$\mathcal{D}$-modules and finite monodromy

Hélène Esnault, Freie Universität Berlin

Sackler Lectures, Nov. 2016

Acknowledgements

Thank you to

the mathematicians of Tel Aviv university for the kind invitation to deliver the Sackler Distinguished Lectures;

Acknowledgements

Thank you to

the mathematicians of Tel Aviv university for the kind invitation to deliver the Sackler Distinguished Lectures;

Mark Kisin for constructive comments and remarks on the slides.

Table of Contents

Differential operators in complex geometry

Complex geometry

• In complex geometry, we know well what is the sheaf $\mathcal{D}$ of *differential operators* on a manifold X: it is a sheaf of $\mathcal{O}$-algebras, generated, locally where one has coordinates $(x_1, \ldots, x_n)$, by differential operators $\mathcal{D}^{\leq N}$ of order $\leq N$, which are written as finite sums with coefficients in the holomorphic functions $\mathcal{O}$ of operators $\partial_{x_1}^{m_1} \circ \ldots \circ \partial_{x_n}^{m_n}$, $\sum_{i=1}^{n} m_i \leq N$. In particular, $\mathcal{D}^{\leq 0} = \mathcal{O}$, $\mathcal{D}^{\leq 1} = \mathcal{O} \oplus \mathcal{T}$, where $\mathcal{T}$ is the sheaf of tangent vectors. The splitting $\mathcal{D} \to \mathcal{O}$ is in fact independent of the choice of the local coordinates, that is it is global, and defined by $P \mapsto P(1)$, where 1 is the global constant holomorphic function equal to 1 everywhere.

Differential operators in complex geometry

Complex geometry

• In complex geometry, we know well what is the sheaf $\mathcal{D}$ of *differential operators* on a manifold X: it is a sheaf of $\mathcal{O}$-algebras, generated, locally where one has coordinates $(x_1, \ldots, x_n)$, by differential operators $\mathcal{D}^{\leq N}$ of order $\leq N$, which are written as finite sums with coefficients in the holomorphic functions $\mathcal{O}$ of operators $\partial_{x_1}^{m_1} \circ \ldots \circ \partial_{x_n}^{m_n}, \ \sum_{i=1}^{n} m_i \leq N$. In particular, $\mathcal{D}^{\leq 0} = \mathcal{O}, \ \mathcal{D}^{\leq 1} = \mathcal{O} \oplus \mathcal{T}$, where $\mathcal{T}$ is the sheaf of tangent vectors. The splitting $\mathcal{D} \to \mathcal{O}$ is in fact independent of the choice of the local coordinates, that is it is global, and defined by $P \mapsto P(1)$, where 1 is the global constant holomorphic function equal to 1 everywhere.

• As we see on the local description, *$\mathcal{D}$ as an $\mathcal{O}$-algebra is spanned by $\mathcal{D}^{\leq 1}$*.

$\mathcal{D}$-modules in complex geometry

Complex geometry

• $\mathcal{D}$ acts on $\mathcal{O}$. Any sheaf of $\mathcal{O}$-modules E which has the property that the $\mathcal{O}$-action factors through a $\mathcal{D}$-action is called a *sheaf of $\mathcal{D}$-modules*. So $\mathcal{O}$ is a $\mathcal{D}$-module, called the *trivial* $\mathcal{D}$-module.

$\mathcal{D}$-modules in complex geometry

Complex geometry

• $\mathcal{D}$ acts on $\mathcal{O}$. Any sheaf of $\mathcal{O}$-modules E which has the property that the $\mathcal{O}$-action factors through a $\mathcal{D}$-action is called a *sheaf of $\mathcal{D}$-modules*. So $\mathcal{O}$ is a $\mathcal{D}$-module, called the *trivial* $\mathcal{D}$-module.

• Writing $\nabla(e) = \sum \partial_{x_i}(e)dx_i$ defines a connection

$$\nabla : E \to \Omega^1 \otimes_{\mathcal{O}} E,$$

and the relation $\partial_{x_i}\partial_{x_j}(e) = \partial_{x_j}\partial_{x_i}(e)$ for all i, j translates into the integrability of ∇.

$\mathcal{D}$-modules in complex geometry

Complex geometry

• $\mathcal{D}$ acts on $\mathcal{O}$. Any sheaf of $\mathcal{O}$-modules E which has the property that the $\mathcal{O}$-action factors through a $\mathcal{D}$-action is called a *sheaf of $\mathcal{D}$-modules*. So $\mathcal{O}$ is a $\mathcal{D}$-module, called the *trivial* $\mathcal{D}$-module.

• Writing $\nabla(e) = \sum \partial_{x_i}(e) dx_i$ defines a connection

$$\nabla : E \to \Omega^1 \otimes_{\mathcal{O}} E,$$

and the relation $\partial_{x_i}\partial_{x_j}(e) = \partial_{x_j}\partial_{x_i}(e)$ for all i, j translates into the integrability of ∇.

• If one assumes E is *$\mathcal{O}$-coherent*, this implies E is *locally free*, that is a vector bundle, so ∇ becomes a *linear differential equation*:

$$\partial_{x_i} e_j = \sum_{\ell=1}^{r} a_{ij}^{\ell} e_\ell, \ \ r = \mathrm{rank}(E).$$

$\mathcal{D}$-modules in complex geometry

Complex geometry

- One sets $M := (E, \nabla)$.

$\mathcal{D}$-modules in complex geometry

Complex geometry

- One sets $M := (E, \nabla)$.
- More is true: there is a tensor product $M \otimes N$, an (internal) dual $M^\vee$, $\mathrm{Hom}(M, N)$ is a complex vector space, $End(\mathcal{O}) = \mathbb{C}$: the category of such M is *tannakian*, and so is the subcategory $\langle M \rangle$ of subquotients of tensors $M^n \otimes (M^\vee)^m$ of M and its dual.

Complex geometry

- One sets $M := (E, \nabla)$.
- More is true: there is a tensor product $M \otimes N$, an (internal) dual $M^\vee$, $\mathrm{Hom}(M, N)$ is a complex vector space, $End(\mathcal{O}) = \mathbb{C}$: the category of such M is *tannakian*, and so is the subcategory $\langle M \rangle$ of subquotients of tensors $M^n \otimes (M^\vee)^m$ of M and its dual.
- If we fix a complex point x, we have the *monodromy representation* $\rho : \pi_1^{\mathrm{top}}(X, x) \to GL(r, \mathbb{C})$ of M.

$\mathcal{D}$-modules in complex geometry

Complex geometry

- One sets $M := (E, \nabla)$.

- More is true: there is a tensor product $M \otimes N$, an (internal) dual $M^\vee$, $\mathrm{Hom}(M, N)$ is a complex vector space, $End(\mathcal{O}) = \mathbb{C}$: the category of such M is *tannakian*, and so is the subcategory $\langle M \rangle$ of subquotients of tensors $M^n \otimes (M^\vee)^m$ of M and its dual.

- If we fix a complex point x, we have the *monodromy representation* $\rho : \pi_1^{\mathrm{top}}(X, x) \to GL(r, \mathbb{C})$ of M.

- The category $\langle M \rangle$ is equivalent to the category of *algebraic representations* in finite dimensional complex vector spaces of a complex group scheme $G(M)$, called its *Tannaka group*, which is the Zariski closure $\overline{\rho(\pi_1^{\mathrm{top}}(X, x))} \subset GL(r, \mathbb{C})$ of the monodromy group. (It is a consequence of the Riemann-Hilbert correspondence.)

Tannakian category over a field k of characteristic 0

Tannaka group

• The same definition in the Zariski topology yields $\mathcal{O}$-coherent $\mathcal{D}$-modules, which are vector bundles with an integrable connection.

Tannakian category over a field k of characteristic 0

Tannaka group

- The same definition in the Zariski topology yields $\mathcal{O}$-coherent $\mathcal{D}$-modules, which are vector bundles with an integrable connection.
- The category $\langle M \rangle$ is Tannakian.

Tannakian category over a field k of characteristic 0

Tannaka group

• The same definition in the Zariski topology yields $\mathcal{O}$-coherent $\mathcal{D}$-modules, which are vector bundles with an integrable connection.

• The category $\langle M \rangle$ is Tannakian.

• Clearly, one can no longer compute its Tannaka group $G(M)$ as before, as one no longer has a topological fundamental group at disposal.

Tannakian category over a field k of characteristic 0

Tannaka group

• The same definition in the Zariski topology yields $\mathcal{O}$-coherent $\mathcal{D}$-modules, which are vector bundles with an integrable connection.

• The category $\langle M \rangle$ is Tannakian.

• Clearly, one can no longer compute its Tannaka group $G(M)$ as before, as one no longer has a topological fundamental group at disposal.

• One computes it by fixing a rational point $x \in X$ (if there are no rational points, one makes a field extension and computes there). Its k-points consist of all automorphisms of $M|_x$ which respect all Homs of subquotients of $M^n \otimes (M^\vee)^m|_x$.

Tannakian category over a field k of characteristic 0

Deligne's Riemann-Hilbert correspondence

says that if $k = \mathbb{C}$, if ∇ extends to a connection with logarithmic poles $\bar{\nabla} : \bar{M} \to \Omega^1(\log(\infty)) \otimes \bar{M}$ along infinity, i.e. if ∇ has regular singular poles, then the two definitions of $G(M)$ coincide.

Example

In particular, it applies when X is projective.

Algebraic solutions

Algebraic solutions

To say M has *algebraic solutions* as in the first lecture is equivalent to saying that $G(M)$ is a 0-dimensional group-scheme, or, equivalent to saying that the monodromy group $\rho(\pi_1^{\text{top}}(X(\mathbb{C}), x))$ is finite. (Here $k \subset \mathbb{C}$).

Proof

The monodromy group $\rho(\pi_1^{\text{top}}(X(\mathbb{C}), x))$ is finite if and only if M trivializes over a finite étale cover $h : Y \to X(\mathbb{C})$. Using that the algebraic closure of $\rho(\pi_1^{\text{top}}(X(\mathbb{C}), x))$ is itself, and the Tannaka formalism over k, one sees that *h is in fact defined over k*. Then solutions trivialize on Y. So solutions of the differential system of equations, locally in a neighborhood on X, consist of some of the regular functions on the pull-back neighborhood on Y.

Recall Grothendieck's p-curvature conjecture

Grothendieck's p-curvature conjecture

Let X be a smooth projective curve defined over a number field K. Then a system of linear differential equations M has *algebraic solutions* if and only if it has a full set of solutions modulo p for almost all p.

Here (X_S, M_S) is a model of (X, M) over a non-trivial open of the spectrum of the number ring $\mathcal{O}_K$.

Tannakian wish

Tannakian thought

In view of the Tannakian characterization of M having algebraic solutions, it is natural to wish to be able to argue as follows.

Tannakian wish

Tannakian thought

In view of the Tannakian characterization of M having algebraic solutions, it is natural to wish to be able to argue as follows.

- One has the model (X_S, M_S).

Tannakian wish

Tannakian thought

In view of the Tannakian characterization of M having algebraic solutions, it is natural to wish to be able to argue as follows.

• One has the model (X_S, M_S).

• On X_K, one has the group-scheme $G(M_K)$ over K, of which one wants to show that it is 0-dimensional, i.e. the set $(G(M_K)(\bar{K})$ of $\bar{K}$-points is finite, where $\bar{K}$ is an algebraic closure of K.

Tannakian wish

Tannakian thought

In view of the Tannakian characterization of M having algebraic solutions, it is natural to wish to be able to argue as follows.

• One has the model (X_S, M_S).

• On X_K, one has the group-scheme $G(M_K)$ over K, of which one wants to show that it is 0-dimensional, i.e. the set $(G(M_K)(\bar{K})$ of $\bar{K}$-points is finite, where $\bar{K}$ is an algebraic closure of K.

• One would like to construct a model $G_S(M_K)$ of $G(M_K)$ over S, such that the fiber at a closed point $s \in S$ is equal or at least related to $G(M_s)$ over $k(s)$.

Tannakian wish

Tannakian thought

In view of the Tannakian characterization of M having algebraic solutions, it is natural to wish to be able to argue as follows.

• One has the model (X_S, M_S).

• On X_K, one has the group-scheme $G(M_K)$ over K, of which one wants to show that it is 0-dimensional, i.e. the set $(G(M_K)(\bar{K})$ of $\bar{K}$-points is finite, where $\bar{K}$ is an algebraic closure of K.

• One would like to construct a model $G_S(M_K)$ of $G(M_K)$ over S, such that the fiber at a closed point $s \in S$ is equal or at least related to $G(M_s)$ over $k(s)$.

• Then one would say: $G_S(M_K)$ has relative dimension 0 over S if and only if $G(M_s)$ has relative dimension 0 over s for all closed points $s \in S$.

Tannakian wish

Tannakian thought

In view of the Tannakian characterization of M having algebraic solutions, it is natural to wish to be able to argue as follows.

- One has the model (X_S, M_S).
- On X_K, one has the group-scheme $G(M_K)$ over K, of which one wants to show that it is 0-dimensional, i.e. the set $(G(M_K)(\bar{K})$ of $\bar{K}$-points is finite, where $\bar{K}$ is an algebraic closure of K.
- One would like to construct a model $G_S(M_K)$ of $G(M_K)$ over S, such that the fiber at a closed point $s \in S$ is equal or at least related to $G(M_s)$ over $k(s)$.
- Then one would say: $G_S(M_K)$ has relative dimension 0 over S if and only if $G(M_s)$ has relative dimension 0 over s for all closed points $s \in S$.
- This would then reduce the problem to a study in pure characteristic $p > 0$, over a finite field.

The bundle with connection M_s does not span a Tannakian category

- Write $M_s = (E_s, \nabla_s)$. If $\nabla_s(\varphi) = 0, \varphi \in E_s$, then $\nabla_s(\lambda^p \varphi) = 0$ for all $\lambda \in \mathcal{O}_{X_s}$, thus the abelian subsheaf $E_s^{\nabla_s} \subset E_s$ of solutions is an $\mathcal{O}_{X_s^{(1)}}$-module $E_s^{(1)}$, where $X_s^{(1)}$ is the Frobenius twist.

From the wish to reality

The bundle with connection M_s does not span a Tannakian category

- Write $M_s = (E_s, \nabla_s)$. If $\nabla_s(\varphi) = 0, \varphi \in E_s$, then $\nabla_s(\lambda^p \varphi) = 0$ for all $\lambda \in \mathcal{O}_{X_s}$, thus the abelian subsheaf $E_s^{\nabla_s} \subset E_s$ of solutions is an $\mathcal{O}_{X_s^{(1)}}$-module $E_s^{(1)}$, where $X_s^{(1)}$ is the Frobenius twist.
- So *having a full set of solutions* (or equivalently having p-curvature 0) is just saying

$$(E_s, \nabla_s) = (F^{-1} E_s^{(1)} \otimes_{F^{-1}\mathcal{O}_{X_s^{(1)}}} \mathcal{O}_{X_s}, \nabla_s = 1 \otimes d) = (F^* E_s^{(1)}, \nabla_{\mathrm{can}}),$$

that is M_s has *one time* Frobenius descent.

From the wish to reality

The bundle with connection M_s does not span a Tannakian category

- Write $M_s = (E_s, \nabla_s)$. If $\nabla_s(\varphi) = 0, \varphi \in E_s$, then $\nabla_s(\lambda^p \varphi) = 0$ for all $\lambda \in \mathcal{O}_{X_s}$, thus the abelian subsheaf $E_s^{\nabla_s} \subset E_s$ of solutions is an $\mathcal{O}_{X_s^{(1)}}$-module $E_s^{(1)}$, where $X_s^{(1)}$ is the Frobenius twist.
- So *having a full set of solutions* (or equivalently having p-curvature 0) is just saying

$$(E_s, \nabla_s) = (F^{-1}E_s^{(1)} \otimes_{F^{-1}\mathcal{O}_{X_s^{(1)}}} \mathcal{O}_{X_s}, \nabla_s = 1 \otimes d) = (F^* E_s^{(1)}, \nabla_{\rm can}),$$

that is M_s has *one time* Frobenius descent.

- So any $\mathcal{O}_{X_s^{(1)}}$-submodule V of $E_s^{(1)}$ yields a subconnection $F^*V \subset M_s$, with quotient $F^*(E^{(1)}/V)$ which is not necessarily locally free. So the subquotients of tensors of M_s and $M_s^\vee$ can *not* span a Tannakian category.

From the wish to reality

The bundle with connection M_s does not span a Tannakian category

- Write $M_s = (E_s, \nabla_s)$. If $\nabla_s(\varphi) = 0, \varphi \in E_s$, then $\nabla_s(\lambda^p \varphi) = 0$ for all $\lambda \in \mathcal{O}_{X_s}$, thus the abelian subsheaf $E_s^{\nabla_s} \subset E_s$ of solutions is an $\mathcal{O}_{X_s^{(1)}}$-module $E_s^{(1)}$, where $X_s^{(1)}$ is the Frobenius twist.
- So *having a full set of solutions* (or equivalently having p-curvature 0) is just saying

$$(E_s, \nabla_s) = (F^{-1}E_s^{(1)} \otimes_{F^{-1}\mathcal{O}_{X_s^{(1)}}} \mathcal{O}_{X_s}, \nabla_s = 1 \otimes d) = (F^* E_s^{(1)}, \nabla_{\mathrm{can}}),$$

that is M_s has *one time* Frobenius descent.
- So any $\mathcal{O}_{X_s^{(1)}}$-submodule V of $E_s^{(1)}$ yields a subconnection $F^*V \subset M_s$, with quotient $F^*(E^{(1)}/V)$ which is not necessarily locally free. So the subquotients of tensors of M_s and $M_s^\vee$ can *not* span a Tannakian category.
- **End of dream. Awakening.**

$\mathcal{D}$-modules in characteristic $p > 0$

Grothendieck's definition of $\mathcal{D}$-modules

• The differential operators are defined as additive endomorphisms $\mathcal{O} \to \mathcal{O}$. It is a sheaf of rings, it is filtered, and $[D_n, D_m]$ has degree $\leq n + m - 1$ if D_n has degree n.

$\mathcal{D}$-modules in characteristic $p > 0$

Grothendieck's definition of $\mathcal{D}$-modules

• The differential operators are defined as additive endomorphisms $\mathcal{O} \to \mathcal{O}$. It is a sheaf of rings, it is filtered, and $[D_n, D_m]$ has degree $\le n + m - 1$ if D_n has degree n.

• In characteristic $p > 0$, not only $\partial_{x_1}^{m_1} \circ \ldots \circ \partial_{x_n}^{m_n}$ span the sheaf of differential operators over $\mathcal{O}$, but also e.g. $\partial_{x_1}^{p}/p$. *Unlike in characteristic* 0, $\mathcal{D}$ is *not* spanned by $\mathcal{D}^{\le 1}$ over $\mathcal{O}$, and in fact not even by $\mathcal{D}^{\le N}$ for any N.

$\mathcal{D}$-modules in characteristic $p > 0$

Grothendieck's definition of $\mathcal{D}$-modules

• The differential operators are defined as additive endomorphisms $\mathcal{O} \to \mathcal{O}$. It is a sheaf of rings, it is filtered, and $[D_n, D_m]$ has degree $\leq n + m - 1$ if D_n has degree n.

• In characteristic $p > 0$, not only $\partial_{x_1}^{m_1} \circ \ldots \circ \partial_{x_n}^{m_n}$ span the sheaf of differential operators over $\mathcal{O}$, but also e.g. $\partial_{x_1}^{p}/p$. *Unlike in characteristic* 0, $\mathcal{D}$ is *not* spanned by $\mathcal{D}^{\leq 1}$ over $\mathcal{O}$, and in fact not even by $\mathcal{D}^{\leq N}$ for any N.

• As a consequence, a $\mathcal{O}_{X_s}$-coherent $\mathcal{D}_{X_S}$-module *is more* than just an integrable connection.

Katz's theorems

Theorem (Riemann-Hilbert correspondence in characteristic $p > 0$)

An integrable connection $M_s = (E_s, \nabla_s)$ is a $\mathcal{D}_{X_s}$-module if

(0) *E_s is a vector bundle;*

(i) *M_s has a full set of solutions;*

(ii) *One has a further infinite Frobenius descent: there are vector bundles $E_s^{(i)}$ on the Frobenius twists $X_s^{(i)}$ such that $F^* E_s^{i+1} \cong E_s^{(i)}$.*

Remark

One should put the data of those isomorphisms in the definitions but as X_s is projective, this is irrelevant (Katz).

Theorem (Tannaka theory)

$\mathcal{O}_{X_s}$-coherent $\mathcal{D}_{X_s}$-modules build a Tannakian category over $k(s)$.

Properties of $\mathcal{O}$-coherent $\mathcal{D}$-modules over a smooth projective variety over a finite field

We study those properties, which are independent of whether the $\mathcal{O}$-coherent $\mathcal{D}$-module comes from characteristic 0 or not. So let (X_s, M_s) be an $\mathcal{O}$-coherent $\mathcal{D}$-module defined over a smooth projective variety X_s over a finite field $k(s)$.

Properties of $\mathcal{O}$-coherent $\mathcal{D}$-modules over a smooth projective variety over a finite field

We study those properties, which are independent of whether the $\mathcal{O}$-coherent $\mathcal{D}$-module comes from characteristic 0 or not. So let (X_s, M_s) be an $\mathcal{O}$-coherent $\mathcal{D}$-module defined over a smooth projective variety X_s over a finite field $k(s)$.

Theorem (E-Kisin)

Then $G(M_s)$ is 0-dimensional and étale. Said differently, there is a finite étale cover $h_s : Y_s \to X_s$ (which is even a torsor under $G(M_s)$) such that $h_s^* M_s$ is trivial, i.e. $h_s^* M_s \cong \oplus_1^r (\mathcal{O}, d)$. In simple terms: (X_s, M_s) is isotrivial.

Proof

It is pure algebraic geometry. Sketch:

(i) The system of Frobenius divided bundles $\{E_s^{(i)},\ F^*E_s^{(i+1)} \cong E_s^i\}_{i\geq 0}$ has the property for some $n_0 \geq 0$, $\{E_s^{(i)},\ F^*E_s^{(i+1)} \cong E_s^i\}_{i\geq n_0}$ is an extension of such objects, with the property that the underlying vector bundles $E_s^{(i)}$ are all stable of degree 0.

(ii) Enough to show isotriviality of one such object (extension of trivial objects by trivial objects are isotrivial).

(iii) Use existence of quasi-projective moduli (Langer) to show there are repetitions on the moduli points of the $E_s^{(i)}$.

(iv) Use that $k(s)$ is a finite field to conclude that there are repetitions among the $E_s^{(i)}$, i.e. $F^a E_s^{(i)} \cong E_s^{(i)}$ for some a and $i \geq i_0$ for some large i_0 (i.e. the Brauer obstruction vanishes).

(v) Then h_s is a Lang torsor trivializing those bundles.

On track for the Tannaka thought

Reinforcing Grothendieck's requirement from full set of solutions to $\mathcal{D}$-module.

Given the previous theorem, one can try to develop the Tannakian dream presented above: compare the group-scheme $G(M_K)$ over the number field K and the group-schemes $G(M_s)$ over the finite fields $k(s)$, *under the stronger assumption* that not only M_s has a full set of solutions for almost all s, that is one has one time Frobenius descent on M_s, but there is a $\mathcal{D}$-module structure on M_s, that is one has a further infinite Frobenius descent.

On track for the Tannaka thought

Reinforcing Grothendieck's requirement from full set of solutions to $\mathcal{D}$-module.

Given the previous theorem, one can try to develop the Tannakian dream presented above: compare the group-scheme $G(M_K)$ over the number field K and the group-schemes $G(M_s)$ over the finite fields $k(s)$, *under the stronger assumption* that not only M_s has a full set of solutions for almost all s, that is one has one time Frobenius descent on M_s, but there is a $\mathcal{D}$-module structure on M_s, that is one has a further infinite Frobenius descent.

Dependency

On the other hand, while the bundle of solutions $E_s^{(1)} = E_s^{\nabla_s}$, which is the first Frobenius descent, depends only on the restriction M_s of M_S, the $\mathcal{D}$-module structure on M_s, i.e. the further Frobenius descents, *is a choice*. Yet one has the following theorem.

Independency

Let again (X_s, M_s) be an $\mathcal{O}$-coherent $\mathcal{D}$-module defined over a smooth projective variety over a finite field $k(s)$.

Independency

Let again (X_s, M_s) be an $\mathcal{O}$-coherent $\mathcal{D}$-module defined over a smooth projective variety over a finite field $k(s)$.

Theorem (E-Kisin)

Then the moment M_s carries a $\mathcal{D}$-module structure, $G(M_s)$ is independent of the choice, thus is prescribed by $G(M_K)$ in case M_s is the reduction at s of $M_K = (E_K, \nabla_K)$ defined over a number field.

Independency

Let again (X_s, M_s) be an $\mathcal{O}$-coherent $\mathcal{D}$-module defined over a smooth projective variety over a finite field $k(s)$.

Theorem (E-Kisin)

Then the moment M_s carries a $\mathcal{D}$-module structure, $G(M_s)$ is independent of the choice, thus is prescribed by $G(M_K)$ in case M_s is the reduction at s of $M_K = (E_K, \nabla_K)$ defined over a number field.

Comment

It is more precise than this: the underlying vector bundle E_s is strongly (semi)-stable, $\langle E_s \rangle$ defines a Tannaka category, and $G(M_s) \cong G(E_s)$.

General assumption in the sequel

(X, M) shall be an $\mathcal{O}$-coherent $\mathcal{D}$-module defined over a smooth projective variety over a number field K. We shall assume that for almost all s of a model (X_S, M_S), M_s descends to a $\mathcal{D}$-module.

First theorem

Theorem (E-Kisin)

If there is a dense set of closed points $s \in S$ such that the order of G_s is prime to p, then M is isotrivial, that is has algebraic solutions.

First theorem

Theorem (E-Kisin)

If there is a dense set of closed points $s \in S$ such that the order of G_s is prime to p, then M is isotrivial, that is has algebraic solutions.

Comment

It would be better not to have to assume this non-divisibility, conjecturally it should be automatic. Still, the assumption is *much much weaker* than assuming that the order of G_s is *bounded* independently of s. Under this latter assumption, Matzat-van der Put conjectured that M is isotrivial. So we give a *positive answer* to their conjecture if X is smooth projective. However, we also give a *negative* answer if X is not proper.

On proof of the First Theorem

• Theorem of Camille Jordan, together with the theorem that the G_s are finite étale, together with the fact that X being proper, its geometric étale fundamental group in characteristic 0 is topologically finitely generated, enables one to replace X by a finite étale cover of X over a finite extension of K and to assume that for a dense set of closed points s, G_s is abelian.

On proof of the First Theorem

• Theorem of Camille Jordan, together with the theorem that the G_s are finite étale, together with the fact that X being proper, its geometric étale fundamental group in characteristic 0 is topologically finitely generated, enables one to replace X by a finite étale cover of X over a finite extension of K and to assume that for a dense set of closed points s, G_s is abelian.

• Algebraic geometry enables one to then go up to characteristic 0 to conclude that M is a successive extension of rank 1 integrable connections M'.

On proof of the First Theorem

• Theorem of Camille Jordan, together with the theorem that the G_s are finite étale, together with the fact that X being proper, its geometric étale fundamental group in characteristic 0 is topologically finitely generated, enables one to replace X by a finite étale cover of X over a finite extension of K and to assume that for a dense set of closed points s, G_s is abelian.

• Algebraic geometry enables one to then go up to characteristic 0 to conclude that M is a successive extension of rank 1 integrable connections M'.

• One then concludes using the theorems of Chudnosky-Chudnovsky-André mentioned in the first lecture.

Second theorem

Theorem (E-Kisin)

If M is an abstract polarized $\mathbb{Z}$-variation of Hodge structure, then it is isotrivial.

Second theorem

Theorem (E-Kisin)

If M is an abstract polarized $\mathbb{Z}$-variation of Hodge structure, then it is isotrivial.

On proof

One uses stability of the E_s and the Simpson correspondence.