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A regulator map for singular varieties

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Introduction

Let X be an algebraic variety over $\mathbb{C}$, the field of complex numbers. If X is smooth, there is a regulator map r from $\mathscr{K}^M_{nX}$, the Zariski sheaf of Milnor K-theory, to $\mathscr{H}^n_{\mathscr{D}}(n)$, the Zariski sheaf of Deligne-Beilinson cohomology. The aim of this article is to construct a similar functorial regulator map ϱ (2.2) from $\mathscr{K}^M_{nX}$ to a Zariski sheaf called $\mathscr{H}^n(n)$ (1.4) if X is not necessarily smooth. For this we assume that $d := $ dimension of the singular locus S verifies $d \leq n-1$ with $n \geq 2$.

If X is smooth, then $\mathscr{H}^n(n) = \mathscr{H}^n_{\mathscr{D}}(n)$ and $\varrho = r$. If not, let $\pi: Y \to X$ be a desingularization. Then ϱ factorizes $\pi_* r$ via the natural map $\mathscr{K}^M_{nX} \to \pi_* \mathscr{K}^M_{nY}$ and a map $\mathscr{H}^n(n) \to \pi_* \mathscr{H}^n_{\mathscr{D}}(n)$ which we construct (1.4) 7).

Taking the cohomology of ϱ, one obtains maps $H^q(\varrho): H^q(X, \mathscr{K}^M_{nX}) \to H^q(X, \mathscr{H}^n(n))$. The cohomology group $H^q(X, \mathscr{H}^n(n))$ is independent of the desingularization choosen as $\mathscr{H}^n(n)$ is. Unfortunately one may only approximate this group by a map t from $H^q(X, \mathscr{H}^n(n))$ to some cohomology group $H^{q+n}(Y, \mathbb{Z}(n)_{\mathrm{an}})$ on Y (2.7).

Srinivas [S] considered a cone X of vertex 0 over a smooth projective curve C. He constructed a map s from

$$H^1(X, \mathscr{K}_{2X}) (= H^0(X, \pi_* \mathscr{K}_{2Y} / K_2(\mathscr{O}_{X,0})) / H^0(Y, \mathscr{K}_{2Y})$$

to $H^0(C, \omega_c(1))$, where ω_c is the dualizing sheaf of C, and $\pi: Y \to X$ is the blowing up of 0, whose non triviality shows that the image of $K_2(\mathscr{O}_{X,0})$ in $K_2(\mathbb{C}(X))$ differs from $\varinjlim_{0 \in U} H^0(\pi^{-1} U, \mathscr{K}_{2Y})$.

Actually s comes from $H^1(\varrho)$ (2.10), Example 2. This fact is the main motivation for this article. I take this opportunity to thank V. Srinivas for getting me acquainted with this topic.

Collino [C] compactified the cone X to a smooth variety $\bar{X}$ (more exactly he considered a normal proper surface $\bar{X}$ with an isolated cone like singularity) and

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lifted s to

$$\bar{s}: H' \to H^3(\bar{Y}, j_! \mathbb{Z}(2) \to \mathcal{O}_{\bar{Y}}(-2C) \to \Omega^1_{\bar{Y}}(\log C)(-2C)),$$

where $\bar{Y} \to \bar{X}$ is the blowing up of $0, H'$ is a subgroup of $H^1(X, \mathcal{K}_{2X})$ and j is the embedding $Y - C \to Y$. In fact $H^1(\varrho)$ factorizes $\bar{s}$ and one has $\bar{s} = t \circ H^1(\varrho)$ (2.8), Example 1.

In this spirit we work out several examples of the cohomology of ϱ (2.7), (2.8), (2.9), (2.10), (2.11), (2.12). However it is not always possible to give a nice answer (2.13).

The construction of $\mathcal{H}^n(n)$ is as follows. Take a desingularisation π such that $E := \pi^{-1} S$ is a divisor with normal crossings and such that $\mathcal{F}_{\mathrm{an}} := \pi^* \Omega^n_X/\text{torsion}$ is a locally free sheaf (0.1). We observe that $\mathcal{F}_{an}$ embedds in $\Omega^n_Y(\log E)(-k \cdot E)$, for some positive integer k (0.3), and therefore the complex $\mathcal{F}^{\geqq n}_{\mathrm{an}}$, where $\mathcal{F}^i_{\mathrm{an}} = 0$ for $i < n$, $\mathcal{F}^n_{\mathrm{an}} = \mathcal{F}_{\mathrm{an}}$, $\mathcal{F}^{n+l}_{\mathrm{an}} = \Omega^{n+l}_Y(\log E)(-k \cdot E)$ for $l \geqq 1$ maps to $j_! \mathbb{C}/\mathbb{Z}(n)$, where j is the embedding from $X - S = Y - E$ to Y (0.4). On each Zariski open set of Y we take those sections of $\mathcal{F}^{\geqq n}_{\mathrm{an}}$ which have logarithmic growth at infinity (0.5). This defines a "subcomplex" $\mathcal{F}^{\geqq n}$ (0.6), with a "map" φ_j from $\mathcal{F}^{\geqq n}$ to $j_! \mathbb{C}/\mathbb{Z}(n)$ (0.7). Taking the n-th cohomology on $\pi^{-1} U$, where U is a Zariski open subset of X, of cone $\varphi_j[-1]$ defines a Zariski sheaf on X (1.4). If $d < n-2$, this is $\mathcal{H}^n(n)$. In general, $\mathcal{H}^n(n)$ is a subquotient of it.

It is easy to prove the independency of $\mathcal{H}^n(n)$ of the desingularization choosen (1.4) 1), and not hard to prove the functoriality (1.4) 7). Then it is straightforward to construct ϱ by lifting the universal situation (2.2).

In order to construct t, one has first to forget the growth condition at infinity (1.5) 2), (1.8), (2.9), a technique used in [E 2] to describe the cycle map from the Chow group to the Deligne-Beilinson cohomology as the cohomology of a forgetful functor.

This paper is organized as follows. In Sect. 0 we construct the complexes on Y and X, whose cohomologies will define the Zariski sheaves wanted in Sect. 1. In Sect. 2 we construct ϱ and compute some examples.

0. Notations and definition of the complexes

(0.1) Let X be a reduced algebraic variety over $\mathbb{C}$. Let S be its singular locus. We assume that $\dim S = d$. We fix in this article an integer n with $n \geqq d+1$ and $n \geqq 2$. Let $S_d := S$ and define by induction S_{d-s}, the singular locus of S_{d-s+1} for $1 \leqq s \leqq d$. S_0 consists of finitely many points.

Let $\pi: Y \to X$ be a desingularization of X such that $E_d := (\pi^{-1} S)_{\mathrm{red}}$ is a normal crossing divisor and such that $\mathcal{F}_{\mathrm{an}} := \pi^* \Omega^n_X/\text{torsion}$ is locally free, where Ω^n_X is the analytic sheaf of Kähler differentials of degree n.

Define $E_{d-s} := (\pi^{-1} S_{d-s})_{\mathrm{red}}$.

(0.2) In this section, we consider a special desingularization Y to give an upper bound on $\mathcal{F}_{\mathrm{an}}$. We will use it just to prove (0.3).

Let $\mathcal{I}_{d-s}$ be the ideal sheaf of S_{d-s} with the reduced structure. This means that $\mathcal{O}_{S_{d-s}} := \mathcal{O}_X/\mathcal{I}_{d-s}$ is a smooth ring away from S_{d-s-1}. We will assume that $(\pi^* \mathcal{I}_{d-s}/\text{torsion})$ is an invertible sheaf $\mathcal{O}_Y(-F_{d-s})$, where F_{d-s} is an effective normal crossing divisor (with multiplicities). We also assume that $\mathscr{F}_{\mathrm{an}}$ is locally free.

Define $F'_{d-s}:=F_{d-s}$-components above S_{d-s-1}

$$F:=\sum_{s=0}^{d}(n-d+s)\cdot F'_{d-s}.$$

Lemma. *One has an embedding*

$$\mathscr{F}_{\mathrm{an}}\to\Omega^n_Y(\log F)(-F).$$

Proof. As both sheaves are locally free, it is enough to prove the injection at the generic point of each component of F.

Let q be a generic point in $F'_{d-s}-F'_{d-s-1}$ and p be $\pi(q)$ lying in $S_{d-s}-S_{d-s-1}$. The exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\pi^*\mathscr{I}_{d-s}/\mathrm{torsion})_q & \longrightarrow & (\pi^*\mathcal{O}_X)_q & \longrightarrow & (\pi^*\mathcal{O}_{S_{d-s}})_q & \longrightarrow & 0\\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathcal{O}(-F'_{d-s})_q & \longrightarrow & (\mathcal{O}_Y)_q & \longrightarrow & (\mathcal{O}_{F'_{d-s}})_q & \longrightarrow & 0
\end{array}$$

splits after passing to the completion $\hat{}$. So for each $f\in\mathcal{O}_{\widehat{X,p}}$ we may write $(\pi^*f)_q=g+h$, where $g\in(\pi^*\mathcal{O}_{\widehat{S_{d-s}}})_q$, $h\in\mathcal{O}(-F'_{d-s})_q$.

The $\mathcal{O}_{X,p}$ module $\Omega^n_{X,p}$ is generated by $df_1\wedge\ldots\wedge df_n$, where $f_i\in\mathcal{O}_{\widehat{X,p}}$. Therefore $(\mathscr{F}_{\mathrm{an}})_p$ is generated by

$$(\pi^*(df_1\wedge\ldots\wedge df_n))_q=\sum_{l=1}^{n}(-1)^{sgn(i_1,\ldots,i_n)}dg_{i_1}\wedge\ldots\wedge dg_{i_l}\wedge dh_{i_{l+1}}\wedge\ldots\wedge dh_{i_n}.$$

For $l>d-s$, one has $dg_{i_1}\wedge\ldots\wedge dg_{i_l}=0$.

For any l, one has $dh_{i_{l+1}}\wedge\ldots\wedge dh_{i_n}\in(\Omega^{n-l}_Y(\log F'_{d-s})(-(n-l)\cdot F'_{d-s}))_q$.

Therefore one has $(\pi^*(df_1\wedge\ldots\wedge df_n))_q\in(\Omega^n_Y(\log F'_{d-s})(-(n-d+s)\cdot F'_{d-s}))_q$.

(0.3) We go back to a general desingularization π as in (0.1).

Lemma. *There is an effective divisor E with support E_d such that $(n-d)\cdot E_d<E$ and such that $\mathscr{F}_{\mathrm{an}}$ embedds in $\Omega^n_Y(\log E)(-E)$.*

Moreover if $S=S_0$, one may take $E=n\cdot F$ where $\mathcal{O}_Y(-F):=(\pi^*\mathscr{I}_0/torsion)$.

Proof. Let $\pi':Y'\to X$ be the desingularization considered in (0.2). If $S=S_0$, we may take π to be π' and apply (0.2).

In general, let $p:Z\to X$ be a desingularization factorizing over $\sigma:Z\to Y$ and $\sigma':Z\to Y'$ such that $p^{-1}S$ is a normal crossing divisor.

Then the conditions (0.1) and (0.2) are fulfilled for p. Call Δ the reduced exceptional locus of σ' in Z, C the locus in Y where σ is not isomorphism. Then C is of codimension $\geqq 2$.

One has injections

$$\begin{aligned}p^*\Omega^n_X/\mathrm{torsion}=\sigma^*\mathscr{F}_{\mathrm{an}}=\sigma'^*\pi'^*\Omega^n_X/\mathrm{torsion}&\to\sigma'^*\Omega^n_{Y'}(\log F)(-F)\\&\to\Omega^n_Z(\log p^{-1}S)(-\Delta)\otimes\sigma'^*\mathcal{O}_{Y'}(-F)=:\mathscr{A}.\end{aligned}$$

As $(n-d)\cdot F_{\mathrm{red}}\subset F$, one has

$$\mathcal{O}_Z((n-d)\cdot(p^{-1}S)_{\mathrm{red}})\subset\sigma'^*\mathcal{O}_{Y'}(F)\subset\sigma'^*\mathcal{O}_{Y'}(-F)\otimes\mathcal{O}_Z(\Delta).$$

Let E be the divisor defined by $E\cap(Y-C):=(\Delta+\sigma'^*F)\cap(Y-C)$. The torsion free sheaf $\sigma_*\mathscr{A}$ embeds on $(Y-C)$ in $\Omega_Y^n(\log E)(-E)_{|Y-C}$. As $\Omega_Y^n(\log E)(-E)$ is locally free everywhere, $\sigma_*\mathscr{A}$ embeds in it everywhere. This gives the map

$$\sigma_*\sigma^*\mathscr{F}_{an}=\mathscr{F}_{an}\to\Omega_Y^n(\log E)(-E).$$

(0.4) We fix now π as in (0.1) and E as in (0.3).

We may differentiate $\mathscr{F}_{an}$ in $\Omega_Y^{n+1}(\log E)(-E)$. This defines a complex $\mathscr{F}_{an}^{\geq n}$ with $\mathscr{F}_{an}^i=0$ for $i<n$, $\mathscr{F}_{an}^n=\mathscr{F}_{an}$ and $\mathscr{F}_{an}^{n+l}=\Omega_Y^{n+l}(\log E)(-E)$ for $l\geq 1$.

One has an injection of complexes

$$\mathscr{F}_{an}^{\geq n}\to\Omega_{\bar{Y}}^{\geq n}(\log E)(-E).$$

(0.5) a) Let π be a desingularization as in (0.1). Fix $\bar{\pi}:\bar{Y}\to\bar{X}$ a good compactification π. This means that $\bar{X}$ is proper, $\bar{Y}$ is proper and smooth; one has a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{i_Y} & \bar{Y} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bar{\pi}} \\ X & \xrightarrow[k_X]{} & \bar{X} \end{array}$$

where $(\bar{Y}-Y)$ and $(\bar{Y}-Y)+\bar{E}$ are normal crossing divisors.

b) Let V be a Zariski open subset of Y. Define $V':=\bar{Y}-(\overline{Y-V})$. Then V' is smooth and $(V'-V)$ is a normal crossing divisor. One has a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{i'} & V' \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau'} \\ Y & \xrightarrow{i_Y} & \bar{Y} \end{array}.$$

Both sheaves $i'_*\mathscr{F}_{an}$ and $\Omega_{V'}^n(\log(V'-V))$ are contained in $i'_*\Omega_V^n$. Define

$$\mathscr{F}_p:=i'_*\mathscr{F}_{an}\cap\Omega_{V'}^n(\log(V'-V)) \quad (p \text{ for partial}).$$

c) Let V be a Zariski open subset of Y. A good compactification $\bar{\tau}:\bar{V}\to\bar{Y}$ of τ is defined by a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{i} & \bar{V} \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\bar{\tau}} \\ Y & \xrightarrow{i_Y} & \bar{Y} \end{array}$$

where $\bar{V}$ is proper and smooth, $(\bar{V}-V)$ and $(\bar{V}-V)+(\overline{E\cap V})$ are normal crossing divisors. If V is of the shape $\pi^{-1}U$, where U is a Zariski open subset of X, one has a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{i} & \bar{V} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bar{\pi}\bar{\tau}} \\ U & \xrightarrow{k} & \bar{X} \end{array}$$

Both sheaves $l_*\mathscr{F}_{\mathrm{an}}$ and $\Omega^n_{\bar V}(\log(\bar V - V))$ are contained in $l_*\Omega^n_V$. Define

$$\mathscr{F} := l_*\mathscr{F}_{\mathrm{an}} \cap \Omega^n_{\bar V}(\log(\bar V - V)).$$

As $\bar\tau_*\Omega^n_{\bar V}(\log(\bar V - V))$ injects into $\tau'_*\Omega^n_{V'}(\log(V' - V))$ one has injections

$$\bar\tau_*\mathscr{F} \to \tau'_*\mathscr{F}_{\mathrm{p}} \to (l_Y\tau)_*\mathscr{F}_{\mathrm{an}}.$$

(0.6) One has injections

$$\mathscr{F}_{\mathrm{p}} \to \Omega^n_{V'}(\log(V' - V) + \overline{E \cap V})(-\overline{E \cap V}),$$
$$\mathscr{F} \to \Omega^n_{\bar V}(\log(\bar V - V) + \overline{E \cap V})(-\overline{E \cap V})$$

which allow one to differentiate $\mathscr{F}_{\mathrm{p}}$ (resp. $\mathscr{F}$) in

$$\Omega^{n+1}_{V'}(\log(V' - V) + \overline{E \cap V})(-\overline{E \cap V})$$

[resp. $\Omega^{n+1}_{\bar V}(\log(\bar V - V) + \overline{E \cap V})(-\overline{E \cap V})$].

Define complexes $\mathscr{F}^{\geqq n}_{\mathrm{p}}$ and $\mathscr{F}^{\geqq n}$ by:

$$\mathscr{F}^i_{\mathrm{p}} = \mathscr{F}^i = 0 \quad \text{for} \quad i < n,$$
$$\mathscr{F}^n_{\mathrm{p}} = \mathscr{F}_{\mathrm{p}}, \qquad \mathscr{F}^n = \mathscr{F},$$
$$\mathscr{F}^{n+l}_p = \Omega^{n+l}_{V'}(\log(V' - V) + \overline{E \cap V})(-\overline{E \cap V}) \quad \text{for} \quad l \geqq 1,$$
$$\mathscr{F}^{n+l} = \Omega^{n+l}_{\bar V}(\log(\bar V - V) + \overline{E \cap V})(-\overline{E \cap V}) \quad \text{for} \quad l \geqq 1.$$

One has injections of complexes.

$$(\bar\tau_*\mathscr{F})^{\geqq n} \to (\tau'_*\mathscr{F}_{\mathrm{p}})^{\geqq n} \to ((l_Y\tau)_*\mathscr{F}_{\mathrm{an}})^{\geqq n}$$

As $\mathscr{F}^{\geqq n}$ is a complex starting in degree n, one has an injection $R^n\bar\tau_*\mathscr{F}^{\geqq n}[-n] \to (\bar\tau_*\mathscr{F})^{\geqq n}$ (and similarly for the others), which gives injections of sheares

$$R^n\bar\tau_*\mathscr{F}^{\geqq n} \to R^n\tau'_*\mathscr{F}^{\geqq n}_{\mathrm{p}} \to R^n(l_Y\tau)_*\mathscr{F}^{\geqq n}_{\mathrm{an}}.$$

(0.7) a) We use the convention $S_\emptyset = \emptyset$, $E_\emptyset = \emptyset$. Define j_s the inclusion $Y - E_s \to Y$ and i_s the inclusion $X - S_s \to X$ for $s = \emptyset, 0, \ldots, d$.

In the derived category $D^b(Y)$ of bounded complexes on Y, one has a map

$$\Omega^{\geqq n}_{\bar Y}(\log E)(-E) \to j_{d!}\mathbb{C}/\mathbb{Z}(n),$$

obtained as the composite map

$$\begin{tikzcd} \Omega^{\geqq n}_{\bar Y}(\log E)(-E) \arrow[r] & \Omega^{\geqq n}_{\bar Y}(\log E_d)(-E_d) \arrow[r] & \Omega^{\cdot}_Y(\log E_d)(-E_d) & j_{d!}\mathbb{C} \arrow[l, "\sim"'] \arrow[d] \\ & & & j_{d!}\mathbb{C}/\mathbb{Z}(n). \end{tikzcd}$$

This defines maps in $D^b(Y)$

$$\varphi^{\mathrm{an}}_{j_s} : \mathscr{F}^{\geqq n}_{\mathrm{an}} \to j_{s!}\mathbb{C}/\mathbb{Z}(n) \quad \text{for } s = \emptyset, 0, \ldots, d.$$

Define in $D^b(Y)$ $\mathbb{Z}(n)_{j_s, \mathrm{an}} := \operatorname{cone} \varphi^{\mathrm{an}}_{j_s}[-1]$ for $s = \emptyset, 0, \ldots, d$.

One has maps

$$\mathbb{Z}(n)_{j_d, \mathrm{an}} \to \ldots \to \mathbb{Z}(n)_{j_0, \mathrm{an}} \to \mathbb{Z}(n)_{j_\emptyset, \mathrm{an}} \to \mathbb{Z}(n)_{\mathscr{D}, \mathrm{an}},$$

where $\mathbb{Z}(n)_{\mathscr{D}, \mathrm{an}} := \operatorname{cone}(\Omega^{\geqq n}_{\bar Y} \to \mathbb{C}/\mathbb{Z}(n))[-1]$ is the Deligne complex.

b) If V is a Zariski open subset of Y as in (0.5), $\varphi_{j_s}^{\mathrm{an}}$ defines in $D^b(V')$

$$\varphi_{j_s}^{\mathrm{p}}: \mathscr{F}_{\mathrm{p}}^{\geqq n} \to Rl'_* j_{s!} \mathbb{C}/\mathbb{Z}(n)$$

and therefore

$$\varphi_{j_s}^{\mathrm{p}}: \mathscr{F}_{\mathrm{p}}^{\geqq n} \to Rl'_* j_{s!} \mathbb{C}/\mathbb{Z}(n)$$

for $s=\emptyset, 0, \ldots, d$.

Define in $D^b(V')$ $\mathbb{Z}(n)_{j_s}^{\mathrm{p}} := \operatorname{cone} \varphi_{j_s}^{\mathrm{p}}[-1]$ for $s=\emptyset, 0, \ldots, d$.

Similarly define a "partial" Deligne-Beilinson complex by

$$\mathbb{Z}(n)_{\mathscr{D}}^{\mathrm{p}} := \operatorname{cone}(\Omega_{V'}^{\geqq n}(\log(V'-V)) \to Rl'_* \mathbb{C}/\mathbb{Z}(n))[-1].$$

One has maps in $D^b(V')$:

$$\mathbb{Z}(n)_{j_d}^{\mathrm{p}} \to \ldots \to \mathbb{Z}(n)_{j_0}^{\mathrm{p}} \to \mathbb{Z}(n)_{j_\emptyset}^{\mathrm{p}} \to \mathbb{Z}(n)_{\mathscr{D}}^{\mathrm{p}}.$$

c) Similarly, one has maps in $D^b(\bar{V})$

$$\varphi_{j_s}: \mathscr{F}^{\geqq n} \to Rl_* j_{s!} \mathbb{C}/\mathbb{Z}(n) \quad \text{for} \quad s=\emptyset, 0, \ldots, d.$$

Define in $D^b(\bar{V})$

$$\mathbb{Z}(n)_{j_s} := \operatorname{cone} \varphi_{j_s}[-1], \quad \text{for} \quad s=\emptyset, 0, \ldots, d.$$

The Deligne-Beilinson complex is defined by

$$\mathbb{Z}(n)_{\mathscr{D}} := \operatorname{cone}(\Omega_{\bar{V}}^{\geqq n}(\log(\bar{V}-V)) \to Rl_* \mathbb{C}/\mathbb{Z}(n)[-1].$$

One has maps in $D^b(\bar{V})$

$$\mathbb{Z}(n)_{j_d} \to \ldots \to \mathbb{Z}(n)_{j_0} \to \mathbb{Z}(n)_{j_\emptyset} \to \mathbb{Z}(n)_{\mathscr{D}}.$$

(0.8) Let U be a Zariski open subset of X.

We consider a compactification of $\pi^{-1}U$ as in (0.5).

As

$$\begin{aligned} R\pi_* j_{d!} &= R\pi_* Rj_{d!} \ (j_{d!} \text{ is exact}) \\ &= R(\pi j_d)_! \ (\pi \text{ is proper}) \\ &= Ri_{d!} = i_{d!} \ (i_{d!} \text{ is exact}) \end{aligned}$$

φ_{j_d} defines

$$\varphi_{i_d}: R(\bar{\pi}\bar{\tau})_* \mathscr{F}^{\geqq n} \to Rk_* i_{d!} \mathbb{C}/\mathbb{Z}(n) \quad \text{in } D^b(\bar{X}).$$

This defines in $D^b(\bar{X})$

$$\varphi_{i_s}: R(\bar{\pi}\bar{\tau})_* \mathscr{F}^{\geqq n} \to Rk_* i_{s!} \mathbb{C}/\mathbb{Z}(n)$$

for $s=\emptyset, 0, \ldots, d$.

Define $\mathbb{Z}(n)_{i_s} := \operatorname{cone} \varphi_{i_s}[-1]$ for $s=\emptyset, 0, \ldots, d$.

One has maps in $D^b(\bar{X})$

$$\mathbb{Z}(n)_{i_d} \to \ldots \to \mathbb{Z}(n)_{i_0} \to \mathbb{Z}(n)_{i_\emptyset}.$$

(0.9) Define $\mathscr{C}_{j_s}$ by the exact triangle in $D^b(\bar{X})$

$$\mathbb{Z}(n)_{j_s} \to \mathbb{Z}(n)_{\mathscr{D}} \to \mathscr{C}_{j_s} \xrightarrow{1} \mathbb{Z}(n)_{j_s},$$

and similarly for $\mathscr{C}^p_{j_s}$ in $D^b(V')$ and $\mathscr{C}_{j_s,\mathrm{an}}$ in $D^b(V)$, for $s=\emptyset, 0, \ldots, d$.

One has

$$\mathscr{C}_{j_s} = \mathrm{cone}(\Omega_{\bar{V}}^{\geqq n}(\log(\bar{V}-V))/\mathscr{F}^{\geqq n} \to Rl_* \mathbb{C}/\mathbb{Z}(n)_{|E_s})[-1],$$

$$\mathscr{C}^p_{j_s} = \mathrm{cone}(\Omega_{\bar{V}'}^{\geqq n}(\log(V'-V))/\mathscr{F}_p^{\geqq n} \to Rl'_* \mathbb{C}/\mathbb{Z}(n)_{|E_s})[-1],$$

$$\mathscr{C}_{j_s,\mathrm{an}} = \mathrm{cone}(\Omega_{\bar{V}}^{\geqq n}/\mathscr{F}_{\mathrm{an}}^{\geqq n} \to \mathbb{C}/\mathbb{Z}(n)_{|E_s})[-1].$$

(0.10) By definition one has $\varphi_{i_d} = R(\bar{\pi}\bar{\tau})_* \varphi_{j_d}$, and one has maps

$$\mathbb{Z}(n)_{i_s} \to R(\bar{\pi}\bar{\tau})_* \mathbb{Z}(n)_{j_s} \quad \text{for} \quad s=\emptyset, 0, \ldots, d,$$

coming from the maps

$$i_{s!} = \pi_* j_{s!} \to R\pi_* j_{s!}.$$

Therefore we have an isomorphism

$$\mathbb{Z}(n)_{i_d} = R(\bar{\pi}\bar{\tau})_* \mathbb{Z}(n)_{j_d}$$

and maps

$$\mathbb{Z}(n)_{i_s} \to R(\bar{\pi}\bar{\tau})_* \mathbb{Z}(n)_{j_s} \quad \text{for} \quad s=\emptyset, 0, \ldots, d-1.$$

(0.11) If Z is any complex algebraic variety, we denote by $\alpha: Z_{\mathrm{an}} \to Z_{\mathrm{zar}}$ the continuous map from Z endowed with the classical topology to Z endowed with the Zariski topology.

1. Definition of the Zariski sheaves

(1.1) Let V be a Zariski open subset of Y as in (0.5).

Define $\mathscr{F}_{\mathrm{an}}(V) = H^0(V, \mathscr{F})$, $\mathscr{F}_{\mathrm{p}}(V) = H^0(V', \mathscr{F}_{\mathrm{p}})$, and $\mathscr{F}(V) = H^0(\bar{V}, \mathscr{F})$.

Lemma. i) *$\mathscr{F}_{\mathrm{p}}(V)$ does not depend on $\bar{Y}$ choosen in* (0.5) a).

ii) *$\mathscr{F}(V)$ does not depend on $\bar{V}$ choosen in* (0.5) c).

It does not require the existence of $\bar{\tau}$.

iii) *One has injections $\mathscr{F}(V) \to \mathscr{F}_{\mathrm{p}}(V) \to \mathscr{F}_{\mathrm{an}}(V)$.*

Proof. i) Let $Y \xrightarrow{\lambda_Y} Z \xrightarrow{\sigma_Y} \bar{Y}$ with $\sigma_Y \lambda_Y = l_Y$ be another good compactification. One has a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\lambda_Y"] & Z \arrow[r, "\sigma_Y"] & \bar{Y} \\ V \arrow[u] \arrow[r, "\lambda'"] & W \arrow[u] \arrow[r, "\sigma'"] & V' \arrow[u] \end{tikzcd}$$

with $W = Z - (\overline{Y-V})$.

One has $\sigma'_*\Omega^n_W(\log(W-V))=\Omega^n_{V'}(\log(V'-V))$.

From the exact sequence

$$0\to\mathcal{G}_p\to\lambda'_*\mathcal{F}_{an}\oplus\Omega^n_W(\log(W-V))\to\lambda'_*\Omega^n_V$$

one obtains the exact sequence

$$0\to\sigma'_*\mathcal{G}_p\to l'_*\mathcal{F}_{an}\oplus\Omega^n_{V'}(\log(V'-V))\to l'_*\Omega^n_V.$$

Therefore one has $\sigma'_*\mathcal{G}_p=\mathcal{F}_p$.

As for any other good compactification $Y\xrightarrow{l^1_Y}\bar{Y}_1$ there is a third one Z as above with $Y\xrightarrow{\lambda_Y}Z\xrightarrow{\sigma^1_Y}\bar{Y}_1$ such that $l^1_Y=\sigma^1_Y\lambda_Y$ and $l_Y=\sigma_Y\lambda_Y$, this proves i).

ii) Let $V\xrightarrow{\lambda}W\xrightarrow{\sigma}\bar{V}$ with $\sigma\lambda=l$ be another good compactification of V (without necessarily assuming that W and $\bar{V}$ map to $\bar{Y}$).

One has $\sigma_*\Omega^n_W(\log(W-V))=\Omega^n_{\bar{V}}(\log(\bar{V}-V))$.

From the exact sequence

$$0\to\mathcal{G}\to\lambda_*\mathcal{F}_{an}\oplus\Omega^n_W(\log(W-V))\to\lambda_*\Omega^n_V$$

one obtains the exact sequence

$$0\to\sigma_*\mathcal{G}\to l_*\mathcal{F}_{an}\oplus\Omega^n_{\bar{V}}(\log(\bar{V}-V))\to l_*\Omega^n_V$$

which proves that $\sigma_*\mathcal{G}=\mathcal{F}$.

One concludes as before

iii) By 0.5 c), one has that

$$H^0(\bar{Y},\bar{\iota}_*\mathcal{F})=H^0(\bar{V},\mathcal{F})=\mathcal{F}(V)$$

injects in

$$H^0(\bar{Y},\tau'_*\mathcal{F}_p)=H^0(V',\mathcal{F}_p)=\mathcal{F}_p(V).$$

(1.2) Define

$$\mathcal{F}_{an}(V)_{cl}:=\operatorname{Ker} d:\mathcal{F}_{an}(V)\to H^0(V,\mathcal{F}^{n+1}_{an}),$$
$$\mathcal{F}_p(V)_{cl}:=\operatorname{Ker} d:\mathcal{F}_p(V)\to H^0(V',\mathcal{F}^{n+1}_p),$$
$$\mathcal{F}(V)_{cl}:=\operatorname{Ker} d:\mathcal{F}(V)\to H^0(\bar{V},\mathcal{F}^{n+1}).$$

Obviously one may replace $H^0(V,\mathcal{F}^{n+1}_{an})$, $H^0(V',\mathcal{F}^{n+1}_p)$, and $H^0(\bar{V},\mathcal{F}^{n+1})$ by $H^0(V,\Omega^{n+1}_V)$, and the three groups defined do not depend on E choosen in (0.3).

Corollary. i) *The groups $\mathcal{F}_{an}(V)_{cl}$, $\mathcal{F}_p(V)_{cl}$, and $\mathcal{F}(V)_{cl}$ depend only on the choice of π in* (0.1) *and on V. They define Zariski sheaves on Y.*

ii) *One has injections*

$$\mathcal{F}(V)_{cl}\to\mathcal{F}_p(V)_{cl}\to\mathcal{F}_{an}(V)_{cl}.$$

(1.3) Let U be a Zariski open subset of X. We consider a good compactification of $V=\pi^{-1}U$ as in (0.5).

Lemma. i) *The group $\mathscr{F}(\pi^{-1}U)_{\mathrm{cl}}$ depends only on U. It defines a Zariski sheaf on X.*
ii) *If U is smooth, then one has $\mathscr{F}(\pi^{-1}U)_{\mathrm{cl}} = F^nH^n(U, \mathbb{C})$, the Hodge filtration.*

Proof. i) Let $\sigma: Z \to Y$ be a birational morphism such that Z is smooth and $F := \sigma^* E$ is a normal crossing divisor. Define $p := \pi\sigma$ and $W := \sigma^{-1}V$. Choose a good compactification $\lambda: W \to \bar{W}$ such that one has a commutative diagram

$$\begin{CD} W @>{\lambda}>> \bar{W} \\ @V{\sigma}VV @VV{\bar{\sigma}}V \\ V @>>{l}> \bar{V}. \end{CD}$$

One has $\bar{\sigma}_*\Omega^n_{\bar{W}}(\log(\bar{W}-W)) = \Omega^n_{\bar{V}}(\log(\bar{V}-V))$.
From the exact sequence

$$0 \to \mathscr{G} \to \lambda_*\sigma^*\mathscr{F}_{\mathrm{an}} \oplus \Omega^n_{\bar{W}}(\log(\bar{W}-V)) \to \lambda_*\Omega^n_W$$

one obtains the exact sequence

$$0 \to \bar{\sigma}_*\mathscr{G} \to l_*\mathscr{F}_{\mathrm{an}} \oplus \Omega^n_{\bar{V}}(\log(\bar{V}-V)) \to l_*\Omega^n_V$$

which shows that $\bar{\sigma}_*\mathscr{G} = \mathscr{F}$.
Therefore one has

$$\begin{aligned} \mathscr{F}(\pi^{-1}U)_{\mathrm{cl}} &= \mathrm{Ker}(\mathscr{F}(\pi^{-1}U) \to H^0(\pi^{-1}U, \Omega^{n+1}_{\pi^{-1}U}) \\ &= \mathrm{Ker}(\mathscr{G}(p^{-1}U) \to H^0(p^{-1}U, \Omega^{n+1}_{p^{-1}U}) \\ &= \mathscr{G}(\pi^{-1}U)_{\mathrm{cl}}. \end{aligned}$$

Now if $\pi_1: Y_1 \to X$ is another desingularization as in (0.1), we find a third one Z as above with $\sigma: Z \to Y$ and $\sigma_1: Z \to Y_1$ such that $p := \pi\sigma = \pi_1\sigma_1$.
ii) If U is smooth, replace in the previous argument V by U, $\mathscr{F}_{\mathrm{an}}$ by Ω^n_U, W by V. Then $\mathscr{F}$ is replaced by $\Omega^n_{\bar{V}}(\log(\bar{V}-V))$.

(1.4) We may now define on X_{zar} the sheaves we are interested in.
Let U be a Zariski open subset of X. Choose a compactification $\bar{X}$ as in (0.5) a). We consider $\mathbb{Z}(n)_{i_s}$ in $D^b(\bar{X})$ as defined in (0.8), which depends on U.
Define

$$H^n(n)_{i_s}(U) := H^n(\bar{X}, \mathbb{Z}(n)_{i_s})$$

and

$$\mathscr{F}_{i_s}(U) := \mathrm{Ker}(\mathscr{F}(\pi^{-1}U)_{\mathrm{cl}} \to H^n(U, i_{s!}\mathbb{C}/\mathbb{Z}(n)) \quad \text{for} \quad s = \emptyset, 0, \ldots, d.$$

Theorem and definition

1) *The groups $H^n(n)_{i_s}(U)$ depend only on U.*
2) *If $\sigma: X' \to X$ is any morphism, then one has a map*

$$\sigma^{-1}: H^n(n)_{i_\emptyset}(U) \to H^n(n)_{i_\emptyset}(\sigma^{-1}U).$$

3) *If σ is the embedding of a Zariski open subset W, one has maps*

$$\sigma^{-1}: H^n(n)_{i_s}(U) \to H^n(n)_{i_s}(U \cap W)$$

for $s = \emptyset, 0, \ldots, d$, and the groups $H^n(n)_{i_s}(U)$ define Zariski presheaves.

4) *Assume U to be affine.*

If $d < n-2$, then $H^n(n)_{i_d}(U) = H^n(n)_{i_\emptyset}(U)$.

If $n = 2$, then $H^2(2)_{i_0}(U) = H^2(2)_{i_\emptyset}(U)$ provided $S_0 \cap U$ is connected.

If $n > 2$, then $H^n(n)_{i_{d-1}}(U) = H^n(n)_{i_\emptyset}(U)$ if $d = n-2$, and $H^n(n)_{i_{d-2}}(U) = H^n(n)_{i_\emptyset}(U)$ if $d = n-1$.

5) *If X is smooth, then $H^n(n)_{i_\emptyset}(U) = H^n_{\mathscr{D}}(U, n) := H^n(\bar{U}, \mathbb{Z}(n)_{\mathscr{D}})$ (0.7) c), the Deligne-Beilinson group.*

6) *Define $\mathscr{H}^n(n)$ to be the Zariski sheaf associated to $H^n(n)_{i}$, and $\mathscr{H}^n(n)_{i_s}$ to be the one associated to $H^n(n)_{i_s}$ for $s = 0, \ldots, d$.*

If $d < n-2$, then $\mathscr{H}^n(n)_{i_d} = \mathscr{H}^n(n)$.

If $n = 2$, then $\mathscr{H}^2(2)_{i_0} = \mathscr{H}^2(2)$.

If $n > 2$, then $\mathscr{H}^n(n)_{i_{d-1}} = \mathscr{H}^n(n)$ if $d = n-2$ and $\mathscr{H}^n(n)_{i_{d-2}} = \mathscr{H}^n(n)$ if $d = n-1$.

At any case, there is always an integer s_0 with $0 \leqq s_0 \leqq d$ such that

$$\mathscr{H}^n(n)_{i_{s_0}} = \mathscr{H}^n(n).$$

If X is smooth, then $\mathscr{H}^n(n) = \mathscr{H}^n_{\mathscr{D}}(n)$, the Deligne-Beilinson sheaf associated to $\mathrm{H}^n_{\mathscr{D}}(U, n)$.

7) *If $\sigma: X' \to X$ is any morphism, one has a map $\sigma^{-1}: \mathscr{H}^n(n) \to \sigma_* \mathscr{H}^n(n)$. In other words, $\mathscr{H}^n(n)$ is functorial. In particular, if σ is any desingularization of X (not necessarily as in* (0.1)*), one has a map $\mathscr{H}^n(n) \to \sigma_* \mathscr{H}^n_{\mathscr{D}}(n)$.*

Proof. 1) One has an exact sequence

$$0 \to H^{n-1}(U, i_{s!}\mathbb{C}/\mathbb{Z}(n)) \to H^n(n)_{i_s}(U) \to \mathscr{F}_{i_s}(U) \to 0.$$

As $\mathscr{F}(\pi^{-1}U)_{\mathrm{cl}}$ depends only on U (1.3) *i*), $\mathscr{F}_{i_s}(U)$ depends only on U as well. This proves 1).

2), 3) Consider a commutative diagram

$$\begin{tikzcd} Y' \arrow[r, "\tau"] \arrow[d, "\pi'"'] & Y \arrow[d, "\pi"] \\ X' \arrow[r, "\sigma"] & X \end{tikzcd}$$

where π' and π are as in (0.1). In case 3) (σ is the embedding of an open set $X' = W$), just take $\pi' = \pi_{|W'}$.

Define $\mathscr{G}_{\mathrm{an}} := \pi'^* \Omega^n_{X'}/\text{torsion}$. Then $\tau^* \mathscr{F}_{\mathrm{an}}$ injects in $\mathscr{G}_{\mathrm{an}}$, and $\tau^* \Omega^{n+l}_Y(\log E)(-E)$ injects in

$$\Omega^{n+l}_{Y'}(\log \tau^{-1} E)(-\tau^* E).$$

Define E' such that $\Omega^n_{Y'}(\log E')(-E')$ *contains both* $\mathscr{G}_{\mathrm{an}}$ and $\tau^* \Omega^n_Y(\log E)(-E)$ (0.3). Define correspondingly $\mathscr{G}^{\geqq n}_{\mathrm{an}}$ (0.4).

If U is Zariski open in X, define $U' := \sigma^{-1}U$, $V' := \pi'^{-1}U'$, $V := \pi^{-1}U$. Take compactifications

$$\begin{tikzcd} V \arrow[r, "l'"] \arrow[d, "\pi'"'] & \bar{V}' \arrow[r, "\bar{\tau}"] \arrow[d, "\bar{\pi}'"] & \bar{V} \arrow[d, "\bar{\pi}"] & V \arrow[l, "l"'] \arrow[d, "\pi"] \\ U' \arrow[r, "k'"'] & \bar{X}' \arrow[r, "\bar{\sigma}"'] & \bar{X} & U \arrow[l, "k"] \end{tikzcd}$$

as in (0.5).

From the exact sequence

$$0 \to \bar{\tau}_* \mathscr{G} \to \bar{\tau}_* l'_* \mathscr{G}_{\mathrm{an}} \oplus \bar{\tau}_* \Omega^n_{\bar{V}'}(\log(\bar{V}' - V')) \to \bar{\tau}_* l'_* \Omega^n_{V},$$

and the maps

$$\mathscr{F}_{\mathrm{an}} \to \tau_* \mathscr{G}_{\mathrm{an}}, \qquad \Omega^n_{\bar{V}}(\log(\bar{V} - V)) \to \bar{\tau}_* \Omega^n_{\bar{V}'}(\log(\bar{V}' - V')),$$

$\mathscr{F}^{n+l} \to \bar{\tau}_* \mathscr{G}^{n+l}$ for $l \geqq 1$, one obtains maps $\mathscr{F} \to \bar{\tau}_* \mathscr{G}$ and

$$\mathscr{F}^{\geqq n} \to (\bar{\tau}_* \mathscr{G})^{\geqq n}.$$

This gives maps in $D^b(\bar{x})$

$$\begin{tikzcd} R\bar{\pi}_* \mathscr{F}^{\geqq n} \arrow[r] & R\bar{\pi}_*(\bar{\tau}_* \mathscr{G})^{\geqq n} \arrow[r] & R(\bar{\pi}\bar{\tau})_* \mathscr{G}^{\geqq n} \arrow[d, equal] \\ & & R\bar{\sigma} R\bar{\pi}'_* \mathscr{G}^{\geqq n} \end{tikzcd}.$$

One also has maps

$$\begin{tikzcd} Rk_* \mathbb{C}/\mathbb{Z}(n) \arrow[r] & Rk_* \sigma_* \mathbb{C}/\mathbb{Z}(n) \arrow[r] & R(k\sigma)_* \mathbb{C}/\mathbb{Z}(n) \arrow[d, equal] \\ & & R(\bar{\sigma}k')_* \mathbb{C}/\mathbb{Z}(n) \end{tikzcd}$$

and if σ is as in 3), maps

$$Rk_* i_{s!} \mathbb{C}/\mathbb{Z}(n) \to Rk_* \sigma_* i_{s!} \mathbb{C}/\mathbb{Z}(n) \to R(\bar{\sigma}k')_* i_{s!} \mathbb{C}/\mathbb{Z}(n).$$

Therefore one has maps

$$\mathbb{Z}(n)_{i_\emptyset} \to R\bar{\sigma}_* \mathbb{Z}(n)_{i_\emptyset} \text{ and if } \sigma \text{ is as in 3)},$$

$$\mathbb{Z}(n)_{i_s} \to R\bar{\sigma}_* \mathbb{Z}(n)_{i_s} \quad \text{for} \quad s = 0, \ldots, d.$$

Then $H^n(n)_i\,(U)$ maps to $H(n)_{i_\emptyset}\,(\sigma^{-1}U)$.

This proves 2).

Also in 3), $H^n(n)_{i_s}(U)$ maps to $H^n(n)_{i_s}(U \cap W)$. This proves 3).

4) If U is affine, then $S_s \cap U$ is affine as well and therefore $H^l(S_s \cap U, \mathbb{C}/\mathbb{Z}(n)) = 0$ for $l > s$. Now $H^n(n)_{i_s}(U)$ surjects onto $H^n(n)_{i_\emptyset}(U)$ if $H^n(S_s \cap U, \mathbb{C}/\mathbb{Z}(n)) = H^{n-1}(S_s \cap U, \mathbb{C}/\mathbb{Z}(n)) = 0$, and is isomorphic to it if moreover $H^{n-2}(U, \mathbb{C}/\mathbb{Z}(n))$ surjects onto $H^{n-2}(S_s \cap U, \mathbb{C}/\mathbb{Z}(n))$.

5) If X is smooth, then $\mathscr{F}$ is just $\Omega^n_{\bar{U}}(\log(\bar{U} - U))$ for a good compactification of U [Proof of (1.3) ii)].

6) By 2), $H^n(n)_{i_\emptyset}(U)$ maps to $H^n(n)_{i_\emptyset}(\sigma^{-1}U)$, which maps to $H^0(\sigma^{-1}U, \mathscr{H}^n(n))$. This proves 7), where one applies 5) if X' is smooth.

1.5) We define on Y_{zar} sheaves to which we will compare $\mathscr{H}^n(n)_{i_*}$ constructed in (1.4). Let V be a Zariski open subset of Y. Choose compactifications as in (0.5).

Define

$$H^n(n)_{j_s,\mathrm{an}}(V):=H^n(V,\mathbb{Z}(n)_{j_s,\mathrm{an}}),$$
$$H^n(n)_{j_s,p}(V):=H^n(V',\mathbb{Z}(n)^p_{j_s}),$$
$$H^n(n)_{j_s}(V):=H^n(\bar{V},\mathbb{Z}(n)_{j_s})$$

for $s=\emptyset, 0,\dots,d, \mathscr{D}$ with the convention $\mathbb{Z}(n)_{j_\mathscr{D}}=\mathbb{Z}(n)_\mathscr{D}$ etc....

Proposition and definition

1) *The groups $H^n(n)_{j_s,\mathrm{an}}(V)$, $H^n(n)_{j_s,\mathrm{p}}(V)$, $H^n(n)_{j_s}(V)$ depend only on V. They define Zariski presheaves on Y for $s=\emptyset, 0,\dots,d,\mathscr{D}$.*

2) *Let $\mathscr{H}^n(n)_{j_s,\mathrm{an}}$, $\mathscr{H}^n(n)_{j_s,\mathrm{p}}$, $\mathscr{H}^n(n)_{j_s}$ be the associated sheaves. There are injectives maps*

$$\mathscr{H}^n(n)_{j_s}\to\mathscr{H}^n(n)_{j_s,\mathrm{p}}\to\mathscr{H}^n(n)_{j_s,\mathrm{an}}$$

for $s=\emptyset, 0,\dots,d,\mathscr{D}$.

3) *There are maps*

$$\mathscr{H}^n(n)_{j_d}\to\dots\to\mathscr{H}^n(n)_{j_0}\to\mathscr{H}^n(n)_{j_\emptyset}\to\mathscr{H}^n(n)_\mathscr{D}$$

and similarly for $\mathscr{H}^n(n)_{j_s,\mathrm{p}}$ and $\mathscr{H}^n(n)_{j_s,\mathrm{an}}$.

Proof. 1) This is by definition for $H^n(n)_{j_s,\mathrm{an}}$. One has an exact sequence

(*) $\quad 0\to H^{n-1}(V,j_{s!}\mathbb{C}/\mathbb{Z}(n))\to H^n(n)_{j_s}(V)\to\mathrm{Ker}(\mathscr{F}(V)_{\mathrm{cl}}\to H^n(V,j_{s!}\mathbb{C}/\mathbb{Z}(n)))\to 0.$

As $\mathscr{F}(V)_{\mathrm{cl}}$ depends only on V (1.2) i), the kernel to $H^n(V,j_{s!}\mathbb{C}/\mathbb{Z}(n))$ depends only on V as well. Similarly for $H^n(n)_{j_s,\mathrm{p}}$.

2) One has

$$R\bar{\tau}_*\mathbb{Z}(n)_{j_s}=\mathrm{cone}(R\bar{\tau}_*\mathscr{F}^{\geqq n}\to R(l_Y\tau)_*j_{s!}\mathbb{C}/\mathbb{Z}(n))\,[-1],$$
$$R\tau_*\mathbb{Z}(n)^p_{j_s}=\mathrm{cone}(R\tau'_*\mathscr{F}^{\geqq n}_{\mathrm{p}}\to R(l_Y\tau)_*j_{s!}\mathbb{C}/\mathbb{Z}(n))\,[-1].$$

As $\mathscr{F}^{\geqq n}$ starts in degree n, one has a map $R^n\bar{\tau}_*\mathscr{F}^{\geqq n}[-n]\to R\bar{\tau}_*\mathscr{F}^{\geqq n}$ whose cone starts in degree $(n+1)$.

Define just for a moment in $D^b(\bar{Y})$

$$K=\mathrm{cone}(R^n\bar{\tau}_*\mathscr{F}^{\geqq n}[-n]\to R(l_Y\tau)_*j_{s!}\mathbb{C}/\mathbb{Z}(n))\,[-1].$$

Then one has an isomorphism

$$H^n(\bar{Y},K)=H^n(\bar{Y},R\bar{\tau}_*\mathbb{Z}(n)_{j_s}).$$

On the other hand, one has an injective map (0.6):

$$R^n\bar{\tau}_*\mathscr{F}^{\geqq n}\to R^n\tau'_*\mathscr{F}^{\geqq n}_{\mathrm{p}}$$

and again a map

$$R^n\tau'_*\mathscr{F}^{\geqq n}_{\mathrm{p}}[-n]\to R\tau'_*\mathscr{F}^{\geqq n}_{\mathrm{p}}.$$

Therefore

$$H^n(n)_{j_s}(V) = H^n(\bar{Y}, R\bar{\tau}_* \mathbb{Z}(n)_{j_s})$$

maps to

$$H^n(\bar{Y}, R\tau'_* \mathbb{Z}(n)^{\mathrm{p}}_{j_s}) = H^n(n)_{j_s,\mathrm{p}}(V).$$

Now write the sequence (*) and the corresponding sequence $(*)_{\mathrm{p}}$ for $H^n(n)_{j_s,\mathrm{p}}(V)$, and apply (1.2) ii).

This gives the injection $\mathcal{H}^n(n)_{j_s} \to \mathcal{H}^n(n)_{j_s,\mathrm{p}}$.

As for the second one consider the restriction map

$$H^n(V', \mathbb{Z}(n)^{\mathrm{p}}_{j_s}) \to H^n(V, \mathbb{Z}(n)^{\mathrm{p}}_{j_s|V}).$$

As $\mathbb{Z}(n)^{\mathrm{p}}_{j_s|V} = \mathbb{Z}(n)_{j_s,\mathrm{an}}$, this gives a map

$$H^n(n)_{j_s,\mathrm{p}} \to H^n(n)_{j_s,\mathrm{an}}(V).$$

One concludes as before. [Actually one could argue via the restriction map to construct the injection $\mathcal{H}^n(n)_{j_s} \to \mathcal{H}^n(n)_{j_s,\mathrm{an}}$.]

3) Apply (0.7).

(1.6) We could have defined on X_{zar} "partial" and "analytic" sheaves in the same way. As we will not use them, we do not give details.

(1.7) **Proposition.** *There is a map*

$$\mathcal{H}^n(n)_{i_s} \to \pi_* \mathcal{H}^n(n)_{j_s}.$$

Proof. By (0.10) there is a map, for each Zariski open set U in X:

$$\begin{array}{ccc} H^n(n)_{i_s}(U) = H^n(\bar{X}, \mathbb{Z}(n)_{i_s}) & \longrightarrow & H^n(\bar{V}, \mathbb{Z}(n)_{j_s}) \\ & & \| \\ & & H^n(n)_{j_s}(\pi^{-1}U), \end{array}$$

and one has a map

$$H^n(n)_{j_s}(\pi^{-1}U) \to H^0(\pi^{-1}U, \mathcal{H}^n(n)_{j_s}).$$

(1.8) **Proposition.** *There is a map*

$$\mathcal{H}^n(n)_{j_s,\mathrm{an}} \to R^n\alpha_* \mathbb{Z}(n)_{j_s,\mathrm{an}}.$$

Proof. One has $H^n(n)_{j_s,\mathrm{an}}(V) = H^n(V, \mathbb{Z}(n)_{j_s,\mathrm{an}})$ which maps to $H^0(V, R^n\alpha_* \mathbb{Z}(n)_{j_s,\mathrm{an}})$.

(1.9) 1) Let V be a Zariski open subset on Y, and take compactifications as in (0.5). One has

$$\begin{aligned} H^{n-1}(V, \mathscr{C}_{j_s,\mathrm{an}}) &= H^{n-1}(V', \mathscr{C}^{\mathrm{p}}_{j_s}) = H^{n-1}(\bar{V}, \mathscr{C}_{j_s}) \\ &= H^{n-2}(V \cap E_s, \mathbb{C}/\mathbb{Z}(n)). \end{aligned}$$

One also has

$$H^{n-1}(V, \mathbb{Z}(n)_{\mathcal{D},\mathrm{an}}) = H^{n-1}(V', \mathbb{Z}(n)^{\mathrm{p}}_{\mathcal{D}}) = H^{n-1}(\bar{V}, \mathbb{Z}(n)_{\mathcal{D}})$$
$$= H^{n-2}(V, \mathbb{C}/\mathbb{Z}(n)).$$

Denoting by $\mathcal{H}^k(\mathbb{C}/\mathbb{Z}(n))$ the Zariski sheaf on Y associated to the Betti cohomology $H^k(\mathbb{C}/\mathbb{Z})$), *we obtain*

Lemma. *There is an exact sequence*

$$0 \to \mathcal{H}^{n-2}(E_s, \mathbb{C}/\mathbb{Z}(n))/\mathcal{H}^{n-2}(\mathbb{C}/\mathbb{Z}(n)) \to \mathcal{H}^n(n)_{j_s} \to \mathcal{H}^n(n)_{\mathcal{D}}$$

for $s = \emptyset, 0, \ldots, d$.

2) As $H^0(\bar{V}, \Omega^{\geqq n}_{\bar{V}}(\log(\bar{V} - V))/\mathcal{F}^{\geqq n})$ might depend on $\bar{V}$, one can not define a sheaf on Y associated to $H^n(\bar{V}, \mathcal{C}_{j_s})$. Similarly for $\mathcal{C}^{\mathrm{p}}_{j_s}$.

But there is a restriction map

$$H^n(\bar{V}, \mathcal{C}_{j_s}) \xrightarrow{\text{rest}} H^n(V, \mathcal{C}_{j_s|V}) = H^n(V, \mathcal{C}_{j_s,\mathrm{an}}).$$

One has an exact sequence

$$0 \to H^{n-1}(V \cap E_s, \mathbb{C}/\mathbb{Z}(n)) \to H^n(V, \mathcal{C}_{j_s,\mathrm{an}})$$
$$\to \operatorname{Ker}(H^n(V, \Omega^{\geqq n}_{\bar{V}}/\mathcal{F}^{\geqq n}_{\mathrm{an}}) \to H^n(V \cap E_s, \mathbb{C}/\mathbb{Z}(n))) \to 0.$$

Define $\mathcal{H}^n(\mathcal{C}_{j_s})$ to be Zariski sheaf on Y associated to $H^n(V, \mathcal{C}_{j_s,\mathrm{an}})$.

Lemma. i) *There is a complex*

$$\mathcal{H}^n(n)_{j_s} \to \mathcal{H}^n(n)_{\mathcal{D}} \to \mathcal{H}^n(\mathcal{C}_{j_s})$$

and a map

$$\mathcal{H}^n(\mathcal{C}_{j_s}) \to R^n\alpha_*(\Omega^{\geqq n}_{\bar{V}}/\mathcal{F}^{\geqq n}_{\mathrm{an}}).$$

ii) *If* $n > \dim X$, *then*

$$\mathcal{H}^n(n)_{j_s} \to \mathcal{H}^n(n)_{\mathcal{D}}$$

is surjective and

$$\mathcal{H}^n(\mathcal{C}_{j_s}) = 0$$

iii) *If* $n = \dim X$, *then*

$$\mathcal{H}^n(\mathcal{C}_{j_s}) \to R^n\alpha_*(\Omega^{\geqq n}_{\bar{V}}/\mathcal{F}^{\geqq n}_{\mathrm{an}}) = \alpha_*\Omega^n_{Y}/\alpha_*\mathcal{F}_{\mathrm{an}}$$

is surjective.

Proof. i) One has an exact sequence

$$H^n(n)_{j_s}(V) \to H^n(n)_{\mathcal{D}}(V) \to H^n(\bar{V}, \mathcal{C}_{j_s}).$$

Applying the map rest, this gives the complex.

The sheaf associated to $H^n(V, \Omega^{\geqq n}_{\bar{V}}/\mathcal{F}^{\geqq n}_{\mathrm{an}})$ is just $R^n\alpha_*(\Omega^{\geqq n}_{\bar{V}}/\mathcal{F}^{\geqq n}_{\mathrm{an}})$.

ii) and iii) If V is affine, then $H^l(V \cap E_s, \mathbb{C}/\mathbb{Z}(n)) = 0$ for $l > \dim E_s$, especially if $l > \dim X - 1$. This proves that $H^n(\bar{V}, \mathcal{C}_{j_s}) = H^n(V, \mathcal{C}_{j_s,\mathrm{an}}) = 0$ if $n > \dim X$, and that $H^n(V, \mathcal{C}_{j_s,\mathrm{an}})$ surjects onto $H^n(V, \Omega^{\geqq n}_{\bar{V}}/\mathcal{F}^{\geqq n}_{\mathrm{an}})$ if $n = \dim X$.

Finally observe that $R^1\alpha_*\mathscr{F}_{\mathrm{an}}=0$ as $\mathscr{F}_{\mathrm{an}}$ is coherent, and therefore

$$R^n\alpha_*(\Omega_Y^{\geqq n}/\mathscr{F}_{\mathrm{an}}^{\geqq n})=\alpha_*\Omega_Y^n/\alpha_*\mathscr{F}_{\mathrm{an}}.$$

(1.10) *Multiplication*. Applying Beilinson's formulae [E–V], Sect. 3, where one replaces the F-filtration by our $\mathscr{F}^{\geqq n}$, one obtains multiplications:

$$\mathbb{Z}(n)_{j_s}\otimes_{\mathbb{Z}}\mathbb{Z}(m)_{j_s}\to\mathbb{Z}(n+m)_{j_s},$$
$$\mathbb{Z}(n)_{i_s}\otimes_{\mathbb{Z}}\mathbb{Z}(m)_{i_s}\to\mathbb{Z}(n+m)_{i_s}$$

which give products:

$$H^n(n)_{j_s}(V)\otimes_{\mathbb{Z}}H^m(m)_{j_s}(V)\to H^{n+m}(n+m)_{j_s}(V),$$
$$H^n(n)_{i_s}(U)\otimes_{\mathbb{Z}}H^m(m)_{i_s}(U)\to H^{n+m}(n+m)_{i_s}(U)$$

and at the sheaf level:

$$\mathscr{H}^n(n)_{j_s}\otimes_{\mathbb{Z}}\mathscr{H}^m(m)_{j_s}\to\mathscr{H}^{n+m}(n+m)_{j_s},$$
$$\mathscr{H}^n(n)\otimes_{\mathbb{Z}}\mathscr{H}^m(m)\to\mathscr{H}^{n+m}(n+m).$$

We observe that in order to perform this construction, one has to take desingularizations π where both $\pi^*\Omega_X^n$/torsion and $\pi^*\Omega_X^m$/torsion are locally free. This is allowed by (1.4) 1) and (1.5) 1).

Of course one obtains also a version for $H^n(n)_{j_s,\mathrm{p}}$, $H^n(n)_{j_s,\mathrm{an}}$ as well as for $R^n\alpha_*\mathbb{Z}(n)_{j_s,\mathrm{an}}$.

2. Definition of the regulator map on the Milnor K-theory

(2.1) We consider Bloch's regulator map

$$r_Z:\mathscr{K}_{nZ}^M\to\mathscr{H}_{\mathscr{D}}^n(n)$$

at the sheaf level from the Milnor K-theory to the Deligne-Beilinson cohomology on a smooth variety Z.

Recall the definition.

Let V be a Zariski open subset of Z, $g_1,\dots,g_n\in\Gamma(V,\mathcal{O}_Z^\times)$, the sheaf of regular invertible functions, and let $\{g_1,\dots,g_n\}$ be their symbol in $\Gamma(V,\mathscr{K}_{nZ}^M)$. Let $g:=(g_1,\dots,g_n):V\to(\mathbb{C}^\times)^n$ be the corresponding morphism, with x_i as coordinate on the i-th factor. Then $x_i\in H_{\mathscr{D}}^1((\mathbb{C}^\times)^n,1)$. The Deligne-Beilinson product $(x_1,\dots,x_n)\in H_{\mathscr{D}}^n((\mathbb{C}^\times)^n,n)$ factorizes over Steinberg symbols (via the existence of the dilogarithm function). Then

$$r_Z\{g_1,\dots,g_n\}:=g^{-1}(x_1,\dots,x_n)\in H_{\mathscr{D}}^n(V,n).$$

Call the situation

$$[x_i\in H_{\mathscr{D}}^1((\mathbb{C}^\times)^n,1),\ (x_1,\dots,x_n)\in H_{\mathscr{D}}^n((\mathbb{C}^\times)^n,n)]$$

the *universal* situation.

(2.2) For any morphism $\sigma: X' \to X$, we consider the natural map $\mathcal{K}^M_{nX'} \to \sigma_* \mathcal{K}^M_{nX'}$. If $\pi: Y \to X$ is any desingularization, we have the map of functoriality $\mathcal{H}^n(n) \to \pi_* \mathcal{H}^n_{\mathcal{D}}(n)$ (1.4) 5). If X is smooth, then $\mathcal{H}^n(n) = \mathcal{H}^n_{\mathcal{D}}(n)$ (1.4) 5).

Theorem. 1) *Let $\pi: Y \to X$ be any desingularization. There is a commutative diagram*

$$\begin{tikzcd} \mathcal{K}^M_{nX} \arrow[r] \arrow[d, "\varrho"'] & \pi_* \mathcal{K}^M_{nY} \arrow[d, "\pi_* r_Y"] \\ \mathcal{H}^n(n) \arrow[r] & \pi_* \mathcal{H}^n_{\mathcal{D}}(n) . \end{tikzcd}$$

2) *If X is smooth, then $\varrho = r_X$.*

3) *If $\sigma: X' \to X$ is any morphism, there is a commutative diagram*

$$\begin{tikzcd} \mathcal{K}^M_{nX} \arrow[r] \arrow[d, "\varrho"'] & \sigma_* \mathcal{K}^M_{nX'} \arrow[d, "\sigma_* \varrho"] \\ \mathcal{H}^n(n) \arrow[r] & \sigma_* \mathcal{H}^n(n) . \end{tikzcd}$$

Proof. 1) Let $p \in X$ be a point, $f_1, \ldots, f_n$ be regular functions in p. Choose a Zariski open neighbourhood U of p such that $f_i \in \Gamma(U, \mathcal{O}^\times)$. Define $V = \pi^{-1} U$, f to be the map $(f_i): U \to (\mathbb{C}^\times)^n$, and $g = f\pi$, with $g_i = \pi^* f_i$. By the functoriality (1.4) 2) f^{-1} maps $(x_1, \ldots, x_n) \in H^n_{\mathcal{D}}((\mathbb{C}^\times)^n, n)$ to an element which we call $\varrho\{f_1, \ldots, f_r\}$ in $H^n(n)_{i_0}(U)$. By definition $\pi^{-1}\{f_1, \ldots, f_n\} = r_Y\{g_1, \ldots, g_n\}$ and it lies in $H^n_{\mathcal{D}}(V, n)$.

2) is by construction.

3) Take the notations of 1). Then one has

$$f^{-1}(x_1, \ldots, x_n) = \varrho\{f_1, \ldots, f_n\} \in H^n(n)_{i_0}(\sigma^{-1} U)$$

which maps to

$$\sigma^{-1} f^{-1}(x_1, \ldots, x_n) = \varrho\{\sigma^{-1} f_1, \ldots, \sigma^{-1} f_n\} \quad \text{in} \quad H^n(n)_{i_0}(\sigma^{-1} U) .$$

(2.3) Following Srinivas [S], define the sheaves $\mathcal{B}$ and $\mathcal{A}$ on X_{zar}, which are supported on S, by the exact sequence

$$0 \to \mathcal{B} \to \mathcal{K}^M_{nX} \to \pi_* \mathcal{K}^M_{nY} \to \mathcal{A} \to 0 .$$

As $\pi_* \mathcal{K}^M_{nY}$ depends on the desingularization chosen in (0.1), $\mathcal{A}$ and $\mathcal{B}$ do too.

Choose s_0 to be the maximum integer with $0 \leqq s_0 \leqq d$ such that $\mathcal{H}^n(n) = \mathcal{H}^n(n)_{i_{s_0}}$ (1.4) 4).

Theorem. *For any s with $0 \leqq s \leqq s_0 \leqq d$, there is a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{B} \arrow[r] \arrow[dd] & \mathcal{K}^M_{nX} \arrow[r] \arrow[d, "\varrho"] & \pi_* \mathcal{K}^M_{nY} \arrow[r] \arrow[dd, "\pi_* r_Y"] & \mathcal{A} \arrow[r] \arrow[dd] & 0 \\ & & \mathcal{H}^n(n) \arrow[d] & & & \\ 0 \arrow[r] & \pi_* \dfrac{\mathcal{H}^{n-2}(E_S, \mathbb{C}/\mathbb{Z}(n))}{\mathcal{H}^{n-2}(\mathbb{C}/\mathbb{Z}(n))} \arrow[r] & \pi_* \mathcal{H}^n(n)_{j_s} \arrow[r] \arrow[dr] & \pi_* \mathcal{H}^n_{\mathcal{D}}(n) \arrow[r] & \pi_* \mathcal{H}^n(\mathcal{C}_{j_s}) \arrow[d] & \\ & & & \pi_* \mathcal{H}^n(n)_{j_0} \arrow[u] & \pi_* R^n \alpha_* (\Omega^{\geqq n}_{\bar{Y}} / \mathcal{F}^{\geqq n}_{\text{an}}) & \end{tikzcd}$$

where the bottom horizontal row is a complex.

Moreover the sequence

$$0 \to \pi_* \frac{\mathscr{H}^{n-2}(E_S, \mathbb{C}/\mathbb{Z}(n))}{\mathscr{H}^{n-2}(\mathbb{C}/\mathbb{Z}(n))} \to \pi_* \mathscr{H}^n(n)_{j_s} \to \pi_* \mathscr{H}^n_{\mathscr{D}}(n)$$

is exact.

Proof. Put together (2.2) and (1.9).

(2.4) *Remark.* This way of mapping $\mathscr{K}^M_{nX}$ in $\pi_* \mathscr{H}^n(n)_{j_s}$ [and a fortiori to $\pi_* \mathscr{H}^n(n)_{j_\emptyset}$] is not as good as considering ϱ itself as $\pi_* \mathscr{H}^n(n)_{j_s}$ depends on the desingularization chosen. However we will now consider the cohomology of ϱ, and it is not clear how to compute the cohomology of $\mathscr{H}^n(n)$. That is the reason why we will "approximate" it by the cohomology of $\mathscr{H}^n(n)_{j_s}$ [or of $\mathscr{H}^n(n)_{j_\emptyset}$].

(2.5) Define $\mathscr{K} := \mathscr{K}^M_{nX}/\mathscr{B}$.
As $\mathscr{B}$ and $\mathscr{A}$ are supported in S of dimension d, one has

$$H^q(X, \mathscr{K}^M_{nX}) = H^q(X, \mathscr{K}) \quad \text{for} \quad q > d,$$
$$H^q(X, \mathscr{K}) = H^q(X, \pi_* \mathscr{K}^M_{nY}) \quad \text{for} \quad q > d+1.$$

Therefore one has exact sequences

$$O \to H^d(\mathscr{B})/H^{d-1}(\mathscr{K}) \to H^d(\mathscr{K}^M_{nX}) \to H^d(\mathscr{K}) \to O,$$
$$O \to H^d(\mathscr{A})/H^d(\pi_* \mathscr{K}^M_{nY}) \to H^{d+1}(\mathscr{K}^M_{nX}) \to H^{d+1}(\pi_* \mathscr{K}^M_{nY}) \to O.$$

(2.6) **Lemma.** *One has*

$$R^m \alpha_* \mathbb{Z}(n)_{j_s, \mathrm{an}} = R^{m-1} \alpha_* j_{s!} \mathbb{C}/\mathbb{Z}(n) \quad \textit{for} \quad m < n,\ s = \mathscr{D}, \emptyset, 0, \ldots, d,$$
$$= \mathscr{H}^{m-1}(\mathbb{C}/\mathbb{Z}(n)) \quad \textit{for} \quad s = \mathscr{D}, \emptyset.$$

Proof. The first equality comes just from the fact that $\mathscr{F}^{\geq n}_{\mathrm{an}}$ and $\Omega^{\geq n}_{\mathrm{an}}$ start in degree n. The second one is due to Deligne [B 2].

(2.7) Consider the spectral sequence

$$E_2^{k,l} = H^k(Y_{\mathrm{zar}}, R^l \alpha_* \mathbb{Z}(n)_{j_s, \mathrm{an}}) \Rightarrow H^{k+l}(Y_{\mathrm{an}}, \quad_{\mathrm{an}}, \mathbb{Z}(n)_{j_s, \mathrm{an}}).$$

By abuse of notation, we write the graded pieces $\sum_{i \geq 1} E_\infty^{k+i, l-i}$ instead of the corresponding filtration on $H^{k+l}(Y_{\mathrm{an}}, \mathbb{Z}(n)_{j_s, \mathrm{an}})$.

Proposition. *Let s be as in* (2.3). *Let* $q \geq n-2$. *Assume that*

$$H^{q+i}(Y, R^{n-i} \alpha_* j_{s!} \mathbb{C}/\mathbb{Z}(n)) = 0 \quad \text{for} \quad i \geq 2.$$

1) *Then one has a commutative diagram*

$$\begin{tikzcd}
H^q(X, \mathscr{K}^M_{nX}) \arrow[rr] \arrow[d, "\varrho"] & & H^q(X, \pi_* \mathscr{K}^M_{nY}) \arrow[d, "\pi_* r_Y"] \\
H^q(X, \mathscr{H}^n(n)) \arrow[rr] \arrow[d] & & H^q(X, \pi_* \mathscr{H}^n_{\mathscr{D}}(n)) \arrow[d] \\
\dfrac{H^{q+n}(Y, \mathbb{Z}(n)_{j_s, \mathrm{an}})}{\sum\limits_{i \geqq 1} E_\infty^{q+i, n-i}} \arrow[rr] \arrow[dr] & & H^{q+n}(Y, \mathbb{Z}(n)_{\mathscr{D}, \mathrm{an}}) \\
& H^{q+n}(Y, \mathbb{Z}(n)_{j_\emptyset, \mathrm{an}}) . \arrow[ur] &
\end{tikzcd}$$

2) $\sum\limits_{i \geqq 1} E_\infty^{q+i, n-i}$ *is contained in* $H^{q+n-2}(E_s, \mathbb{C}/\mathbb{Z}(n))/H^{q+n-2}(Y, \mathbb{C}/\mathbb{Z}(n))$ *which maps to*

$$H^{q+n-1}(Y, \mathscr{C}_{j_s, \mathrm{an}})/H^{q+n}(Y, \mathbb{Z}(n)_{j_s, \mathrm{an}}) .$$

Proof. 1) Consider the diagram (2.3).

One has maps

$$H^q(X, \pi_* \mathscr{H}^n(n)_{j_s}) \to H^q(Y, \mathscr{H}^n(n)_{j_s}) \xrightarrow[(1.5)2)]{} H^q(Y, \mathscr{H}^n(n)_{j_s, \mathrm{an}})$$
$$\xrightarrow[(1.8)]{} H^q(Y, R^n \alpha_* j_{s!} \mathbb{C}/\mathbb{Z}(n)) .$$

One has $E_2^{q+i, n-i+1} = H^{q+i}(Y, R^{n-i} \alpha_* j_{s!} \mathbb{C}/\mathbb{Z}(n))$ for $i \geqq 2$ (2.6). This vanishes by hypothesis. Therefore

$$H^q(Y, R^n \alpha_* j_{s!} \mathbb{Z}(n)_{j_s, \mathrm{an}})$$

maps to $H^{q+n}(Y, \mathbb{Z}(n)_{j_s, \mathrm{an}}) \Big/ \sum\limits_{i \geqq 1} E_\infty^{q+i, n-i}$.

On the other hand as $H^j(\mathscr{H}^{m-1}(\mathbb{C}/\mathbb{Z}(n))) = 0$ for $j \geqq m$ [B 1], one has $E_2^{q+i, n-i+1} = 0$ for $i \geqq 2$, and $E_2^{q+i, n-i} = E_\infty^{q+i, n-i} = 0$ for $i \geqq 1$ and $s = \mathscr{D}$ or $\emptyset$.

2) As $E_2^{q+i, n-i} = H^{q+i}(Y, R^{n-i-1} \alpha_* j_{s!} \mathbb{C}/\mathbb{Z}(n))$ for $i \geqq 1$, (2.6), $\sum\limits_{i \geqq 1} E_\infty^{q+i, n-i}$ maps to

$$H^{q+n-1}(Y, j_{s!} \mathbb{C}/\mathbb{Z}(n))$$

which maps to $H^{q+n}(Y, \mathbb{Z}(n)_{j_s, \mathrm{an}})$. For $i \geqq 1$, one has $q+i > n-i-1$. Therefore $H^{q+i}(Y, R^{n-i-1} \alpha_* j_{s!} \mathbb{C}/\mathbb{Z}(n))$, and $\sum\limits_{i \geqq 1} E_\infty^{q+i, n-i}$ maps to 0 in $H^{q+n-1}(E_s, \mathbb{C}/\mathbb{Z}(n))$; in other words it is contained in

$$H^{q+n-2}(E_s, \mathbb{C}/\mathbb{Z}(n))/H^{q+n-2}(Y, \mathbb{C}/\mathbb{Z}(n)) .$$

(2.8) *Example 1.* Assume $n = 2$, $d = 0$ or 1, $q = 1$; then $s_0 = 0$. Then

$$R^0 \alpha_* j_{0!} \mathbb{C}/\mathbb{Z}(2) = j_{0!} \mathbb{C}/\mathbb{Z}(2) .$$

From the exact sequence

$$0 \to j_{0!} \mathbb{C}/\mathbb{Z}(2) \to \mathbb{C}/\mathbb{Z}(2) \to \mathbb{C}/\mathbb{Z}(2)_{|S = S_0} \to 0$$

one obtains $H^i_{\mathrm{zar}}(j_{0!} \mathbb{C}/\mathbb{Z}(2)) = 0$ for $i \geqq 2$.

Therefore one has $H^{q+i}(Y, R^{n-i}\alpha_* j_{s!}\mathbb{C}/\mathbb{Z}(2))=0$ for $i \geqq 2$ and $E_2^{3,0}=E_2^{2,1}=0$. One obtains a commutative diagram

$$\begin{array}{ccc} H^1(X, \mathscr{K}_{2X}) & \longrightarrow & H^1(X, \pi_* \mathscr{K}_{2Y}) \\ \downarrow & & \downarrow \\ H^3(Y, \mathbb{Z}(2)_{j_0, \mathrm{an}}) & \longrightarrow & H^3(Y, \mathbb{Z}(2)_{\mathscr{D}, \mathrm{an}}) . \end{array}$$

If $d=0$, one has a map (0.3):

$$\mathscr{F}_{\mathrm{an}} \to \Omega_Y^2(\log F)(-2F),$$
$$\mathcal{O}_Y(-F) := \pi^* \mathscr{I}_0/\text{torsion}.$$

Therefore $\mathbb{Z}(2)_{j_0, \mathrm{an}} \to \mathbb{Z}(2)_{\mathscr{D}, \mathrm{an}}$ factorizes over

$$\mathbb{Z}(2)' := j_{0!}\mathbb{Z}(2) \to \mathcal{O}_Y(-2F) \to \Omega_Y^1(\log F)(-2F)$$

and one obtains a diagram

$$\begin{array}{ccc} H^1(X, \mathscr{K}_{2X}) & \xrightarrow{\pi_*} & H^1(X, \pi_* \mathscr{K}_{2Y}) \\ \downarrow & & \downarrow \\ H^3(Y, \mathbb{Z}(2)') & \longrightarrow & H^3(Y, \mathbb{Z}(2)_{\mathscr{D}, \mathrm{an}}) . \end{array}$$

If X is a proper surface with one isolated conelike singularity, (in this case $F = F_{\mathrm{red}}$ is a smooth curve), the left vertical arrow was constructed by Collino [C] [on a subgroup of $H^1(X, \mathscr{K}_{2X})$].

(2.9) Let $(d\log)^q$ be the map

$$(d\log)^q : H^q(Y, \mathscr{K}_{nY}^M) \to H^{q+n}(Y, \Omega_Y^{\geqq n})$$

and α be the map

$$\alpha : H^{d+n}(Y, \mathbb{Z}(n)_{\mathscr{D}, \mathrm{an}}) \to H^{d+n}(Y, \Omega_Y^{\geqq n}).$$

If $d \geqq n-2$, then $(d\log)^d$ factorizes α (2.7).

Proposition. 1) *If* $(d\log)^q = 0$ *one has a map*

$$H^q(X, \mathscr{A})/H^q(Y, \pi_* \mathscr{K}_{nY}^M) \to H^{q+n}(Y, \Omega_Y^{\geqq n}/\mathscr{F}_{\mathrm{an}}^{\geqq \mathrm{an}}),$$

2) *if* $\alpha = 0$ *and* $d \geqq n-3$, *one has a commutative diagram*

$$\begin{array}{ccccc} 0 \longrightarrow & H^d(X, \mathscr{A})/H^d(X, \pi_* \mathscr{K}_{nY}^M) & \longrightarrow & H^{d+1}(X, \mathscr{K}_{nX}^M) \\ & \downarrow & & \downarrow \\ 0 \longrightarrow & H^{d+n}(Y, \Omega_Y^{\geqq n}/\mathscr{F}_{\mathrm{an}}^{\geqq \mathrm{an}}) & \longrightarrow & H^{d+1+n}(Y, \mathbb{Z}(n)_{j_{\mathscr{D}}, \mathrm{an}}), \end{array}$$

where the two sequences are exact.

3) *If* $\alpha = 0$, $d \geqq n-3$, *take s as in* (2.3). *Assume moreover that*

$$H^{d+1+i}(Y, R^{n-i}\alpha_* j_{s!}\mathbb{C}/\mathbb{Z}(n)) = 0 \quad \text{for} \quad i \geqq 2.$$

Then the diagram in 2) *factorizes over the exact sequence*

$$0 \to \frac{H^{d+n}(Y, \mathscr{C}_{j_s, \mathrm{an}})}{H^{d+n}(Y, \mathbb{Z}(n)_{\mathscr{D}, \mathrm{an}}) + \sum_{i \geqq 1} E_\infty^{d+1+i, n-i}} \to \frac{H^{d+1+n}(Y, \mathbb{Z}(n)_{j_s, \mathrm{an}})}{\sum_{i \geqq 1} E_\infty^{d+1+i, n-i}}.$$

Proof. 1) Apply (2.3) and notice that one has maps

$$H^q(X, \pi_* R^n \alpha_* (\Omega_{\bar{Y}}^{\geqq n} / \mathscr{F}_{\mathrm{an}}^{\geqq n})) \to H^q(Y, R^n \alpha_* (\Omega_{\bar{Y}}^{\geqq n} / \mathscr{F}_{\mathrm{an}}^{\geqq n})) \to H^q(Y, \Omega_{\bar{Y}}^{\geqq n} / \mathscr{F}_{\mathrm{an}}^{\geqq n})$$

as the complex $\Omega_{\bar{Y}}^{\geqq n} / \mathscr{F}_{\mathrm{an}}^{\geqq n}$ starts in degree n.

2) Apply (2.7) and notice that

$$\mathrm{cone}(\mathbb{Z}(n)_{j_s, \mathrm{an}} \to \mathbb{Z}(n)_{\mathscr{D}, \mathrm{an}}) = \mathrm{cone}(\Omega_{\bar{Y}}^{\geqq n} / \mathscr{F}_{\mathrm{an}}^{\geqq n})[-1].$$

3) Apply (2.7) again.

(2.10) *Example 2.* Assume that X is an affine cone over a smooth projective variety E_0 of dimension $< n$. Set $\pi: Y \to X$ be the blow up of the vertex $0 = S_0 = S$, and $p: Y \to E_0$ be the corresponding $\mathbb{A}^1$-bundle.

Then $F_{\mathbb{Z}}^n(Y) := \mathrm{Ker}(F^n H^n(Y, \mathbb{C}) \to H^n(Y, \mathbb{C}/\mathbb{Z}(n))$ is vanishing as it embeds in $Gr_n^W H^n(Y, \mathbb{C})$, and this last group is zero since Y has a good compactification with a smooth divisor at infinity. (Here W is the weight filtration.)

As $(d\log)^0 : H^0(X, \pi_* \mathscr{K}_{nY}^M) = H^0(Y, \mathscr{K}_{nY}^M) \to H^0(Y, \Omega_Y^n)$ factorizes over $F_{\mathbb{Z}}^n(Y)$, it is zero as well. Therefore one obtains (2.9) 1) for $q = 0$: one has a map

$$H^0(X, \mathscr{A}) / H^0(X, \pi_* \mathscr{K}_{nY}^M) \to H^0(Y, \Omega_Y^n / \mathscr{F}_{\mathrm{an}}^n).$$

By (0.3), $\mathscr{F}_{\mathrm{an}}^n$ embedds in $\Omega_Y^n(\log E_0)(-n \cdot E_0)$.

As $\Omega_Y^n / \Omega_Y^n(\log E_0)(-n \cdot E_0) = \omega_{(n-1)E_0}(-(n-1) \cdot E_0)$, where ω is the dualizing sheaf, one obtains a map

$$H^0(X, \mathscr{A}) / H^0(X, \pi_* \mathscr{K}_{nY}^M) \to H^0(Y, \omega_{(n-1)E_0}(-(n-1) \cdot E_0)).$$

If E_0 is a curve, this is Srinivas map.

Actually in this case, Srinivas proves that

$$H^1(X, \mathscr{K}_{2X}) = H^0(X, \mathscr{A}) / H^0(X, \pi_* \mathscr{K}_{2Y}^M),$$

where $\mathscr{A}$ is by definition $\varinjlim_{0 \in U} H^0(\pi^{-1} U, \mathscr{K}_{2Y}) / K_2(\mathcal{O}_{X,0})$.

(2.11) *Example 3.* Assume X proper. As α factorizes over

$$\mathrm{Ker}(H^{d+n}(Y, \Omega_{\bar{Y}}^{\geqq n}) \to H^{d+n}(Y, \mathbb{C}/\mathbb{Z}(n))),$$

which is 0 for $d \leqq n-1$, one obtains the diagram (2.9) 2).

(2.12) *Example 4.* 1) Assume $n = 2$, $d = 0$ or 1 as in Example 1, (2.8), and assume moreover that X is proper. Then one has (2.9) 3) with

$$H^{d+2}(Y, \mathbb{Z}(2)_{\mathscr{D}, \mathrm{an}}) = H^{d+1}(Y, \mathbb{C}/\mathbb{Z}(2)) / F^2 H^{d+1}(Y, \mathbb{C}),$$

$$\sum_{i \geqq 1} E_\infty^{d+1+i, 2-i} = 0.$$

If $d=0$, then $\mathbb{Z}(2)_{\mathscr{D},\mathrm{an}}$ maps to $\mathbb{Z}(2)'$ as in (2.8), and $\mathscr{C}_{j_0,\mathrm{an}}$ maps to

$$\mathscr{C}'=\mathrm{cone}(\Omega_{\bar{Y}}^{\geq 2}/\Omega_{\bar{Y}}^{\geq 2}(\log F)(-2F)\to\mathbb{C}/\mathbb{Z}(2)\,|E_0)[-1].$$

One may map the sequence of (2.9) 3) to the similar one replacing $\mathbb{Z}(2)_{j_0,\mathrm{an}}$ by $\mathbb{Z}(2)'$, $\mathscr{C}_{j_0,\mathrm{an}}$ by $\mathscr{C}'$.

2) Let X be a singularity of type A_2, of equation t^3-xy. One knows (letter of Collino), that $\mathscr{A}$ contains $\mathbb{C}\oplus\mathbb{C}$ if one takes $\pi: Y\to X$ to be the blow up of the singularity 0.

We first define candidates α and β in $\pi_*(\mathscr{K}_{2Y})_0$ for those two elements (as we do not know exactly how Collino constructs them ...), and then we prove via (2.3) that they contribute to $\mathscr{A}$.

A) Cover Y by three Zariski open sets Y_0, Y_1, Y_2 of coordinates and equations

$$Y_0:(a,b,t),\qquad x=at\,,y=bt\,;\,t-ab\,,$$
$$Y_1:(x,b',T),\qquad y=b'x,\qquad t=Tx;\qquad T^3x-b'\,,$$
$$Y_2:(a',y,T'),\qquad x=a'y,\qquad t=T'y;\qquad T'^3y-a'\,.$$

Consider $Y'=Y-\{t^2=1\}$. The exceptional locus of π is contained in Y'. Define $Y_i':=Y'\cap Y_i$.

We consider the two Loday symbols in $K_2(Y_0')$ [see Be] for the definition:

$$\alpha_0:=\{1-ab,b\},\qquad \beta_0:=\{1-(ab)^2,b^2\}\,.$$

In $K_2(Y_0'\cap Y_1')$, one has

$$\alpha_{0|Y_0'\cap Y_1'}=\{1-Tx,T^2x\}\,.$$

As T is a unit on $Y_0'\cap Y_1'$, $\alpha_{0|Y_0'\cap Y_1'}$ is the sum of the normal Steinberg symbol $\{1-Tx,T\}$ and of the Loday symbol $\{1-Tx,Tx\}$. The later is zero as it is zero on $Y_0'\cap Y_1'\cap(Tx\neq 0)$ where it is a Steinberg symbol, and it is uniquely determined by its restriction on $Y_0'\cap Y_1'\cap(Tx\neq 0)$.

Therefore $\alpha_{0|Y_0\cap Y_1}=\alpha_{1|Y_0\cap Y_1}$ where $\alpha_1\in Y_1'$ is the Steinberg symbol $\{1-Tx,T\}$. Similarly, as T' is a unit on $Y_0'\cap Y_2'$, one has $\alpha_{0|Y_0\cap Y_2}=\alpha_{2|Y_0\cap Y_2}$ where $\alpha_2\in K_2(Y_2')$ is the Steinberg symbol $-\{1-T'y,T'\}$. One computes in the same way that

$$\alpha_{1|Y_1\cap Y_2}=\alpha_{2|Y_1\cap Y_2}\in K_2(Y_1'\cap Y_2')\,.$$

Define $\alpha\in H^0(Y',\mathscr{K}_{2Y})$ to be α_i on Y_i'.

In $K_2(Y_0'\cap Y_1')$ one has $\beta_{0|Y_0\cap Y_1}=\{1-(Tx)^2,(T^2x)^2\}$.

Similarly as before, $\beta_{0|Y_0'\cap Y_1'}$ is equal to the Steinberg symbol

$$\{1-(Tx)^2,T^2\}\in K_2(Y_0'\cap Y_1')\,,$$

restriction of the Lorelei symbol $\beta_1=\{1-(Tx)^2,T^2\}\in K_2(Y_1')$.

One also has $\beta_{0|Y_0'\cap Y_2}=\beta_{2|Y_0'\cap Y_0}$ where $\beta_2\in K_2(Y_2')$ is the Loday symbol $-\{1-(T'y)^2,T'^2\}$, and $\beta_{1|Y_1\cap Y_2}=\beta_{2|Y_1\cap Y_2}$ in $K_2(Y_1'\cap Y_2')$. Define $\beta\in H^0(Y',\mathscr{K}_{2Y})$ to be β_i on Y_i'.

B) One has $\pi^*\mathcal{I}_0/\text{torsion} = \mathcal{O}_Y(-E)$ with $E = E_1 + E_2$, $E_i^2 = -2$ and $E_1 \cap E_2 = : p$. One has $\pi^*\Omega_X^2/\text{torsion} = m_p\Omega_Y^2(-E)$, where m is the maximal ideal of p. Moreover, as $\pi^*\Omega_X^2/\text{torsion}$ is generated by global sections and $(X, 0)$ is a rational singularity, one has $R^1\pi_*(\pi^*\Omega_X^2/\text{torsion}) = 0$. If $\sigma: Z \to Y$ is the blow up of p with exceptional line F, one has

$$\mathcal{F}_{\text{an}} = \sigma^*\pi^*\Omega_X^2/\text{torsion} = \sigma^*\Omega_Y^2(-E) \otimes \mathcal{O}_Z(-F).$$

As $R^1\sigma_*\mathcal{O}_Z(-F) = 0$, one obtains

$$\pi_*\sigma_*(\Omega_Z^2/\mathcal{F}_{\text{an}}) = \pi_*(\Omega_Y^2/m\Omega_Y^2(-E)) = \mathbb{C}_p \oplus \mathbb{C}.$$

where $\mathbb{C}_p$ is $\Omega_Y^2(-E)/m\Omega_Y^2(-E)$ and $\mathbb{C} = H^0(E, \omega_E(-E))$.

C) We consider the map

$$\begin{array}{l} d\log = H^0(Y', \mathcal{K}_{2Y'}) \longrightarrow H^0(\pi(Y'), \pi_*\Omega_Y^2) \\ \qquad\qquad \| \\ H^0(\pi(Y'), \pi_*\mathcal{K}_{2Y'}). \end{array}$$

One has

$$d\log\alpha = -\frac{da \wedge db}{1-ab} = -\frac{dx \wedge dT}{1-xT} = \frac{dy \wedge dT'}{1-yT'},$$

$$\frac{1}{4} d\log\beta = -ab\frac{da \wedge db}{1-(ab)^2} = -xT\frac{dx \wedge dT}{1-(xT)^2} = yT'\frac{dy \wedge dT'}{1-(yT')^2}.$$

On Y_0', $m\Omega_Y^2(-E)$ is generated by

$$a^2b\frac{da \wedge db}{1-ab} \quad \text{and} \quad ab^2\frac{da \wedge db}{1-ab}.$$

Therefore $d\log\alpha$, $d\log\beta$ define two linearly independent elements of

$$H^0(\pi(Y'), \pi_*(\Omega_Y^2/m\Omega_Y^2(-E))).$$

(2.13) One may also consider the map

$$H^q(X, \mathcal{B}) \to H^q(Y, \mathcal{H}^{n-2}(E_s, \mathbb{C}/\mathbb{Z}(n))/\mathcal{H}^{n-2}(Y, \mathbb{C}/\mathbb{Z}(n))). \tag{2.3}$$

Of course if $n = 2$, and E_s is connected, the second group is trivial. In general I do not know how to compute it. This is related to finding good assumptions under which the conditions (2.7) are fulfilled.

(2.14) Levine [L] defines another presheaf on X. If U is a Zariski subset of X, such that a compactification $\bar{U}$ exists with the property that $\bar{U} - U$ is supported by a Cartier divisor, he defines $\Omega_{\bar{U}}^{\cdot}(\log(\bar{U} - U))$ as those forms which have logarithmic growth along $\bar{V} - V$ where V and $\bar{V}$ are as in (0.5). Further, he takes the cone of $\Omega_{\bar{U}}^{\geq n}(\log(\bar{U} - U))$ with values in the cone of $\mathbb{Z}(n)$ in the de Rham complex $\Omega_U^{\cdot}$.

As I kill the torsion of $\Omega_{\bar{U}}^{\cdot}(\log(\bar{U} - U))$ by taking a desingularization for which the Kähler differentials become locally free, "his" forms lift "mine". As I take the cone with values in $\mathbb{C}/\mathbb{Z}(n)$, which maps to $\Omega_U^{\cdot}/\mathbb{Z}(n)$, "my" Betti part lifts "his". So one does not obtain a map in either direction.

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