1.3) and 4.1. a) above imply

$$(*) \qquad e(X, P) = e(X_t, P_t) = \frac{1}{24}\{\chi(E_t) + K_{Y_t}^2 + 12 p_g(P_t)\} \mod \mathbb{Z},$$

where $\chi(E_t)$ is the topological Euler-Poincaré characteristic of $E_t = \pi_t^{-1}(P_t)$. Since $\chi(E_t)$ and K_Y^2 are both determined by the graph of E_t, it follows that if (X_t, P_t) is orientation preserving homeomorphic to (X, P), then their minimal good resolutions are homeomorphic, by [8], and therefore $\chi(Y_t) = \chi(Y_0)$ and $K_{Y_t}^2 = K_{Y_0}^2$, hence $(*)$ implies

$$p_g(P) \equiv p_g(P_t) \mod 2 .$$

Thus one has

4.4. Corollary. *Let (X, P) and (X', P') be isolated Gorenstein surface singularities, and assume they are orientation preserving homeomorphic. If these two singularities can be put together in a flat family of isolated surface singularities, then the genus of P is congruent to the genus of P' modulo 2.*

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