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HÉLÈNE ESNAULT
ECKART VIEHWEG

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Effective bounds for semipositive sheaves and for the height of points on curves over complex function fields*

HÉLÈNE ESNAULT[1] & ECKART VIEHWEG[2]

[1]*Max-Planck-Institut für Mathematik, Gottfried-Claren-Str. 26, D-5300 Bonn 3, FRG;* [2]*FB6, Mathematik, Universität-GH-Essen, Universitätsstr 3, D-4300 Essen 1, FRG*

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In this note we prove an effective version of the positivity theorems for certain direct image sheaves for fibre spaces over curves and apply it to obtain bounds for the height of points on curves of genus $g \geqslant 2$ over complex function fields. Similar positivity theorems over higher dimensional basis and their applications to moduli spaces [13] were presented by the second author at the conference on algebraic geometry, Humboldt Universität zu Berlin, 1988.

Let X be a complex projective surface, Y be a curve and $f: X \to Y$ be a surjective, non isotrivial morphism with connected general fibre F. In 1963 Y. Manin [6] showed that the number of $\mathbb{C}(Y)$ rational points of F is finite if the genus g of F is larger than 1. A $\mathbb{C}(Y)$ rational point $p \in F$ gives rise to a section $\sigma: Y \to X$ of f. If one assumes that the fibres of f do not contain exceptional curves the height of p with respect to ω_F is $h(p) = h(\sigma(Y)) = \deg(\sigma^* \omega_{X/Y})$.

It is well known that Manin's theorem "the Mordell conjecture over function fields" can be proved by bounding $h(\sigma(Y))$ from above for semistable morphisms f. The main result of this note is:

THEOREM 1. *Assume that $f: X \to Y$ is relatively minimal. Let q be the genus of Y, $g \geqslant 2$ the genus of F and s the number of singular fibres of f. Then for all sections σ of f one has*

$$h(\sigma(Y)) < 2 \cdot (2g - 1)^2 \cdot (2q - 2 + 2s).$$

If moreover f is semistable, then

$$h(\sigma(Y)) < 2 \cdot (2g - 1)^2 \cdot (2q - 2 + s).$$

In fact, if f is not semistable, a closer look to the semistable reduction of f gives a slightly better bound (see Corollary 4.10).

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Effective bounds for the height were first given by A. Parshin [8] and, in arbitrary characteristics, by L. Szpiro [9]. In [15] A. Parshin announced that, using H. Grauert's proof [4], it is possible to bound $h(\sigma(Y))$ by a polynomial of degree 13 in g. As S. Lang and Y. Miyaoka pointed out, one can use the Miyaoka-Yau inequality to get a bound, linear in g.

The proof of Theorem 1 presented in this paper is given in two steps: First we show, that the sheaf $f_*\omega^2_{X/Y}(\sigma(Y))$ can not have an invertible quotient sheaf of low degree, if $h(\sigma(Y))$ is large. Then, in Section 4, we use the Kodaira-Spencer map to show that this sheaf always has a quotient of degree $2q - 2 + 2s$. The existence of global one forms is only used in this second step. Of course, it would be quite interesting to get along without using Ω^1_X at all. May be, combining methods from P. Vojta's proof of Manin's theorem in [14] and from this paper, this could be done. In fact, the methods used in the first part are overlapping with those used in [1] to prove Dyson's lemma in several variables. Hence the relation between [14] and this paper might be quite close.

The experts will see immediately that the second step in our proof is not too different from the arguments used by Y. Manin [6], H. Grauert [4] and L. Szpiro [9]. The "Parshin-construction" used in [8] and [9], however, is replaced by the effective bounds on the "positivity of certain direct image sheaves". This part (see 2.4 for the exact statement) is presented in the first two sections of this article. Without having any other application, we took Theorem 1 as a pretext allowing us to work out for fibre spaces over curves an effective version of the results of [11]. The reader not familiar with the notations used there should have a look to S. Mori's survey article [7].

In Section 3 we just evaluate the constants obtained for general fibre spaces in the special case of families of curves and we verify the assumptions made in 2.4 in this special case.

The motivation to write this note grew out of discussions with A. Parshin during his and our stay at the Max-Planck-Institute for Mathematics in Bonn. The details were worked out during our stay at the TATA-Institute in Bombay.

1. The lower degree of direct images of sheaves

Let Y be a nonsingular compact curve defined over $\mathbb{C}$ and $\mathscr{F}$ be a coherent locally free sheaf on Y.

DEFINITION 1.1. (a) The *lower degree* of $\mathscr{F}$ is defined as

$$\mathrm{ld}(\mathscr{F}) = \min\{\deg(\mathscr{N});\ \mathscr{N} \text{ invertible quotient sheaf of } \mathscr{F}\}.$$

(b) The *stable lower degree* of $\mathscr{F}$ is

$$\operatorname{sld}(\mathscr{F}) = \inf\left\{\frac{\operatorname{ld}(\tau^*\mathscr{F})}{\deg(\tau)};\ \tau\colon Y' \to Y \text{ a finite map of non singular curves}\right\}.$$

If $\mathscr{F} = 0$ we put $\operatorname{ld}(\mathscr{F}) = \operatorname{sld}(\mathscr{F}) = \infty$.

(c) $\mathscr{F}$ is called *semi-positive* if $\operatorname{sld}(\mathscr{F}) \geqslant 0$ (Fujita, [3]).

1.2. Some properties

(a) If $\mathscr{L}$ is an invertible sheaf of degree d, then

$$\operatorname{ld}(\mathscr{F} \otimes \mathscr{L}) = \operatorname{ld}(\mathscr{F}) + d \quad \text{and}$$
$$\operatorname{sld}(\mathscr{F} \otimes \mathscr{L}) = \operatorname{sld}(\mathscr{F}) + d.$$

(b) If $\rho\colon Y'' \to Y$ is a non singular covering then

$$\operatorname{sld}(\rho^*\mathscr{F}) = \deg(\rho)\cdot\operatorname{sld}(\mathscr{F})$$

(c) $\mathscr{F}$ is ample if and only if $\operatorname{sld}(\mathscr{F}) > 0$.

(d) The following three conditions are equivalent:

(i) $\operatorname{sld}(\mathscr{F}) \geqslant 0$.

(ii) $\mathscr{F}$ is weakly positive over Y (see [11]).

(iii) If $\mathscr{H}$ is an ample invertible sheaf on Y then for all $\gamma > 0$ the sheaf $S^\gamma(\mathscr{F}) \otimes \mathscr{H}$ is ample.

(e) For all $\gamma > 0$ $\operatorname{ld}(\otimes^\gamma(\mathscr{F})) \leqslant \operatorname{ld}(S^\gamma\mathscr{F}) \leqslant \gamma\cdot\operatorname{ld}(\mathscr{F})$ and

$$\operatorname{sld}(\otimes^\gamma(\mathscr{F})) \leqslant \operatorname{sld}(S^\gamma\mathscr{F}) \leqslant \gamma\cdot\operatorname{sld}(\mathscr{F}).$$

Proof. (a) and (e) follow directly from the definition. (c) If $\mathscr{F}$ is ample then $S^\gamma(\mathscr{F}) \otimes \mathscr{H}^{-1}$ will be ample for some $\gamma \gg 0$ and $\operatorname{sld}(\mathscr{F}) \geqslant 1/\gamma\cdot\deg(\mathscr{H})$. If $\operatorname{sld}(\mathscr{F}) \geqslant \varepsilon > 0$ then $\mathcal{O}_{\mathbb{P}}(1)$ on $\mathbb{P} = \mathbb{P}(\mathscr{F})$ satisfies the Seshadri criterium for ampleness (see for example: R. Hartshorne, Ample subvarieties of algebraic varieties, Lecture Notes in Math. *156*, springer 1970, or [3] §2). The proof of (d) is similar. In [11] Section 1 and [13] Section 3 the reader can find some generalizations for higher dimensional Y.

In (b) it is obvious that $\operatorname{sld}(\rho^*\mathscr{F}) \geqslant \deg(\rho)\cdot\operatorname{sld}(\mathscr{F})$. On the other hand, if $\tau\colon Y' \to Y$ is another nonsingular covering we can find $\tau'\colon Y''' \to Y$ dominating both, τ and ρ. One as

$$\operatorname{sld}(\rho^*\mathscr{F}) \leqslant \frac{\operatorname{ld}(\tau'^*\mathscr{F})}{\deg(Y''' \to Y'')} \leqslant \frac{\operatorname{ld}(\tau^*\mathscr{F})\cdot\deg(Y''' \to Y')}{\deg(Y''' \to Y'')} = \frac{\operatorname{ld}(\tau^*\mathscr{F})\cdot\deg(\rho)}{\deg(\tau)}.$$

(1.3) In our estimations of the stable lower degree we will frequently use vanishing theorems for integral parts of $\mathbb{Q}$ divisors. Let $g\colon Z \to X$ be a morphism of complex projective manifolds. For a normal crossing divisor $D = \Sigma \nu_i D_i$ and $e \in \mathbb{R}$ we write $[e \cdot D] = \Sigma [e \cdot \nu_i] \cdot D_i$ where $[e \cdot \nu_i]$ is the integral part of $e \cdot \nu_i$. Recall that an invertible sheaf $\mathscr{L}$ on Z is called numerically effective if $\deg(\mathscr{L}|_C) \geqslant 0$ for all curves C in Z. We will say that $\mathscr{L}$ is *g-numerically effective* if $\deg(\mathscr{L}|_C) \geqslant 0$ for all curves C in Z with $\dim(g(C)) = 0$. $\kappa(\mathscr{L})$ denotes the Iitaka dimension of $\mathscr{L}$ (see [7] for example) and $\omega_{Z/X} = \omega_Z \otimes g^*\omega_X^{-1}$ the difference of the canonical sheaves.

LEMMA 1.4. (a) *Assume that there exists an effective normal crossing divisor D on Z such that $\mathscr{L}^N(-D)$ is g-numerically effective. If for a general fibre G of g $\kappa(\mathscr{L}(-[D/N]) \otimes \mathcal{O}_G) = \dim G$ then for $i > 0$, $R^i g_*(\omega_{Z/X} \otimes \mathscr{L}(-[D/N])) = 0$.*

(b) *Assume that g is birational.*

If Γ is an effective divisor on X, $D = g^*\Gamma$ a normal crossing divisor and $N > 0$, then $R^i g_* \omega_{Z/X} \otimes \mathcal{O}_Z(-[D/N]) = 0$ for $i > 0$. If moreover Γ is a normal crossing divisor then $g_*\omega_{Z/X} \otimes \mathcal{O}_Z(-[D/N]) = \mathcal{O}_X(-[\Gamma/N])$.

Proof. (a) The assumptions imply that $\mathscr{L}^N(-D) \otimes g^*\mathscr{H}^N$ will be numerically effective and $\kappa(\mathscr{L}(-[D/N]) \otimes g^*\mathscr{H}) = \dim Z$ for all "very very" ample invertible sheaves $\mathscr{H}$ on X. As in [10], 2.3, the vanishing theorem due to Kawamata [5] and the second author ([10] and [2], 2.13) implies (a) by using the Leray spectral sequence. (b) Is shown in [10], 2.3.

(1.5) Let in the sequel Y be a nonsingular compact curve, X a projective manifold of dimension n and $f\colon X \to Y$ a surjective morphism. The general fibre of f will be denoted by F. Let $\mathscr{L}$ be an invertible sheaf on X.

All estimates of $\mathrm{sld}(f_*(\mathscr{L} \otimes \omega_{X/Y}))$ will follow from the following corollary of Fujita's positivity theorem [3] (see [11] 5.1):

LEMMA 1.6. *Let D be an effective normal crossing divisor on X and $N > 0$ such that $\mathscr{L}^N = \mathcal{O}_X(D)$. Then $f_*(\omega_{X/Y} \otimes \mathscr{L}(-[D/N]))$ is semi-positive.*

Proof. The case $\mathscr{L} = \mathcal{O}_X$ is Fujita's original theorem. It may be easily obtained by Hodge theory on cyclic covers ([12], 6 and 8).

The general case follows from this one, applied to the cyclic cover given by $\mathscr{L}^N = \mathcal{O}_X(D)$ (as in [2], 2.7 or [11], 2.2).

Recall the following notation: If $U \subset X$ is open, we call $\mathscr{L}$ *very ample with respect to U* if $H^0(X, \mathscr{L}) \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathscr{L}$ is surjective over U and the natural map $U \to \mathbb{P}(H^0(X, \mathscr{L}))$ is an embedding. Correspondingly we call $\mathscr{L}$ *ample with respect to U* if for some $a > 0$ $\mathscr{L}^a$ is very ample with respect to U (see [13], 1.16).

We will call $\mathscr{L}$ *numerically effective with respect to U* if there exists a birational morphism $g\colon X' \to X$, isomorphic over U, and an invertible numerically effective sheaf $\mathscr{L}'$ and an inclusion $\mathscr{L}' \to g^*\mathscr{L}$, isomorphic over U.

Obviously, if $\mathcal{L}$ is ample with respect to U it is numerically effective with respect to U. Moreover, if $\mathcal{H}$ is ample with respect to U and $\mathcal{L}$ numerically effective with respect to U then $\mathcal{H} \otimes \mathcal{L}^a$ is ample with respect to U, for all $a \geqslant 0$.

COROLLARY 1.7. *Let D be a normal crossing divisor on X and $N > 0$, such that $\mathcal{L}^N(-D)$ is numerically effective with respect to a neighbourhood of F and $\kappa(\mathcal{L}^N(-D)) = n$. Then $f_*(\omega_{X/Y} \otimes \mathcal{L}(-[D/N]))$ is semi-positive.*

Proof. By 1.4(b) the statement is compatible with blowing ups. As $\kappa(\mathcal{L}^N(-D)) = n$, we may assume (for example as in [2], 2.12) that there exists an effective divisor Γ on X such tha $D + \Gamma$ has normal crossings and such that $\mathcal{L}^N(-D - \Gamma)$ is ample. Then for $M \geqslant 0$ $\mathcal{L}^{N \cdot M}(-M \cdot D - \Gamma)$ will be ample with respect to a neighbourhood of F. Replacing N, D and Γ by some common multiple we can find a divisor H, smooth on F such that (blowing up a little bit more) $D' = H + M \cdot D + \Gamma$ has normal crossings and $\mathcal{L}^{N \cdot M} = \mathcal{O}_X(D')$. For M big enough

$$\left[\frac{D'}{M \cdot N}\right]\Bigg|_F = \left[\frac{H + M \cdot D + \Gamma}{M \cdot N}\right]\Bigg|_F = \left[\frac{D}{N}\right]\Bigg|_F$$

and therefore 1.7 follows from 1.6.

DEFINITION 1.8. Let Z be a manifold $\mathcal{M}$ be an invertible sheaf and Γ be an effective divisor.

(a) Let $\tau: Z' \to Z$ be a blowing up such that $\Gamma' = \tau^*\Gamma$ is a normal crossing divisor. We define

$$e(\Gamma) = \min\left\{e \in \mathbb{N} - \{0\}; \tau_*\omega_{Z'/Z}\left(-\left[\frac{\Gamma'}{e}\right]\right) = \mathcal{O}_Z\right\}.$$

(b) $e(\mathcal{M}) = \max\{e(\Gamma);\ \Gamma \text{ zero divisor of } s \in H^0(Z, \mathcal{M})\}$.

By 1.4(b) the definition of $e(\Gamma)$ is independent of the blowing up choosen. In Section 2 we will give upper bounds for $e(\Gamma)$.

1.8.1. Especially one obtains $e(\mathcal{M}) < \infty$.

For $e \geqslant e(\Gamma)$ and τ as above one has $\tau_*\omega_{Z'/Z}(-[\Gamma'/e]) = \mathcal{O}_Z$. If $\Gamma = \sum \nu_i \Gamma_i$ is a normal crossing divisor, then $e(\Gamma) = \max\{\nu_i\} + 1$.

COROLLARY 1.9. *Assume that $\mathcal{L}$ is numerically effective with respect to some neighbourhood of F and $\kappa(\mathcal{L}) = n$. Assume moreover that for some invertible sheaf $\mathcal{H}$ on Y of degree h and some $N > 0$ one has an inclusion $\varphi: f^*\mathcal{H} \to \mathcal{L}^N$. Then*

$$\operatorname{sld}(f_*(\mathcal{L} \otimes \omega_{X/Y})) \geqslant h \cdot \min\left\{\frac{1}{N}, \frac{1}{e(\mathcal{L}^N|_F)}\right\}.$$

Remark. Especially, if $h > 0$ the sheaf $f_*(\mathscr{L} \otimes \omega_{X/Y})$ will be ample by 1.2(c) and 1.8.1. A similar result for higher dimensional Y can be found in [11] 5.4.

Proof. Let $M = \max\{N, e(\mathscr{L}^N|_F)\}$ (1.8.1). We can choose a cover $\sigma\colon Y' \to Y$ of degree M such that the fibres of f over the ramification locus are non singular. Then $X' = X \times_Y Y'$ is non singular and

$$pr_{2*}(pr_1^* \mathscr{L} \otimes \omega_{X'/Y'}) = \sigma^* f_*(\mathscr{L} \otimes \omega_{X/Y}).$$

By 1.2(b) it s enough to show 1.9 for $X' \to Y'$. Hence, by abuse of notations we may assume that M divides h. Let $\mathscr{N} \in \mathrm{Pic}^\circ(Y)$. One has for example by Seshadri's criterion $\kappa(\mathscr{L} \otimes f^*\mathscr{N}) = \kappa(\mathscr{L})$. Using 1.2(a) we may replace $\mathscr{L}$ by $\mathscr{L} \otimes f^*\mathscr{N}$ and $\mathscr{H}$ by $\mathscr{H} \otimes \mathscr{N}^N$. Therefore we may assume that $\mathscr{H} = \mathscr{O}_Y(h \cdot p)$ for some point $p \in Y$. Let Γ be the zero divisor of φ and let $\tau\colon X' \to X$ be a blowing up such that $\tau^*(\Gamma + h \cdot f^{-1}(p))$ becomes a normal crossing divisor. Let $f' = f \circ \tau$, $\Gamma' = \tau^*\Gamma$ and $\mathscr{L}' = \tau^*\mathscr{L}$. Since $M \geqslant e(\Gamma|_F)$ the inclusion

$$f'_*\left(\mathscr{L}'\left(-\left[\frac{\Gamma'}{M}\right]\right) \otimes \omega_{X'/Y}\right) \to f_*(\mathscr{L} \otimes \omega_{X/Y})$$

is surjective at the general point of Y. This implies

$$\mathrm{sld}(f_*(\mathscr{L} \otimes \omega_{X/Y})) \geqslant \mathrm{sld}\left(f'_*\left(\mathscr{L}'\left(-\left[\frac{\Gamma'}{M}\right]\right) \otimes \omega_{X'/Y}\right)\right).$$

One has $\mathscr{L}'^M(-\Gamma' - h \cdot f'^{-1}(p)) = \mathscr{L}'^{M-N}$ and by 1.6 (if $M = N$) or 1.7 (if $M > N$) the sheaf

$$f'_*\left(\mathscr{L}'\left(-\left[\frac{\Gamma'}{M}\right] - \frac{h}{M} \cdot f'^{-1}(p)\right) \otimes \omega_{X'/Y'}\right)$$

$$= f'_*\left(\mathscr{L}'\left(-\left[\frac{\Gamma'}{M}\right]\right) \otimes \omega_{X'/Y}\right) \otimes \mathscr{O}_Y\left(-\frac{h}{M} \cdot p\right)$$

is semipositive. From 1.2, a we obtain $\mathrm{sld}(f_*(\mathscr{L} \otimes \omega_{X/Y})) \geqslant h/M$.

2. Bounds for $e(\mathscr{L})$ and the main theorem

(2.1) Consider a complex projective manifold V and an effective divisor Γ on V. We write $\mathscr{M} = \mathscr{O}_V(\Gamma)$ and choose a blowing up $\tau\colon V' \to V$ such that $\tau^*\Gamma = \Gamma'$ has normal crossings. Let us write $\mathscr{C}(e) = \operatorname{coker}(\tau_* \omega_{V'}(-[\Gamma'/e]) \to \omega_V)$. By 1.4(b) $\mathscr{C}(e)$ is independent of the blowing up choosen.

LEMMA 2.2 *Let H be a smooth prime divisor of V which is not a component of Γ. Then* $\operatorname{Supp}(\mathscr{C}(e)) \cap H = \emptyset$ *for $e \geqslant e(\Gamma|_H)$.*

Proof. We may assume by 1.4(b) that Γ' intersects the proper transform H' of H transversally. Then $[\Gamma'/e]|_{H'} = [\Gamma'|_{H'}/e]$. One has a commutative diagram

$$\begin{array}{ccccc}
\tau_*\omega_{V'}\left(-\left[\dfrac{\Gamma'}{e}\right]+H'\right) & \xrightarrow{\alpha} & \tau_*\omega_{H'}\left(-\left[\dfrac{\Gamma'|_{H'}}{e}\right]\right) & \xrightarrow{\beta_H} & \omega_H \\
\downarrow & & & & \| \\
\tau_*\omega_{V'}\left(-\left[\dfrac{\Gamma'}{e}\right]\right)\otimes\mathcal{O}(H) & \xrightarrow{\beta} & \omega_V(H) & \longrightarrow & \omega_H
\end{array}$$

By the vanishing Theorem 1.4(b), α is surjective. If $e \geqslant e(\Gamma|_H)$, β_H is surjective. Then β has to be surjective in a neighbourhood of H.

PROPOSITION 2.3. *Let Z_i, $i = 1, \ldots, r$, be projective manifolds, $\mathscr{H}_i$ be a very ample invertible sheaf on Z_i and $m, d > 0$ such that $c_1(\mathscr{H}_i)^{\dim(Z_i)} \leqslant d/m$. Let V be the r-fold product $Z_1 \times \cdots \times Z_r$ and $\mathscr{M} = \otimes_{i=1}^r pr_i^* \mathscr{H}_i^m$. Then $e(\mathscr{M}) \leqslant d + 1$.*

Proof. Consider first the case $r = 1$. Set $Z = Z_1$, $\mathscr{H} = \mathscr{H}_1$. We prove 2.3 by induction on $\dim Z$. If Z is a curve Γ is an effective divisor on it of degree $\leqslant d$. Therefore $[\Gamma/d + 1] = 0$. Assume $\dim Z \geqslant 2$. Choose, Γ, τ as in 2.1 and H a smooth hyperplane section with $\mathscr{H} = \mathcal{O}(H)$. Then $c_1(\mathscr{H}|_H)^{\dim Z - 1} \leqslant d/m$. If H is not a component of Γ, then by induction and 2.2 $\operatorname{Supp} \mathscr{C}(d + 1)$ does not meet H. As we may find such a H containing any given point, we obtain $\mathscr{C}(d + 1) = 0$.

We proceed by induction on r. We assume that 2.3 holds for $T = Z_1 \times \cdots \times Z_{r-1}$ and $\mathscr{L} = \otimes_{i=1}^{r-1} pr_i^* \mathscr{H}_i^m$.

If Z_r is a point, then 2.3 holds by induction. Assume that Z_r is a curve. Choose Γ and τ as in 2.1. Take a point $p \in Z_r$, and define $D = T \times p \simeq T$. Let ν be the maximal integer such that $\nu \cdot D \leqslant \Gamma$. As $\deg \mathscr{H}_r^m \leqslant d$, one has $0 \leqslant \nu \leqslant d$. We may assume that the proper transform D' of D in V' meets $\Delta' = \Gamma' - \nu \cdot \tau^* D$ transversally. From the inequality

$$\Delta' \geqslant \Gamma' - (d + 1) \cdot (\tau^* D - D') - \nu \cdot D' = \Delta' - (d + 1 - \nu) \cdot (\tau^* D - D')$$

one obtains $-[\Delta'/d + 1] + D' \leqslant -[\Gamma'/d + 1] + \tau^* D$. The multiplicity of D' is one on both sides of the inequality. One has thereby a commutative diagram

$$\begin{array}{ccccc} \tau_*\omega_{V'}\left(-\left[\dfrac{\Delta'}{d+1}\right]+D'\right) & \xrightarrow{\alpha} & \tau_*\omega_{D'}\left(-\left[\dfrac{\Delta'|_{D'}}{d+1}\right]\right) & \xrightarrow{\beta_D} & \omega_D \\ \downarrow & & & & \Vert \\ \tau_*\omega_{V'}\left(-\left[\dfrac{\Gamma'}{d+1}\right]\right)\otimes\mathcal{O}(D) & \xrightarrow{\beta} & \omega_V(D) & \longrightarrow & \omega_D \end{array}$$

By the vanishing Theorem 1.4(b) α is surjective. As $\mathcal{O}_D(\Gamma - v\cdot D) = \mathcal{O}_D(\Gamma) \simeq \mathcal{L}$, β_D is an isomorphism by induction. Therefore Supp $\mathscr{C}(d+1)$ does not meet D. Moving p, we obtain 2.3 for dim $Z_r = 1$. Assume dim $Z_r \geqslant 2$. Choose F a general hyperplane section in Z_r with $\mathcal{O}(F) = \mathcal{H}_r$. As $c_1(\mathcal{H}_{r|F})^{\dim F} \leqslant d/m$, we have $e(\mathcal{M}|_H) \leqslant d + 1$ by induction for $H = T \times F$, and 2.2 implies that for all Γ with $\mathcal{M} = \mathcal{O}_V(\Gamma)$, supp $\mathscr{C}$ $(d + 1)$ does not meet H. As we may find such a F such that $H = T \times F$ is not a component of Γ and contains any given point, we obtain 2.3.

The main result of this note is the following theorem, which for $\mathcal{L} = \omega_{X/Y}^{k-1}$ is an effective version of a special case of [11], 6.2.

THEOREM 2.4. *Let Y be a nonsingular compact curve; X be a projective manifold of dimension n and $f\colon X \to Y$ be a surjective morphism. Let $\mathcal{L}$ be an invertible sheaf on X with $\kappa(\mathcal{L}) = \dim X$. Assume that for some $N > 0$ the sheaf $\mathcal{L}^N|_F$ is very ample on the general fibre F of f and that $\mathcal{L}$ is numerically effective with respect to some neighbourhood of F. Write $d = c_1(\mathcal{L}|_F)^{n-1}$. Then for all $m > 0$*

$$\mathrm{sld}(f_*(\mathcal{L}\otimes\omega_{X/Y}))\cdot(m\cdot N^{n-1}\cdot d + 1) \geqslant \frac{\deg(f_*(\mathcal{L}^{m\cdot N}))}{\mathrm{rank}(f_*(\mathcal{L}^{m\cdot N}))}.$$

Proof. Let us start with the case $m = 1$:

Let $r = \mathrm{rank}(f_*\mathcal{L}^N)$, X^r the r-fold product $X \times_Y X \cdots \times_Y X$ and $f^r\colon X^r \to Y$ the induced map. If $\mathcal{N}$ is any locally free sheaf on X we obtain by flat base change $f^r_*(\otimes_{i=1}^r pr_i^*\mathcal{N}) = \otimes^r f_*\mathcal{N}$. f is a flat Gorenstein morphism and $\omega_{X/Y}$ the same as the dualizing sheaf of f. Therefore $\omega_{X^r/Y} = \otimes_{i=1}^r pr_i^*\omega_{X/Y}$ (see [11], 3.5, for similar constructions). Let $\sigma\colon X^{(r)} \to X^r$ be a desingularization, isomorphic on the general fibre $F \times \cdots \times F$, and $f^{(r)} = f^r \circ \sigma$. For $\mathcal{M} = \sigma^*(\otimes_{i=1}^r pr_i^*\mathcal{L})$ we have inclusions $\otimes^r{}_{i=1} pr_i^*\mathcal{L}^N \to \sigma_*\mathcal{M}^N$ and

$$\sigma_*(\mathcal{M}\otimes\omega_{X^{(r)}/Y}) = \left(\bigotimes_{i=1}^r pr_i^*\mathcal{L}\right)\otimes \mathcal{H}om_{\mathcal{O}_{X^r}}(\sigma_*\mathcal{O}_{X^{(r)}}, \omega_{X^r/Y}) \to$$

$$\to \left(\bigotimes_{i=1}^r pr_i^*\mathcal{L}\right)\otimes\omega_{X^r/Y} = \bigotimes_{i=1}^r pr_i^*(\mathcal{L}\otimes\omega_{X/Y}).$$

The induced inclusions

$$\bigotimes^{r} f_* \mathscr{L}^N \to f_*^{(r)} \mathscr{M}^N \text{ and } f_*^{(r)}(\mathscr{M} \otimes \omega_{X^{(r)}/Y}) \to \bigotimes^{r} f_*(\mathscr{L} \otimes \omega_{X/Y})$$

are both isomorphisms at the general point of Y. Especially one has

$$\operatorname{sld} \bigotimes^{r} f_*(\mathscr{L} \otimes \omega_{X/Y}) \geqslant \operatorname{sld} f_*^{(r)}(\mathscr{M} \otimes \omega_{X^{(r)}/Y}).$$

One has $\kappa(\mathscr{M}) = \dim X^{(r)}$ and $\mathscr{M}^N$ is very ample on the general fibre $F \times \cdots \times F$ of $f^{(r)}$. We have a natural inclusion

$$\det(f_* \mathscr{L}^N) \to \bigotimes^{r} f_* \mathscr{L}^N \to f_*^{(r)} \mathscr{M}^N$$

$$\text{and} \quad f^{(r)*} \det(f_* \mathscr{L}^N) \to f^{(r)*} f_*^{(r)} \mathscr{M}^N \to \mathscr{M}^N.$$

By 1.9 (applied to $f_*^{(r)}(\mathscr{M} \otimes \omega_{X^{(r)}/Y})$ and $\mathscr{H} = \det(f_* \mathscr{L}^N)$ one has

$$\operatorname{sld}(f_*^{(r)}(\mathscr{M} \otimes \omega_{X^{(r)}/Y})) \geqslant \deg(f_* \mathscr{L}^N) \cdot \min\left\{\frac{1}{N}, \frac{1}{e(\mathscr{M}^N|_F)}\right\}.$$

Up to now we did not use that $\mathscr{L}^N$ is very ample and the last inequality holds for all exponents. Especially replacing N by $m \cdot N$ we find that (using 1.2(e))

$$\operatorname{rank}(f_*(\mathscr{L}^{m \cdot N})) \cdot \operatorname{sld}(f_*(\mathscr{L} \otimes \omega_{X/Y})) \geqslant \deg(f_*(\mathscr{L}^{m \cdot N})) \cdot \min\left\{\frac{1}{m \cdot N}, \frac{1}{e(\mathscr{L}^{m \cdot N}|_F)}\right\}.$$

On the other hand we have shown in 2.3 that

$$e(\mathscr{L}^{m \cdot N}) \leqslant m \cdot c_1(\mathscr{L}^N|_F)^{n-1} + 1 = m \cdot N^{n-1} \cdot d + 1.$$

COROLLARY 2.5. *Under the assumptions of* 2.4 *let* $\mathscr{L}$ *be even numerically effective on* X. *Then*

$$\operatorname{sld}(f_*(\mathscr{L} \otimes \omega_{X/Y})) \geqslant \frac{c_1(\mathscr{L})^n}{n \cdot N^{n-2} \cdot d^2}.$$

Proof. If $\mathscr{L}$ is numerically effective, the dimension of the higher cohomology groups of $\mathscr{L}^{m \cdot N}$ is bounded from above by a polynomial of degree $n - 1$ in m.

Since the Leray spectral sequence gives an inclusion

$$H^1(Y, f_*\mathscr{L}^{m\cdot N})) \to H^1(X, \mathscr{L}^{m\cdot N}),$$

the same holds true for $h^1(Y, f_*\mathscr{L}^{m\cdot N})$.

The Riemann-Roch-Theorem for vector-bundles on Y and for invertible sheaves on X implies that

$$\frac{1}{n!}\cdot c_1(\mathscr{L})^n\cdot N^n\cdot m^n = h^0(X, \mathscr{L}^{N\cdot m}) + O(m^{n-1}) = h^0(Y, f_*\mathscr{L}^{N\cdot m}) + O(m^{n-1})$$
$$= O(m^{n-1}) + \deg(f_*\mathscr{L}^{N\cdot m}) - \operatorname{rank}(f_*\mathscr{L}^{N\cdot m})\cdot(q-1).$$

Then, using in the same way the Riemann-Roch on F and taking the limit over m we get 2.5 from 2.4.

REMARK 2.6. Especially for those, mostly interested in the case that f is a family of curves, it might look quite complicated that the proof of 2.5 and 2.4 forced us to consider higher dimensional fibre spaces. In fact, if one is just interested in 2.5 this is not necessary and we sketch in the sequel a proof which is avoiding the products in 2.3 and 2.4: If p is a point on Y the Riemann-Roch theorem shows that $h^0(Y, f_*\mathscr{L}^{N\cdot m}\otimes\mathcal{O}_Y(-h\cdot p))$ is larger than or equal to $\deg(f_*\mathscr{L}^{N\cdot m}) - \operatorname{rank}(f_*\mathscr{L}^{N\cdot m})\cdot(h+q-1)$. Therefore, whenever we have

$$h < \deg(f_*\mathscr{L}^{N\cdot m})\cdot\operatorname{rank}(f_*\mathscr{L}^{N\cdot m})^{-1} - q + 1,$$

we will find an inclusion of $\mathcal{O}_Y(+h\cdot p)$ in $f_*\mathscr{L}^{N\cdot m}$.

Applying 1.9 and 2.3 (for $r = 1$) we obtain the same inequality as in 2.4, except that we have to add a $-q$ on the right hand side. Since in the proof of 2.5 we were taking the limit over m anyway, this is enough to obtain 2.5.

3. Examples and applications

The first application of 2.4 is not really needed in the proof of theorem one and it is just added for historical reasons.

THEOREM 3.1. *Let $f\colon X\to Y$ be a surjective morphism with general fibre F, where X is a projective manifold of dimension n and Y a non singular curve, and let $v > 1$. Assume that for $N > 0$ ω_F^N is very ample and that f is non isotrivial. Then, for all multiples m of $v-1$*

$$\operatorname{sld}(f_*\omega_{X/Y}^{v})\cdot(m\cdot(N\cdot c_1(\omega_F))^{n-1}+1) \geqslant \frac{\deg(f_*\omega_{X/Y}^{N\cdot m})}{\operatorname{rank}(f_*\omega_{X/Y}^{N\cdot m})}.$$

Moreover for $\nu > 1$, $f_*\omega^{\nu}_{X/Y}$ *is ample, (if it is not trivial).*

Proof. Results due to J. Kollár and the second author show that $\kappa(\omega_{X/Y}) = \dim(X)$ (see [7] or [13], §1(c) for example). In fact, one first shows that $\deg(f_*\omega^{N\cdot m}_{X/Y}) > 0$ for $m \gg 0$ and then one uses methods similar to those used in the proof of 2.4 to show that $f_*\omega^{\mu}_{X/Y}$ is ample for $\mu \gg 0$. Then $\omega_{X/Y}$ will be ample with respect to a neighbourhood of F and the inequality follows then from 2.4 for $\mathscr{L} = \omega^{\nu-1}_{X/Y}$. By 1.4(a) $f_*\omega^{\nu}_{X/Y}$ will be ample whenever it is not trivial.

EXAMPLE 3.2. Assume that X is a surface, and moreover that f is a non isotrivial family of curves of genus $g \geqslant 2$.

(a) For $N > 1$ one has rank $f_*\omega^N_{X/Y} = (2N-1)\cdot(g-1)$. Applying 3.1 for $\nu = 2$, one obtains

$$\mathrm{sld}(f_*\omega^2_{X/Y})\cdot(N\cdot(2g-2)+1)\cdot(2N-1)\cdot(g-1) \geqslant \deg f_*\omega^N_{X/Y}.$$

(b) Let $\tilde{f}\colon \tilde{X} \to Y$ be the relative minimal model of f. By definition the fibres of $\tilde{f}$ do not contain any (-1)-curves and $\omega_{\tilde{X}/Y}$ is $\tilde{f}$-numerically effective. On the other hand, if B is a curve on X which dominates Y, then $\tilde{f}^*\tilde{f}_*\omega_{\tilde{X}/Y} \to \omega_{\tilde{X}/Y} \to \omega_{\tilde{X}/Y}|_B$ is non trivial. Since the sheaf on the left hand side is the pullback of a semipositive sheaf, $\deg(\omega_{\tilde{X}/Y}|_B) \geqslant 0$. Therefore $\omega_{\tilde{X}/Y}$ is numerically effective.
From 2.5 we obtain that $\mathrm{sld}(f_*\omega^2_{X/Y})\cdot(2g-2)^2 \geqslant \frac{1}{2}c_1(\omega_{\tilde{X}/Y})^2$.

(c) One has $c_1(\omega_{\tilde{X}/Y})^2 > 0$ as $\kappa(\omega_{\tilde{X}/Y}) = 2$ and since $\omega_{\tilde{X}/Y}$ is numerically effective (see [10], §3). If one does not want to use this non trivial fact, one can get along with $c_1(\omega_{\tilde{X}/Y})^2 \geqslant 0$ if one replaces all strict inequalities in the sequel by "$\geq$". The weak inequality follows directly from (b).

(3.3) From now on $f\colon X \to Y$ will denote a non isotrivial family of curves of genus $g \geqslant 2$ and $\sigma\colon Y \to X$ a section. Let $C = \sigma(Y)$ and let $\tilde{f}\colon \tilde{X} \to Y$ be the relative minimal model. The image $\tilde{C}$ of C in $\tilde{X}$ intersects the fibres of $\tilde{f}$ in smooth points. Therefore we may assume that all fibres of f are normal crossing divisors, and that C does not meet any exceptional divisor contained in the fibres. Of course, $h(\tilde{C}) = c_1(\omega_{\tilde{X}/Y})\cdot\tilde{C}$ is the same as $c_1(\omega_{X/Y})\cdot C$ under this assumption.

LEMMA 3.4. *For* $N > 1$ *one has*

$$\deg(f_*\omega_{X/Y}(C)^N) = \deg(f_*\omega^N_{X/Y}) + \tfrac{1}{2}N(N-1)\cdot h(C).$$
$$\mathrm{rank}(f_*\omega_{X/Y}(C)^N) = N\cdot(2g-1)-(g-1).$$

Proof. We may assume here that f is relatively minimal, i.e. $\tilde{X} = X$. Then $\omega_{X/Y}$ as well as $\omega_{X/Y}(C)$ are f-numerically effective (see 1.3). Then by 1.4(a) we have for $0 \leqslant \mu < N$ exact sequences

$$0 \to f_*\omega^N_{X/Y}(\mu\cdot C) \to f_*\omega^N_{X/Y}((\mu+1)C) \to \omega^N_{X/Y}((\mu+1)C)|_C \to 0.$$

Since $\omega_{X/Y}(C)|_C = \mathcal{O}_C$, the sheaf on the right hand side is

$$\mathcal{O}_C(-(N-\mu-1)\cdot C) = \omega_{X/Y}^{N-\mu-1}|_C.$$

Adding up we obtain

$$\deg(f_*\omega_{X/Y}(C)^N) - \deg(f_*\omega_{X/Y}^N) = \sum_{\mu=0}^{N-1} (N-\mu-1)\cdot h(C).$$

The second equality is trivial.

COROLLARY 3.5. *Under the assumptions made in* 3.3 *we have for* $N \geqslant 2$

$$\begin{aligned} &\operatorname{sld}(f_*\omega_{\bar{X}/Y}^2(C))\cdot(N\cdot(2g-1)+1)\cdot(N\cdot(2g-1)-(g-1)) \\ &\quad \geqslant \deg(f_*\omega_{\bar{X}/Y}^N) + \frac{N(N-1)}{2}\cdot h(C) \end{aligned}$$

and

$$\operatorname{sld}(f_*\omega_{\bar{X}/Y}^2(C))\cdot(2g-1)^2 \geqslant \tfrac{1}{2}(c_1(\omega_{\bar{X}/Y})^2 + h(C)) > \tfrac{1}{2}h(C).$$

Proof. Since $\omega_{\bar{X}/Y}$is numerically effective, the same holds for $\omega_{\bar{X}/Y}(\tilde{C})$. Moreover, if $h(C) \neq 0, c_1(\omega_{\bar{X}/Y}(\tilde{C}))^2 = c_1(\omega_{\bar{X}/Y})^2 + h(C) > 0$ and hence $\kappa(\omega_{\bar{X}/Y}(\tilde{C})) = 2$. The first inequality follows from 3.4 and 2.4 applied to $\mathcal{L} = \omega_{\bar{X}/Y}(C)$ and the second one from 2.5 applied to $\mathcal{L} = \omega_{\bar{X}/Y}(\tilde{C})$.

REMARK. Since the arguments used in 3.4 also show that $\omega_{X/Y|C}$ is a quotient of $f_*\omega_{\bar{X}/Y}^2(C)$ we can state as well $h(C) \geqslant \operatorname{sld}(f_*\omega_{\bar{X}/Y}^2(C))$ and

COROLLARY 3.6. *Using the notations and assumptions made in* 3.3

$$h(C)\cdot(2\cdot(2g-1)^2-1) \geqslant c_1(\omega_{\bar{X}/Y})^2.$$

4. Effective bounds for the height

We want to finish the proof of Theorem 1.

(4.1) Let $f: X \to Y$ be a family of curves. Let $S = \{y \in Y; f^{-1}(y) \text{ singular}\}$ and $D = f^*(S)$. We assume that D is a normal crossing divisor (i.e. an effective divisor, locally in the analytic topology with nonsingular components meeting transversally). Recall that f is called semistable when D is a reduced divisor.

Let $\Omega_X^1\langle D\rangle = \Omega_X^1\langle D_{\text{red}}\rangle$ be the sheaf of differential forms with logarithmic poles along D. The natural inclusion $f^*\Omega_Y^1\langle S\rangle \to \Omega_X^1\langle D\rangle$ splits locally. In fact, f is locally given by $t = x^\alpha \cdot y^\beta$, where x and y are parameters on X and

t a parameter on Y. Then $\mathrm{d}t/t = \alpha \cdot \mathrm{d}x/x + \beta \cdot \mathrm{d}y/y$ is part of a local bases of $\Omega^1_X\langle D\rangle$. The quotient sheaf, denoted by $\Omega^1_{X/Y}\langle D\rangle$, is therefore invertible. Comparing the highest wedge products one finds

$$\Omega^1_{X/Y}\langle D\rangle = \omega_X(D_{\mathrm{red}}) \otimes f^*\omega_Y(S)^{-1} = \omega_{X/Y}(D_{\mathrm{red}} - D).$$

As $C = \sigma(Y)$ meets D only in points which are smooth on D, the cokernel $\Omega^1_{X/Y}\langle D + C\rangle$ of $f^*\Omega^1_Y\langle S\rangle \to \Omega^1_X\langle D + C\rangle$ will be invertible as well and one has $\Omega^1_{X/Y}\langle D + C\rangle = \Omega^1_{X/Y}\langle D\rangle \otimes \mathcal{O}_X(C)$.

LEMMA 4.2. *If f is non isotrivial, there is a nonzero map*

$$\gamma : f_*(\omega_{X/Y} \otimes \Omega^1_{X/Y}\langle D\rangle) \to \Omega^1_Y\langle S\rangle.$$

Moreover, if C is a section γ factors through

$$\gamma' : f_*(\omega_{X/Y} \otimes \Omega^1_{X/Y}\langle D + C\rangle) \to \Omega^1_Y\langle S\rangle.$$

Proof of Theorem 1. If f is semistable then $\Omega^1_{X/Y}\langle D\rangle = \omega_{X/Y}$. 4.2 implies that $\mathrm{sld}(f_*(\omega^2_{X/Y}(C)) \leqslant 2q - 2 + s$. In general one has an inclusion

$$f_*(\omega^2_{X/Y}(C)) \to f_*(\omega_{X/Y} \otimes \Omega^1_{X/Y}\langle D + C\rangle) \otimes \mathcal{O}_Y(S)$$

and using 4.2 and 1.2(a) one obtains $\mathrm{sld}(f_*\omega^2_{X/Y}(C)) \leqslant 2 \cdot q - 2 + 2 \cdot s$. In both cases 3.5 gives $(2 \cdot g - 1)^{-2} \cdot h(C) \leqslant 2 \cdot \mathrm{sld}(f_*\omega^2_{X/Y}(C))$.

The construction of γ': (4.3) We have a commutative diagram of exact sequences:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & f^*\Omega^1_Y\langle S\rangle \arrow[r] \arrow[d, "="] & \Omega^1_X\langle D\rangle \arrow[r] \arrow[d] & \Omega^1_{X/Y}\langle D\rangle \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f^*\Omega^1_Y\langle S\rangle \arrow[r] & \Omega^1_X\langle D + C\rangle \arrow[r] \arrow[d] & \Omega^1_{X/Y}\langle D + C\rangle \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{O}_C \arrow[r, "="] \arrow[d] & \mathcal{O}_C \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

(4.4) Applying Rf_* to the diagram 4.3 we obtain

$$\begin{tikzcd} f_*\Omega^1_{X/Y}\langle D\rangle \arrow[r, "\delta"] \arrow[d] & R^1f_*\mathcal{O}_X \otimes \Omega^1_Y\langle S\rangle \arrow[d, "="] \\ f_*\Omega^1_{X/Y}\langle D+C\rangle \arrow[r, "\delta^1"] & R^1f_*\mathcal{O}_X \otimes \Omega^1_Y\langle S\rangle \end{tikzcd}$$

δ is the Kodaira-Spencer class and, since f is not isotrivial, both δ and δ' are non zero. If we tensorize 4.3 with $\omega_{X/Y}$ and apply Rf_* again we get

(4.5)

$$\begin{tikzcd} f_*(\omega_{X/Y} \otimes \Omega^1_{X/Y}\langle D\rangle) \arrow[r, "\gamma"] \arrow[d] & R^1f_*\omega_{X/Y} \otimes \Omega^1_Y\langle S\rangle \arrow[d, "="] \\ f_*(\omega_{X/Y} \otimes \Omega^1_{X/Y}\langle D+C\rangle) \arrow[r, "\gamma'"'] & R^1f_*\omega_{X/Y} \otimes \Omega^1_Y\langle S\rangle = \Omega^1_Y\langle S\rangle \end{tikzcd}$$

Since the left vertical arrow is injective γ' cannot be zero if γ is non zero.

LEMMA 4.6. *There is a commutative diagram*

$$\begin{tikzcd} f_*\omega_{X/Y} \otimes f_*\Omega^1_{X/Y}\langle D\rangle \arrow[r, "id \times \delta"] \arrow[d, "m"] & f_*\omega_{X/Y} \otimes R^1f_*\mathcal{O}_X \otimes \Omega^1_Y\langle S\rangle \arrow[d, "\bigcup \otimes id"] \\ f_*(\omega_{X/Y} \otimes \Omega^1_{X/Y}\langle D\rangle) \arrow[r, "\gamma"'] & R^1f_*\omega_{X/Y} \otimes \Omega^1_Y\langle S\rangle \end{tikzcd}$$

where m is multiplication and $\bigcup$ the cup product.

Proof. One has a natural map $f^*f_*\omega_{X/Y} \to \omega_{X/Y}$ and taking the tensor product with the first row of 4.3 we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & f^*f_*\omega_{X/Y} \otimes f^*\Omega^1_Y\langle S\rangle \arrow[r] \arrow[d] & f^*f_*\omega_{X/Y} \otimes \Omega^1_X\langle D\rangle \arrow[r] \arrow[d] & f^*f_*\omega_{X/Y} \otimes \Omega^1_{X/Y}\langle D\rangle \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \omega_{X/Y} \otimes f^*\Omega^1_Y\langle S\rangle \arrow[r] & \omega_{X/Y} \otimes \Omega^1_X\langle D\rangle \arrow[r] & \omega_{X/Y} \otimes \Omega^1_{X/Y}\langle D\rangle \arrow[r] & 0 \end{tikzcd}$$

Applying Rf_* we obtain the diagram in 4.6.

Proof of 4.2. It is enough to show that γ is non zero. Let U be some open subvariety of Y such that $\Omega^1_Y\langle S\rangle$ is generated by a differential form α. Since δ is non zero we may choose $s \in f_*\Omega^1_{X/Y}\langle D\rangle(U)$ with $\delta(s) \neq 0$. For some

$0 \neq \lambda \in R^1 f_* \mathcal{O}_X(U)$ we can write $\delta(s) = \lambda \otimes \alpha$. Since $\bigcup$ induces a perfect pairing there is some $\mu \in f_* \omega_{X/Y}(U)$ with $\mu \bigcup \lambda \neq 0$. Then $\mu \bigcup \lambda \otimes \alpha \neq 0$ and by 4.6 this is the same as $\gamma(m(\mu \otimes s))$.

(4.7) Of course, 4.2 also implies that for non isotrivial families

$$\mathrm{sld}(f_*(\omega_{X/Y} \otimes \Omega^1_{X/Y}\langle D\rangle)) \leqslant 2q - 2 + s.$$

If f is semistable and minimal this together with 3.2 shows (see [9] for better bounds):

COROLLARY 4.8. *If f is semistable, relatively minimal and non isotrivial and if q is the genus of Y and s the number of degenerate fibres of f, then for $N > 1$*

$$(2q - 2 + s)(N \cdot (2g - 2) + 1)(2N - 1)(g - 1) \geqslant \deg(f_* \omega^N_{X/Y})$$

and

$$(2q - 2 + s)(2g - 2)^2 \geqslant \tfrac{1}{2} c_1(\omega_{X/Y})^2 > 0.$$

If f is not semistable let $\tau\colon Y' \to Y$ be a non singular Galois cover such that the ramification index of $s \in Y$ is divisible by the multiplicities of $f^{-1}(s)$. Let $\delta\colon X' \to X \times_Y Y'$ be the normalization, $f' = pr_2 \circ \delta\colon X' \to Y'$ and $\tau = pr_1 \circ \delta\colon X' \to X$. If the fibres of f are normal crossing divisors, then X' has at most rational Gorenstein singularities. Especially $\omega_{X'}$ is invertible and, if $\tau'\colon X'' \to X'$ is a desingularization, then $\tau'_* \omega_{X''} = \omega_{X'}$. Moreover X' is non singular in a neighbourhood of a section C'. In fact, under the assumptions made, we may choose X'' such that $X'' \to Y'$ is semistable.

LEMMA 4.9. *Let $\mathcal{L}$ be an invertible sheaf on X and $\mathcal{L}' = \tau^* \mathcal{L}$. Then there is an inclusion $f'_*(\mathcal{L}' \otimes \omega_{X'/Y'}) \to \tau^* f_*(\mathcal{L} \otimes \omega_{X/Y})$ isomorphic in the general point of Y'.*

Proof. As in the proof of 2.4 one obtains by duality theory an inclusion

$$\delta_*(\mathcal{L}' \otimes \omega_{X'/Y'}) \to pr_1^*(\mathcal{L} \otimes \omega_{X/Y}).$$

4.9 follows by flat base change.

COROLLARY 4.10. *If $f\colon X \to Y$ is not semistable, then the bound for $h(\sigma(Y))$ given in Theorem 1 can be improved to*

$$h(\sigma(Y)) < 2 \cdot (2g - 1)^2 \cdot (2q - 2 + s) + s.$$

Sketch of proof. If $f\colon X \to Y$ is any morphism, not necessarily relatively minimal, and $C=\sigma(Y)$, then $h(C)\leqslant c_1(\omega_{X/Y})\cdot C$. Let $\tau\colon Y'\to Y$ be a nonsingular Galois cover with Galois-group G, such that $X\times_Y Y'\to Y'$ is birational to a semistable family of curves $\varphi\colon T\to Y'$, relatively minimal over Y'. Let B be the image of the section of φ lifting σ. The action of G on Y' extends to an action of G on T, such that X is birational to T/G.

CLAIM 4.11. There exists a sequence of at most $s\cdot\operatorname{ord}(G)$ blowing ups of points $\eta\colon X''\to T$ such that G acts on X'' and such that the quotient X''/G is non singular in some neighbourhood of C''/G, where C'' is the proper transform of B.

Using the notations from 4.11 one has $c_1(\omega_{X''/Y'})\cdot C''\leqslant s\cdot\deg(\tau)+h(B)$. We choose X to be a desingularisation of X''/G, isomorphic to X''/G near C''/G, such that $f\colon X\to Y$ has fibres with normal crossings. As above, let $f'\colon X'\to Y'$ be a minimal desingularization of $X\times_Y Y'\to Y'$, $\tau'\colon X'\to X$ the induced map, $C'=\tau'^{-1}(C)$ and $D'=\tau'^{-1}(D)$. One has $\tau'^*\Omega^1_{X'/Y'}\langle D'+C'\rangle=\Omega^1_{X'/Y'}\langle D'+C'\rangle$ and, since f' has a semistable model, both $f'_*\omega_{X'/Y'}(C')$ and

$$f'_*(\Omega^1_{X'/Y'}\langle D'+C'\rangle\otimes\omega_{X'/Y'})$$

are independent of the model chosen for f'. Especially both sheaves coincide. Using 3.5, 4.9, 1.2(b) and 4.2 one obtains

$$\begin{aligned}\tfrac{1}{2}(2g-1)^{-2}\cdot h(C')=\tfrac{1}{2}(2g-1)^{-2}\cdot h(B)&\leqslant \operatorname{sld}(f'_*\omega^2_{X'/Y'}(C))\\&\leqslant \operatorname{sld}(f_*\omega_{X/Y}\otimes\Omega^1_{X/Y}\langle D+C\rangle)\cdot\deg(\tau)\\&\leqslant (2q-2+s)\cdot\deg(\tau).\end{aligned}$$

On the other hand, near C the sheaves $\Omega^1_{X/Y}\langle D\rangle$ and $\omega_{X/Y}$ coincide and, using the notations introduced above, one has

$$\begin{aligned}h(C)&\leqslant c_1(\omega_{X/Y})\cdot C=c_1(\Omega^1_{X/Y}\langle D\rangle)\cdot C=\frac{1}{\deg(\tau)}c_1(\omega_{X'/Y'})\cdot C'\\&\leqslant\frac{1}{\deg(\tau)}c_1(\omega_{X''/Y'})\cdot C''\leqslant s+\frac{h(B)}{\deg(\tau)}\leqslant 2\cdot(2g-1)^2(2q-2+s)+s.\end{aligned}$$

Proof of 4.11. The question is local in Y and we may replace G by the ramification group of some $p\in Y'$. Hence we assume Y to be a small disk and $G=\langle\sigma\rangle$ to be cyclic of order N. Let $Q=B\cap\varphi^{-1}(P)\in T$. We can find local coordinates x and y near Q, such that x is the pullback of a coordinate on Y', and such that the zero set of y is B. Moreover we can assume that $\sigma(x)=e\cdot x$ and $\sigma(y)=e^{\mu}\cdot$, where e is a primitive Nth root of unit and $O\leqslant\mu<N$. Blowing up Q we obtain T' and local coordinates x' and y' near $Q'=\varphi'^{-1}(p)\cap B'$ with $x'=x$ and $y'=y/x$. Therefore $\sigma(x')=e\cdot x'$ and $\sigma(y')=e^{\mu'}\cdot y'$ for $\mu'=\mu-1$. After at most N blowing ups we may assume that $\mu'=0$. Then, however, the quotient is non singular.

COROLLARY 4.12. *Assume that f is not isotrivial. Let q be the genus of Y and s be the number of degenerate fibres of f. Then one has*

$$0 < \mathrm{sld}(f_*(\omega_{X/Y} \otimes \Omega^1_{X/Y}\langle D\rangle)) \leqslant 2q - 2 + s.$$

4.12 implies the well known fact that a non isotrivial family of curves of genus $g \geqslant 2$ over Y must have at least three degenerate fibres, if $Y = \mathbb{P}^1$, and at least one degenerate fibre, if Y is an elliptic curve.

(4.13) If $f: X \to Y$ is a family of higher dimensional canonically polarized manifolds with degenerate fibres one can consider the iterated Kodaira–Spencer map $f_*\Omega^{n-1}_{X/Y}\langle D\rangle \xrightarrow{\delta^{n-1}} (\Omega^1_Y\langle S\rangle)^{n-1} \otimes R^{n-1}f_*\mathcal{O}_X$ and

$$f_*(\omega_{X/Y} \otimes \Omega^{n-1}_{X/Y}\langle D\rangle) \xrightarrow{\gamma^{(n-1)}} (\Omega^1_Y\langle S\rangle)^{n-1}.$$

With 3.1 and the same arguments we used above one obtains

PROPOSITION 4.14. *Assume that δ^{n-1} is non trivial then* $(2q - 2 + s) > 0$.

However, we do not know any reasonable criterion implying that $\delta^{n-1} \neq 0$.

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