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# Hodge type of subvarieties of $\mathbb{P}^n$ of small degrees

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Let $S$ be a subvariety of a proper complex manifold $X$ and $j: U \to X$ be the embedding of its complement. One says that the *Hodge type* of $S$ ($Ht(S)$) is $\geqq a$ if $a$ is a natural number such that $F^a H_c^t(U) = H_c^t(U)$ for all $t \geqq 0$, where $F$ is the Hodge-Deligne (decreasing) filtration [D II].

Deligne and Dimca conjecture in [DD] that if $X = \mathbb{P}^n$, the projective space, and if $S$ is defined by $r$ equations of degrees $d_i, d_1 \geqq \sup\{d_i\}$, then $Ht(S) \geqq \kappa := \left[n - \sum_2^r d_i/d_1\right]$, where [] denotes the integral part of a number. This bound $\kappa$ is coming from a theorem of Katz [K], improving a former one of Ax: if $S$ is defined over the finite field $\mathbb{F}_q$, then card $U(\mathbb{F}_q) = 0$ modulo $q^\kappa$. If $S$ is a smooth (complex) complete intersection, P. Deligne had already proved in [D] that $Ht(S) \geqq \kappa$. In [DD] it is proved that $Ht(S) \geqq \kappa$ if $r = 1$, that is if $S$ is an hypersurface.

In this article we reduce the question to a vanishing theorem (1.3), which we are able to prove if $S$ is a complete intersection (2.1) to obtain

**Theorem 0.** *Let $S$ be a complete intersection of multidegree $(d_1, \ldots, d_r)$, $d_1 \geqq \sup\{d_i\}$, in $\mathbb{P}^n$ over $\mathbb{C}$. Then $Ht(S) \geqq \kappa$.*

## 1. Cohomology with compact supports of an open complex manifold $U$

**1.1.** Let $X$ be an analytic manifold, $S$ be a subvariety of $X$, and $j: U \to X$ be the embedding of its complement.

Let $\Omega^{\cdot}_{X,S}$ be any complex mapping to the de Rham complex $\Omega^{\cdot}_X$ with the following properties:

i) $\Omega^j_{X,S} \to \Omega^j_X$ is an isomorphism over $U$ for all $j$.

ii) Let $f: \tilde{X} \to X$ be an embedded desingularization of $S$ such that $\tilde{S} := f^{-1}(S)$

is a normal crossing divisor. Then the map

$f^*\Omega^j_{X,S} \to \Omega^j_{\tilde{X}}$ factorizes through

$f^*\Omega^j_{X,S} \to \Omega^j_{\tilde{X}}(\log \tilde{S})(-\tilde{S})$, where $\mathcal{O}_{\tilde{X}}(-\tilde{S})$ has the reduced structure.

Actually, ii) does not depend on the desingularization $f$ choosen. If $f_1: X_1 \to X$ is another one, with $S_1 := f_1^{-1}(S)$, choose a third one $Y$:

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\sigma_1"'] \arrow[dr, "g"] & \tilde{X} \arrow[d, "f"] \\ X_1 \arrow[r, "f_1"'] & X \end{tikzcd}$$

with $T := g^{-1}S$. Then one has maps

$$f_1^*\Omega^j_{X,S} \to \sigma_{1*}\sigma_1^* f_1^* \Omega^j_{X,S} \to \sigma_{1*}\Omega^j_Y(\log T)(-T) = \Omega^j_{X_1}(\log S_1)(-S_1) .$$

Denote by $i: j_!\mathbb{C} \to \Omega^{\cdot}_{X,S}$ the natural map coming from $j_!\mathbb{C} \to \Omega^0_{X,S}$, and by $D^b(X)$ the derived category of bounded complexes on $X$. In [E, (3-2)] we proved the following.

**Lemma.** *There is a map $f^{-1}: \Omega^{\cdot}_{X,S} \to j_!\mathbb{C}$ in $D^b(X)$ such that $f^{-1} \circ i$ is a quasi-isomorphism.*

*Proof.* Let $\tilde{j}: U \hookrightarrow \tilde{X}$ be the embedding with $f \circ \tilde{j} = j$. Then ii) defines a map $f^{-1}: \Omega^{\cdot}_{X,S} \to Rf_*\Omega^{\cdot}_{\tilde{X}}(\log \tilde{S})(-\tilde{S}) = Rf_*\tilde{j}_!\mathbb{C} = j_!\mathbb{C}$ in $D^b(X)$, and $f^{-1} \circ i$ is coming from $j_!\mathbb{C} \to \Omega^0_{X,S} \to f_*\mathcal{O}_{\tilde{X}}(-\tilde{S})$.

**1.2.** We assume now that $X$ is proper. Let $\Omega'^j_{X,S}$ be another sheaf as in (1.1) i), ii), defined for $j \leqq a-1$, where $a$ is a given positive natural number, such that $\Omega^j_{X,S} \to \Omega^j_X$ factorizes through $\Omega^j_{X,S} \to \Omega'^j_{X,S}$.

**Proposition.** *For any $t \geqq 0$, there is a commutative diagramm*

$$\begin{tikzcd} H^t(X, \Omega^{\cdot}_{X,S}) \arrow[r, "f^{-1}"] \arrow[d] & H^t_c(U) \arrow[d] \\ H^t(X, \Omega^{\leqq a-1}_{X,S}) \arrow[r, "(f^{-1})^{\leqq a-1}"] \arrow[d] & H^t_c(U)/F^a H^t_c(U) \\ H^t(X, \Omega'^{\leqq a-1}_{X,S}) \arrow[ur, "(f^{-1})^{\leqq a-1}"'] & \end{tikzcd}$$

*where the two maps denoted by $(f^{-1})^{\leqq a-1}$ are surjective.*

*Proof.* By (1.1), $f^{-1}$ is surjective, whereas by Hodge theory [DII] one has:

$$H^t_c(U)/F^a H^t_c(U) = H^t(\tilde{X}, \Omega^{\leqq a-1}_{\tilde{X}}(\log \tilde{S})(-\tilde{S})).$$

**1.3.** *Conclusion of this section.*

In order to prove that $F^a H^t_c(U) = H^t_c(U)$ for any $t$, it is enough to find $\Omega'^{\leqq a-1}_{X,S}$ as in (1.2) such that $H^t(X, \Omega'^{\leqq a-1}_{X,S}) = 0$, or, if one is not able to compute precisely this hypercohomology, such that $H^t(X, \Omega'^j_{X,S}) = 0$ for any $t$ and $j \leqq a-1$.

## 2. Vanishing theorems for complete intersections in $\mathbb{P}^n$

**2.1.** Let $S \subset \mathbb{P}^n$ be a scheme theoretic complete intersection of multidegree $(d_1, \ldots, d_r)$, $d_1 \geqq \sup\{d_i\}$, and let $\mathcal{J}$ be the corresponding ideal sheaf of $S$.

**Proposition.** *One has $H^t(\mathbb{P}^n, \mathcal{J}^a(-k)) = 0$ for any $t$ if $a$ and $k$ are natural numbers such that $0 \leqq k \leqq n - (ad_1 + d_2 + \ldots + d_r)$.*

*Proof.* If $r=1$, then $H^t(\mathcal{O}(-ad_1 - k)) = 0$ for $0 \leqq ad_1 + k \leqq n$, and if $a=0$, $H^t(\mathcal{O}(-k)) = 0$ for $0 \leqq k \leqq n - \sum_{2}^{r} d_i$. Otherwise, denote by $\mathcal{J}_{r-1}$ the ideal defined by the $(r-1)$ first equations, and by $f_r$ the last equation. One has an exact sequence if $a \geqq 1$:

$$0 \to \mathcal{J}^a_{r-1}(-k) \cap f_r \mathcal{J}^{a-1}(-k) \to \mathcal{J}^a_{r-1}(-k)$$
$$\oplus \mathcal{J}^{a-1}(-k-d_r) \xrightarrow[\oplus f_r]{\text{inclusion}} \mathcal{J}^a(-k) \to 0$$

where the left term is $\mathcal{J}^a_{r-1}(-k-d_r)$ is $\mathcal{J}$ is a complete intersection. One argues by induction on $a+r$.

**2.2.** *Proof of theorem 0.*

With the notation of (1.3) and (2.1), we take $\Omega'^{j}_{X,S} := \mathcal{J}^{\kappa - j} \oplus \Omega^j_{\mathbb{P}^n}$ for $j \leqq \kappa - 1$.

As $\Omega^j_{\mathbb{P}^n}$ has a resolution by the sheaves $\mathcal{O}(-k)$, $0 \leqq k \leqq j$, we have just to see by (1.3) that $H^t(\mathcal{J}^{\kappa - j}(-k)) = 0$ for $0 \leqq k \leqq j \leqq \kappa - 1$.

By (2.1), it is enough to verify that $j \leqq n - ((\kappa - j)d_1 + d_2 + \ldots + d_r)$.

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