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On the Loday Symbol in the Deligne–Beilinson Cohomology

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Abstract. Let A be an algebraic smooth variety over $\mathbb{C}$, $X_0, \ldots, X_n$ be global regular functions on A, ϕ be a global invertible regular function on X of value 1 on the divisor Y defined by X_0 and Z defined by $X_1 \ldots X_n$. Then the cup product $\{\phi|_{A-Z}, X_1, \ldots, X_n\}$ in the Deligne–Beilinson cohomology group $H_{\mathscr{D}}^{n+1}(A - Z, Y; \mathbb{Z}(n + 1))$ extends across Z to a so-called Loday symbol, denoted by $\{\phi, X_1, \ldots, X_n\}$, in $H_{\mathscr{D}}^{n+1}(A, Y; \mathbb{Z}(n+1))$. In this article, we give explicit formulae for $\{\phi, X_1, \ldots, X_n\}$ as a Čech cocycle. Thereby, one obtains a proof of Beilinson's formula for the evaluation of the Loday symbol along certain homology cycles.

Key words. Deligne–Beilinson cohomology, Loday symbol, cup product.

This article is thought as a complement to the volume *Beilinson's Conjectures on Special Values of L-Functions*, where [3] and [4] are two of the contributions. It gives an explicit formula for the Loday symbol in the Deligne–Beilinson cohomology. Thereby, one obtains the proof of the 'crucial lemma' 2.4 in [4], II, which is a formula for the evaluation of the Lody symbol on certain cycles. This formula was stated by A. Beilinson in [1], 7.0.2, and – together with some very useful comments and really necessary assumptions – in [4], II, 2.4, both times, however, without proof. Note that the explicit description of the regulator map for Spec $\mathbb{Q}(\mu_N)$, where μ_N is the group of Nth roots of unity, given by Beilinson in [1], 7.1, relies on this crucial lemma.

Let $\mathbf{A}_{\mathbb{C}}^{n+1}$ be the affine space of dimension $n + 1$ of coordinates X_i over the complex numbers $\mathbb{C}$. Let

$$\phi = 1 - X_0 \ldots X_n, \qquad A = \mathbf{A}_{\mathbb{C}}^{n+1} - (\phi = 0), \qquad U = A - \bigcup_{i \geqslant 1} (X_i = 0).$$

Then $\phi|_U \in H_{\mathscr{D}}^1(U, (X_0 = 0); \mathbb{Z}(1))$, the group of invertible regular functions on U which are 1 on $(X_0 = 0)$ and $X_i \in H_{\mathscr{D}}^1(U, \mathbb{Z}(1))$, the group of invertible regular functions on U. One considers the cup product $\{\phi|_U, X_1, \ldots, X_n\}$ in the Deligne–Beilinson cohomo-

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logy group $H_{\mathscr{D}}^{n+1}(U, (X_0 = 0); \mathbb{Z}(n+1))$. *As*

$$H_{\mathscr{D}}^{\cdot}(U, (X_0 = 0); \cdot) \xleftarrow[\text{rest}]{} H_{\mathscr{D}}^{\cdot}(A, (X_0 = 0); \cdot)$$

is an isomorphism, this defines an element $\text{rest}^{-1}\{\phi|_U, X_1, \ldots, X_n\}$ in $H_{\mathscr{D}}^{n+1}(A, (X_0 = 0); \mathbb{Z}(n+1))$. This is the *Loday symbol in the Deligne–Beilinson cohomology*. In this article, we give *explicit formulae* for the Loday symbol as a *Čech cocycle* (1.8), (2.3), (2.5i).

Let $h\colon X \to A$ be an algebraic morphism, with X smooth. This gives explicit formulae for $h^*\text{rest}^{-1}\{\phi|_U, X_1, \ldots, X_n\}$ in $H_{\mathscr{D}}^{n+1}(X, S; \mathbb{Z}(n+1))$ if $h(S) \subset (X_0 = 0)$. If complex dimension $X \leqslant n$, then $H_{\mathscr{D}}^{n+1}(X, S; \mathbb{Z}(n+1)) = H^n(X, S; \mathbb{C}/\mathbb{Z}(n+1))$, the Betti cohomology group. Therefore, we may evaluate $h^*\text{rest}^{-1}\{\phi|_U, X_1, \ldots, X_n\}$ along relative homology classes $[\gamma] \in H_n(X, S; \mathbb{Z})$. The previous explicit formulae give an expression (3.9) for this evaluation under certain assumptions on a representative γ of $[\gamma]$.

Our method consists of reducing the problem to the analytic Deligne cohomology (1.3), and there to define a substitute for the cup product if the functions X_i, $i \geqslant 1$ are not invertible (1.4), (1.5). As this definition makes sense for analytic varieties as well, we define in this way a sort of Loday symbol in the analytic case (1.6), (1.7), which is no longer unique (2.5ii), (2.5iii).

In Section 4 we weaken the condition on the dimension of the algebraic variety X by an assumption on the curvature of a sum of pull-backs of the Loday symbol. This allows us to define it as the class of a global closed holomorphic n-form (4.2). In (4.4) and (4.5), we give the evaluation of this class along relative cycles with some assumptions which are milder than in (3.9).

Finally, in (4.7) we explain the relationship with Bloch's regulator map $K_2(X) \to H_{\mathscr{D}}^2(X, \mathbb{Z}(2))$ in any dimension.

I cordially thank M. Rapoport with whom I discussed the above points several times. In the first version of this article, I considered all the cohomology groups only over $\mathbb{Q}$, which, as O. Gabber pointed out to me, was more careful than necessary. I thank S. Bloch for suggesting several improvements, especially (2.6) is due to him.

1. Construction of a Class x in $H_{\mathscr{D}}^{n+1}(A, Y; \mathbb{Z}(n+1))$

1.1. Let A be a smooth algebraic variety over $\mathbb{C}$, $Y + Z$ be a normal crossing divisor on A where Z is defined by $X_1 \ldots X_m$, X_i being a global regular reduced function on A. We define the natural embeddings

$$\begin{tikzcd}
A - Y \arrow[r, "i"] & A \\
A - Y - Z \arrow[u] \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

Let ϕ be in

$$H_{\mathscr{D}}^1(A, Y + Z; \mathbb{Z}(1)) = \ker \mathscr{O}(A)^* \to \mathscr{O}(Y + Z)^*.$$

Define $U = A - Z$, $Y_U = Y \cap U$. Then $\phi|_U$ lies in

$$H^1_{\mathscr{D}}(U, Y_U; \mathbb{Z}(1)) = \ker \mathscr{O}(U)^* \to \mathscr{O}(Y_U)^*,$$

and X_i lies in $H^1_{\mathscr{D}}(U, \mathbb{Z}(1)) = \mathscr{O}(U)^*$. Choose $1 \leqslant n \leqslant m$. Then the cup product $\{\phi|_U, X_1, \ldots, X_n\}$ is defined as an element in $H^{n+1}_{\mathscr{D}}(U, Y_U; \mathbb{Z}(n+1))$. In Section 1, we construct a specific element $x \in H^{n+1}_{\mathscr{D}}(A, Y; \mathbb{Z}(n+1))$ from which we show in Section 2 that its restriction to U, $x|_U \in H^{n+1}_{\mathscr{D}}(U, Y_U; \mathbb{Z}(n+1))$, is precisely $\{\phi|_U, X_1, \ldots, X_n\}$. In other words, we define a lifting of the cup product across Z.

1.2. Here we show that the problem is reduced to a problem in the analytic Deligne cohomology. Recall [3], 2.9, that

$$\begin{aligned} &H^{q+1}_{\mathscr{D}}(A, Y; \mathbb{Z}(p+1)) \\ &\quad = H^{q+1}(\bar{A}, \mathrm{cone}[(Rk_* i_! \mathbb{Z}(p+1) + F^{p+1}(\log(H + \bar{Y}))(-\bar{Y})) \to \\ &\quad\quad \to \Omega^{\cdot}_{\bar{A}}(*H + \log \bar{Y})(-\bar{Y})][-1]) \\ &\quad = H^{q+1}(\bar{A}, \mathrm{cone}[F^{p+1}(\log H + \bar{Y})(-\bar{Y}) \to Rk_* i_! \mathbb{C}/\mathbb{Z}(p+1)][-1]), \end{aligned}$$

where $k\colon A \to \bar{A}$ is a good compactification such that $H := \bar{A} - A$, $\bar{Y} :=$ closure of Y in $\bar{A}$ and $H + \bar{Y}$ are divisors with normal crossings.

Forgetting the growth condition along H on the F^{p+1} part, one obtains a morphism in the analytic Deligne cohomology [3], 2.13:

$$\begin{aligned} &H^{q+1}_{\mathscr{D},\mathrm{an}}(A, Y; \mathbb{Z}(p+1)) \\ &\quad = H^{q+1}(A, \mathrm{cone}[(i_! \mathbb{Z}(p+1) + \Omega^{\geqslant p+1}_A(\log Y)(-Y)) \to \\ &\quad\quad \to \Omega^{\cdot}_A(\log Y)(-Y)][-1]) \\ &\quad = H^{q+1}(A, \mathrm{cone}[\Omega^{\geqslant p+1}_A(\log Y)(-Y) \to i_! \mathbb{C}/\mathbb{Z}(p+1)][-1]) \\ &\quad = H^{q+1}(A, i_! \mathbb{Z}(p+1) \to \Omega^{\leqslant p}_A(\log Y)(-Y)). \end{aligned}$$

One obtains a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 \to & \dfrac{H^q(A, Y; \mathbb{C}/\mathbb{Z}(p+1))}{F^{p+1}H^q(A, Y; \mathbb{C})} & \to & H^{q+1}_{\mathscr{D}}(A, Y; \mathbb{Z}(p+1)) & \xrightarrow{d} & F^{p+1,q+1}_{\mathbb{Z}}(A, Y) & \to 0 \\ & \big\downarrow & & \big\downarrow {\scriptstyle f_{p+1,q+1}} & & \big\downarrow & \\ 0 \to & \dfrac{H^q(A, Y; \mathbb{C}/\mathbb{Z}(p+1))}{H^q(A, \Omega^{\geqslant p+1}_A(\log Y)(-Y))} & \to & H^{q+1}_{\mathscr{D},\mathrm{an}}(A, Y; \mathbb{Z}(p+1)) & \xrightarrow{d_{\mathrm{an}}} & \mathrm{Hol}^{p+1}_{\mathbb{Z}}(A, Y) & \to 0 \end{array}$$

where

$$\begin{aligned} F^{p+1,q+1}_{\mathbb{Z}}(A, Y) := \{\omega \in F^{p+1}H^{q+1}(A, Y; \mathbb{C}) &\text{ such that the image of} \\ &\omega \text{ in } H^{q+1}(A, Y; \mathbb{C}/\mathbb{Z}(p+1)) \text{ vanishes}\} \end{aligned}$$

and

$$\mathrm{Hol}_{\mathbb{Z}}^{p+1,q+1}(A, Y) := \{\omega \in H^{q+1}(A, \Omega_A^{\geqslant p+1}(\log Y)(-Y)) \text{ such that the image of } \omega \text{ in } H^{q+1}(A, Y; \mathbb{C}/\mathbb{Z}(p+1)) \text{ vanishes}\}.$$

LEMMA (see also [3], 2.13 and [1], 1.6.1). (i) $f_{n+1,n+1}$ *is injective. One has*

$$H_{\mathscr{D}}^{n+1}(A, Y; \mathbb{Z}(n+1)) = \{x \in H_{\mathscr{D},\mathrm{an}}^{n+1}(A, Y; \mathbb{Z}(n+1)) \textit{ such that } \mathrm{d}x \in F^{n+1}H^{n+1}(A, Y; \mathbb{C})\}$$

and

$$H^n(A, Y; \mathbb{C}/\mathbb{Z}(n+1)) = Ker\, d = Ker\, d_{\mathrm{an}}.$$

(ii) $f_{p+1,q+1}$ *is an isomorphism for* $q < p$. *One then has*

$$H_{\mathscr{D}}^{q+1}(A, Y; \mathbb{Z}(p+1)) = H^q(A, Y; \mathbb{C}/\mathbb{Z}(p+1))$$

(iii) $f_{p+1,q+1}$ *is an isomorphism for dim* $A < p + 1$. *One then has*

$$H_{\mathscr{D}}^{q+1}(A, Y; \mathbb{Z}(p+1)) = H^q(A, Y; \mathbb{C}/\mathbb{Z}(p+1))$$

Proof. (i) One has

$$F^{n+1}H^n(A, Y; \mathbb{C}) = 0 = H^n(A, \Omega_A^{\geqslant n+1}(\log Y)(-Y))$$

and

$$F^{n+1}H^{n+1}(A, Y; \mathbb{C}) = H^0(\bar{A}, \Omega_{\bar{A}}^{n+1}(\log(H + \bar{Y})(-\bar{Y}))_{d\ \mathrm{closed}}$$

is embedded in

$$H^{n+1}(A, \Omega_A^{\geqslant n+1}(\log Y)(-Y)) = H^0(A, \Omega_A^{n+1}(\log Y)(-Y))_{d\ \mathrm{closed}}.$$

(ii), (iii) In both cases, the cohomology of F^{p+1} and $\Omega^{\geqslant p+1}$ appearing in the exact sequences vanishes.

1.3. COROLLARY. *In order to construct an element* $x \in H_{\mathscr{D}}^{n+1}(A, Y; \mathbb{Z}(n+1))$, *it is enough to construct it as an element of* $H_{\mathscr{D},\mathrm{an}}^{n+1}(A, Y; \mathbb{Z}(n+1))$ *and to verify that its curvature* $\mathrm{d}x$ *is algebraic, that is, in* $F^{n+1}H^{n+1}(A, Y; \mathbb{C})$.

Therefore, in (1.4), (1.5), (1.6), and (1.7), we assume only A, $Y + Z$ to be analytic, X_i to be global holomorphic on A, and ϕ to be global holomorphic invertible on A such that $\phi|_{Y\cup Z} = 1$.

1.4. Consider $\phi: A \to \mathbb{C}^*$, with $\phi(Y \cup Z) = 1$. Let $\mathscr{A}_0 \cup \mathscr{A}_1$ be an analytic open cover of $\mathbb{C}^*$ such that $1 \in \mathscr{A}_1 - \mathscr{A}_0$, $\log \phi|_{\phi^{-1}(\mathscr{A}_i)}$is single-valued and

$$\log \phi|_{\phi^{-1}(\mathscr{A}_1)\cap(Y\cup Z)} = 0.$$

One has

$$\log \phi|_{\phi^{-1}(\mathscr{A}_i)} \in H^0(\phi^{-1}(\mathscr{A}_i), \mathcal{O}_A(-Y-Z)).$$

Then for any refinement $(A_i)_{i\in I}$ of $\phi^{-1}(\mathscr{A}_i)$, defined by a map $\sigma: I \to \{0, 1\}$, one has

(α) $\log_i \phi := \log \phi|_{A_i \subset \phi^{-1}(\mathscr{A}_{\sigma(i)})}$
$\qquad \in H^0(A_i, \mathcal{O}_{A_i}(-Y-Z))$.

(β) $z^{n-1}_{i_0 i_0} := (\delta \log \phi)_{i_0 i_1} = \log_{i_1} \phi - \log_{i_0} \phi$
$\qquad \in H^0(A_{i_0 i_1}, \lambda_! \mathbb{Z}(1))$,

$z^{n-1}_{i_0 i_1} = 0$, if $A_{i_0 i_1} \cap (Y \cup Z) \neq \phi$, even if $A_{i_0 i_1}$ is not connected and $(\delta z^{n-1}) = 0$.

Take such a refinement with

(γ) if $A_{i_0 \ldots i_k} \cap (Y \cup Z) = \phi$,
$\log_{i_0 \ldots i_k} X_k \in H^0(A_{i_0 \ldots i_k}, \mathcal{O}_A)$.

For example, we choose (A_i) such that all the $A_{i_0 \ldots i_k}$ with $A_{i_0 \ldots i_k} \cap (Y \cup Z) = \phi$ are simply connected. (It is enough, of course to ensure that all the $A_{i_0 \ldots i_k}$ are simply connected.)

Define

$$g_{i_0 \ldots i_k} := \begin{cases} \log_{i_0 \ldots i_k} X_k, & \text{if } A_{i_0 \ldots i_k} \cap (Y \cup Z) = \phi, \\ 0 & \text{if } A_{i_0 \ldots i_k} \cap (Y \cup Z) \neq \phi. \end{cases}$$

One has

$$g_{i_0 \ldots i_k} \in H^0(A_{i_0 \ldots i_k}, \mathcal{O}_A(-Y-Z)).$$

We want to construct

$$\bar{x} \in H^{n+1}(A, \lambda_! \mathbb{Z}(n+1) \to \Omega_A^{\leq n}(\log(Y+Z))(-Y-Z))$$

as a cocycle $\bar{x} = (x^{-1}, x^0, \ldots, x^n)$ in the Čech complex

$$(\mathscr{C}^\cdot(A_i, \lambda_! \mathbb{Z}(n+1) \to \Omega_A^{\leq n}(\log(Y+Z))(-Y-Z), (-1)^\cdot \delta + d):$$

$$x^{-1} \in \mathscr{C}^{n+1}(\lambda_! \mathbb{Z}(n+1)),$$
$$x^0 \in \mathscr{C}^n(\mathcal{O}_A(-Y-Z)),$$
$$\vdots$$
$$x^n \in \mathscr{C}^0(\Omega_A^n(\log Y)(-Y)),$$

with $(-1)^{n+1} \delta x^j + \mathrm{d}x^{j-1} = 0$.

1.5. The condition (1.4α) implies that

$$x_i^n := \log_i \phi \frac{\mathrm{d}X_1}{X_1} \wedge \ldots \wedge \frac{\mathrm{d}X_n}{X_n} \quad \text{is in } H^0(A_i, \Omega_A^n(\log Z)(-Y-Z)),$$

which is embedded in $H^0(A_i, \Omega_A^n(\log(Y+Z))(-Y-Z))$. This defines x_i^n. We have to resolve the equation

$$(\mathrm{d}x^{n-1})_{i_0i_1} = (-1)^n(\delta x^n)_{i_0i_1} = (-1)^n z_{i_0i_1}^{n-1}\frac{\mathrm{d}X_1}{X_1}\wedge\ldots\wedge\frac{\mathrm{d}X_n}{X_n}.$$

Define

$$x_{i_0i_1}^{n-1} := (-1)^n z_{i_0i_1}^{n-1} g_{i_0i_1}\frac{\mathrm{d}X_2}{X_2}\wedge\ldots\wedge\frac{\mathrm{d}X_n}{X_n}$$
$$\in H^0(A_{i_0i_1}, \Omega_A^{n-1}(\log(Y+Z))(-Y-Z)).$$

Assume by induction that we may define for $1 \leqslant l \leqslant k$

$$z_{i_0\ldots i_l}^{n-l} \in H^0(A_{i_0\ldots i_l}, \lambda_!\mathbb{Z}(l)),$$

with

$$(\delta z^{n-1}) = 0 \quad \text{and} \quad z_{i_0\ldots i_l}^{n-1} = 0, \quad \text{if } A_{i_0\ldots i_l}\cap(Y\cup Z)\neq\phi,$$

$$x_{i_0\ldots i_l}^{n-l} = (-1)^{ln} z_{i_0\ldots i_l}^{n-l} g_{i_0\ldots i_l}\frac{\mathrm{d}X_{l+1}}{X_{l+1}}\wedge\ldots\wedge\frac{\mathrm{d}X_n}{X_n},$$

$$\mathrm{d}x^{n-l} = (-)^n\delta x^{n-l+1}, \quad l \leqslant k.$$

Define

$$z_{i_0\ldots i_{k+1}}^{n-(k+1)} := \delta(z_{i_0\ldots i_k}^{n-k} g_{i_0\ldots i_k}).$$

If, for all $l \in \{0, \ldots, k+1\}$,

$$A_{i_0\ldots\hat{i}_l\ldots i_{k+1}}\cap(Y\cup Z)\neq\phi, \quad \text{then } z_{i_0\ldots i_{k+1}}^{n-(k+1)} = 0$$

(especially if $A_{i_0\ldots i_{k+1}}\cap(Y\cup Z)\neq\phi$).

Otherwise, $A_{i_1\ldots i_{k+1}}\cap(Y\cup Z) = \phi$ (say). Then

$$z_{i_0\ldots i_{k+1}}^{n-(k+1)} = \sum_{l=1}^{k+1}(-1)^l z_{i_0\ldots\hat{i}_l\ldots i_{k+1}}^{n-k}(g_{i_0\ldots\hat{i}_l\ldots i_{k+1}} - g_{i_1\ldots i_{k+1}}) +$$
$$+ (\delta z^{n-k})_{i_0\ldots i_{k+1}} g_{i_1\ldots i_{k+1}}.$$

If

$$z_{i_0\ldots\hat{i}_l\ldots i_{k+1}} \neq 0,$$

then

$$A_{i_0\ldots\hat{i}_l\ldots i_{k+1}}\cap(Y\cup Z) = \phi,$$

therefore

$$g_{i_0\ldots\hat{i}_l\ldots i_{k+1}} - g_{i_1\ldots i_{k+1}} \in \mathbb{Z}(1).$$

Therefore, one has

$$z_{i_0\ldots i_{k+1}}^{n-(k+1)} \in H^0(A_{i_0\ldots i_{k+1}}, \lambda_!\mathbb{Z}(k+1)).$$

We may define

$$x^{n-(k+1)}_{i_0\dots i_{k+1}} := (-1)^{(k+1)\cdot n} z^{n-(k+1)}_{i_0\dots i_{k+1}} g_{i_0\dots i_{k+1}} \frac{\mathrm{d}X_{k+2}}{X_{k+2}} \wedge \cdots \wedge \frac{\mathrm{d}X_n}{X_n}$$

$$\in H^0(A_{i_0\dots i_{k+1}}, \Omega^{n-(k+1)}_A(\log(Y+Z))(-Y-Z)),$$

with

$$\mathrm{d}x^{n-(k+1)} = (-1)^n \,\delta x^{n-k}, \quad \text{if } k < n.$$

If $k = n$,

$$x^{-1}_{i_0\dots i_{n+1}} = (-1)^{(n+1)n} z^{-1}_{i_0\dots i_{n+1}}.$$

1.6. PROPOSITION. *The Čech cocycle* $\bar{x} = (x^{-1}, x^0, \dots, x^n)$ *constructed in* (1.5) *defines a cohomology class*

$$\bar{x} \in H^{n+1}(A, \lambda_! \mathbb{Z}(n+1)) \to \Omega^{\leq n}_A(\log(Y+Z)(-Y-Z)).$$

1.7. Let Z_l be a smooth component of Z. We consider the morphism of restriction

$$\begin{array}{ccc} i_!\mathbb{Z}(n+1) & \to & \Omega^{\leq n}_A(\log Y)(-Y) \\ \Big\downarrow \frac{1}{2i\pi} & & \Big\downarrow \text{restriction}_l \\ i|_{Z_l !}\mathbb{Z}(n) & \longrightarrow & \Omega^{\leq n-1}_{Z_l}(\log Y)(-Y) \end{array}$$

whose kernel contains

$$\lambda_! \mathbb{Z}(n+1) \to \Omega^{\leq n}_A(\log(Y+Z))(-Y-Z),$$

and whose cohomology reads

$$H^{n+1}_{\mathcal{D},\mathrm{an}}(A, Y; \mathbb{Z}(n+1)) \xrightarrow[\text{restriction}_l]{} H^{n+1}_{\mathcal{D},\mathrm{an}}(Z_l, Y; \mathbb{Z}(n)).$$

THEOREM. *There is a class* $x \in H^{n+1}_{\mathcal{D},\mathrm{an}}(A, Y; \mathbb{Z}(n+1))$, *such that* $\text{restriction}_l\, x = 0$ *and such that*

$$\mathrm{d}x = \frac{\mathrm{d}\phi}{\phi} \wedge \frac{\mathrm{d}X_1}{X_1} \wedge \cdots \wedge \frac{\mathrm{d}X_n}{X_n} \in \mathrm{Hol}^{n+1,n+1}_{\mathbb{Z}}(A, Y).$$

Proof. Define x as the image of $\bar{x}$ via

$$\begin{array}{l} H^{n+1}(A, \lambda_!\mathbb{Z}(n+1) \to \Omega^{\leq n}_A(\log(Y+Z))(-Y-Z)) \\ \qquad\quad\downarrow \\ H^{n+1}_{\mathcal{D},\mathrm{an}}(A, Y; \mathbb{Z}(n+1)) \end{array}$$

given by the same cocycle. One has $\mathrm{d}x = \mathrm{d}x^n_i$.

1.8. Go back to the algebraic situation described in (1.1). Then

$$\mathrm{d}x = \frac{\mathrm{d}\phi}{\phi} \wedge \frac{\mathrm{d}X_1}{X_1} \wedge \cdots \wedge \frac{\mathrm{d}X_n}{X_n} \in F^{n+1}H^{n+1}(A, Y; \mathbb{C}).$$

By (1.2i), we obtain the following theorem.

THEOREM. *The class x of (1.7) is in*

$$H^{n+1}_{\mathscr{D}}(A, Y; \mathbb{Z}(n+1)) \quad \textit{and} \quad \mathrm{d}x = \frac{\mathrm{d}\phi}{\phi} \wedge \frac{\mathrm{d}X_1}{X_1} \wedge \cdots \wedge \frac{\mathrm{d}X_n}{X_n}.$$

2. Restriction of x to U

2.1. In this section, we want to show that the restriction to U of the class x constructed in (1.8) is

$$y := \{\phi_{|U}, X_1, \ldots, X_n\} \in H^{n+1}_{\mathscr{D}}(U, Y_U; \mathbb{Z}(n+1)).$$

As

$$\mathrm{d}y = \frac{\mathrm{d}\phi}{\phi} \wedge \frac{\mathrm{d}X_1}{X_1} \wedge \cdots \wedge \frac{\mathrm{d}X_n}{X_n} \text{ [3], (3.7), we have by (1.2i):}$$

LEMMA. $(x_{|U} - y) \in H^n(U, Y_U; \mathbb{C}/\mathbb{Z}(n+1))$.

Therefore we may assume, as in (1.4), (1.5), (1.6) and (1.7), that A, and therefore U, are only analytic manifolds.

2.2. We take a refinement U_j of $X_j \cap U$ such that $\log X_{i|U_j} := \log_j X_i$ is single-valued, that is $\log_j X_i \in H^0(U_j, \mathcal{O}_U)$ for $i \leqslant n$. Define $\mu = i|_U: U - Y_U \to U$. Define y as a cocycle $y = (y^{-1}, y^0, \ldots, y^n)$ in the Čech complex

$$(\mathscr{C}^{\cdot}(U_j, \mu_! \mathbb{Z}(n+1) \to \Omega_U^{\leqslant n}(\log Y_U)(-Y_U)), (-1)^{\cdot}\delta + d)$$

with

$$\begin{aligned}
&y^{-1} \in \mathscr{C}^{n+1}(\mu_! \mathbb{Z}(n+1)),\\
&y^0 \in \mathscr{C}^n(\mathcal{O}_U(-Y_U)),\\
&\vdots\\
&y^n \in \mathscr{C}^0(\Omega_U^n(\log Y_U)(-Y_U)),
\end{aligned}$$

with $(-1)^{n+1}\delta y^j + \mathrm{d}y^{j-1} = 0$.

One has [3] (3.2):

$$y_j^n = \log_j \phi \frac{\mathrm{d}X_1}{X_1} \wedge \cdots \wedge \frac{\mathrm{d}X_n}{X_n},$$

$$y^{n-1}_{j_0j_1} = (-1)^n Z^{n-1}_{j_0j_1} \log_{j_1} X_1 \frac{\mathrm{d}X_2}{X_2} \wedge \cdots \wedge \frac{\mathrm{d}X_n}{X_n},$$

$$\vdots$$

$$y^{n-k}_{j_0\ldots j_k} = (-1)^{kn} Z^{n-k}_{j_0\ldots j_k} \log_{j_k} X_k \frac{\mathrm{d}X_{k+1}}{X_{k+1}} \wedge \cdots \wedge \frac{\mathrm{d}X_n}{X_n},$$

$$y^{-1}_{j_0 j_{n+1}} = (-1)^{(n+1)n} Z^{-1}_{j_0\ldots j_{n+1}},$$

with

$$Z^{n-1}_{j_0j_1} := z^{n-1}_{j_0j_1} = (\delta \log \phi)_{j_0j_1} \in H^0(U_{j_0j_1}, \mu_! \mathbb{Z}(1)),$$

$$Z^{n-k}_{j_0\ldots j_k} := \delta(Z^{n-k+1}_{j_0\ldots j_{k-1}} \log_{j_{k-1}} X_{k-1})$$
$$\in H^0(U_{j_0\ldots j_k}\ \mu_! \mathbb{Z}(k)).$$

Therefore, one has $x^n - y^n = 0$ and for $1 \leqslant k \leqslant n$:

$$(x^{n-k} - y^{n-k})_{i_0\ldots i_k}$$
$$= (-1)^{n-k}(z^{n-k}_{i_0\ldots i_k} g_{i_0\ldots i_k} - Z^{n-k}_{i_0\ldots i_k} \log_{i_k} X_k). \frac{\mathrm{d}X_{k+1}}{X_{k+1}} \wedge \cdots \wedge \frac{\mathrm{d}X_n}{X_n}$$

and

$$x^{-1} - y^{-1} = (-1)^{(n+1)n}(z^{-1} - Z^{-1}).$$

2.3. Define

$$N^{n-1}_{i_0i_1} := z^{n-1}_{i_0i_1} g_{i_0i_1} - Z^{n-1}_{i_0i_1} \log_{i_1} X_1$$
$$= z^{n-1}_{i_0i_1}(g_{i_0i_1} - \log_{i_1} X_1) \in H^0(U_{i_0i_1}, \mu_! \mathbb{Z}(2)),$$
$$(\delta N^{n-1}) = z^{n-2} - Z^{n-2}.$$

Define

$$r^{n-2}_{i_0i_1} := (-1)^n N^{n-1}_{i_0i_1} \log_{i_1} X_2 \frac{\mathrm{d}X_3}{X_3} \wedge \cdots \wedge \frac{\mathrm{d}X_n}{X_n}$$
$$\in H^0(U_{i_0i_1}, \Omega^{n-2}_U(\log Y_U)(-Y_U)).$$

One has

$$x^{n-1} - y^{n-1} - \mathrm{d}r^{n-2} = 0.$$

Define, by induction, $1 \leqslant l \leqslant k$:

$$N^{n-l}_{i_0\ldots i_l} \in H^0(U_{i_0\ldots i_l}, \mu_! \mathbb{Z}(l+1)),$$

with $\delta N^{n-l} = z^{n-l-1} - Z^{n-l-1}$,

$$r^{n-l-1}_{i_0\ldots i_l} = (-1)^{ln} N^{n-l}_{i_0\ldots i_l} \log_{i_l} X_{l+1} \frac{\mathrm{d}X_{l+2}}{X_{l+2}} \wedge \cdots \wedge \frac{\mathrm{d}X_n}{X_n}$$
$$\in H^0(U_{i_0\ldots i_l}, \Omega^{n-(l+1)}_U(\log Y_U)(-Y_U))$$

such that

$$x^{n-l} - y^{n-l} - ((-1)^n \delta r^{n-l} + \mathrm{d}r^{n-(l+1)}) = 0, \quad l < k.$$

Define

$$N^{n-k}_{i_0\dots i_k} := z^{n-k}_{i_0\dots i_k} g_{i_0\dots i_k} - Z^{n-k}_{i_0\dots i_k} \log_{i_k} X_k - \\ - \delta(N^{n-k+1}_{i_0\dots i_{k-1}} \log_{i_{k-1}} X_k)_{i_0\dots i_k}.$$

One has

$$\delta N^{n-k} = z^{n-k-1} - Z^{n-k-1}$$

and

$$N^{n-k}_{i_0\dots i_k} = z^{n-k}_{i_0\dots i_k}(g_{i_0\dots i_k} - \log_{i_k} X_k) - \\ - (-1)^{k-1} N^{n-k+1}_{i_0\dots i_{k-1}} (\delta \log X_k)_{i_{k-1} i_k} \\ \in H^0(U_{i_0\dots i_k}, \mu_! \mathbb{Z}(k+1)).$$

Define

$$r^{n-k-1}_{i_0\dots i_k} := (-1)^{kn} N^{n-k}_{i_0\dots i_k} \log_{i_k} X_{k+1} \frac{\mathrm{d}X_{k+2}}{X_{k+2}} \wedge \cdots \wedge \frac{\mathrm{d}X_n}{X_n} \\ \in H^0(U_{i_0\dots i_k}, \Omega_U^{n-(k+1)}(\log Y_U)(-Y_U)),$$

then

$$x^{n-k} - y^{n-k} - ((-1)^n \delta r^{n-k} - \mathrm{d}r^{n-k-1}) = 0.$$

Therefore, one has $x - y - ((-1)^n\delta + d)r = 0$, and $x - y$ is a coboundary.

PROPOSITION (see (2.6) for another proof in the universal situation). *One has*

$$x|_U = y, \quad \text{in } H^{n+1}_{\mathcal{D},\mathrm{an}}(U, Y_U; \mathbb{Z}(n+1))$$

and

$$x|_U = y, \quad \text{in } H^{n+1}_{\mathcal{D}}(U, Y_U; \mathbb{Z}(n+1)).$$

2.4. Consider the morphisms

$$\text{rest: } H^{n+1}_{\mathcal{D}}(A, Y; \mathbb{Z}(n+1)) \to H^{n+1}_{\mathcal{D}}(U, Y_U; \mathbb{Z}(n+1)),$$

respectively if A is analytic

$$\text{rest}^{\mathrm{an}}\colon H^{n+1}_{\mathcal{D},\mathrm{an}}(A, Y; \mathbb{Z}(n+1)) \to H^{n+1}_{\mathcal{D},\mathrm{an}}(U, Y_U; \mathbb{Z}(n+1)),$$

and

$$\cup\colon H^1_{\mathcal{D}}(A, Y+Z; \mathbb{Z}(1)) \to H^{n+1}_{\mathcal{D}}(U, Y_U; \mathbb{Z}(n+1)),$$

respectively if A is analytic

$$\cup^{\mathrm{an}}\colon H^1_{\mathcal{D},\mathrm{an}}(A, Y+Z; \mathbb{Z}(1)) \to H^{n+1}_{\mathcal{D}}(U, Y_U; \mathbb{Z}(n+1)),$$

defined by

$$\cup\phi = \{\phi|_U, X_1, \dots, X_n\}.$$

Then (1.7), (1.8) and (2.3) prove the following theorem.

THEOREM. image $\cup \subset$ image rest (respectively, image $\cup^{an} \subset$ image rest^{an}).

2.5. *Remarks.*

(i) *The universal situation*:

Consider

$B := \mathbb{A}_{\mathbb{C}}^{n+1} - (\Psi = 0)$, $\Psi = 1 - Y_0 \dots Y_n$, where Y_i are the coordinates. Then one has [4], (2.1):

$$H_{\mathscr{D}}^{n+1}(B, (Y_0 = 0); \mathbb{Z}(n+1)) \xrightarrow{\text{rest}} H_{\mathscr{D}}^{n+1}\left(B - \bigcup_1^n (Y_i = 0), (Y_0 = 0); \mathbb{Z}(n+1)\right)$$

is an isomorphism. Take A as in (1.1). Then $(1-\phi)/X_1 \dots X_n \in H^0(A, \mathscr{O}(-Y))$. Define $X_0 := (1-\phi)/X_1 \dots X_n$. One defines a morphism

$$\begin{aligned} h_\phi \colon A &\to B \\ X_i &\leftarrow\!\shortmid Y_i, \quad 0 \leqslant i \leqslant n \end{aligned}$$

with $h_\phi^* \Psi = \phi$.

Then

$$h_\phi^* \mathrm{rest}^{-1}\{\Psi|_{B - \cup_1^n (Y_i = 0)}, Y_1, \dots, Y_n\} = x'$$

is in $H_{\mathscr{D}}^{n+1}(A, Y; \mathbb{Z}(n+1))$, of restriction

$$\begin{aligned} \mathrm{rest}\ x' &= h_\phi^*\{\Psi|_{B - \cup_1^n (Y_i = 0)}, Y_1, \dots, Y_n\} \\ &= \{\phi|_U, X_1, \dots, X_n\}. \end{aligned}$$

In (1.5), we have given explicit formulae for x as a Čech cocycle. This applies for

$$\mathrm{rest}^{-1}\{\Psi|_{B - \cup_1^n (Y_i = 0)}, Y_1, \dots, Y_n\},$$

and, therefore, by pull-back for x'. Of course, we could have worked directly on B, the universal case.

(ii) If A is only analytic, there is no universal situation. One observes the following: [4], (2.1) and (1.2) imply that

$$\begin{aligned} &H^n(B, (Y_0 = 0); \mathbb{C}/\mathbb{Z}(n+1)) \\ &= H^n\left(B - \bigcup_1^n (Y_i = 0), (Y_0 = 0); \mathbb{C}/\mathbb{Z}(n+1)\right), \end{aligned}$$

and therefore that

$$H^{n+1}_{\mathscr{D},\mathrm{an}}(B, (Y_0 = 0); \mathbb{Z}(n+1)) \text{ injects into}$$

$$H^{n+1}_{\mathscr{D},\mathrm{an}}\left(B - \bigcup_1^n (Y_i = 0), (Y_0 = 0); \mathbb{Z}(n+1)\right).$$

The class x of (1.5) is then uniquely defind by (2.3):

$$x|_{B - \cup_1^n (Y_i = 0)} = y$$

in

$$H^{n+1}_{\mathscr{D},\mathrm{an}}\left(B - \bigcup_1^n (Y_i = 0), (Y_0 = 0); \mathbb{Z}(n+1)\right).$$

(iii) More generally, whenever $H^n(A, Y; \mathbb{C}/\mathbb{Z}(n+1))$ injects into $H^n(U, Y_U; \mathbb{C}/\mathbb{Z}(n+1))$, then $\mathrm{rest}^{\mathrm{an}}$ is injective via (1.2). Therefore, in this case x constructed in (1.5) is uniquely defined by $x|_U$ via (2.3).

2.6. In this section, we give another proof of (2.3) in the universal situation case (2.5i), *due to* S. Bloch. Applying (3.8.1), this proves $x_{|U} = y$ is general. We call A the universal situation and keep the notation of Section 1.

Let W be the open set in $\mathbf{A}^{n+1}_{\mathbb{C}}$ defined by $X_1 \ldots X_n \neq 0$, and D be the hypersurface $\phi = 0$. Then D lies in W and is isomorphic via the projection $p: D \to \mathbf{A}^n_{\mathbb{C}}, (X_0, \ldots, X_n) \to (X_1, \ldots, X_n)$ to $(\mathbb{C}^*)^n$. The pair (W, Y_U) is isomorphic to $(\mathbf{A}^1, 0) \times (\mathbb{C}^*)^n$ via the projection $(X_0, \ldots, X_n; X_1, \ldots, X_n) \to (X_0; X_1, \ldots, X_n)$. Therefore, $H^k(W, Y; \mathbb{Z}) = 0$ for all k. From the exact sequence

$$H^k(W, Y_U) \to H^k(U, Y_U) \to H^{k-1}(D) \to H^{k+1}(W, Y_U)$$

and, from (2.1), one obtains

$$z := x_{|U} - y \in H^{n-1}(D, \mathbb{C}/\mathbb{Z}(n+1)) = H^{n-1}(D, \mathbb{C}/\mathbb{Z}(n-1)) \bigotimes_{\mathbb{Z}} \mathbb{C}/\mathbb{Z}.$$

For r with $1 \leqslant r \leqslant n$, define the morphism $p(r): W \to W$ sending X_0 to $X_0 | X_r$, X_r to X_r^2 and fixing X_i for $i \neq 0, r$. It fixes ϕ and defines a morphism $p(r): D \to D$ with $pp(r)X_i = X_i$, if $i \neq r$ and $pp(r)X_r = X_r^2$.

Consider

$$p(r)^* x_{|U} \quad \text{and} \quad p(r)^* y \quad \text{in } H^{n+1}_{\mathscr{D}}(U, Y_U; \mathbb{Z}(n+1))$$

given as Čech cocyles in $H^{n+1}_{\mathscr{D},\mathrm{an}}(U, Y_U; \mathbb{Z}(n+1))$ by

$$(p(r)^* x^{-1}, \ldots, p(r)^* x^n) \quad \text{and} \quad (p(r)^* y^{-1}, \ldots, p(r)^* y^n).$$

By (1.4) one has $p(r)^* z^{n-1} = z^{n-1}$

$$\begin{aligned} p(r)^* g_{i_0 \ldots i_k} &= g_{i_0 \ldots i_k}, \quad \text{if } k \neq r, \\ &= 2 g_{i_0 \ldots i_k}, \quad \text{if } k = r \end{aligned}$$

and by (2.2),

$$\begin{aligned} p(r)^*\log_{i_k} X_k &= \log_{i_k} X_k, \quad \text{if } k \neq r, \\ &= 2\log_{i_k} X_k, \quad \text{if } k = r. \end{aligned}$$

Assume by induction that for l with $1 \leqslant l \leqslant k$

$$\begin{aligned} p(r)^* z^{n-l} &= 2z^{n-l}, \quad \text{if } r < l, \\ &= z^{n-l}, \quad \text{if } r \geqslant l, \\ p(r)^* Z^{n-l} &= 2Z^{n-l}, \quad \text{if } r < l, \\ &= Z^{n-l}, \quad \text{if } r \geqslant l. \end{aligned}$$

Then $p(r)^*$ acts on

$$z^{n-(k+1)} = \delta(z^{n-k} \cdot g_{i_0 \ldots i_k}) \quad \text{or} \quad Z^{n-(k+1)} = \delta(Z^{n-k} \cdot \log_{i_k} X_k)$$

via 1 on the z or Z factor and 2 on the g or log factor if $r = k$, via 2 on the z or Z factor and 1 on the g or log factor if $r < k$, via 1 if $r > k$. This proves the induction. As $p(r)^*$ acts on

$$\frac{\mathrm{d}X_{k+2}}{X_{k+2}} \wedge \cdots \wedge \frac{\mathrm{d}X_n}{X_n}$$

via 1 if $r < k + 2$, 2 if $r \geqslant k + 2$, one finds

$$p(r)^* x = 2x, \qquad p(r)^* y = 2y \quad \text{and} \quad p(r)^* z = 2z.$$

As $\mathrm{d}X_i/X_i$ is the orientation class of $H^1(\mathbb{C}^*)_i, \mathbb{Z}(1))$ written as a de Rham class, then z may be uniquely written as

$$z(\underline{\lambda}) = (\lambda_1, \ldots, \lambda_n) := \sum_{i=1}^{n} \lambda_i \frac{\mathrm{d}X_1}{X_1} \wedge \cdots \wedge \frac{\mathrm{d}\hat{X}_i}{X_i} \wedge \cdots \wedge \frac{\mathrm{d}X_n}{X_n},$$

with $\lambda_i \in \mathbb{C}$ modulo $z(\underline{\eta})$, for $\eta_i \in \mathbb{Z}$.

Therefore,

$$p(r)^* z(\underline{\lambda}) = (2\lambda_1, \ldots, 2\lambda_{r-1}, \lambda_r, 2\lambda_{r+1}, \ldots, 2\lambda_n)$$

and $p(r)^* z(\underline{\lambda}) = 2z(\underline{\lambda})$ implies that $\lambda_r \in \mathbb{Z}$. As this is true for all r with $1 \leqslant r \leqslant n$, one has $z = 0$.

3. Pull-Back of x to X and Formula [4], II.2.4

3.1. Let X be a smooth algebraic variety over $\mathbb{C}$ of a dimension $\leqslant n$, equipped with a morphism $h: X \to A$, where now A is the universal situation described in (2.5i), with coordinates X_i, and with $\phi = 1 - X_0 \ldots X_n$.

Define

$$h^*X_i = a_i \in H^0(X, \mathcal{O}_X), \quad \text{for } i \geqslant 1,$$
$$h^*\phi = f \in H^1_{\mathcal{D}}(X, S + T; \mathbb{Z}(1)),$$

where T is defined by $t := a_1 \dots a_n$ and S is a closed subvariety of X contained in h^*Y, of the ideal sheaf $\mathcal{J}_S$. Denote by $\mathcal{O}_X(-T)$ the reduced ideal sheaf of T.

Define

$$\begin{tikzcd} X - S \arrow[r, "j"] & X \\ X - S - T \arrow[u] \arrow[ur, "v"] & \end{tikzcd}$$

One has

$$\begin{aligned} h^*\mathrm{rest}^{-1}\{\phi|_U, X_1, \dots, X_n\} &\in H^{n+1}_{\mathcal{D}}(X, S; \mathbb{Z}(n+1)) \\ &= H^n(X, S; \mathbb{C}/\mathbb{Z}(n+1)) \quad (1.2\text{iii}). \end{aligned}$$

As S is not necessarily a normal crossing divisor, we will explain this more precisely (3.2), (3.3), (3.4), (3.5), and (3.6). Then we want to evaluate this class along relative homology classes $[\gamma] \in H_n(X, S; \mathbb{Z})$ (3.4).

3.2. We assume in (3.2), (3.3), (3.4) that X is smooth analytic, T is a divisor defined by $a_1 \dots a_n = t = 0$, $a_i \in H^0(X, \mathcal{O}_X)$, and S is a closed subvariety of X.

We define the subcomplexes $\Omega^{\cdot}_{X,S+T}$ and $\Omega^{\cdot}_{X,S}$ of the holomorphic de Rham complex $\Omega^{\cdot}_X$ by: for each open set U

$$\Omega^i_{X,S}(U) = \{\omega \in \Omega^i_X(U),\ \omega_{|S \cap U} = 0\},$$
$$\Omega^i_{X,S+T}(U) = \{\omega \in \Omega^i_{X,S}(U),\ \omega|_{a_j=0} = 0 \text{ for any } 1 \leqslant j \leqslant n\}.$$

The sheaves $\Omega^i_{X,S}$ and $\Omega^i_{X,S+T}$ are coherent. As $\Omega^0_{X,S} = \mathcal{J}_S$, one has a natural inclusion

$$j_!\mathbb{C} \xrightarrow{\text{incl}} \Omega^{\cdot}_{X,S}$$

which defines a map in cohomology

$$H^{\cdot}(X, S; \mathbb{C}) \xrightarrow{\text{incl}} H^{\cdot}(X, \Omega^{\cdot}_{X,S}).$$

If S is a divisor with normal crossings then $\Omega^{\cdot}_{X,S}$ is the complex $\Omega^{\cdot}_X(\log S)(-S)$, and incl is a quasi isomorphism. In general we construct a 'splitting' of incl.

LEMMA. *There is a morphism p in $D^b(X)$, $p: \Omega^{\cdot}_{X,S} \to j_!\mathbb{C}$ such that $p \circ incl$ is an isomorphism.*

Proof. Let $\sigma: \tilde{X} \to X$ be an embedded resolution of S. This means $\sigma^{-1}S = \tilde{S}$ is a divisor with normal crossings, σ is proper and $\sigma|_{X-S}$ is an isomorphism.

Consider

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\sigma"] & & X \\ & X - S \arrow[ul, "\tilde{j}"] \arrow[ur, "j"'] & \end{tikzcd}$$

One has $\sigma^* \Omega^i_{\tilde{X},S} \subset \Omega^i_{\tilde{X}}(\log \tilde{S})(-\tilde{S})$, and $\sigma^{-1} j_! \mathbb{C} \to \tilde{j}_! \mathbb{C}$. Therefore, one has a diagram in $D^b(X)$

$$\begin{tikzcd} \Omega^{\bullet}_{X,S} \arrow[r, "\sigma^*"] & R\sigma_* \Omega^{\bullet}_{\tilde{X}}(\log \tilde{S})(-\tilde{S}) \\ j_! \mathbb{C} \arrow[u, "\mathrm{incl}"] \arrow[r, "\sigma^{-1}"] & R\sigma_* \tilde{j}_! \mathbb{C}. \arrow[u, "R\sigma_* \widetilde{\mathrm{incl}}"'] \end{tikzcd}$$

As σ is proper, and j and $\tilde{j}$ are exact, one has $R\sigma_* \tilde{j}_! = R\sigma_! \tilde{j}_! = R(\sigma \circ \tilde{j})_! = Rj_! = j_!$ in $D^b(X)$, and σ^{-1} is an isomorphism in $D^b(X)$. As $\widetilde{\mathrm{incl}}$ is a quasi-isomorphism, $R\sigma_*$ $\widetilde{\mathrm{incl}}$ is an isomorphism in $D^b(X)$.

Define

$$\mathrm{p} = (\sigma^{-1})^{-1} \circ (R\sigma_* \ \widetilde{\mathrm{incl}})^{-1} \circ \sigma^*.$$

3.3 Define

$$K^{\bullet} = j_! \mathbb{Z}(n+1) \to \Omega^{\bullet}_{X,S} \quad \text{and} \quad K'^{\bullet} = v_! \mathbb{Z}(n+1) \to \Omega^{\bullet}_{X,S+T},$$

which is a subcomplex of $K^{\bullet}$. One has

$$\begin{tikzcd} j_! \mathbb{Z}(n+1) \to j_! \mathbb{C} \arrow[d, "\mathrm{incl}"] \\ K^{\bullet} \arrow[d, "p"] \\ j_! \mathbb{Z}(n+1) \to j_! \mathbb{C} \end{tikzcd}$$

with $p \circ \mathrm{incl}$ as an isomorphism (3.2).

COROLLARY. *There are morphisms*

$$\begin{tikzcd} H^{\bullet -1}(X, S; \mathbb{C}/\mathbb{Z}(n+1)) \arrow[r, "\mathrm{incl}"] & H^{\bullet}(X, K^{\bullet}) \arrow[d, "p"] \\ & H^{\bullet -1}(X, S; \mathbb{C}/\mathbb{Z}(n+1)) \end{tikzcd}$$

with $p \circ \mathrm{incl}$ as an isomorphism.

3.4. Let $\bar{z}$ be a cohomology class in

$$\frac{H^0(X, \Omega^n_{X,S+T})}{H^n(X, v_!\mathbb{Z}(n+1) \to \Omega^{\leq n-1}_{X,S+T})} \subset H^{n+1}(X, K'^{\cdot})$$

of representative $\omega \in H^0(X, \Omega^n_{X,S+T})$.

Its image z in $H^{n+1}(X, K^{\cdot})$ lies in

$$\frac{H^0(X, \Omega^n_{X,S})}{H^n(X, j_!\mathbb{Z}(n+1) \to \Omega^{\leq n-1}_{X,S})} \subset H^{n+1}(X, K^{\cdot})$$

and is of representative ω. Then for any n-chain γ with $\partial\gamma \subset S$ representing the homology class $[\gamma] \in H_n(X, S; \mathbb{Z})$, one has $\langle [\gamma], pz \rangle = \int_\gamma \omega$ modulo $\mathbb{Z}(n+1)$.

3.5. *Remark.* If X is affine one has

$$H^{n+1}(X, v_!\mathbb{Z}(n+1)) = H^{n+1}(X, j_!\mathbb{Z}(n+1)) = 0$$

[2]. Then one is always in the situation of (3.4).

3.6. We go back to the situation (3.1). One has morphisms

$$h^*\Omega^i_A(\log(Y+Z))(-Y-Z) \to \Omega^i_{X,S+T},$$
$$h^*\Omega^i_A(\log Y)(-Y) \to \Omega^i_{X,S},$$
$$h^{-1}\lambda_!\mathbb{Z}(n+1) \to v_!\mathbb{Z}(n+1),$$
$$h^{-1}i_!\mathbb{Z}(n+1) \to j_!\mathbb{Z}(n+1).$$

Therefore, one has morphisms in $D^b(A)$:

$$\begin{tikzcd} \lambda_!\mathbb{Z}(n+1) \to \Omega^{\leq n}_A(\log(Y+Z))(-Y-Z) \arrow[d, "h^*"] \\ Rh_*K'^{\cdot} \end{tikzcd}$$

and

$$\begin{tikzcd} i_!\mathbb{Z}(n+1) \to \Omega^{\leq n}_A(\log Y)(-Y) \arrow[d] \\ Rh_*K^{\cdot}. \end{tikzcd}$$

This proves the following lemma.

LEMMA. *One has commutative diagrams*

$$\begin{tikzcd} H^{n+1}_{\mathscr{D}}(A, Y; \mathbb{Z}(n+1)) \arrow[r] \arrow[d, "(1.2\mathrm{i})"'] & H^n(X, S; \mathbb{C}/\mathbb{Z}(n+1)) \\ H^{n+1}_{\mathscr{D},\mathrm{an}}(A, Y; \mathbb{Z}(n+1)) \arrow[r, "h^*"] & H^{n+1}(X, K^{\cdot}) \arrow[u, "p"'] \\ & H^{n+1}(X, K'^{\cdot}) \arrow[u] \\ H^{n+1}(A, \lambda_!\mathbb{Z}(n+1) \longrightarrow \Omega^{\leq n}_A(\log(Y+Z))(-Y-Z)) \arrow[uu] \arrow[ur, "h^*"'] & \end{tikzcd}$$

3.7. Consider the open cover $h^{-1}A_j$ of X (1.4). Then $h^*\bar{x}$ is represented by the cocycle

$$h^*\bar{x} = (h^{-1}x^{-1}, h^*x^0, \dots, h^*x^n) \quad \text{in} \quad (\mathscr{C}^{n+1}(h^{-1}A_i, K'^{\bullet}), (-1)^{n+1}\delta + d)$$

with

$$h^{-1}x^{-1} = (-1)^{(n+1)n}z^{-1},$$

$$h^*x^{n-k} = (-1)^{kn}z^{n-k}_{i_0\dots i_k}h^*g_{i_0\dots i_k}\frac{\mathrm{d}q_{k+1}}{a_{k+1}} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n}, \quad 1 \leqslant k \leqslant n,$$

.

.

.

$$h^*x^n = \log_i f\frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n}, \quad \text{with } \log_i f := h^*\log_i\phi.$$

Define for simplicity of notations

$$G_{i_0\dots i_k} := h^*g_{i_0\dots i_k} \in H^0(h^{-1}A_{i_0\dots i_k}, t\mathscr{I}_S).$$

3.8. Let X_j be a refinement of $h^{-1}A_j$ such that another determination $\ln_j f$ of $\log_i f$ on X_j exists with

$$\ln_j f \in H^0(X_j, \mathscr{I}_S\mathscr{O}_X(-T)).$$

Observe that this implies that if $X_j \cap (S \cup T) \neq \phi$, then $\ln_j f = \log_j f$ and, therefore,

$$(\ln_{i_1} f - \ln_{i_0} f) \in H^0(X_{i_0 i_1}, v_! \mathbb{Z}(1))$$

and vanishes if $X_{i_0 i_1} \cap (S \cup T) \neq \phi$.

Define the element

$$u = (u^{-1}, u^0, \dots, u^n) \textit{ in } (\mathscr{C}^{n+1}(X_j, K'^{\bullet}), (-1)^{n+1}\delta + d) \text{ by}$$

$$u^{-1} := (-1)^{(n+1)\cdot n} Z^{-1};$$

$$u^{n-k} := (-1)^{kn}Z^{n-k}_{i_0\dots i_k}G_{i_0\dots i_k}\frac{\mathrm{d}a_{k+1}}{a_{k+1}} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n}, \quad 1 \leqslant k \leqslant n,$$

.

.

.

$$u^n := \ln_i f\frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n},$$

with

$$Z^{n-1}_{i_0 i_1} := (\delta \ln f)_{i_0 i_1},$$
$$Z^{n-k}_{i_0 \ldots i_k} := \delta(Z^{n-k+1}_{i_0 \ldots i_{k-1}} G_{i_0 \ldots i_{k-1}})_{i_0 \ldots i_k}.$$

As in (1.5), the condition $(\ln_{i_1} f - \ln_{i_0} f) = 0$ if $(X_{i_0 i_1} \cap (S \cup T) \neq \phi$ implies that

$$Z^{n-k}_{i_0 \ldots i_k} \in H^0(X_{i_0 \ldots i_k}, v_! \mathbb{Z}(k))$$

and that u is a Čech cocycle defining a cohomology class u in $H^{n+1}(X, K'^{\cdot})$.

PROPOSITION. *One has*

$$h^* \bar{x} = u \quad in \quad H^{n+1}(X, K'^{\cdot}).$$

Proof. Choose a refinement X'_j of X_j such that if $X'_{i_0 \ldots i_k} \cap (S \cap T) = \phi$, then $\log_{i_0 \ldots i_k} a_{k+1}$ is single-valued on $X'_{i_0 \ldots i_k}$, that is, in $H^0(X'_{i_0 \ldots i_k}, \mathcal{O}_X)$.

Define

$$h_{i_0 \ldots i_k} := \begin{cases} \log_{i_0 \ldots i_k} a_{k+1}, & \text{if } X'_{i_0 \ldots i_k} \cap (S \cup T) = \phi, \\ 0, & \text{if } X'_{i_0 \ldots i_k} \cap (S \cup T) \neq \phi. \end{cases}$$

In this refinement X'_j, one has

$$h^* x^n - u^n = (\log_i f - \ln_i f) \frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n}.$$

Define

$$N^n_i = (\log_i f - \ln_i f) \in H^0(X'_i, v_! \mathbb{Z}(1)).$$

One then has

$$(\delta N^n)_{i_0 i_1} = z^{n-1}_{i_0 i_1} - Z^{n-1}_{i_0 i_1}.$$

Define

$$r^{n-1}_i = N^n_i h_i \frac{\mathrm{d}a_2}{a_2} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n} \in H^0(X'_i, \Omega^{n-1}_{X, S+T}).$$

One has $h^* x^n - u^n = \mathrm{d}r^{n-1}_i$.

Define, by induction for $1 \leqslant l \leqslant k$,

$$N^{n-l}_{i_0 \ldots i_l} = (z^{n-l}_{i_0 \ldots i_l} - Z^{n-l}_{i_0 \ldots i_l}) G_{i_0 \ldots i_l} - \delta(N^{n-l+1}_{i_0 \ldots i_{l-1}} h_{i_0 \ldots i_{l-1}})_{i_0 \ldots i_l}$$
$$\in H^0(X'_{i_0 \ldots i_l}, v_! \mathbb{Z}(l+1)),$$

with $(\delta N^{n-l}) = z^{n-l-1} - Z^{n-l-1}$

and

$$r^{n-l-1}_{i_0\ldots i_l} = (-1)^{ln} N^{n-l}_{i_0\ldots i_l} h_{i_0\ldots i_l} \frac{\mathrm{d}a_{l+2}}{a_{l+2}} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n}$$
$$\in H^0(X'_{i_0\ldots i_l}, \Omega^{n-(l+1)}_{X,S+T})$$

with

$$(h^* x^{n-l} - u^{n-l}) - [(-1)^n \delta r^{n-1} + \mathrm{d}r^{n-(l+1)}] = 0.$$

Define

$$N^{n-k}_{i_0\ldots i_k} = (z^{n-k}_{i_0\ldots i_k} - Z^{n-k}_{i_0\ldots i_k}) G_{i_0\ldots i_k} - \delta(N^{n-k+1}_{i_0\ldots i_{k-1}} h_{i_0\ldots i_{k-1}})_{i_0\ldots i_k}.$$

One has $\delta N^{n-k} = z^{n-k-1} - Z^{n-k-1}$.

If $X'_{i_0\ldots\hat{i}_l\ldots i_k} \cap (S \cup T) \neq \phi$ for all $l \in \{0,\ldots,k\}$, then $N^{n-k}_{i_0\ldots i_k} = 0$, especially if $X'_{i_0\ldots i_k} \cap (S \cup T) \neq \phi$. Otherwise $X'_{i_1\ldots i_k} \cap (S \cup T) = \phi$ (say). Then

$$N^{n-k}_{i_0\ldots i_k} = (z^{n-k}_{i_0\ldots i_k} - Z^{n-k}_{i_0\ldots i_k})(G_{i_0\ldots i_k} - h_{i_1\ldots i_k}) -$$
$$- \sum_{l=1}^{k} (-1)^l N^{n-k+1}_{i_0\ldots\hat{i}_l\ldots i_k} (h_{i_0\ldots\hat{i}_l\ldots i_k} - h_{i_1\ldots i_k}).$$

If $(z^{n-k} - Z^{n-k})_{i_0\ldots i_k} \neq 0$, then

$$X'_{i_0\ldots i_k} \cap (S \cup T) = \phi, \quad \text{and} \quad (G_{i_0\ldots i_k} - h_{i_1\ldots i_k}) \in \mathbb{Z}(1).$$

If $N^{n-k+1}_{i_0\ldots\hat{i}_l\ldots i_k} \neq 0$, then

$$X'_{i_0\ldots\hat{i}_l\ldots i_k} \cap (S \cup T) = \phi, \quad \text{and} \quad (h_{i_0\ldots\hat{i}_l\ldots i_k} - h_{i_1\ldots i_k}) \in \mathbb{Z}(1)).$$

Therefore

$$N^{n-k}_{i_0\ldots i_k} \in H^0(X'_{i_0\ldots i_k}, v_!\mathbb{Z}(k+1)).$$

Define

$$r^{n-k-1}_{i_0\ldots i_k} = (-1)^{kn} N^{n-k}_{i_0\ldots i_k} h_{i_0\ldots i_k} \frac{\mathrm{d}a_{k+2}}{a_{k+2}} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n}.$$

One has

$$(h^* x^{n-k} - u^{n-k}) - [(-1)^n \delta r^{n-k} + \mathrm{d}r^{n-k-1}] = 0.$$

Therefore,

$$(h^* \bar{x} - u) - [(-1)^n \delta + d] r = 0,$$

and $(h^* \bar{x} - u)$ is a coboundary in $\mathscr{C}^{\boldsymbol{\cdot}}(K'^{\boldsymbol{\cdot}})$.

3.8.1. If we do not assume that dimension $X \leqslant n$ and we define $u \in H^{n+1}(X, v_! \mathbb{Z}(n+1) \to \Omega^{\cdot}_{X,S+T})$ by formulae (3.8) for a determination $ln_j f$, the proof of Proposition (3.8) just shows $h^* \bar{x} = u$.

In other words, one has the following theorem.

THEOREM *The analytic element*

$$\bar{x} \in H^{n+1}(A, \lambda_! \mathbb{Z}(n+1) \to \Omega_A^{\leqslant n}(\log(Y+Z))(-Y-Z))$$

defined in (1.6) *does not depend on the choice of log's made in* (1.4) α, β, γ.

In particular, if $h_\phi \colon A \to B$ is the analytic (or algebraic) morphism defined in (2.5i), then one has $\bar{x} = h_\phi^* \bar{x}_B$, where $\bar{x}_B$ is the element defined in (1.6) in the universal situation B.

3.9. Let γ be an n-chain with support $\gamma \subset \mathscr{U}, \mathscr{U}$ open analytic, $\partial\gamma \subset S$, of homology class $[\gamma]$ $H_n(X, S; \mathbb{Z})$ such that there is a determination $\ln f$ of $\log f$ on $\mathscr{U}$ with $\ln f \in H^0(\mathscr{U}, \mathscr{I}_S \mathscr{O}_X(-T))$.

By 3.8, one has

$$h^* \bar{x} = \text{class of } \omega = \ln f \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}$$

in $H^{n+1}(\mathscr{U}, K'^{\cdot})$.

By (3.4), one obtains the following theorem.

THEOREM (see [1], 7.0.2, and [4], II, (2.4)). *Let X be a smooth algebraic complex variety of dimension $\leqslant n$, equipped with a morphism $h \colon X \to A$, where A is the universal situation described in* (2.5i). *Let $t := h^*(X_1 \ldots X_n)$, $\mathscr{O}_X(-T)$ be the reduced ideal sheaf associated to t, $f := h^* \phi$. Let S be a closed subvariety of X contained in $h^*(X_0 = 0)$ of ideal sheaf $\mathscr{I}_S$. Let $\mathscr{U}$ be an analytic open subset of X on which there is a single-valued determination* $\ln f$ *of f with* $\ln f \in H^0(\mathscr{U}, \mathscr{I}_S \mathscr{O}_X(-T))$. *Let γ be a n-chain supported in $\mathscr{U}$ with $\partial\gamma \subset S$, of homology class $[\gamma] \in H_n(X, S; \mathbb{Z})$. Then one has*

$$\langle [\gamma], h^* \mathrm{rest}^{-1} \{\phi_{|U}, X_1, \ldots, X_n\} \rangle$$
$$= \int_\gamma \ln f h^* \left(\frac{dX_1}{X_1} \wedge \cdots \wedge \frac{dX_n}{X_n} \right) \quad \text{modulo } \mathbb{Z}(n+1).$$

3.10. *Remark.* The condition X affine of [4], II, (2.4) does not appear in (3.9). This is just because the assumption on the existence of $\ln f$ is sufficient to ensure that ph^*x is represented by a global n-form on $\mathscr{U}$ (via (3.8)).

3.11. *Comment.* Formula (3.9) depends on the existence of a representative γ of the homology class $[\gamma] \in H_n(X, S; \mathbb{Z})$ along which there is a single-valued determination of $\log f$ which vanishes on support $\gamma \cap S$ and support $\gamma \cap (a_i = 0)$ for $1 \leqslant i \leqslant n$. So it is not valid in general. In Section 4, we weaken the assumptions on dimension X and on γ in order to write a slightly more general formula in the case $n = 1$.

4. Other Formulae on X and Relationship with Bloch's Regulator Map

4.1. Let X be a smooth affine variety over $\mathbb{C}$ equipped with morphisms $h^\alpha\colon X \to A$, $\alpha = 1, \ldots, N$, where A is the universal situation as in (3.1). We define

$$h^{\alpha *}\phi = f^\alpha \in H^1_{\mathscr{D}}(X, S + T^\alpha; \mathbb{Z}(1)), \qquad h^{\alpha *}X_i = a_i^\alpha \in H^0(X, \mathcal{O}_X),$$

where $t^\alpha := a_1^\alpha \cdots a_n^\alpha$ defines T^α and S is a closed subvariety of X contained in $\bigcap_1^N h^{\alpha-1}Y$ of ideal sheaf $\mathscr{I}_S$. This defines

$$u := \sum_1^N h^{\alpha *}\mathrm{rest}^{-1}\{\phi|_U, X_1, \ldots, X_n\} \in H^{n+1}_{\mathscr{D}}(X, S; \mathbb{Z}(n+1)).$$

Define $j\colon \mathrm{X} - \mathrm{S} \to \mathrm{X}$.

Recall (3.6) that we have defined

$$h^{\alpha *}\colon (i_!\mathbb{Z}(n+1) \to \Omega_A^{\leqslant n}(\log Y)(-Y)) \to Rh^\alpha_*(j_!\mathbb{Z}(n+1) \to \Omega^{\leqslant n}_{X,S})$$

in $D^b(A)$.

This defines

$$\bar{u} := \sum_1^N h^{\alpha *}\mathrm{rest}^{-1}\{\phi|_U, X_1, \ldots, X_n\}$$

as a class in

$$H^{n+1}(X, j_!\mathbb{Z}(n+1) \to \Omega^{\leqslant n}_{X,S}).$$

LEMMA. *The natural morphism*

$$H^{n+1}(X, K^\cdot) \to H^{n+1}(X, j_!\mathbb{Z}(n+1) \to \Omega^{\leqslant n}_{X,S})$$

is injective. The class $\bar{u}$ lies in $H^{n+1}(X, K^\cdot)$ if and only if

$$\mathrm{d}\bar{u} = \sum_1^N \frac{\mathrm{d}f^\alpha}{f^\alpha} \wedge \frac{\mathrm{d}a_1^\alpha}{a_1^\alpha} \wedge \cdots \wedge \frac{\mathrm{d}a_n^\alpha}{a_n^\alpha} = 0.$$

Proof. The kernel of

$$H^{n+1}(X, K^\cdot) \to H^{n+1}(X, j_!\mathbb{Z}(n+1) \to \Omega^{\leqslant n}_{X,S})$$

comes from

$$H^{n+1}(X, \Omega^{\geqslant n+1}_{X,S}[-1]) = 0 \quad \text{and} \quad \bar{u} \in H^{n+1}(X, K^\cdot)$$

if and only if it maps to 0 under

$$d\colon H^{n+1}(X, j_!\mathbb{Z}(n+1) \to \Omega^{\leqslant n}_{X,S}) \to$$
$$H^{n+1}(X, \Omega^{\geqslant n+1}_{X,S}) = H^0(X, \Omega^{n+1}_{X,S})_{d\,\mathrm{closed}}.$$

One has

$$\begin{aligned} \mathrm{d}\bar{u} &= \sum_1^N h^{\alpha *} \frac{\mathrm{d}\phi}{\phi} \wedge \frac{\mathrm{d}X_1}{X_1} \wedge \cdots \wedge \frac{\mathrm{d}X_n}{X_n} \\ &= \sum_1^N \frac{\mathrm{d}f^\alpha}{f^\alpha} \wedge \frac{\mathrm{d}a_1^\alpha}{a_1^\alpha} \wedge \cdots \wedge \frac{\mathrm{d}a_n^\alpha}{a_n^\alpha}. \end{aligned}$$

4.2. COROLLARY. *There is $\omega \in H^0(X, \Omega^n_{X,S})_{d\text{ closed}}$ representing u modulo torsion via the composed morphism*

$$\begin{tikzcd} H^0(X, \Omega^n_{X,S})_{d\text{ closed}} \arrow[r] & H^{n+1}(X, K^\bullet) \arrow[d, "p(3.2)"] \\ & H^n(X, S; \mathbb{C}/\mathbb{Q}(n+1)) \arrow[d, "(1.2)"] \\ & H^{n+1}_{\mathscr{D}}(X, S; \mathbb{Q}(n+1)) \end{tikzcd}$$

if

$$\mathrm{d}u = \mathrm{d}\bar{u} = \sum_1^N \frac{\mathrm{d}f^\alpha}{f^\alpha} \wedge \frac{\mathrm{d}a_1^\alpha}{a_1^\alpha} \wedge \cdots \wedge \frac{\mathrm{d}a_n^\alpha}{a_n^\alpha} = 0.$$

Proof. One has an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \dfrac{H^n(X, \Omega^\bullet_{X,S})}{H^n(X, S; \mathbb{Q}(n+1))} \arrow[r] \arrow[d, equal, "X\text{ affine}"] & H^{n+1}(X, K^\bullet_{\mathbb{Q}}) \arrow[d, "e'"] \\ & \text{quotient of } H^0(X, \Omega^n_{X,S})_{d\text{ closed}} & H^{n+1}(X, S; \mathbb{Q}(n+1)) \arrow[d, "e"] \\ & & H^{n+1}(X, \Omega^\bullet_{X,S}) \arrow[d, equal, "X\text{ affine}"] \\ & & \dfrac{H^0(X, \Omega^{n+1}_{X,S})_{d\text{ closed}}}{\mathrm{d}H^0(X, \Omega^n_{X,S})} \end{tikzcd}$$

As $d = e \circ e'$ and $H^{n+1}(X, S; \mathbb{Q}(n+1))$ is torsion free, (3.2) implies that e is injective.

4.3. Let γ be an n-chain on X with $\partial\gamma \subset S$, of homology class $[\gamma] \in H_n(X, S; \mathbb{Z})$. One has

$$\langle [\gamma], u \rangle = \int_\gamma \omega \quad \text{modulo } \mathbb{Q}(n+1).$$

4.4. We assume now $n = 1$ in (4.4) and (4.5). Given $[\gamma]$ as in 4.3, then there is a representative γ of $[\gamma]$ as a chain as in [4], II, 2.4:

$$\gamma = \gamma_0 + \sum_{i \geqslant 1} \gamma_i \quad \text{with } \partial\gamma_0 = \phi,\ \partial\gamma_i \neq \phi \subset S \text{ for } i \geqslant 1.$$

We first compute $\langle[\gamma_0], u\rangle$.

PROPOSITION. *Let $p_0 \in$ support γ_0 be a point such that* $\log f^\alpha$ *is single-valued along $\gamma_0 - p_0$, and vanishes along $t^\alpha = 0$ and S, for $\alpha = 1, \ldots, N$.*

(1) *Assume $p_0 \notin \bigcup_1^N T^\alpha$. Then if $p_0 \notin S$ or if p_0 is an isolated point of $S \cap$ support γ_0, one has*

$$\langle[\gamma_0], u\rangle = \int_{\gamma_0} \sum_\alpha \log f^\alpha \frac{da_1^\alpha}{a_1^\alpha} - \sum_\alpha \log a_1^\alpha(p_0) \int_{\gamma_0} \frac{df^\alpha}{f^\alpha} \quad \textit{modulo } \mathbb{Q}(2).$$

(2) *If $p_0 \in S$ is not isolated in $S \cap$ support γ_0, or if $p_0 \in \bigcap_1^N T^\alpha$ is not isolated in $\bigcap_1^N T^\alpha \cap$ support γ_0, one has*

$$\langle[\gamma_0], u\rangle = \int_{\gamma_0} \sum_\alpha \log f^\alpha \frac{da_1^\alpha}{a_1^\alpha} \quad \textit{modulo } \mathbb{Q}(2).$$

(3) *If log f^α is single-valued along γ_0 and vanishes along $t^\alpha = 0$ and S for $\alpha = 1, \ldots, N$, one has*

$$\langle[\gamma_0], u\rangle = \int_{\gamma_0} \sum_\alpha \log f^\alpha \frac{da_1^\alpha}{a_1^\alpha} \quad \textit{modulo } \mathbb{Z}(2).$$

Proof. In (1) and (2), there is an open set $\mathscr{U}$ containing γ_0, I a segment in $\mathscr{U}$ with $p_0 = I \cap$ support γ_0, and a determination $\ln_1 f^\alpha$ on $\mathscr{U}_1 = \mathscr{U} - \mathrm{I}$ with $\ln_1 f^\alpha \in H^0(\mathscr{U}_1, t^\alpha \mathscr{I}_S)$. For any $\varepsilon > 0$, define an open set $\mathscr{U}_{0\varepsilon}$ containing p_0 such that

(*) is fulfilled in case (1),
(**) is fulfilled in case (2),

with

(*) $\log a_1^\alpha$ is single valued along $\mathscr{U}_{0\varepsilon} \cap$ support γ_0 and verifies

$$\sup_{x, y \in \mathscr{U}_{0\varepsilon} \cap \text{support } \gamma_0} |\log a_1^\alpha(x) - \log a_1^\alpha(y)| < \varepsilon,$$

(**) $\mathscr{U}_{0\varepsilon} \cap$ support $\gamma_0 \subset S$ or $\bigcap_1^N T^\alpha$.

(As support $\gamma_0 \cap S$ (or support $\gamma_0 \cap \bigcap_1^N T^\alpha$) is compact, condition (2) says that a subsegment of γ_0 centered at p_0 is contained in S (or in $\bigcap_1^N T^\alpha$). Therefore, one may realize (**).)

Let $\mathscr{V}_\varepsilon = \mathscr{U}_1 \cup \mathscr{U}_{0\varepsilon}$. Take a common refinement of the covers $\mathscr{U}_1 \cup \mathscr{U}_{0\varepsilon}$ and $\mathscr{V}_\varepsilon \cap h^{\alpha-1} A_i$ of $\mathscr{V}_\varepsilon$. By (3.8), $\bar{u}|_{\mathscr{V}_\varepsilon}$ is represented by the Čech cocycle in this cover

$$(u^{-1}, u^0, u^1) \in \mathscr{C}^2(\mathscr{V}_{\varepsilon, J!}\mathbb{Q}(2)) \times \mathscr{C}^1(\mathscr{V}_\varepsilon, \mathscr{I}_S) \times \mathscr{C}^0(\mathscr{V}_\varepsilon, \Omega^1_{X,S,d\,\text{closed}}),$$

with

$$u^{-1} = \sum_\alpha Z^\alpha_{i_0 i_1 i_2}, \qquad u^0 = -\sum_\alpha Z^\alpha_{i_0 i_1} G^\alpha_{i_0 i_1}, \qquad u^1 = \sum_\alpha \ln_i f^\alpha \frac{\mathrm{d}a_1^\alpha}{a_1^\alpha}$$

with

$$G^\alpha_{i_0 i_1} = h^{\alpha *} g_{i_0 i_1}, \qquad Z^\alpha_{i_0 i_1} = (\delta \ln f^\alpha)_{i_0 i_1},$$

$$Z^\alpha_{i_0 i_1 i_2} = \delta(z^\alpha_{i_0 i_1} G^\alpha_{i_0 i_1})_{i_0 i_1 i_2}.$$

By (4.2), there is a refinement $(\mathscr{V}_i)$ $i = 0, \ldots, l$ of the open cover, there are

$$\omega \in H^0(X, \Omega^1_{X,S})_{d \text{ closed}}, \quad s \in \mathscr{C}^1(\mathscr{V}_i, j_! \mathbb{Q}(2))$$

and $r \in \mathscr{C}^0(\mathscr{V}_i, \mathscr{I}_S)$ with

$$u^{-1} = -\delta s, \qquad u^0 = -\delta r + s, \qquad u^1 = \omega + \mathrm{d}r.$$

Following the orientation of γ_0, take an order $\mathscr{V}_i$ with

$$p_0 \in \mathscr{V}_0 - \bigcup_{i \geqslant 1} \mathscr{V}_i,$$
$$p_1 \in \mathscr{V}_0 \cap \mathscr{V}_1 \cap \gamma_0,$$
$$p_l \in \mathscr{V}_{l-1} \cap \mathscr{V}_l \cap \gamma_0,$$
$$p_{l+1} \in \mathscr{V}_l \cap \mathscr{V}_0 \cap \gamma_0$$

One has $\int_{\gamma_0} \omega = F - R_\varepsilon$ with

$$F = \int_{p_{l+1}}^{p_1} \sum_\alpha \ln_0 f^\alpha \frac{\mathrm{d}a_1^\alpha}{a_1^\alpha} + \int_{p_1}^{p_{l+1}} \sum_\alpha \ln_1 f^\alpha \frac{\mathrm{d}a_1^\alpha}{a_1^\alpha},$$

$$\begin{aligned} R_\varepsilon &= \int_{p_{l+1}}^{p_1} \mathrm{d}r_0 + \int_{p_1}^{p_2} \mathrm{d}r_1 + \cdots + \int_{p_l}^{p_{l+1}} \mathrm{d}r_l \\ &= r_0|_{p_{l+1}}^{p_1} + r_1|_{p_1}^{p_2} + \cdots + r_l|_{p_l}^{p_{l+1}} \text{ (Stokes)} \\ &= \sum_\alpha [Z^\alpha_{10} G^\alpha_{10}(p_1) + Z^\alpha_{21} G^\alpha_{21}(p_2) + \cdots + Z^\alpha_{l,l-1} G^\alpha_{l,l-1}(p_l) + \\ &\quad + Z^\alpha_{0l} G^\alpha_{0l}(p_{l+1})] \quad \text{modulo } \mathbb{Q}(2). \end{aligned}$$

One has

$$Z^\alpha_{21} = \cdots = Z^\alpha_{l,l-1} = 0.$$

In (1), $G^\alpha_{10}(p_1)$ and $G^\alpha_{0l}(p_{l+1})$ are two determinations of $\log a_1^\alpha$ by (1.4), γ. Therefore, one has

$$R_\varepsilon = \sum_\alpha Z^\alpha_{10} \log a_1^\alpha(p_1) + Z^\alpha_{0l} \log a_1^\alpha(p_{l+1}) \quad \text{modulo } \mathbb{Q}(2).$$

As Z^{α}_{10} and Z^{α}_{0l} do not depend on ε, one has

$$\left|\sum_{\alpha} Z^{\alpha}_{10}(\log a^{\alpha}_{1}(p_1) - \log a^{\alpha}_{1}(p_0)) + Z^{\alpha}_{0l}(\log a^{\alpha}_{1}(p_{l+1}) - \log a^{\alpha}_{1}(p_0))\right|$$
$$\leqslant \text{constant}.\,\varepsilon \text{ by } (*).$$

Therefore, R_ε tends to

$$R = \sum_{\alpha}(Z^{\alpha}_{10} + Z^{\alpha}_{0l}) \log a^{\alpha}_{1}(p_0) = \sum_{\alpha} \log a^{\alpha}_{1}(p_0) \int_{\gamma_0} \frac{df^{\alpha}}{f^{\alpha}}$$

as ε tends to zero.

In (2), R_ε does not depend on ε, and $G^{\alpha}_{10}(p_1) = G^{\alpha}_{0l}(p_{l+1}) = 0$ by (**) and 1.4) γ. This proves cases (1) and (2).

In case (3), consider an open set $\mathscr{U}$ containing γ_0 such that a determination $\ln f^{\alpha}$ of $\log f^{\alpha}$ exists and is single-valued on $\mathscr{U}$ with

$$\ln f^{\alpha} \in H^0(\mathscr{U}, t^{\alpha}\mathscr{I}_S).$$

Then $u_{|\mathscr{U}} \in H^2_{\mathscr{D},\mathrm{an}}(\mathscr{U}, K^{\cdot})$ is the class of

$$\omega := \sum_{\alpha} \ln f^{\alpha} \frac{da^{\alpha}_{1}}{a^{\alpha}_{1}} \in H^0(\mathscr{U}, \Omega^1_{X,S})_{d\text{ closed}} \quad (3.8.1).$$

4.5. Take γ_1 with $\partial\gamma_1 \neq \phi \subset S$. Let $p_0 \in \text{support } \gamma_1 \cap S$. If for all $\alpha = 1 \ldots, N$ there is a single-valued determination of $\log f^{\alpha}$ along $\gamma_1 - p_0$ which vanishes along $t^{\alpha} = 0$ and S, then $\log f^{\alpha}$ is single-valued along γ_1 as well.

PROPOSITION. *Let $p_0 \in \text{support } \gamma_1 - S$ be a point such that $\log f^{\alpha}$ is single-valued along $\gamma_1 - p_0$, and vanishes along $t^{\alpha} = 0$ and S for $\alpha = 1, \ldots, N$.*

(1) *Assume $p_0 \notin \bigcup_1^N T^{\alpha}$. Then one has*

$$\langle[\gamma_1],u\rangle = \int_{\gamma_1} \sum_{\alpha} \log f^{\alpha} \frac{da^{\alpha}_{1}}{a^{\alpha}_{1}} - \sum_{\alpha} \log a^{\alpha}_{1}(p_0) \int_{\gamma_1} \frac{df^{\alpha}}{f^{\alpha}} \quad \text{modulo } \mathbb{Z}(2).$$

(2) *If $p_0 \in \bigcap_1^N T^{\alpha}$ and is not isolated in $\bigcap_1^N T^{\alpha} \cap \text{support } \gamma_1$, then one has*

$$\langle[\gamma_1], u\rangle = \int_{\gamma_1} \sum_{\alpha} \log f^{\alpha} \frac{da^{\alpha}_{1}}{a^{\alpha}_{1}} \quad \text{modulo } \mathbb{Z}(2).$$

(3) *If $\log f^{\alpha}$ is single-valued along γ_1 and vanishes along $t^{\alpha} = 0$ and S for $\alpha = 1, \ldots, N$ then one has*

$$\langle[\gamma_1], u\rangle = \int_{\gamma_1} \sum_{\alpha} \log f^{\alpha} \frac{da^{\alpha}_{1}}{a^{\alpha}_{1}} \quad \text{modulo } \mathbb{Q}(2).$$

Proof. For (1), (2), (3) define $(\mathscr{V}_i)_{i=0,\ldots,l}$ as in the proof of (4.4,1) and (4.4,2). Write $\partial_{\gamma_1} = \{s_0, \ldots, s_k\} \subset S$.

One has to take

$$p_0 \in \mathscr{V}_0 - \bigcup_{i \geqslant 1} \mathscr{V}_i,$$
$$p_1 \in \mathscr{V}_0 \cap \mathscr{V}_1 \cap \gamma_1,$$
$$p_{l_1} \in \mathscr{V}_{l_1-1} \cap \mathscr{V}_{l_1} \cap \gamma_1,$$
$$s_1 \in \mathscr{V}_{l_1},$$
$$s_2 \in \mathscr{V}_{l_1+1},$$
$$p_{l_1+2} \in \mathscr{V}_{l_1+1} \cap \mathscr{V}_{l_1+2} \cap \gamma_1,$$
$$p_{l_1+3} \in \mathscr{V}_{l_1+2} \cap \mathscr{V}_{l_1+3} \cap \gamma_1,$$
$$\vdots$$
$$p_{l_1+l_2} \in \mathscr{V}_{l_1+l_2-1} \cap \mathscr{V}_{l_2} \cap \gamma_1,$$
$$s_3 \in \mathscr{V}_{l_2}$$
$$\vdots$$
$$p_l \in \mathscr{V}_{l-1} \cap \mathscr{V}_l \cap \gamma_1$$
$$p_{l+1} \in \mathscr{V}_l \cap \mathscr{V}_0 \cap \gamma_1.$$

Note that the corresponding R is defined by

$$R = \int_{p_{l+1}}^{p_1} \mathrm{d}r_0 + \int_{p_1}^{p_2} \mathrm{d}r_1 + \cdots + \int_{p_{l_1}}^{s_1} \mathrm{d}r_{l_1} + \\ + \int_{s_2}^{p_{l_1+2}} \mathrm{d}r_{l_1+1} + \int_{p_{l_1+2}}^{p_{l_1+3}} \mathrm{d}r_{l_1+2} + \cdots + \int_{p_l}^{p_{l+1}} \mathrm{d}r_l.$$

As $r \in \mathscr{C}^1(\mathscr{I}_S)$, one has

$$R = (r_0 - r_1)(p_1) + (r_1 - r_2)(p_2) + \cdots + (r_{l_1-1} - r_{l_1})(p_{l_1}) + \\ + (r_{l_1+1} - r_{l_1+2})(p_{l_1+2}) + \cdots + (r_l - r_0)(p_{l+1}).$$

One concludes as in (4.4).

4.6. Let now X be a smooth affine variety over $\mathbb{C}$. Let $f_0^\alpha, \ldots, f_n^\alpha$ be a global invertible algebraic function on X, for $\alpha = 1, \ldots, N$. We consider the cup product

$$u = \sum_1^N \{f_0^\alpha, \ldots, f_n^\alpha\} \in H_{\mathscr{D}}^{n+1}(X, \mathbb{Q}(n+1)).$$

Assuming

$$\mathrm{d}u = \sum_1^N \frac{\mathrm{d}f_0^\alpha}{f_0^\alpha} \wedge \cdots \wedge \frac{\mathrm{d}f_n^\alpha}{f_n^\alpha} = 0,$$

we have ((1.2i), with $Y = \phi$)

$$u \in H^n(X, \mathbb{C}/\mathbb{Q}(n+1)).$$

Now, X being affine, we have as in (4.2)

$$H^n(X, \mathbb{C}/\mathbb{Q}(n+1)) = \frac{H^0(X, \Omega_X^n)_{d \text{ closed}}}{H^n(X, \mathbb{Q}(n+1)) + \mathrm{d}H^0(X, \Omega_X^{n-1})}$$

and if $\omega \in H^0(X, \Omega_X^n)_{d \text{ closed}}$ represents u, one has for any $[\gamma] \in H_n(X, \mathbb{Z})$ of representative γ:

$$\langle [\gamma], u \rangle = \int_\gamma \omega \quad \text{modulo } \mathbb{Q}(n+1).$$

4.7. Take $n = 1$, and X as no longer affine. As explained by R. Hain in his talk at the Max-Planck-Institut, fall 1987, one has Bloch's regular map

$$r: K_2(X) \to H^2_{\mathscr{D}}(X, \mathbb{Z}(2)).$$

This is defined as follows. Let $x = \Pi_1^N \{f_0^\alpha, f_1^\alpha\}$ be in $K_2(\mathbb{C}(X))$. Let U be an affine subset of X such that $f_i^\alpha \in \mathcal{O}(U)^*$. Then the cup product

$$\sum_1^N f_0^\alpha \cup f_1^\alpha \text{ lies in } H^2_{\mathscr{D}}(U, \mathbb{Z}(2)) \subset \xrightarrow[\substack{V \text{ Zariski} \\ \text{open in } X}]{\lim} H^2_{\mathscr{D}}(V, \mathbb{Z}(2)).$$

The existence of the dilogarithm function tells us that

$$\sum_1^N f_0^\alpha \cup f_1^\alpha \in \xrightarrow[\substack{V \text{ Zariski} \\ \text{open in } X}]{\lim} H^2_{\mathscr{D}}(V, \mathbb{Z}(2)).$$

does not depend on the decomposition chosen of x as symbols $\{f_0^\alpha, f_1^\alpha\}$. The existence of a Gersten–Quillen resolution for $H^2_{\mathscr{D}}(2)$ tells us that if $x \in H^0(X, \mathscr{K}_2) \subset K_2(\mathbb{C}(X))$, where $\mathscr{K}_2$ is the Zariski sheaf associated to K_2, then $r(x) := \Sigma_1^N f_0^\alpha \cup f_1^\alpha$ lies in

$$H^2_{\mathscr{D}}(X, \mathbb{Z}(2)) \subset \xrightarrow[V]{\lim} H^2_{\mathscr{D}}(V, \mathbb{Z}(2)).$$

Assume $\mathrm{d}r(x) = 0$.

PROPOSITION. *Let $[\gamma] \in H_1(U, \mathbb{Z})$, of representative γ. Let $p_0 \in$ support γ such that $\log f_0^\alpha$ is single-valued along $\gamma - p_0$. Then*

$$\langle [\gamma], r(x) \rangle = \int_\gamma \sum_\alpha \log f_0^\alpha \frac{\mathrm{d}f_1^\alpha}{f_1^\alpha} - \\ - \sum_\alpha \log f_1^\alpha(p_0) \int_\gamma \frac{\mathrm{d}f_0^\alpha}{f_0^\alpha} \quad \text{modulo } \mathbb{Q}(2).$$

If X is a curve, this is true modulo $\mathbb{Z}(2)$.

The *proof* is word-by-word the same as in (4.4,1), where one replaces $G^\alpha_{i_0 i_1}$ by $\log_{i_1} f_1^\alpha$. If X is a curve, apply (3.5).

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