# A NON-ABELIAN VERSION OF DELIGNE'S FIXED PART THEOREM

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Abstract. We formulate and prove a non-abelian analog of Deligne's Fixed Part theorem on Hodge classes, revisiting previous work of Jost–Zuo, Katzarkov–Pantev and Landesman–Litt. To this aim we study algebraically isomonodromic extensions of local systems and we relate them to variations of Hodge structures, for example we show that the Mumford-Tate group at a generic point stays constant in an algebraically isomonodromic extension of a variation of Hodge structure.

## 1. Introduction

On a quasi-projective complex manifold, a rational Betti class is called Hodge if it is integral and lies as a de Rham class in the right levels of the Hodge and the weight filtrations. Let $f\colon X \to S$ be a projective morphism between quasi-projective complex manifolds. Deligne's Fixed Part Theorem asserts that a Hodge class on a fibre $X_s$ has finite orbit under the fundamental group of $S$ based at $s$ if and only if it extends to a Hodge class on $X$ after base change to a finite étale cover of $S$. Here we have to make the technical assumption that $S$ topologically has the properties of Artin's elementary neighborboods, see Theorem 3.1 for a precise formulation.

Replacing above the Hodge class with a rational polarized variation of Hodge structure which admits an integral structure, Deligne's Fixed Part Theorem has a perfect non-abelian analog, which is the topic of our note.

We introduce some notation in order to give a precise formulation of the main theorem. On a complex manifold $X$, there is the notion of a *polarized variation of $K$-Hodge structure* ($K$-$\mathsf{PVHS}$), for $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, see Subsection 5.1. It consists of a triple $(\mathbb{L}, \mathsf{F}, Q)$, where $\mathbb{L}$ is a $K$-local system on $X$, $\mathsf{F}$ is a finite filtration by holomorphic subbundles of $\mathbb{L} \otimes_K \mathcal{O}_X$ and $Q\colon \mathbb{L} \otimes \overline{\mathbb{L}} \to K$ is a sesquilinear perfect pairing of local systems. This triple is asked to satisfy the Hodge-Riemann relations and Griffiths transversality.

If $R \subset K$ is a subring we denote by $(R)K$-$\mathsf{PVHS}$ those polarized variations of $K$-Hodge structure $(\mathbb{L}, \mathsf{F}, Q)$ such that the local system $\mathbb{L}$ can be defined over $R$.

Let $\overline{f}\colon \overline{X} \to S$ be a projective holomorphic submersion of connected complex manifolds with connected fibres. Let $X \subset \overline{X}$ be the complement of a relative simple normal crossings divisor $D \subset \overline{X}$ over $S$. We call $f = \overline{f}|_X$ a *good holomorphic map*. Throughout the introduction we assume that $S$ is quasi-projective and that one of the following properties holds for $f$.

I) $S$ is an Artin $K\pi 1$, i.e. its fundamental group is a successive extension of finitely generated free groups and $\pi_i(S) = 0$ for $i > 1$,

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II) $X_s$ is a hyperbolic Riemann surface,
III) $f$ has a continuous section $S \to X$.

For the precise notion of an Artin $K\pi 1$ see Subsection 2.2. A hyperbolic Riemann surface $Y$ for us is the complement of finitely many points in a compact connected Riemann surface $\overline{Y}$ such that the Euler characteristic $\chi(Y)$ is negative.

Our main theorem is the following non-abelian Fixed Part Theorem, see Section 6 for the proof.

**Theorem 1.1.** *Let $f : X \to S$ be a good holomorphic map, $s$ be a point in $S$. Let $(\mathbb{L}_s, \mathsf{F}_s, Q_s)$ be a $(\mathbb{Z})K$-PVHS on the fibre $X_s$, where $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. Then the following conditions are equivalent:*

1) *the orbit $\pi_1(S, s) \cdot [\mathbb{L}_s]$ in the set of isomorphism classes of $K$-local systems on $X_s$ is finite;*
2) *$\mathbb{L}_s$ essentially extends to a $K$-local system $\mathbb{L}$ on $X$;*
3) *$(\mathbb{L}_s, \mathsf{F}_s, Q_s)$ essentially extends to a $(\mathbb{Z})K$-PVHS $(\mathbb{L}, \mathsf{F}, Q)$ on $X$;*
4) *$(\mathbb{L}_s, \mathsf{F}_s, Q_s)$ extends to a $K$-PVHS on $X_\Delta = f^{-1}(\Delta)$ for some contractible open neighborhood $\Delta \subset S$ of $s$.*

*If we furthermore assume $f$ to be proper then these conditions are also equivalent to:*

5) *the filtration $\mathsf{F}_s$ extends to a (relative) Griffiths transversal filtration by subbundles $\hat{\mathsf{F}}$ of the formal isomonodromic extension $(\hat{\mathcal{E}}, \hat{\nabla} : \hat{\mathcal{E}} \to \hat{\Omega}^1_{\hat{X}/\hat{S}}(\hat{\mathcal{E}}))$.*

Here "essentially extends" means that it extends after base change with respect to a finite, étale, surjective covering of $S$, see Subsection 3.1 for a precise formulation. In 5) $\hat{S}$ is the formal scheme of $S$ along $s$ and $\hat{X}$ the formal scheme of $X$ along $X_s$. The formal isomonodromic deformation $(\hat{\mathcal{E}}, \hat{\nabla})$ is the formal completion of the flat bundle on $X_\Delta$ associated to the local system on $X_\Delta$ which canonically extends $\mathbb{L}_s$.

The idea of the proof is to show the implication 1) $\Rightarrow$ 3) by constructing an algebraically isomonodromic extension of $\mathbb{L}_s$ to $X$, see Theorem 1.4, and to use T. Mochizuki's non-abelian Hodge correspondence, see Subsection 5.2, which enables one to relate the action of the $\mathbb{C}^\times$ flow on the Higgs bundle associated to $\mathbb{L}_s$ to the one on an algebraically isomonodromic extension $\mathbb{L}$.

In order to show the implication 4) $\Rightarrow$ 1) we use Simpson's non-abelian Hodge loci. In fact he considers only the case $f$ projective and we generalize his theory to the case of a good holomorphic map $f$ in Appendix A. The analogous implication 5) $\Rightarrow$ 1) was suggested to us by D. Litt, see also [Lit24, Cor. 4.4.3].

**Remark 1.2.** Theorem 1.1 has a well-known "Lefschetz hyperplane version", which is deduced from T. Mochizuki's work in the same way. Let us assume $\overline{X}$ is a projective compactification of $X$ with a simple normal crossings divisor $D = \overline{X} \setminus X$ at infinity. Consider a hyperplane section $\overline{Y} \subset \overline{X}$ for a projective embedding $\overline{X} \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$ which is transversal to the stratification induced by $D$, i.e. such that the intersection of $\overline{Y}$ and of irreducible components of $D$ is smooth of the right dimension. Assume that $\dim X \geq 2$ and recall that then the map $\pi_1(Y) \to \pi_1(X)$ is surjective [GM88, II.1.1], where $Y = \overline{Y} \cap X$. Let $\mathbb{L}$ be a $K$-local system on $X$ such that $\mathbb{L}|_Y$ underlies a $(\mathbb{Z})K$-PVHS $(\mathbb{L}|_Y, \mathsf{F}_Y, Q_Y)$. Then the filtration $\mathsf{F}_Y$ and the polarization $Q_Y$ uniquely extend to a $(\mathbb{Z})K$-PVHS $(\mathbb{L}, \mathsf{F}, Q)$.

**Remark 1.3.** For $K = \mathbb{Q}$ one can choose the extension in 3) such that the Hodge-generic locus of $(\mathbb{L}, \mathsf{F}, Q)$ meets $X_s$, or stated differently such that the Mumford-Tate groups of $(\mathbb{L}_s, \mathsf{F}_s, Q_s)$ and of $(\mathbb{L}, \mathsf{F}, Q)$ are the same, see Section 6.

The same argument implies that in Remark 1.2 the Hodge-generic locus of $(\mathbb{L}, \mathsf{F}, Q)$ meets $Y$.

As mentioned above our new method is to study *algebraically isomonodromic extensions* and to relate them to rigidity properties of the Hodge data on local systems. The notion of an isomonodromic extension of a bundle with a flat connection on a fibre $X_s$ of a good holomorphic map $f\colon X \to S$ to a relative flat bundle over a neighborhood of $s \in S$ is well-established in the theory of non-linear differential equations; its connection to Hodge theory has beed studied in particular in [LL24a].

It turns out that the right notion for our global study of local systems is not the discrete monodromy group itself, but the identity component of its Zariski closure. For a $K$-local system $\mathbb{L}$ on $X$ and $x \in X$ we denote by $\mathrm{Mo}(\mathbb{L}, x)$ the Zariski closure in $\mathrm{GL}(\mathbb{L}_x)$ of the image of the monodromy representation $\pi_1(X, x) \to \mathrm{GL}(\mathbb{L}_x)$ and by $\mathrm{Mo}(\mathbb{L}, x)^0$ its identity component. One calls $\mathrm{Mo}(\mathbb{L}, x)^0$ the *algebraic monodromy group.*

We say that an extension of a local system to a bigger manifold is *algebraically isomonodromic* if the two algebraic monodromy groups are the same. We first prove by group theoretic means, developed in Sections 2 and 4 and Appendix B, the following theorem, which is shown in Section 6. We fix a field $K$ of characteristic zero.

**Theorem 1.4.** *Let $f\colon X \to S$ be a good holomorphic map as above. Let $\mathbb{L}_s$ be a $K$-local system on $X_s$ with semi-simple algebraic monodromy group. Then the following conditions are equivalent:*

a) *the orbit $\pi_1(S, s) \cdot [\mathbb{L}_s]$ in the set of isomorphism classes of $K$-local systems on $X_s$ is finite;*
b) *$\mathbb{L}_s$ essentially extends to a local system $\mathbb{L}$ on $X$;*
c) *$\mathbb{L}_s$ essentially extends to an algebraically isomonodromic local system $\mathbb{L}$ relative to $X_s$.*

*Moreover, the following holds:*

i) *if $\mathbb{L}$ as in c) exists it is essentially unique;*
ii) *if $K$ is a number field and $\mathbb{L}_s$ has an integral lattice $\mathbb{L}_{s,\mathcal{O}_K}$, then $\mathbb{L}$ as in c) essentially has an integral lattice $\mathbb{L}_{\mathcal{O}_K}$;*
iii) *if $\mathbb{L}_s$ is absolutely simple, then an extension $\mathbb{L}$ to $X$ which has finite determinant is algebraically isomonodromic relative to $X_s$.*

Recall that if $\mathbb{L}_s$ underlies a $(\mathbb{Z})\mathbb{C}$-$\mathsf{PVHS}$ then by [Del71, Sec. 4.2] its algebraic monodromy group is semi-simple, see Theorem 5.1(v). We do not know whether the lattice $\mathbb{L}_{\mathcal{O}_K}$ in ii) can be chosen as an extension of $\mathbb{L}_{s,\mathcal{O}_K}$.

It is a classical observation due to Clifford that in the setting of Theorem 1.4 one can canonically extend the representation of the fundamental group associated to an absolutely simple local system $\mathbb{L}_s$ to a projective representation of $\pi_1(X)$. The problem is to lift the latter to a proper representation. Our observation is that this lifting process, which a priori has an obstruction in $H^2(\Gamma, K^\times)$, is more manageable in the world of pro-finite groups. Our proof of Theorem 1.4 in fact relies on passing to continuous representations over the

$\ell$-adic numbers of the profinite completion of fundamental groups. In the profinite world the continuous $H^2$-obstruction can be killed by passing to an open subgroup.

We can formulate an application of Theorem 1.4 to representations of mapping class groups which refines [LL24b, Prop. 2.3.4], see Section 6 for a proof.

**Corollary 1.5.** *Let $S$ be a finite étale covering of the moduli stack of hyperbolic punctured compact Riemann surfaces with universal curve $f\colon X \to S$. Fix $s \in S$. Then a semi-simple local system $\mathbb{L}_s$ on $X_s$ such that the orbit $\pi_1(S, s) \cdot [\mathbb{L}_s]$ is finite, essentially extends to $X$.*

Many of the results of our note are known to the experts in one form or another. In particular we refer to the previous work by Jost–Zuo [JZ01] and Katzarkov–Pantev [KP02]. Our motivation to write down a complete account of their theory stems from our ongoing work on the arithmetic of $\ell$-adic local systems over $p$-adic local fields. In that work we for example study an arithmetic variant of non-abelian Hodge loci for $\ell$-adic local systems.

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## 2. Profinite completion of groups and Artin $K\pi 1$ spaces

In this section we recall some classical properties of profinite group completion in the context of fundamental groups of quasi-projective complex manifolds.

2.1. **Group theory.** We recall after Anderson [And74] some criteria when a finitely generated normal subgroup $H \subset G$ of a finitely generated abstract group $G$ induces an injective homomorphism $\widehat{H} \to \widehat{G}$ after profinite completion.

Note first that

$\widehat{H} \to \widehat{G}$ is injective if and only if for any normal subgroup of finite index $H' \subset H$ there exists a subgroup of finite index $G' \subset G$ such that $G' \cap H \subset H'$.

We remark that we could equally request the existence of a normal subgroup of finite index $G' \subset G$ as any subgroup of finite index contains a normal such. We denote by $\Gamma$ the quotient group $G/H$.

We say that $\Gamma$ is *a successive extension of finitely generated free groups* if there is a filtration

$$(\star) \qquad \{1\} = \Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_s = \Gamma$$

by finitely generated subgroups $\Gamma_i$ such that $\Gamma_{i-1}$ is normal in $\Gamma_i$ and such that $\Gamma_i/\Gamma_{i-1}$ is a free group for all $1 \leq i \leq s$. Note that if $\Gamma$ is a successive extension of finitely generated free groups and $\Gamma' \subset \Gamma$ is a subgroup of finite index, then $\Gamma'$ is itself a successive extension of finitely generated free groups.

**Lemma 2.1.** *If $\Gamma$ is a successive extension of finitely generated free groups, $N > 0$ and $i > 0$ are integers, there exists a subgroup of finite index $\Gamma'$ such that the restriction map*

$$H^i(\Gamma, \mathbb{Z}/N\mathbb{Z}) \to H^i(\Gamma', \mathbb{Z}/N\mathbb{Z})$$

*vanishes.*

*Proof.* This is Exercise 2)(d) in [Ser94, I.2.6]. □

**Proposition 2.2.** *If the surjective homomorphism $G \to \Gamma$ has a section or if $\Gamma$ is a successive extension of finitely generated free groups, then $\widehat{H} \to \widehat{G}$ is injective.*

*Proof.* The first part is [And74, Prop. 4]. For the second part let us assume that $\Gamma$ is an extension of $s > 0$ finitely generated free groups as above. We first assume that $s = 1$ that is that $\Gamma$ is free. Then the surjection $G \to \Gamma$ has a section and we can conclude as before. For $s \geq 2$, we define $G_{s-1}$ as the inverse image in $G$ of $\Gamma_{s-1}$. By induction on $s$ we can assume that $\hat{H} \to \hat{G}_{s-1}$ is injective. By the case $s = 1$, $\hat{G}_{s-1} \to \hat{G}_s$ is injective. Thus the composite $\hat{H} \to \hat{G}_{s-1} \to \hat{G}_s$ is injective as well. □

The next proposition is [And74, Prop. 3].

**Proposition 2.3.** *If $\widehat{H}$ has trivial center then $\widehat{H} \to \widehat{G}$ is injective.*

2.2. **Geometric examples.** Let $X$ and $S$ be connected complex manifolds and let $f : X \to S$ be a good holomorphic map as defined in Section 1. Fix a point $x \in X$ and set $s = f(x)$. By Thom's first isotopy lemma [GM88, Thm. I.1.5] or Ehresmann's theorem in the proper case, $f$ is a topological fibration. We obtain an exact sequence of homotopy groups

$$\ldots \to \pi_2(X, x) \to \pi_2(S, x) \to \pi_1(X_s, x) \to \pi_1(X, x) \to \pi_1(S, s) \to 1 \tag{2.1}$$

We say that $S$ is *an Artin $K\pi 1$* if $\pi_i(S) = 0$ for $i > 1$ and $\pi_1(S)$ is a successive extension of finitely generated free groups. Recall that Artin showed [Art72, 4.6] that if $S$ underlies a smooth algebraic variety, Artin $K\pi 1$'s form a base of the Zariski topology.

**Proposition 2.4.** *Let $\mathbb{L}$ be a local system on $S$ with finite fibres. If $S$ is an Artin $K\pi 1$ and $i > 0$ is an integer, there exists a finite étale cover $S' \to S$ such that the pull-back map*

$$H^i(S, \mathbb{L}) \to H^i(S', \mathbb{L})$$

*vanishes.*

*Proof.* As $S$ is an Artin $K\pi 1$ the morphism $H^i(\pi_1(S, s), \mathbb{L}_s) \xrightarrow{\sim} H^i(S, \mathbb{L})$ stemming from the Hochschild-Serre spectral sequence is an isomorphism. We apply Lemma 2.1 to the left term. □

**Proposition 2.5.** *If $S$ is an Artin $K\pi 1$, then $\pi_1(X_s, x) \to \pi_1(X, x)$ is injective and remains injective after profinite completion.*

*Proof.* By definition $\pi_2(S, s) = 0$ so $\pi_1(X_s, x) \to \pi_1(X, x)$ is injective in view of (2.1). We apply Proposition 2.2.

□

**Proposition 2.6.** *If $X_s$ is a hyperbolic Riemann surface, the map $\pi_1(X_s, x) \to \pi_1(X, x)$ is injective and remains injective after profinite completion.*

*Proof.* By [And74, Prop. 18], the centers of $\pi_1(X_s, x)$ and of $\widehat{\pi_1(X_s, x)}$ are trivial. Thus by Proposition 2.3 it is sufficient to see that the homomorphism $\pi_2(S, s) \to \pi_1(X_s, x)$ is trivial, which follows as by hyperbolic geometry one shows that $\pi_1(X_s, s)$ does not contain any normal non-trivial abelian subgroup.

□

## 3. An integral version of Deligne's Fixed Part Theorem

3.1. **Integral Fixed Part Theorem.** In this section we recall the integral version of Deligne's Fixed Part Theorem in order to motivate our non-abelian version, Theorem 1.1. The reason why we discuss the integral version of this classical result here is that while in the abelian case we can always make a rational class integral by multiplication with a positive integer, this is not possible in the non-abelian case. So one has to understand the integral results in the abelian world in order to see what one can hope for in the non-abelian world.

Let $X$ and $S$ be quasi-projective complex manifolds and let $f\colon X \to S$ be a projective holomorphic submersive map with connected fibres. Fix a point $\tilde{s}$ in the universal cover $\tilde{S}$ of $S$ and let $s \in S$ be its image. We say that *a property essentially holds* if the property holds after replacing $S$ by a finite quotient cover $S' \to S$ of $\tilde{S} \to S$, $s$ by the image of $\tilde{s}$ in $S'$ and $X$ by $X \times_S S'$. We denote by $X_s$ the fibre over $s$.

Let $Z$ be a quasi-projective complex manifold. A class $\xi \in H^{2i}(Z, \mathbb{Q})$ is said to be a *Hodge class* if

$$\xi \in \operatorname{im} H^{2i}(Z, \mathbb{Z}) \cap \operatorname{im} F^i H^{2i}(\overline{Z}, \mathbb{C}) \subset H^{2i}(Z, \mathbb{C})$$

where $F$ is the Hodge filtration and $\overline{Z}$ is a good compactification of $Z$, i.e. a complex manifold which is projective and contains $Z$ as a Zariski open submanifold [Del74, Thm. 3.2.5].

As $f$ is a topological fibration, $\pi_1(S, s)$ acts on the cohomology $H^i(X_s, \mathbb{Q})$. We denote by $\widehat{S}$ the formal completion of $S$ along $s$ and by $\widehat{X}$ the formal completion of $X$ along $X_s$.

**Theorem 3.1** (Deligne's Fixed Part Theorem). *Let $f : X \to S$ be a proper good holomorphic map with $S$ as in I) in Section 1. Let $s$ be a complex point of $S$. Let $\xi \in H^{2i}(X_s, \mathbb{Q})$ be a Hodge class on $X_s$. Then the following conditions are equivalent:*

1) *the orbit $\pi_1(S, s) \cdot \xi$ is finite;*
2) *$\xi$ essentially extends to a class in $H^{2i}(X, \mathbb{C})$;*
3) *$\xi$ essentially extends to a Hodge class in $H^{2i}(X, \mathbb{Q})$;*
4) *$\xi$ extends to $H^{2i}(X_\Delta, \Omega^{\geq i}_{X_\Delta})$, where $\Delta$ is a contractible open neighborhood of $s$;*
5) *the Gauß-Manin flat deformation of $\xi$ in $H^{2i}_{dR}(\hat{X}/\hat{S})$ lies in the subbundle*

$$F^i H^{2i}(\hat{X}/\hat{S}) := H^{2i}(\hat{X}/\hat{S}, \Omega^{\geq i}_{\hat{X}/\hat{S}}) \subset H^{2i}_{dR}(\hat{X}/\hat{S}).$$

*Proof.* The implications 3) ⇒ 2) ⇒ 1) and 3) ⇒ 4) ⇒ 5) are clear.

1) ⇒ 3) holds rationally by Deligne's Fixed Part Theorem [Del71, Thm. 4.1.1], that is for any good compactification $\overline{X}$ of $X$ there exists an element in $F^i H^{2i}(\overline{X}, \mathbb{C}) \cap H^{2i}(\overline{X}, \mathbb{Q})$ extending $\xi$; let $\zeta$ be its image in $H^{2i}(X, \mathbb{Q})$.

We have to check that after replacing $S$ by a finite quotient covering $S'$ of $\tilde{S}$ the class $\zeta$ becomes integral. The obstruction for integrality is the image of $\zeta$ in $H^{2i}(X, \mathbb{Q}/\mathbb{Z})$. There exists an $N > 0$ such that this image is induced by an element of

$$\ker[H^{2i}(X, \frac{1}{N}\mathbb{Z}/\mathbb{Z}) \to H^{2i}(X_s, \frac{1}{N}\mathbb{Z}/\mathbb{Z})] \tag{3.1}$$

We claim that there exists a finite étale covering $S' \to S$ such that the pullback along $X \times_S S' \to X$ kills the group (3.1). By the Leray spectral sequence it is sufficient to show that we can kill the groups $H^p(S, R^q f_*(\frac{1}{N}\mathbb{Z}/\mathbb{Z}))$ for $p + q = 2i$ and $p = 1$, then for $p = 2$ etc. by a finite étale covering of $S$. This is Proposition 2.4.

We now address 5) $\Rightarrow$ 1). Recall that the *Hodge* or *Noether-Lefschetz locus* NL is the closed analytic subset of the étale space associated to $R^{2i} f_* \mathbb{Q}$ consisting of Hodge cycles, see [CDK95, Intro.]. Restricted to any irreducible component of NL the morphism $\mathrm{NL} \to S$ is finite and unramified. The assumption 5) implies that there exists only one irreducible component $W$ of NL containing the given Hodge cycle $\xi$ and that $W \to S$ is étale at $\xi$. Then $W \to S$ is a finite, surjective, unramified holomorphic map of reduced, irreducible analytic spaces of the same dimension with a complex manifold as codomain, so it is a finite étale covering of complex manifolds. We conclude that the orbit $\pi_1(S, s) \cdot \xi$ consists of the finitely many elements of the fibre of $W \to S$ over $s$. See also [BEK14, Intro.] and references therein. □

## 4. Extensions of representations of discrete groups

This section contains the main group theoretic results of our note, which are related to Clifford theory [Cli37]. Let $G$ and $H$ be finitely generated groups such that $H \subset G$ is a normal subgroup. We write $\Gamma$ for the group $G/H$. Let $K$ be a field of characteristic zero and $V$ be a finite dimensional $K$-vector space.

### 4.1. Main group theoretic result.

We say that a property holds *essentially* for a representation $\rho\colon G \to \mathrm{GL}(V)$ if it holds after restriction to a subgroup $G'$ of finite index in $G$ with $H \subset G'$. We say that a representation $\rho_H\colon H \to \mathrm{GL}(V)$ *essentially extends* to $G$ if it extends to a representation of such a $G'$.

For a representation $\rho\colon G \to \mathrm{GL}(V)$ we denote by $\mathrm{Mo}(\rho)$ the Zariski closure of the image of $\rho$ as an algebraic group over $K$. Its identity component $\mathrm{Mo}(\rho)^0$ is called the *algebraic monodromy group* of $\rho$. We call $\rho$ *algebraically isomonodromic relative to* $H$ if the embedding

$$\mathrm{Mo}(\rho|_H)^0 \to \mathrm{Mo}(\rho)^0$$

defined by $H \to G$ is an isomorphism.

We denote by $[\rho_H] \in \mathrm{Rep}(H)$ the isomorphism class of a representation $\rho_H\colon H \to \mathrm{GL}(V)$. There is a natural action of $\Gamma$ on the set of isomorphism classes of representations $\mathrm{Rep}(H)$ by conjugation by lifts to $G$ of elements in $\Gamma$.

**Theorem 4.1.** *Assume that the map of profinite completions $\widehat{H} \to \widehat{G}$ is injective. Let $\rho_H\colon H \to \mathrm{GL}(V)$ be a representation with semi-simple algebraic monodromy group. The following condition are equivalent.*

a) *the orbit $\Gamma \cdot [\rho_H]$ is finite;*
b) *$\rho_H$ essentially extends to a representation $\rho\colon G \to \mathrm{GL}(V)$;*

c) *$\rho_H$ essentially extends to an algebraically isomonodromic representation $\rho\colon G \to \mathrm{GL}(V)$.*

*Moreover,*

i) *if a representation $\rho$ as in c) exists it is essentially unique;*
ii) *if $K$ is a number field and $\rho_H$ is integral then a representation $\rho$ as in c) is integral;*
iii) *if $\rho_H$ is absolutely simple, then an extension $\rho : G \to \mathrm{GL}(V)$ of $\rho_H$ is algebraically isomonodromic relative to $H$ if and only if $\rho$ has finite determinant.*

The proof of Theorem 4.1 is given in Subsection 4.3.

4.2. **Admissible representations.** In this subsection we assume that $K$ is algebraically closed of characteristic zero. A representation $\rho_H\colon H \to \mathrm{GL}(V)$ is called *admissible* if it is semi-simple and all simple constituents of $\rho_H$ have finite determinant. A representation $\rho\colon G \to \mathrm{GL}(V)$ is called *admissible relative to $H$* (or an *admissible extension of $\rho|_H$*) if $\rho$ is admissible and all its simple constituents stay simple when restricted to $H$.

The two following lemmata ought to be well known, but we could not find a precise reference.

**Lemma 4.2.** *The algebraic monodromy group $\mathrm{Mo}(\rho)^0$ of $\rho$ is semi-simple if and only if $\rho|_{G'}$ is admissible for all subgroups of finite index $G' \subset G$.*

*Proof.* Assume the algebraic monodromy group $\mathrm{Mo}(\rho)^0$ is semi-simple. Then for all finite index subgroups $G' \subset G$, $\rho(G') \subset \rho(G)$ has finite index, thus $\mathrm{Mo}(\rho|_{G'}) \subset \mathrm{Mo}(\rho)$ has finite index as well. This implies the equality $\mathrm{Mo}(\rho|_{G'})^0 = \mathrm{Mo}(\rho)^0$.

In particular $\mathrm{Mo}(\rho|_{G'})^0$ is reductive, thus by Weyl's theorem, $\rho|_{G'}$ as a representation of the algebraic group $\mathrm{Mo}(\rho|_{G'})^0$ is semi-simple, thus $\rho|_{G'}$ itself is semi-simple. As $\mathrm{Mo}(\rho)^0 = [\mathrm{Mo}(\rho)^0, \mathrm{Mo}(\rho)^0]$, all rank one algebraic representations of $\mathrm{Mo}(\rho|_{G'})$ are finite. Thus $\rho|_{G'}$ is admissible.

Assume conversely that $\rho|_{G'}$ is admissible for all subgroups of finite index $G' \subset G$. Upon replacing $G$ by the preimage of $\mathrm{Mo}(\rho)^0(K)$, which is a finite index subgroup in $G$ (recall $K$ is algebraically closed), we may assume that $\mathrm{Mo}(\rho)$ is connected. As the unipotent radical $U$ of $\mathrm{Mo}(\rho)$ is normal, by Clifford's theorem $\rho|_U$ is still semi-simple. So by Engel's theorem, we conclude that $U$ is trivial. The radical $R$ of $\mathrm{Mo}(\rho)$ is then a central torus, so by Schur's lemma it acts by a character $\lambda\colon R \to \mathbb{G}_m$ on each simple $G$-subrepresentation $V_i \subset V$. As the action on $\det(V_i)$ is finite, a power of $\lambda$ is trivial, so $\lambda$ is trivial. As the action of $R$ on $V$ is faithful, $R$ is trivial and $\mathrm{Mo}(\rho)^0$ is semi-simple. □

We draw the following consequence of Lemma 4.2 which ought to be well known, but we could not find a reference.

**Lemma 4.3.** *Finite direct sums of representations with semi-simple algebraic monodromy groups have semi-simple algebraic monodromy groups. Direct summands of representations with semi-simple algebraic monodromy group have a semi-simple algebraic monodromy group.*

*Proof.* By Lemma 4.2 it is sufficient to prove this for "semi-simple algebraic monodromy group" replaced by "admissible in restriction to all finite index subgroups", for which it is trivial. □

**Lemma 4.4.** *Let $\rho\colon G \to \mathrm{GL}(V)$ be an admissible representation relative to $H$, then the restriction homomorphism*

$$\mathrm{End}_\rho(V) \xrightarrow{\sim} \mathrm{End}_{\rho|_H}(V).$$

*is an isomorphism.*

*Proof.* Let $V = \bigoplus_i V_i \otimes W_i$ be the canonical decomposition with $V_i$ non-isomorphic simple representations of $G$ and $W_i$ $K$-vector spaces as trivial representations. By Schur's lemma

$$\mathrm{End}_\rho(V) = \prod_i \mathrm{End}(W_i).$$

As $V_i$ as a representation of $H$ is simple as well, the same formula holds for $\mathrm{End}_{\rho|_H}(V)$. □

**Remark 4.5.** In Lemma 4.4 we do not need the condition on the determinants of the constituents being torsion. We won't need this fact.

**Proposition 4.6.** *Let $\rho, \tilde{\rho}\colon G \to \mathrm{GL}(V)$ be two admissible representations relative to $H$ with $\rho|_H = \tilde{\rho}|_H$. Then $\rho$ is essentially isomorphic to $\tilde{\rho}$.*

*Proof.* A vector subspace $V' \subset V$ is a simple $\rho$-subrepresentation if and only if it is the image of $V$ by a minimal right ideal of $\mathrm{End}_\rho$ and the same for $\rho$ replaced with $\tilde{\rho}$. So by Lemma 4.4 the simple $\rho$-subrepresentations are exactly the simple $\tilde{\rho}$-subrepresentations. We can therefore assume that $\rho$ and $\tilde{\rho}$ are simple.

Set $\tau(g) = \tilde{\rho}(g)\rho(g)^{-1}$ for $g \in G$. As

$$\tau(g)\rho(h)\tau(g)^{-1} = \tilde{\rho}(g)\rho(g^{-1}hg)\tilde{\rho}(g^{-1}) = \tilde{\rho}(g)\tilde{\rho}(g^{-1}hg)\tilde{\rho}(g^{-1}) = \rho(h)$$

for all $h \in H$ and $g \in G$, and as $\rho|_H$ is simple, Schur's lemma implies that $\tau(g) \in K^\times$ for all $g \in G$. In fact $\tau(g) \in \mu_N(K)$ for some $N > 0$ as the determinants of $\rho$ and $\tilde{\rho}$ are finite. This implies that $\tau\colon G \to \mu_N(K) \subset \mathrm{GL}(V)$ is a homomorphism. Thus $\rho = \tilde{\rho}$ in restriction to the kernel $G' \subset G$ of $\tau$, which is a finite index subgroup. □

The next proposition is the reason why we have to pass to profinite completions.

**Proposition 4.7.** *Assume that the map of profinite completions $\widehat{H} \to \widehat{G}$ is injective. For an admissible representation $\rho_H\colon H \to \mathrm{GL}(V)$ the following are equivalent:*

1) *the orbit $\Gamma \cdot [\rho_H]$ is finite;*
2) *$\rho_H$ essentially extends to a representation of $G$;*
3) *$\rho_H$ essentially extends to an admissible representation $\rho$ of $G$ (relative to $H$).*

*Proof.* 3) ⇒ 2) ⇒ 1) is obvious. We have to show 1) ⇒ 3). After replacing $G$ by a subgroup $G'$ of finite index with $H \subset G'$ the orbit $\Gamma \cdot [\rho_H]$ is trivial. As $\Gamma$ permutes the finitely many isomorphism classes of simple constituents of $\rho_H$, after replacing $G$ by a subgroup $G'$ of finite index with $H \subset G'$, we may assume that this permutation action is trivial. Then we may and do assume that $\rho_H$ is simple. Choose a basis of $V$ and identify $\mathrm{GL}_r(K)$ with $\mathrm{GL}(V)$. As $H$ is finitely generated there exists a finitely generated $\mathbb{Z}$-subalgebra $R \subset K$ with $\mathrm{im}(\rho_H) \subset \mathrm{GL}_r(R)$. Choose an injective ring homomorphism $R \to \mathcal{O}_E$ where $\mathcal{O}_E \subset E$ is the ring of integer of an $\ell$-adic field $E$ for some prime number $\ell$. The induced representation $\rho_H\colon H \to \mathrm{GL}_r(E)$ factors through a continuous representation of $\widehat{H}$, which is simple as well. By Proposition B.2 we can essentially extend it to a continuous representation $\widehat{\rho}\colon \widehat{G} \to \mathrm{GL}_r(E)$ with finite determinant. As $G$ is finitely generated the image of $\widehat{\rho}|_G$ has values in $\mathrm{GL}_r(A)$, where $A$ is a finitely generated $\mathrm{Frac}(R)$-subalgebra of $E$. Choose a $K$-point of $\mathrm{Spec}\,(A)$ over $\mathrm{Frac}(R)$ (recall that $K$ is assumed to be algebraically closed). It induces an extension $\rho\colon G \to \mathrm{GL}_r(K)$ of $\rho_H$ with finite determinant. □

4.3. **Algebraically isomonodromic representations.** In this subsection we collect some facts about algebraically isomonodromic representations relative to $H$.

**Lemma 4.8.** *If $\rho$ is algebraically isomonodromic relative to $H$, then $\mathrm{Mo}(\rho|_H) \to \mathrm{Mo}(\rho)$ is essentially an isomorphism. In particular $\mathrm{End}_\rho(V) \xrightarrow{\sim} \mathrm{End}_{\rho|_H}(V)$ is essentially an isomorphism.*

*Proof.* Under the isomonodromic assumption, $\mathrm{Mo}(\rho|_H) \subset \mathrm{Mo}(\rho)$ is a (normal) finite index algebraic subgroup, which induces a finite index (normal) embedding $\pi_0(\mathrm{Mo}(\rho|_H)) \subset \pi_0(\mathrm{Mo}(\rho))$ of finite groups. After replacing $G$ by the preimage of $\pi_0(\mathrm{Mo}(\rho|_H))$ the map $\mathrm{Mo}(\rho|_H) \to \mathrm{Mo}(\rho)$ is an isomorphism. □

**Lemma 4.9.** *Assume $K$ is algebraically closed. If $\rho\colon G \to \mathrm{GL}(V)$ is algebraically isomonodromic relative to $H$ and $\mathrm{Mo}(\rho)^0$ semi-simple, then $\rho$ is essentially admissible relative to $H$.*

*Proof.* By Lemma 4.8 we may assume that $\mathrm{Mo}(\rho|_H) \to \mathrm{Mo}(\rho)$ is an isomorphism. In that case the lemma is clear. □

**Proposition 4.10.** *Assume $K$ is algebraically closed. Let $\rho\colon G \to \mathrm{GL}(V)$ be a represenation such that $\mathrm{Mo}(\rho|_H)$ is connected and semi-simple. Then $\rho$ is essentially admissible relative to $H$ if and only if $\rho$ is algebraically isomonodromic relative to $H$.*

*Proof.* The implication "$\Leftarrow$" follows from Lemma 4.9.

For "$\Rightarrow$" we first observe that by replacing $G$ by the preimage of $\mathrm{Mo}(\rho)^0$, which is a subgroup of finite index containing $H$, we may assume that $\mathrm{Mo}(\rho)$ is connected. The argument in the proof of Lemma 4.2 shows that $\mathrm{Mo}(\rho)$ is semi-simple. If the embedding $\mathrm{Mo}(\rho|_H) \to \mathrm{Mo}(\rho)$ is not an isomorphism then there exists a non-trivial normal connected algebraic subgroup $W \subset \mathrm{Mo}(\rho)$ centralizing $\mathrm{Mo}(\rho|_H)$ such that

$$W \times \mathrm{Mo}(\rho|_H) \to \mathrm{Mo}(\rho)$$

is surjective with finite kernel, see e.g. [Hum75, 14.2]. The $W$ action stabilizes every simple $\rho$-subrepresentation $V' \subset V$ and commutes with the action of $H$ on $V'$, which is simple. So by Schur's lemma $W$ acts by a character on $V'$, but all characters are trivial as $W$ is semi-simple. So $W$ has to be trivial as it acts faithfully on $V$. □

**Proposition 4.11.** *Assume that $K$ is algebraically closed. Let $\rho_H\colon H \to \mathrm{GL}(V)$ be a representation with semi-simple algebraic monodromy group. Let $G' \subset G$ be a subgroup of finite index, $H' = H \cap G'$. If $\mathrm{Mo}(\rho|_{H'})$ is connected and $\rho|_{H'}$ has an admissible extension $\rho'\colon G' \to \mathrm{GL}(V)$ relative to $H'$, then $\rho$ essentially has an algebraically isomonodromic extension $\rho\colon G \to \mathrm{GL}(V)$ relative to $H$.*

*Proof.* After replacing $G'$ by a subgroup of finite index we may assume that $G' \subset G$ is normal, thus $H' \subset H$ is normal as well. Replace $G$ by the preimage of $H/H' \subset G/G'$ via $G \to G/G'$. Then by definition $[G : G'] = [H : H']$. We set

$$\tilde{\rho} = \mathrm{Ind}_{G'}^{G}(\rho').$$

We show in the sequel that there is a direct summand of $\tilde{\rho}$ which extends $\rho_H$. First observe that $\tilde{\rho}$ is algebraically isomonodromic relative to $H$. Indeed, it suffices to check that $\tilde{\rho}|_{G'}$ is algebraically isomonodromic relative to $H'$. Noting that $\tilde{\rho}|_{H'}$ is a direct sum of copies of $\rho_H|_{H'}$, so $\mathrm{Mo}(\tilde{\rho}|_{H'})$ is connected and semi-simple, by Proposition 4.10 we have to show that $\tilde{\rho}|_{G'}$ is admissible relative to $H'$. This holds as $\tilde{\rho}|_{G'}$ it is a direct sum of $H$-conjugates of $\rho'$ and $\rho'$ is admissible relative to $H'$. On the other hand,

$$\tilde{\rho}|_H \cong \mathrm{Ind}_{H'}^{H}(\rho_H|_{H'})$$

contains $\rho_H$ as a direct summand. So by Lemma 4.8 after replacing $G$ by a subgroup of finite index containing $H$ we find a direct summand of $\tilde{\rho}$ which extends $\rho_H$. □

*Proof of Theorem 4.1.* Essential uniqueness in i) follows from Lemma 4.9 and Proposition 4.6.

We clearly have c) ⇒ b) ⇒ a). We now reduce the proof of the implication a) ⇒ c) to the case where $K$ is algebraically closed. Let $\overline{K}$ be an algebraic closure of $K$. Consider an algebraically isomonodromic representation $\rho\colon G \to \mathrm{GL}(V \otimes_K \overline{K})$ relative to $H$ with $\rho|_H$ stabilizing $V$ (thus defined over $K$). Then, as $G$ is finitely generated, there exists a finite Galois subextension $\tilde{K} \subset \overline{K}$ of $K$ such that $\rho$ stabilizes $V \otimes_K \tilde{K}$. The Galois group $\mathrm{Gal}(\tilde{K}/K)$ acts on $V \otimes_K \tilde{K}$ via its action on $\tilde{K}$, thus on $\mathrm{GL}(V \otimes_K \tilde{K})$. For $\gamma \in \mathrm{Gal}(\tilde{K}/K)$ we define $\gamma \cdot \rho$ by the formula $\gamma \cdot \rho(g) = \gamma \cdot (\rho(g))$. Then $\mathrm{Mo}(\gamma \cdot \rho) = \gamma \cdot \mathrm{Mo}(\rho)$ thus both $\gamma \cdot \rho$ and $\rho$ are algebraically isomonodromic extensions relative to $H$ of $\rho|_H = \gamma \cdot \rho|_H$. Then i) implies that there is a finite index subgroup $G^\gamma \subset G$ containing $H$ such that $\rho|_{G^\gamma} = \gamma \cdot \rho|_{G^\gamma}$. We define $G' = \cap_{\gamma \in \mathrm{Gal}(\tilde{K}/K)} G^\gamma$ which contains $H$ and is a subgroup of finite index in $G$. Then $\rho|_{G'}$ is $\mathrm{Gal}(\overline{K}/K)$ invariant thus descends to $K$.

Now we prove a) ⇒ c) for $K = \overline{K}$. The condition $\widehat{H} \to \widehat{G}$ injective allows us to find a subgroup of finite index $G' \subset G$, $H' = H \cap G'$, such that $\mathrm{Mo}(\rho_H|_{H'})$ is connected. After replacing $G'$ by a subgroup of finite index containing $H'$, Proposition 4.7 implies that there exists an admissible extension $\rho'\colon G' \to \mathrm{GL}(V)$ of $\rho_H|_{H'}$ relative to $H'$. Proposition 4.11 shows then the essential existence of an algebraically isomonodromic extension $\rho\colon G \to \mathrm{GL}(V)$ of $\rho_H$.

We now prove ii). After replacing $K$ by a finite extension, we may assume that all simple constituents of $\rho_H$ are absolutely simple. By Lemma 4.8 we can replace $G$ by a subgroup of finite index containing $H$ such that $\mathrm{End}_\rho(V) \xrightarrow{\sim} \mathrm{End}_{\rho_H}(V)$ is an isomorphism. Then we have to show ii) only for the simple subrepresentations $V' \subset V$. So we assume now that $\rho$ is absolutely simple and that $\mathcal{V} \subset V$ is a $\rho_H$-stable lattice.

For $g \in G$ the lattice $\rho(g)\mathcal{V}$ is stabilized by the action of $H$ via $\rho_H$ as $\rho(h)\rho(g)\mathcal{V} = \rho(g)\rho(g^{-1}hg)\mathcal{V} = \rho(g)\mathcal{V}$. By the the Jordan-Zassenhaus theorem [CR62, Sec. 79] there are only finitely many such lattices up to isomorphism, which by Schur's lemma means up to homothety. So after replacing $G$ by a subgroup of finite index containing $H$ we can assume that for all $g \in G$ there is $\lambda_g \in K^\times$ with $\rho(g)\mathcal{V} = \lambda_g \mathcal{V}$. As $\rho$ has finite determinant, $\lambda_g \in \mathcal{O}_K^\times$ and $\mathcal{V}$ is stabilized by the $\rho$-action of $G$.

We now prove iii). Without loss of generality $K = \overline{K}$. By assumption $\rho_H$ is absolutely simple and has finite determinant, thus is admissible and the extension $\rho\colon G \to \mathrm{GL}(V)$ of $\rho_H$ with finite determinant is admissible relative to $H$. Such an extension is essentially unique by Proposition 4.6. But an algebraically isomonodromic extension of $\rho_H$, which essentially exists by the above argument, has finite determinant, so has to essentially agree with $\rho$. □

**Proposition 4.12.** *Assume that $\widehat{H} \to \widehat{G}$ is injective. Then finite direct sums of algebraically isomonodromic representations with semi-simple algebraic monodromy are algebraically isomonodromic. Direct summands of algebraically isomonodromic representations with semi-simple algebraic monodromy are algebraically isomonodromic.*

*Sketch of proof.* The second part is easy. For the first part we can assume without loss of generality that $K$ is algebraically closed and we can argue as in the proof of Theorem 4.1 reducing to the observation that direct sums of admissible representations relative to $H$ are admissible relative to $H$. □

## 5. Reminder on variations of Hodge structure

In this section we summarize what we need about variations of Hodge structure. We could not find some of the formulations in the literature, so we provide a few details.

5.1. **Polarized variations of $\mathbb{C}$-Hodge structure.** For our purpose it is useful to define polarized $\mathbb{C}$-Hodge structure in a form without fixing a weight. Let $V$ be a finite dimensional $\mathbb{C}$-vector space, $\mathsf{F}$ a finite decreasing filtration of $V$ and $Q\colon V \times \overline{V} \to \mathbb{C}$ a perfect Hermitian pairing. We call the triple $(V, \mathsf{F}, Q)$ a *polarized $\mathbb{C}$-Hodge structure* ($\mathbb{C}$-PHS) if the *Hodge-Riemann relations* hold for all $a \in \mathbb{Z}$:

- $V = \mathsf{F}^a \oplus (\mathsf{F}^a)^\perp$ and
- $(-1)^a Q|_{\mathsf{F}^a \cap (\mathsf{F}^{a+1})^\perp}$ is positive definite.

If $N\colon V \to V$ is a nilpotent endomorphism with $Q(Nv, w) = Q(v, Nw)$ for all $v, w \in V$ then we call $(V, \mathsf{F}, Q, N)$ a *polarized mixed $\mathbb{C}$-Hodge structure* if $N(\mathsf{F}^a) \subset \mathsf{F}^{a-1}$ and if the filtrations induced by $\mathsf{F}$ on the monodromy graded pieces $\mathrm{gr}_i^{\mathsf{M}} V$ are compatible with the Lefschetz decomposition of $\oplus_i \mathrm{gr}_i^{\mathsf{M}} V$ with respect to $\overline{N}\colon \mathrm{gr}_i^{\mathsf{M}} V \to \mathrm{gr}_{i-2}^{\mathsf{M}} V$ and furthermore for $i > 0$ the primitive part

$$\ker(\mathrm{gr}_i^{\mathsf{M}} V \xrightarrow{\overline{N}^{i+1}} \mathrm{gr}_{-i-2}^{\mathsf{M}} V)$$

endowed with the pairing

$$\mathrm{gr}_i^{\mathsf{M}} V \times \mathrm{gr}_i^{\mathsf{M}} V \to \mathbb{C}, \quad (v, w) \mapsto Q(v, \overline{N}^i w)$$

and the induced filtration is a $\mathbb{C}$-PHS .

Let us recall how this notion of a $\mathbb{C}$-PHS is related to the more common version of a $\mathbb{R}$-PHS. Let $V$ be a finite dimensional $K$-vector space with $K \in \{\mathbb{Q}, \mathbb{R}\}$, $\mathsf{F}$ a finite decreasing filtration on $V_{\mathbb{C}} = V \otimes_K \mathbb{C}$ and $Q_\circ \colon V \times V \to K$ a $(-1)^w$-symmetric perfect pairing with $w \in \mathbb{Z}$. We call the triple $(V, \mathsf{F}, Q)$ a *polarized $K$-Hodge structure* ($K$-PHS) of *weight $w$* if

$$(V_{\mathbb{C}}, \mathsf{F}, (v, w) \overset{Q}{\mapsto} \sqrt{-1}^{-w} Q_\circ(v, \overline{w}))$$

is a $\mathbb{C}$-PHS and $(\mathsf{F}^a)^\perp = \overline{\mathsf{F}}^{w+1-a}$ for all $a \in \mathbb{Z}$, where the orthogonal complement is with respect to the Hermitian pairing $Q$.

A *polarized variation of $\mathbb{C}$-Hodge structure* ($\mathbb{C}$-PVHS) on a complex manifold $X$ is given by a triple $(\mathbb{L}, \mathsf{F}, Q)$, where $\mathbb{L}$ is a complex local system on $X$, $\mathsf{F} \subset \mathbb{L} \otimes_{\mathbb{C}} \mathcal{O}_X$ is a finite filtration by holomorphic subbundles and $Q \colon \mathbb{L} \times \overline{\mathbb{L}} \to \mathbb{C}$ is a Hermitian perfect pairing, where we assume that the Hodge-Riemann relations at each point of $X$ are statisfied and Griffiths transversality $\nabla(\mathsf{F}^a) \subset \Omega^1_X(\mathsf{F}^{a-1})$ holds.

**Theorem 5.1.** *Let $X$ be a quasi-projective connected complex manifold. Let $(\mathbb{L}, \mathsf{F}, Q)$ and $(\mathbb{L}', \mathsf{F}', Q')$ be $\mathbb{C}$-PVHS. The following properties hold:*

i) *if $\phi \colon \mathbb{L} \xrightarrow{\sim} \mathbb{L}'$ is an isomorphism of local systems such that for one point $x \in X$ the Hodge filtrations are preserved, i.e. $\phi_x(\mathsf{F}^a_x) = (\mathsf{F}')^a_x$ for all $a \in \mathbb{Z}$, and such that $\phi^*(Q') = Q$, then $\phi$ is an isomorphism of $\mathbb{C}$-PVHS;*

ii) *if $\mathbb{L}$ is simple then $Q$ is unique up to a factor in $\mathbb{R}_{>0}$ and $\mathsf{F}$ is unique up to shift. Moreover, there are no gaps in $\mathsf{F}$, i.e. we cannot have $\mathrm{gr}^{\mathsf{F}}_i = 0$ and $\mathrm{gr}^{\mathsf{F}}_j, \mathrm{gr}^{\mathsf{F}}_k \neq 0$ for $j < i < k$;*

iii) *there is a canonical structure of a $\mathbb{C}$-PHS on $H^0(X, \mathbb{L})$ such that $H^0(X, \mathbb{L}) \to \mathbb{L}_x$ is a polarized Hodge substructure for all $x \in X$;*

iv) *there are $\mathbb{C}$-PVHS $(\mathbb{L}_i, \mathsf{F}_i, Q_i)$ for $1 \leq i \leq s$ with $\mathbb{L}_i$ simple local systems such that the canonical map*

$$\bigoplus_i (\mathbb{L}_i, \mathsf{F}_i, Q_i) \otimes_{\mathbb{C}} H^0(X, \mathbb{L}_i^\vee \otimes \mathbb{L}) \xrightarrow{\sim} (\mathbb{L}, \mathsf{F}, Q)$$

*is an isomorphim in $\mathbb{C}$-PVHS. Here $H^0$ is endowed with the $\mathbb{C}$-PHS from* iii)*;*

v) *if $\mathbb{L}$ underlies a $\mathbb{Z}$-local system then the algebraic monodromy group $\mathrm{Mo}(\mathbb{L}, x)^0$ is semi-simple and the monodromy at infinity of $\mathbb{L}$ is quasi-unipotent.*

The theorem is classical [Del87].

**Proposition 5.2.** *Let $f \colon Y \to X$ be either a finite étale covering or a Zariski open embeding (both with dense image) of quasi-projective complex manifolds. Let $\mathbb{L}$ be a $\mathbb{C}$-local system on $X$. Then $\mathbb{L}$ underlies a $\mathbb{C}$-PVHS if and only if $f^*\mathbb{L}$ underlies a $\mathbb{C}$-PVHS.*

*Proof.* The case of a finite étale covering is a consequence of Theorem 5.1(iv) which implies that for $\mathbb{L}_1 \subset \mathbb{L}_2$ an inclusion of complex local systems such that $\mathbb{L}_2$ underlies a $\mathbb{C}$-PVHS then $\mathbb{L}_1$ underlies a $\mathbb{C}$-PVHS.

The case of a Zariski open embeding follows from the nilpotent orbit theorem [CK89, (2.1)]. □

Let us recall for later reference one of the main results about *degeneration of pure $\mathbb{C}$-Hodge structure.* Let $\mathbb{L}$ be a unipotent complex local system on $\Delta^\times = \Delta \setminus \{0\}$ with $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$. Let $Q \colon \mathbb{L} \times \overline{\mathbb{L}} \to \mathbb{C}$ be a Hermitian perfect pairing. Let $\mathcal{E}$ be the holomorphic bundle on $\Delta$ which is the Deligne extension of the flat bundle $\mathbb{L} \otimes \mathcal{O}_{\Delta^\times}$ [Del70, Prop. II.5.2].

Let $N\colon \mathcal{E}_0 \to \mathcal{E}_0$ be the residue of the flat logarithmic connection $\nabla\colon \mathcal{E} \to \frac{1}{z}\Omega^1_X(\mathcal{E})$ on $\Delta$. By continuous extension, $Q$ induces a pairing $\mathcal{E} \times \overline{\mathcal{E}} \to \mathcal{C}^\infty$ which restricts over the point $0 \in \Delta$ to the canonical Hermitian pairing $Q_0\colon \mathcal{E}_0 \times \overline{\mathcal{E}}_0 \to \mathbb{C}$. Note that our $N$ differs from the one in [SS22] by a sign.

**Proposition 5.3.** *Let $\mathsf{F} \subset \mathcal{E}|_{\Delta^\times}$ be a Griffiths transversal finite filtration by holomorphic subbundles. Then the following are equivalent:*

1) *for $0 < |z| \ll 1$ the filtration $\mathsf{F}_z$ is a pure Hodge structure polarized by $Q_z$;*
2) *$\mathsf{F}$ continuously extends to $\mathcal{E}$ and its fibre over $0$ is a mixed Hodge structure polarized by $N$ and $Q_0$.*

*Proof.* 1) $\Rightarrow$ 2) is due to Schmid, see [CK89, (2.1)].
2) $\Rightarrow$ 1) is shown in [CK89, Thm. 2.8] even in the several variable case. □

**Remark 5.4.** There is a version of Proposition 5.3 with continuous dependence on parameters. We later need 2) $\Rightarrow$ 1) with parameters, i.e. if the family of holomorphic filtrations $(\mathsf{F}(s))_{s\in V}$ of $\mathcal{E}$ depends continuously on parameters $s \in V$ with $V \subset \mathbb{R}^m$ open, $0 \in V$, and for every $s \in V$ the filtration $\mathsf{F}_0(s)$ is a mixed Hodge structure on $\mathcal{E}_0$ polarized by $N$ and $Q_0$, then there exists $\epsilon > 0$ and a neighborhood $U \subset V$ of $0$ such $\mathsf{F}_z(s)$ is a pure Hodge structure polarized by $Q_z$ for $0 < |z| < \epsilon$ and $s \in U$, see the argument in [CK89, Sec. 4] which extends immediately to the parametrized situation.

**Remark 5.5.** Note that the classical literature, see [CK89], the degeneration of Hodge structure, and as a consequence Deligne's semi-simplicity theorem, are studied for $\mathbb{R}$-PVHS with quasi-unipotent monodromy at infinity. Deligne remarked that the theory generalizes to arbitrary $\mathbb{C}$-PVHS [Del87, 1.11]. Recently, this was documented in the one variable case in [SS22].

5.2. **Characterization of a polarized variation of $\mathbb{C}$-Hodge structure via the non-abelian Hodge correspondence.** Let $\overline{X}$ be a projective complex manifolds (with fixed ample line bundle). Let $D \subset \overline{X}$ be a simple normal crossings divisor with complement $X$. We recall following Simpson [Sim92, Lem. 4.1] and T. Mochizuki [Moc06, 10.1] a citerion for a simple complex local system $\mathbb{L}$ on $X$ to underly a $\mathbb{C}$-PVHS. Let $E$ be the $\mathcal{C}^\infty$-bundle underlying $\mathbb{L}$ with its flat connection $\nabla$.

T. Mochizuki and Jost–Zuo construct a tame, purely imaginary pluri-harmonic metric $h$ on $E$ which is unique up to a factor in $\mathbb{R}_{>0}$, see [Moc07, Part 5] and [Moc09]. This allows us to write $\nabla = \partial_E + \overline{\partial}_E + \theta + \theta^\dagger$. Here $\theta$ is a $(1,0)$-form with values in $End(E)$ with adjoint $\theta^\dagger$ with respect to $h$ and $\partial_E + \overline{\partial}_E$ is a connection compatible with $h$. The associated Higgs bundle $(E, \overline{\partial}_E, \theta)$ has a canonical parabolic structure at infinity such that $\theta$ has only simple poles and purely imaginary eigenvalues of its residues.

For $\lambda \in S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ one considers the flat bundle $(E, \nabla^\lambda)$ with associated local system $\mathbb{L}^\lambda$, where $\nabla^\lambda = \partial_E + \overline{\partial}_E + \lambda\theta + \overline{\lambda}\theta^\dagger$, see [Moc09, 2.2.1].

**Proposition 5.6.** *Fix $\lambda \in S^1$ which is not a root of unity. Then $\mathbb{L} \cong \mathbb{L}^\lambda$ if and only if $\mathbb{L}$ underlies a $\mathbb{C}$-PVHS.*

*Proof.* The Higgs bundle of $\mathbb{L}^\lambda$ is $(E, \overline{\partial}_E, \lambda\theta)$ with the same parabolic structure as the Higgs bundle of $\mathbb{L}$. So $\mathbb{L} \cong \mathbb{L}^\lambda$ if and only if there is an isomorphism of Higgs bundles

$$\phi\colon (E, \overline{\partial}_E, \theta) \xrightarrow{\sim} (E, \overline{\partial}_E, \lambda\theta)$$

compatible with the parabolic structure. Decomposing $E$ according to generalized eigenspaces of $\phi$ induces an $h$-orthogonal bundle decomposition $E = E_1 \oplus \cdots \oplus E_r$ with $\theta(E_i) \subset \Omega^1_X(E_{i-1})$, which itself corresponds to a $\mathbb{C}$-$\mathsf{PVHS}$ as in [Sim92, Lem. 4.1]. □

5.3. **The Mumford-Tate group of a polarizable variation of $\mathbb{Q}$-Hodge structure.** To a polarizable $\mathbb{Q}$-Hodge structure $(V, \mathsf{F})$ of weight $w$ one associates the *Mumford-Tate group* $\mathrm{MT}(V, \mathsf{F})$. It can be characterized as the smallest linear algebraic subgroup of $\mathrm{GL}(V)$ over $\mathbb{Q}$ the complex points of which comprise the maps $\phi_\lambda\colon V_\mathbb{C} \to V_\mathbb{C}$ defined by $\lambda^i\overline{\lambda}^j$ on $\mathsf{F}^i \cap \overline{\mathsf{F}}^j$, where $i + j = w$. In particular $\mathrm{MT}(V, \mathsf{F})$ is connected.

Let now $X$ be a quasi-projective complex manifold and let $(\mathbb{L}, \mathsf{F})$ be a polarizable variation of $\mathbb{Q}$-Hodge structure on $X$. We also assume that $\mathbb{L}$ underlies a $\mathbb{Z}$-local system, which however should not be necessary for most arguments in view of Remark 5.5.

**Proposition 5.7.** *The Mumford-Tate group* $\mathrm{MT}(\mathbb{L}_x, F_x)$ *normalizes the Zariski closure of the monodromy group* $\mathrm{Mo}(\mathbb{L}, x)$ *in* $\mathrm{GL}(\mathbb{L}_x)$.

A slightly weaker result has been observed in [And92, Sec. 5]. We argue in terms of Tannaka groups.

**Remark 5.8.** The Mumford-Tate group $\mathrm{MT}(V, \mathsf{F})$ and the group $\mathrm{Mo}(\mathbb{L}, x)$ are Tannaka groups in view of the following observation. Let $V$ be a $\mathbb{Q}$-vector space and $G \subset \mathrm{GL}(V_\mathbb{C})$ a abstract subgroup. There is an associated Tannaka category $\mathbf{T}$. Its objects are all subquotients of $V^{\otimes n} \otimes (V^\vee)^{\otimes m}$ preserved over $\mathbb{C}$ by the $G$-action and its morphisms are $\mathbb{Q}$-linear maps compatible with the $G$-action over $\mathbb{C}$. Then the Tannaka group $\mathrm{Tann}(\mathbf{T})$ with respect to the obvious fibre funtor over $\mathbb{Q}$ is the same as the smallest linear algebraic subgroup of $\mathrm{GL}(V)$ over $\mathbb{Q}$ such that its complexification contains $G$.

*Proof of Proposition 5.7.* Recall that polarizable variations of mixed $\mathbb{Q}$-Hodge structure form a Tannaka category for which every point $x \in X$ induces a fibre functor. We think of any polarizable pure variation of $\mathbb{Q}$-Hodge structure as an object of this Tannaka category. We denote by $\mathrm{Tann}((\mathbb{L}, \mathsf{F}), x) \subset \mathrm{GL}(\mathbb{L}_x)$ its Tannaka group with respect to the fibre functor at $x$. Note that $\mathrm{Mo}(\mathbb{L}, x)$ is the Tannaka group of $\mathbb{L}$ in the Tannaka category of $\mathbb{Q}$-local systems, and that by Remark 5.8 the Tannaka group of a polarizable $\mathbb{Q}$-Hodge structure is its Mumford-Tate group. The inclusion of Tannaka groups

$$\mathrm{Mo}(\mathbb{L}, x) \subset \mathrm{Tann}((\mathbb{L}, \mathsf{F}), x)$$

which corresponds to the functor of Tannaka categories defined by forgetting the Hodge filtration is normal by [dAE22, Cor. 1.2] and [EHS07, Thm. A.1]. The restriction to $x$ of a variation of Hodge structure induces an inclusion of Tannaka groups

$$\mathrm{MT}(\mathbb{L}_x, \mathsf{F}_x) \subset \mathrm{Tann}((\mathbb{L}, \mathsf{F}), x). \tag{5.1}$$

This finishes the proof.

□

We define the *Mumford-Tate group of the polarizable variation of* $\mathbb{Q}$*-Hodge structure* $(\mathbb{L}, \mathsf{F})$ at $x \in X$ as the product

$$\mathrm{MT}((\mathbb{L}, \mathsf{F}), x) = \mathrm{Mo}(\mathbb{L}, x)^0 \cdot \mathrm{MT}(\mathbb{L}_x, \mathsf{F}_x) \subset \mathrm{GL}(\mathbb{L}_x),$$

which is a connected subgroup as by Proposition 5.7 the second factor normalizes the first factor.

For two polarizable variations of $\mathbb{Q}$-Hodge structure $(\mathbb{L}_1, \mathsf{F}_1)$ and $(\mathbb{L}_2, \mathsf{F}_2)$ a $\mathbb{Q}$-linear map $\psi\colon \mathbb{L}_{1,x} \to \mathbb{L}_{2,x}$ induces a morphism of variations of Hodge structure if and only if $\psi$ is a morphism of Hodge structures at $x$ and if $\psi$ commutes with the action of $\pi_1(X, x)$, see Theorem 5.1(i). So by Remark 5.8 we deduce that $\mathrm{MT}((\mathbb{L}, \mathsf{F}), x)$ is the neutral component of

$$\mathrm{Tann}((\mathbb{L}, \mathsf{F}), x) = \mathrm{Mo}(\mathbb{L}, x) \cdot \mathrm{MT}(\mathbb{L}_x, \mathsf{F}_x) \subset \mathrm{GL}(\mathbb{L}_x).$$

As an isomorphism of two fibre functors induces an isomorphism of Tannaka groups this shows that the isomorphism $\mathrm{GL}(\mathbb{L}_x) \cong \mathrm{GL}(\mathbb{L}_y)$, induced by a path between $x, y \in X$, gives us an isomorphism $\mathrm{MT}((\mathbb{L}, \mathsf{F}), x) \cong \mathrm{MT}((\mathbb{L}, \mathsf{F}), y)$. In this sense *the Mumford-Tate groups at different points of* $X$ *of a variation of Hodge structure on* $X$ *form a local system*, i.e. up to isomorphism it is independent of the choice of the base point.

**Remark 5.9.** Assume that $\mathbb{L}$ underlies a $\mathbb{Z}$-local system. The set $\Sigma$ of points $x \in X$ with $\mathrm{MT}(\mathbb{L}_x, \mathsf{F}_x) \neq \mathrm{MT}((\mathbb{L}, \mathsf{F}), x)$, or equivalently with $\mathrm{Mo}(\mathbb{L}, x)^0 \not\subset \mathrm{MT}(\mathbb{L}_x, \mathsf{F}_x)$, is called the *Hodge-exceptional locus*. Its complement is called the *Hodge-generic locus*. By [CDK95] and [Del72, Prop. 7.5], $\Sigma$ is a countable union of proper closed algebraic subsets of $X$; for an o-minimal proof see [BKT20].

5.4. **Reminder on Simpson's non-abelian Hodge locus.** In this section we formulate a generalization of Simpson's results in [Sim97, Sec. 12] to good holomorphic maps $f\colon X \to S$ in the sense of Section 1. In contrast to Simpson we formulate the result only "on the Betti side" of the non-abelian Hodge correspondence.

Let $f\colon X \to S$ be a good holomorphic map. As recalled in Subsection 2.2, $f$ is a topological fibration, so the sheaf of pointed sets $R^1 f_* \mathrm{GL}_r(\mathbb{C})$ is a local system on $S$, with fibre $H^1(X_s, \mathrm{GL}_r(\mathbb{C}))$, the set of isomorphism classes of rank $r$ complex local systems on $X_s$. Its associated étale space is a covering $\epsilon : T \to S$ ([Bou16, I.86 Prop. 9]). We endow $T$ with the usual complex manifold structure. As a topolocial space $T$ can be constructed as

$$T = \big(\tilde{S} \times H^1(X_{s_0}, \mathrm{GL}_r(\mathbb{C}))\big)/\pi_1(S, s_0)$$

where $\tilde{S} \to S$ is a universal covering and the $\pi_1(S, s_0)$-action is diagonal, for a fixed base point $s_0 \in S$.

We write $[\mathbb{L}_s] \in H^1(X_s, \mathrm{GL}_r(\mathbb{C}))$ for the isomorphism class of a local system $\mathbb{L}_s$ on $X_s$. Simpson defines the *non-abelian Hodge locus* or *Noether-Lefschetz locus* $\mathrm{NL} = \mathrm{NL}(f, r)$ of rank $r$ as the subset of $T$ consisting of those $[\mathbb{L}_s]$ such that $\mathbb{L}_s$ underlies a $(\mathbb{Z})\mathbb{C}$-$\mathsf{PVHS}$ on $X_s$.

Simpson [Sim97, Thm. 12.1] proved the following theorem for $f$ proper. The extension to non-proper $f$ is explained in Appendix A.

**Theorem 5.10.** *For a good holomorphic map* $f\colon X \to S$ *the non-abelian Hodge locus* $\mathrm{NL}(f, r)$ *is a closed analytic subset of* $T$ *which is finite over* $S$.

In fact the finiteness part of Theorem 5.10 is due to Deligne [Del87].

## 6. Proof of the main results

6.1. **Setting.** Let $f\colon X \to S$ be a good holomorphic map which satisfies one of the conditions I) - III) from Section 1.

In case III) the existence of a continuous section implies that $\pi_2(X,s) \to \pi_2(S,s)$ is surjective, so by the exact sequence (2.1) the map $\pi_1(X_s,x) \to \pi_1(X,x)$ is injective. From Propositions 2.5 and 2.6 it follows that under the assumptions I) or II) the map $\pi_1(X_s,x) \to \pi_1(X,x)$ is injective as well. Furthermore, the map of profinite completions $\widehat{\pi_1(X_s,x)} \to \widehat{\pi_1(X,x)}$ is injective by using Proposition 2.2 in case III) and Propositions 2.5 and 2.6 in cases I) and II).

6.2. **Proof of Theorem 1.4 and Corollary 1.5.** Theorem 1.4 is a direct consequence of Theorem 4.1 by using the equivalence between representations of the fundamental group and local systems.

In order to deduce Corollary 1.5 one only has to observe that semi-simple representations of surface groups which have finite orbit under the mapping class group, or what is the same finite orbit under the fundamental of the moduli of those Riemann surfaces, have semi-simple algebraic monodromy groups. This follows from Lemma 4.2 and the fact that in rank one such a representation has finite monodromy, see e.g. [BKMS18, Thm. 1.1].

6.3. **Proof of Theorem 1.1.** The implications 3) $\Rightarrow$ 2) $\Rightarrow$ 1), 3) $\Rightarrow$ 4) and 4) $\Rightarrow$ 5) are clear.

We prove 1) $\Rightarrow$ 3). By Theorem 5.1(v) the algebraic monodromy group of $\mathbb{L}_s$ is semi-simple, so we can apply Theorem 1.4 to deduce the existence of an algebraically isomonodromic extension $\mathbb{L}$ of $\mathbb{L}_s$, which can be defined over $\mathbb{Z}$. The polarization $Q_s$ essentially extends uniquely to a perfect pairing $Q\colon \mathbb{L} \otimes \overline{\mathbb{L}} \to K$. In order to show that the Hodge filtration $\mathsf{F}_s$ extends uniquely to a filtration of $\mathbb{L}\otimes_{\mathbb{C}}\mathcal{O}_X$ by holomorphic subbundles $\mathsf{F}^i$ which make $(\mathbb{L}, \mathsf{F}, Q)$ a $K$-$\mathsf{PVHS}$ we may assume without loss of generality that $K = \mathbb{C}$. Let

$$\mathbb{L} \cong \bigoplus_i \mathbb{L}_i \otimes_{\mathbb{C}} V_i \tag{6.1}$$

be the canonical decomposition, where $\mathbb{L}_i$ are non-isomorphic simple local systems. By Lemma 4.9 we obtain that $\mathbb{L}_i|_{X_s}$ is simple for all indices $i$, after replacing $S$ by a finite étale covering. Then by Theorem 5.1(iv) there are $\mathbb{C}$-$\mathsf{PVHS}$ $(\mathbb{L}_i|_{X_s}, F_{(i),s}, Q_{(i),s})$ and $\mathbb{C}$-$\mathsf{PHS}$ $(V_i, F_{(i),V}, Q_{(i),V})$ such that $(\mathbb{L}_s, F_s, Q_s)$ is the direct sum of their tensor products via the isomorphism (6.1).

With the notation of Proposition 5.6 for any $\lambda \in S^1$ we essentially obtain that $\mathbb{L}_i^\lambda \cong \mathbb{L}_i$ as such an isomorphism exists after restricting to $X_s$ and as admissible extensions are essentially unique by Proposition 4.6. So by Proposition 5.6 we see that $\mathbb{L}_i$ underlies a $\mathbb{C}$-$\mathsf{PVHS}$ $(\mathbb{L}_i, F_{(i)}, Q_{(i)})$ which by Theorem 5.1(ii) can be chosen to extend $(\mathbb{L}_i|_{X_s}, F_{(i),s}, Q_{(i),s})$. We can finally endow $\mathbb{L}$ with the $\mathbb{C}$-$\mathsf{PVHS}$ induced via the isomorphism (6.1).

We finally prove 5) $\Rightarrow$ 1), the implication 4) $\Rightarrow$ 1) is analogous and we omit the details. Using Artin approximation [Art69] we can approximate the formal filtration $\hat{\mathsf{F}}$ by a formal filtration with the same fibre over $s$ but which extends to a (relative) Griffiths transversal filtration by subbundles $\mathsf{F}$ of the isomonodromic extension

$$(\mathcal{E}_\Delta, \nabla\colon \mathcal{E}_\Delta \to \Omega^1_{X_\Delta/\Delta}(\mathcal{E}_\Delta))$$

over $X_\Delta$, where $\Delta \subset S$ is a suitable contractible open neighborhood of $s$. Let $\mathbb{L}_\Delta$ be the extension of the local system $\mathbb{L}$ to $X_\Delta$, let similarly $Q_\Delta$ denote the extension of the polarization. As in Appendix B the set of $t \in \Delta$ such that $(\mathbb{L}_\Delta|_{X_t}, \mathsf{F}|_{X_t}, Q_\Delta|_{X_t})$ is a $\mathbb{C}$-$\mathsf{PVHS}$ is open in $\Delta$, here we need the properness of $f$. By the same reasoning as in the proof of Theorem 3.1 the connected component of the non-abelian Hodge locus $\mathrm{NL}(f,r)$ containing $[\mathbb{L}_s]$, see Subsection 5.4, is a finite étale covering of $S$. So the monodromy orbit of $[\mathbb{L}_s]$ is finite.

6.4. **Proof of Remark 1.3.** By the argument in Subsection 6.3 one can find an extension of variations of Hodge structure as in Theorem 1.1(3) which is algebraically isomonodromic, i.e. such that

$$\mathrm{Mo}(\mathbb{L}_s, x)^0 \xrightarrow{\sim} \mathrm{Mo}(\mathbb{L}, x)^0$$

is an isomorphism. Then by Subsection 5.3 we deduce an isomorphism of Mumford-Tate groups of variations

$$\mathrm{MT}((\mathbb{L}_s, F_s), x) \xrightarrow{\sim} \mathrm{MT}((\mathbb{L}, F), x).$$

This shows that the Hodge-generic locus of $(\mathbb{L}, F)$ intersected with $X_s$ coincides with the Hodge-generic locus of $(\mathbb{L}_s, F_s)$. But the latter is non-empty by Remark 5.9. This finishes the proof.

## Appendix A. Non-abelian Hodge loci (after Simpson)

In this appendix we explain how to generalize Simpson's proof of Theorem 5.10 from the proper case to good holomophic maps $f$. We assume in the following that $f\colon X \to S$ is good holomorphic, where $S$ can be an arbitrary complex manifold. As the statement is local on $S$, we can fix $s_0 \in S$ and allow ourselves to replace $S$ by an open neighborhood of $s_0$. We assume without loss of generality that $S$ is contractible.

Simpson observes that Deligne's proof of his finiteness theorem [Del87] immediately generalizes to the following proposition.

**Proposition A.1.** *After possibly shrinking $S$ around $s_0$ there are only finitely many leaves of the (trivial) fibration $T \to S$ which meet $\mathrm{NL}(f,r)$.*

In view of Proposition A.1 we can fix a $\mathbb{C}$-local system $\mathbb{L}$ on $X$ of rank $r$ which underlies a $\mathbb{Z}$-local system. Then to prove Theorem 5.10 it is sufficient to show that the set of $s \in S$ such that $\mathbb{L}_s$ underlies a $\mathbb{C}$-$\mathsf{PVHS}$ is closed analytic in $S$.

*A reduction.*

We reduce to $f$ of relative dimension one and $\mathbb{L}$ unipotent along $\overline{X}\setminus X$. The first condition can be achieved by choosing a sufficiently generic relative hyperplane section $Y \hookrightarrow X$ after shrinking $S$ around $s_0$. By Bertini, $Y \to S$ is then good holomorphic and Remark 1.2 tells us that for $\dim(X_s) > 1$

$$\mathbb{L}_s \text{ underlies a } \mathbb{C}\text{-}\mathsf{PVHS} \quad \Leftrightarrow \quad \mathbb{L}_s|_Y \text{ underlies a } \mathbb{C}\text{-}\mathsf{PVHS}$$

for any $s \in S$. After performing this Bertini argument successively we can assume without loss of generality that $f$ is of relative dimension one.

By Theorem 5.1(v) the monodromy of $\mathbb{L}$ along $\overline{X} \setminus X$ is quasi-unipotent. By pullback along a finite ramified covering of $\overline{X}$ we can reduce to the case in which this monodromy is unipotent and then conclude for the original $X$ using Proposition 5.2. This argument also allows us to assume that $\overline{X} \setminus X$ consists of images of finitely many disjoint sections of $\overline{X} \to S$.

*Fixing Hodge numbers.*

By Theorem 5.1(iv) we have to show that for each simple constituent $\mathbb{M}$ of $\mathbb{L}$ the set of $s \in S$ such that $\mathbb{M}_s$ underlies a $\mathbb{C}$-$\mathsf{PVHS}$ is closed analytic in $S$.

We fix such a simple constituent $\mathbb{M}$ and a Hermitian perfect pairing $Q \colon \mathbb{M} \times \overline{\mathbb{M}} \to \mathbb{C}$, which is unique up to a factor in $\mathbb{R}^\times$. In view of Theorem 5.1(ii) there are only finitely many numbers $h^i \geq 0$ ($i \in \mathbb{Z}$) which can occur up to shift as Hodge numbers of a $\mathbb{C}$-$\mathsf{PVHS}$ on $\mathbb{M}_s$ for some $s \in S$. In the following we fix such numbers $h^i$. For each component $E$ of $\overline{X} \setminus X$ we also fix numbers $h^i_{E,j} \geq 0$ for $|j| \leq r$ such that $h^i = \sum_j h^i_{E,j}$ for all $i \in \mathbb{Z}$. Clearly, also here there are only finitely many choices. Let $\mathrm{NL}_{\mathbb{M}}$ be the set of $s \in S$ such that $\mathbb{M}_s$ underlies a $\mathbb{C}$-$\mathsf{PVHS}$ with Hodge numbers $h^i$ and polarization $Q|_{X_s}$ and such that the $j$-th monodromy graded piece $\mathrm{gr}^{\mathsf{M}}_j$ of $\mathbb{M}_s[E]$ has Hodge numbers $h^i_{E,j}$ for all components $E$ of $\overline{X} \setminus X$ and $j \in \mathbb{Z}$. Here we use the notation for the monodromy filtration as in [**?**, 1.7.8].

We have now reduced the proof of Theorem 5.10 to the construction of a proper holomorphic map of analytic spaces $\psi^\circ \colon G^\circ \to S$ such that $\mathrm{NL}_{\mathbb{M}} = \mathrm{im}(\psi^\circ)$, since then we conclude by Remmert's proper mapping theorem. Let $\mathcal{E}$ be the Deligne extension of $\mathbb{M} \otimes_{\mathbb{C}} \mathcal{O}_X$ to a holomorphic bundle on $\overline{X}$.

Let $\mathrm{Gr} \to S$ be the projective map of analytic spaces, where Gr is the Grassmannian a point of which consists of $s \in S$ together with a filtration by holomorphic subbundles $\mathsf{F}^i \subset \mathcal{E}_s$ with $\mathrm{rank}(\mathsf{F}^i/\mathsf{F}^{i+1}) = h^i$ for all $i \in \mathbb{Z}$. Let $G \hookrightarrow \mathrm{Gr}$ be the closed analytic subset consisting of those filtrations on fibres of $f$ which satisfy Griffiths transversality $\nabla \mathsf{F}^i \subset \Omega^1_{X_s}(\mathsf{F}^{i-1})$ and such that for each point $\{x\} = E_s$ at infinity and for each $j \in \mathbb{Z}$ the Hodge numbers of the filtrations induced on the monodromy graded pieces satisfy $\mathrm{rank}(\mathrm{gr}^i_{\mathsf{F}} \mathrm{gr}^{\mathsf{M}}_j \mathcal{E}_x) = h^i_{E,j}$ for all $i$ and $j$.

**Claim A.2.** The set $G^\circ$ consisting of pairs $(s, \mathsf{F}) \in G$ which define a polarizable variation of $\mathbb{C}$-Hodge structure on $X_s$ is open in $G$.

*Proof of Claim A.2.* Note that the points $(s, \mathsf{F})$ of $G$ in the claim correspond to those filtrations $\mathsf{F}$ such that there exists a perfect Hermitian pairing $Q_s$ for which the Hodge-Riemann relations are satisfied at every point of $X_s$, see Subsection 5.1. For a variation of complex Hodge structure the Hodge-Riemann relations are an open condition at every point of $X$, so the claim is easy in case $f$ is proper as then an open neighborhood of a fibre $X_s$ of $f$ contains the preimage of an open neighborhood of $s$ in $S$.

When $f$ is not proper the argument is more complicated. Consider a sequence $(s(n), \mathsf{F}(n)) \in G$ converging to $(s, \mathsf{F}) \in G^\circ$. In the sequel we fix local coordinates for the divisors at infinity on $\overline{X}$ in order to be able to speak about the limiting Hodge structure. Then by Proposition 5.3 we see that $\mathsf{F}_x$ for $\{x\} = E_s$ is the limiting mixed Hodge structure polarized by the induced pairing $Q_x \colon \mathcal{E}_x \times \overline{\mathcal{E}}_x \to \mathbb{C}$ and the nilpotent endomorphism $N_x$ on $\mathcal{E}_x$.

So by the openness of the mixed Hodge locus in the Grassmannian for $n \gg 0$ also $\mathsf{F}(n)_{E_{s(n)}}$ is a polarized mixed Hodge structure with respect to the polarization induced by $N$ and $Q$. Then by Proposition 5.3 and Remark 5.4 there is an open neighborhood $U_x$ of $x \in \overline{X}$ such that $\mathsf{F}(n)$ is a polarizable pure variation of Hodge structure on $U_x \cap X_{s(n)}$ for $n \gg 0$.

A similar open neighborhood $U_x$ of $x \in X$ exists for each $x \in X_s$ as explained at the beginning of the proof. By compactness of $\overline{X}_s$ there is finite union $U$ of such $U_x$ with $\overline{X}_s \subset U$. Then for $n \gg 0$ we obtain that $\overline{X}_{s(n)} \subset U$ and that $\mathsf{F}(n)$ is a polarizable pure Hodge structure at each point of $X_{s(n)}$. □

**Claim A.3.** The map $G^\circ \to S$ is proper.

In the proof of Claim A.3 we need the following continuity result for a fibrewise pluri-harmonic metric due to T. Mochizuki [Moc09, Prop. 4.2]. We note that $\det \mathbb{M}$ is finite by Theorem 5.1(v), so we can fix a positive flat Hermitian metric $h^{\det}$ on $\det \mathbb{M}$, which is automatically pluri-harmonic.

**Proposition A.4.** *Let* $h\colon (\mathbb{M} \otimes C^\infty_X) \times (\overline{\mathbb{M} \otimes C^\infty_X}) \to C^\infty_X$ *be the unique fibrewise pluri-harmonic, tame and purely imaginary metric with determinant* $h^{\det}$*. Then* $h$ *is continuous.*

*Proof of Claim A.3.* Consider a sequence $(s(n), \mathsf{F}(n)) \in G^\circ$ such that $s(n)$ converges to $s \in S$. We have to show that a subsequence converges in $G^\circ$. By the properness of the Grassmannian we can assume that $(s(n), \mathsf{F}(n))$ converges to $(s, \mathsf{F}) \in G$. We have to prove that $(s, \mathsf{F}) \in G^\circ$, i.e. we have to check the Hodge-Riemann relations at all points $x \in X_s$. Note that the orthogonal complement $(\mathsf{F}^i_x)^\perp$ of $\mathsf{F}^i_x$ in $\mathbb{L}_x$ with respect to $Q_x$ and with respect to the fibrewise pluri-harmonic metric $h$ from Proposition A.4 coincide by continuity since they coincide for $\mathsf{F}(n)^i$ by the description of the metric $h$ as a Hodge metric in terms of $Q$ on the fibre over $s(n)$. As $h$ is positive definite we deduce that $\mathbb{L}_x = \mathsf{F}^i_x \oplus (\mathsf{F}^i_x)^\perp$. In other words $Q_x$ restricted to $\mathsf{F}^i$ is a perfect pairing. For the same reason $(-1)^i Q_x$ restricted to $\mathsf{F}^i_x \cap (\mathsf{F}^{i+1}_x)^\perp$ is positive definite as by continuity it agrees with $h_x$ there up to a positive factor. □

## Appendix B. Extensions of representations of profinite groups (after Simpson)

Let $E$ be a finite field extension of $\mathbb{Q}_\ell$. Let $H \subset G$ be a closed normal subgroup of a profinite group $G$. Set $\Gamma = G/H$. Let $\rho_H\colon H \to \mathrm{GL}_r(E)$ be a continuous representation and denote by $[\rho_H]$ the isomorphism class of $\rho_H$. Then $\Gamma$ acts on the set of isomorphism classes of continuous representations $H \to \mathrm{GL}_r(E)$ by conjugation.

**Lemma B.1.** *If* $\rho_H$ *is semi-simple then the stabilizer group* $\Gamma_{[\rho_H]}$ *is closed in* $\Gamma$.

For a proof of the lemma see for example [Zoc24, Sect. 5] in which the author endows the space of isomorphism classes of $\ell$-adic representations of rank $r$ with a uniform $\ell$-adic topology on which the action of $\Gamma$ is continuous.

We say that a property related to $G$ holds *essentially* if it holds after replacing $G$ by an open subgroup $G'$ with $H \subset G'$.

**Proposition B.2.** *If $\rho_H$ is absolutely simple,* $\det(\rho_H)$ *is finite and* $\Gamma$ *fixes* $[\rho_H]$*, there essentially exists a unique extension of $\rho_H$ to a continuous representation $\rho\colon G \to \mathrm{GL}_r(E)$ with finite determinant.*

This result originates in the work of Simpson [Sim92, Proof of Thm. 4]. Variants of it can be found in [EG18, Proof of Prop. 3.1] and [Lit21, Prop. 3.1.1]. We are not aware of a reference for Proposition B.2 in the literature, so we sketch an argument the details of which can be found in Hugo Zock's master thesis [Zoc24].

*Proof sketch.* For $M \in \mathfrak{gl}_n(E)$ and $P = [W] \in \mathrm{PGL}_r(E)$ we denote the conjugation action by ${}^{P}M = WMW^{-1}$. For each $g \in G$ there exists a unique $P_g \in \mathrm{PGL}_r(E)$ with ${}^{P_g}\rho_H(h) = \rho_H(ghg^{-1})$. We claim that the map $g \mapsto P_g$ is continuous. In order to check this we have to show that for any $M \in \mathfrak{gl}_n(E)$, the map $g \mapsto {}^{P_g}M$ is continuous. As $\rho_H$ is absolutely simple, every such matrix $M$ is an $E$-linear combination of matrices $\rho_H(h)$ for varying $h \in H$, so it is sufficient to observe that for any $h \in H$ the map

$$g \mapsto {}^{P_g}\rho_H(h) = \rho_H(ghg^{-1})$$

is continuous. So we see that we obtain a continuous extension

$$\mathbb{P}\rho\colon G \to \mathrm{PGL}_r(E) \quad \text{of} \quad \mathbb{P}\rho_H\colon H \to \mathrm{PGL}_r(E).$$

In order to prove essential uniqueness in Proposition B.2 let $\rho$ and $\tilde{\rho}$ be two lifts of $\mathbb{P}\rho$ to two continuous representations of $G$ with finite determinants and which agree on $H$. Then $\tau(g) = \rho(g)\tilde{\rho}(g)^{-1} \in K^\times$ defines a continuous homomorphism $\tau\colon \Gamma \to K^\times$ with finite image. We deduce $\rho = \tilde{\rho}$ after restricting to the kernel of $\tau$. This proves the unicity part. See Proposition 4.6 for a similar method.

For the existence part in Proposition B.2 we first check that it is sufficient to construct $\rho$ with values in $\mathrm{GL}_r(\tilde{E})$, where $\tilde{E}$ is a finite Galois extension of $E$. Indeed, then for any $\eta \in \mathrm{Gal}(\tilde{E}/E)$ the extensions $\eta \cdot \rho$ and $\rho$ are essentially equal by the uniqueness part. So, as $\mathrm{Gal}(\tilde{E}/E)$ is finite, we can replace $G$ by an open subgroup containing $H$ so they are equal for all $\eta$, i.e. $\rho$ takes values in $\mathrm{GL}_r(E)$.

Consider a finite extension $\tilde{E}$ of $E$ in which every element of $E^\times$ is an $r$-power of an element of $\tilde{E}^\times$. Replace $E$ by $\tilde{E}$. Then $\mathbb{P}\rho$ has values in the closed subgroup $\mathrm{PSL}_r(E)$ of $\mathrm{PGL}_r(E)$. Consider the strict exact sequence of topological groups

$$1 \to \mu_r(E) \to \mathrm{SL}_r(E) \to \mathrm{PSL}_r(E) \to 1$$

in which the surjection on the right has a continuous splitting in topological spaces. So the standard cocycle argument, see [Zoc24, App. B], gives an exact sequence

$$H^1_{\mathrm{cont}}(G, \mathrm{SL}_r(E)) \to H^1_{\mathrm{cont}}(G, \mathrm{PSL}_r(E)) \xrightarrow{\mathrm{Ob}} H^2_{\mathrm{cont}}(G, \mu_r(E)).$$

As the restriction of $\mathrm{Ob}(\mathbb{P}\rho)$ to $H$ vanishes, the Hochschild-Serre spectral sequence implies that $\mathrm{Ob}(\mathbb{P}\rho)$ lies in a subgroup of $H^2_{\mathrm{cont}}(G, \mu_r(E))$ involving $H^a(\Gamma, H^{2-a}(H, \mu_r(E)))$ for $a = 1, 2$ only. Those two groups die on a finite index subgrouop of $\Gamma$. So we can assume that $\mathrm{Ob}(\mathbb{P}\rho) = 0$.

So we can lift $\mathbb{P}\rho$ to a continuous representation $\check{\rho}\colon G \to \mathrm{SL}_r(E)$. Now

$$\tau\colon H \to E^\times, \quad \tau(h) = \rho_H(h)\check{\rho}(h)^{-1} \in E^\times$$

is a continuous homomorphism with values in $\mu_N(E)$ for some $N > 0$ by the condition on finite determinants. Note that $\tau$ is fixed by the $G$-action by conjugation. Using the exact inflation-restriction sequence

$$H^1_{\text{cont}}(G, \mu_N(E)) \to H^1_{\text{cont}}(H, \mu_N(E))^G \to H^2_{\text{cont}}(\Gamma, \mu_N(E))$$

we see that after replacing again $G$ by an open subgroup $G'$ with $H \subset G'$ we can lift $\tau$ to a continuous homomorphism $\check{\tau}\colon G \to \mu_N(E)$. Finally, we deduce that $\rho = \check{\tau}\check{\rho}\colon G \to \mathrm{GL}_r(E)$ is the requested continuous extension of $\rho_H$. □

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