# Connections and Symmetric Differential Forms

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# over $\mathbb{C}$

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- Simpson: $M_B(X,r) \leftrightsquigarrow^{\rm top} M_{dR}(X,r) \leftrightsquigarrow^{\rm top} M_{\rm Dol}(X,r)$
- $M_B(X,r)$ affine
- $M_{\rm Dol}(X,r) \xrightarrow{\rm Hitchin} \mathbb{A}^N$, $N = \oplus_{i=1}^r h^0(X, {\rm Sym}^i\Omega^1)$ proper
- **Van:** $h^0(X, {\rm Sym}^i\Omega^1) = 0 \ \forall i \in \mathbb{N}_{>0}$
- $\Longrightarrow^{\rm Arapura}$: **Van** $\Rightarrow$ **Fin** with **Fin**: $[M_B(X,r)$ 0-dim'l$]$
- **Fin** $\Rightarrow$ all complex local systems are rigid.

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- Katzarkov-Zuo: **Van** $\Rightarrow$ is in fact a $\bar{\mathbb{Z}}$-factor of a $\mathbb{Z}$-VHS
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## **Fin** $\not\Rightarrow$ **Van**, **Fin** $\not\Rightarrow$ Thm

Margulis superrigidity: Shimura var of rk $\geq 2$: has **Fin** but by far not **Van** and has inftly many loc syst with infinite monodromy.

# Integrality and $F$-isocrystals

**Theorem (EG'18)** in (partial) answer to Simpson's integrality conjecture

**Fin** (in a given rank $r$) $\Rightarrow$

- (rank $r$) complex loc syst are integral.
- $(E, \nabla)_W \otimes_W K$ are isoc with a Frobenius structure

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So: BKT $\Leftrightarrow$ unitary mon $\Leftrightarrow$ Higgs field $= 0$, seen in char. $p > 0$.

# Problems addressed

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$\mathcal{L}$ connection of rank 1; **Fin** $\Rightarrow \{\mathcal{L}^n\}_{n\in\mathbb{Z}}$ finite $\Rightarrow \mathcal{L}^m \cong \mathcal{L}^n$ for some $m \neq n \in \mathbb{N}$ (preperiodicity) $\Rightarrow \mathcal{L}^{n-m} = 1$ $(m-n) \neq 0$ so $\mathcal{L}$ torsion.

# Isocrystals

**Proposition (EG'20)**

$X = X_W \otimes_W k, k = \bar{k}$ sm proj, **Van**$/K \Rightarrow$

1) all $\ell$-adic loc. syst have fin mon
2) if $k = \bar{\mathbb{F}}_p$ $\exists h : Y \to X$ fin ét trivializing conv isoc

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## Proof

- 1): $\pi_1^{\text{ét}}(X_{\mathbb{C}}) \twoheadrightarrow \pi_1^{\text{ét}}(X_k)$ + BKT
- $(E_K, \nabla_K) = (E_W, \nabla_W) \otimes_W K, (E_W, \nabla_W) \otimes_W k$, nilp $p$-curv
- $F$ acts on isoc, **Fin** $\Rightarrow$ preperiodicity $F$-orbit of any isoc
- $\Rightarrow$ given $(E_K, \nabla_K)$ (not nec conv), $\exists N, (F^N)^*(E_K, \nabla_K)$ $F$-str
- $\Rightarrow$ (Abe-E +K) $\exists \ell$-adic companion so 1) $\Rightarrow \exists\ h : Y \to X$ st $h^*(F^N)^*(E_K, \nabla_K)$ trivial
- conv $\Rightarrow$ 2): $h^*(E_K, \nabla_K)$ trivial as well.

# Problem on isocrystals

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- $X/k, k = \bar{k}$ sm proj, **Van**$/k \Rightarrow^?$ all $\bar{\mathbb{Q}}_\ell$ loc syst have fin mon
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## Remark

$X/k$ lifts to $X_W$: Proposition $\Rightarrow$ both problems have a $> 0$ answer

# $/\mathbb{F}_q$

**Theorem (EG'20)**

$X = X_{W_2(\mathbb{F}_q)} \otimes \mathbb{F}_q$ sm proj, **Van** $/\mathbb{F}_q \Rightarrow$ rk 2 loc free ss deg 0 $(E, \nabla)$ are étally trivializable.

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**Proof**

- **Van** $\Rightarrow$ Hitchin base$= \{0\} \Rightarrow$ $p$-curv nilpotent
- preperiodic Higgs-dR flow (OV corr, Lan-Sheng-Zuo)
- assume periodic period 1 (for talk): then
- either $(E, \nabla) = (F^*E, \nabla_{\mathrm{can}}) \Rightarrow$ (Lang torsor) ét triv, or
- $0 \to (F^*L^{<0}, \mathrm{can}) \to (E, \nabla) \to (F^*L^{>0}, \mathrm{can}) \to 0$ ($p$-curv nil)
- $0 \to L^{>0} \to E \to L^{<0} \to 0$ ($F$-Filt)
- $\rightsquigarrow KS : L^{>0} \otimes (L^{<0})^{-1} \hookrightarrow \Omega^1$, $(L^{<0})^{p-1} \hookrightarrow \mathcal{O} \hookrightarrow (L^{>0})^{p-1}$
- $\rightsquigarrow \mathcal{O}_X \hookrightarrow \mathrm{Sym}^{p-1}\Omega^1 \perp$ to **Van**.