Some Fundamental Groups in Arithmetic Geometry

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Acknowledgements

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Thank you to

Moritz Kerz, Lars Kindler, Takeshi Saito and Atsushi Shiho for their constructive comments and remarks on the slides.

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Finiteness theorem over $\mathbb{C}$

Theorem (Deligne '87)

$X/\mathbb{C}$ smooth connected variety, $r \in \mathbb{N}_{>0}$ given. Then there are finitely many rank r $\mathbb{Q}$-local systems which are direct factors of $\mathbb{Q}$-variations of polarisable pure Hodge structures of a given weight, definable over $\mathbb{Z}$.

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Example (Faltings' finiteness of abelian schemes on X, '83)

In general, this is a generalisation the version over $\mathbb{C}$ of Faltings' theorem.

Finiteness theorem over $\mathbb{F}_q$

Theorem (Deligne '12)

$X/\mathbb{F}_q$ smooth quasi-projective variety, $r \in \mathbb{N}_{>0}$ given; $D \subset \bar{X}$ an effective Cartier divisor of a normal compactification with support $\bar{X} \setminus X$, and $r \in \mathbb{N}_{>0}$ given. Then there are finitely many irreducible Weil (resp. étale) rank r lisse $\bar{\mathbb{Q}}_\ell$-sheaves with ramification bounded by D, up to twist with Weil (resp. étale) characters of $\mathbb{F}_q$. The number does not depend on the choice of ℓ.

Corollary (Deligne '07, Deligne's conjecture, Weil II, 1.2.10)

Given an étale lisse $\bar{\mathbb{Q}}_\ell$-sheaf V with finite determinant, the subfield of $\bar{\mathbb{Q}}_\ell$ spanned by the EV of the Frobenii F_x at closed points $x \in |X|$ acting on $V_{\bar{x}}$ is a number field.

Lefschetz theorem over $\mathbb{C}$

Theorem over $\mathbb{C}$ is in fact a theorem on X of dimension 1: fixing a good compactification $\bar{X} \supset X$, with a s.n.c.d. at infinity, then a curve $\bar{C}$, complete intersection of ample divisors in $\bar{X}$ in good position, fulfils the Lefschetz theorem

$$\pi_1^{\mathrm{top}}(C := X \cap \bar{C}) \twoheadrightarrow \pi_1^{\mathrm{top}}(X).$$

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Yet one has:

Tame Lefschetz theorem in char. $p > 0$

Theorem (Drinfeld '11)

Let $\bar{X} \supset X$ be a projective normal compactification of X smooth over a field k, $\Sigma \subset \bar{X}$ be closed of codimension ≥ 2 such that $(\bar{X} \setminus \Sigma)$ and $(\bar{X} \setminus \Sigma) \cap (\bar{X} \setminus X)$ are smooth, $\bar{C} \subset \bar{X} \setminus \Sigma$ be a smooth projective curve, complete intersection of ample divisors, meeting $\bar{X} \setminus X$ transversally. Then

$$\pi_1^t(C = \bar{C} \cap X) \twoheadrightarrow \pi_1^t(X).$$

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$$\pi_1^t(C = \bar{C} \cap X) \twoheadrightarrow \pi_1^t(X).$$

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Proof.

Bertini to get that restriction to C of connected finite étale cover of X is connected, tameness and transversality to keep smoothness, thus irreducibility. □

Tame Lefschetz theorem in char. $p > 0$ II

If $\bar{X} \setminus X$ is a s.n.c.d. compactification, Kindler *enhances* the theorem: if $\bar{S} \subset \bar{X}$ is a smooth projective surface, complete intersection of divisors in good position, then

$$\pi_1^t(S = \bar{S} \cap X) \xrightarrow{\cong} \pi_1^t(X).$$

Wild Lefschetz theorems in char. $p > 0$

Theorem (Wiesend '06, Drinfeld '11)

Over X quasi-projective smooth over $\mathbb{F}_q$, with $S \subset |X|$ finite:

1) *let V be an irreducible $\bar{\mathbb{Q}}_\ell$-Weil or -étale lisse sheaf, then there is a smooth curve $C \to X$ with $S \subset |C|$, such that $V|_C$ is irreducible;*
2) *let $H \subset \pi_1(X)$ be an open normal subgroup, then there is a smooth curve $C \to X$ with $S \subset |C|$, such that $\pi_1(C) \twoheadrightarrow \pi_1(X)/H$.*

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Proof.

Uses Hilbert irreducibility à la Wiesend. □

Corollaries of the wild Lefschetz theorems: weights and companions

Corollary (Drinfeld '11, Deligne's conjecture in Weil II, 1.2.10)

1) *if* $\det(V)$ *is torsion, then* V *has weight* 0*;*
2) *if* V *is an irreducible Weil lisse* $\bar{\mathbb{Q}}_\ell$*-sheaf with determinant of finite order, and* $\sigma \in \mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$*, there is an irreducible Weil lisse* $\bar{\mathbb{Q}}_\ell$*-sheaf* V_σ*, called the* σ-companion *of* V*, with determinant of finite order, such that the characteristic polynomials* $f_V, f_{V_\sigma} \in \bar{\mathbb{Q}}_\ell[t]$ *of the local Frobenii* F_x *satisfy* $f_{V_\sigma} = \sigma(f_V)$*.*

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Proof.

Reduce the problem to curves. Then consequence of Lafforgue's Langlands duality:

1) existence of weights on curves;
2) existence of companions on curves.

□

Abelian Lefschetz theorem

Theorem (Kerz-S.Saito '14)

Let X be a smooth quasi-projective variety over a perfect field k, let $X \subset \bar{X}$ be a projective s.n.c.d. compactification, D be an effective divisor with support in $\bar{X} \setminus X$. Define $\pi_1^{\mathrm{ab}}(X, D)$ by the condition that a character $\chi : \pi_1(X) \to \mathbb{Q}/\mathbb{Z}$ factors through $\pi_1^{\mathrm{ab}}(X, D)$ iff the Artin conductor of χ pulled-back to any curve $C \to X$ is bounded by the pull-back of D via $\bar{C} \to \bar{X}$. Then Lefschetz holds: for $i : \bar{Y} \subset \bar{X}$ very very ample and in good position w.r.t. $\bar{X} \setminus X$, one has:

$$i_* : \pi_1^{\mathrm{ab}}(Y, \bar{Y} \cap D) \to \pi_1^{\mathrm{ab}}(X, D)$$

is an isomorphism if dim $Y \geq 2$, surjective if dim $Y = 1$.

Corollary of Abelian Lefschetz theorem: abelian finiteness over $\mathbb{F}_q$

Corollary (Raskind '92, this formulation by Kerz-S.Saito '14)

$k = \mathbb{F}_q$, then $\mathrm{Ker}\big(\pi_1^{\mathrm{ab}}(X, D) \to \pi_1^{\mathrm{ab}}(k)\big)$ is finite. (So in particular, this implies Deligne's finiteness for sums of rank 1 lisse sheaves).

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Proof.

Reduce to curves via de Jong's alterations (in the more general case $\bar{X}$ is a normal compactification) plus the Theorem and apply then CFT. □

Right fundamental group with ramification bounded by a $\mathbb{Q}_{\geq 0}$-divisor

Questions

One has the notion of a lisse étale $\bar{\mathbb{Q}}_\ell$-sheaf $\pi_1(X) \to \mathrm{Aut}(V)$ with ramification bounded by D, a positive $\mathbb{Q}$-divisor (Hu-Yang: does not need a good compactification; as for Drinfeld's Lefschetz theorem for $\pi_1^t(X)$). How does one define a quotient $\pi_1(X) \twoheadrightarrow \pi_1(X, D)$ generalising $\pi_1^{\mathrm{ab}}(X, D)$? Then one could ask for a Lefschetz theorem $\pi_1(C, D_C) \twoheadrightarrow \pi_1(X, D)$ for a suitable curve C which would reflect Deligne's finiteness theorem.

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X smooth geometrically connected over a perfect field k, $W := W(k)$ the ring of Witt vectors, $K = \mathrm{Frac}(W)$ its field of fractions.

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One defines the crystalline sites X/W_n as PD-thickenings $(U \hookrightarrow T/W_n, \delta)$, coverings coming from $U \subset X$ Zariski; X/W as limit.

The various categories of isocrystals under consideration II

• category of *crystals* (i.e. sheaves of $\mathcal{O}_{X/W}$-modules of finite presentation, with transition maps which are isomorphisms) $\mathrm{Crys}(X/W)$, which is W-linear;

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• $\mathbb{Q}$-linearisation $\mathrm{Crys}(X/W)_{\mathbb{Q}}$=category of *isocrystals*, which is K-linear, tannakian;

• absolute Frobenius F acts on $\mathrm{Crys}(X/W)_{\mathbb{Q}}$;

• largest full subcategory on which every object is F^∞-divisible is $\mathrm{Conv}(X/K) \subset \mathrm{Crys}(X/W)_{\mathbb{Q}}$, the K-tannakian subcategory of *convergent* isocrystals (Berthelot-Ogus); (Ogus defines the site of enlargements from X/W, then convergent isocrystals are crystals of $\mathcal{O}_{X/K}$-modules of finite presentation).

The various categories of isocrystals under consideration III

- $F-\mathrm{Conv}(X/K) \to \mathrm{Conv}(X/K),\ (\mathcal{E}, \Phi : F^*\mathcal{E} \xrightarrow{\cong} \mathcal{E}) \mapsto \mathcal{E};$

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- $F - \mathrm{Overconv}(X/K) \xrightarrow{\text{fully faithful Kedlaya}} F - \mathrm{Conv}(X/K)$;
- $F - \mathrm{Overconv}(X/K)$ consists of those E which have "unipotent local monodromy" after alteration (Kedlaya);

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• Grothendieck over $\mathbb{F}_q$: lisse $\bar{\mathbb{Q}}_\ell$-sheaves have quasi-unipotent local monodromies (action of local inertia $\mathbb{Z}_\ell(1)$).

The various categories of isocrystals under consideration V

• Kedlaya over k (not necessarily perfect): $\mathcal{E} \in F - \mathrm{Overconv}(X/K)$ has 'unipotent monodromy' (in a suitable sense) at infinity after an alteration (uses André-Kedlaya-Mebkhout local result).

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There are blow-ups at infinity: analog to resolution of turning points over $\mathbb{C}$ (Kedlaya/T. Mochizuki).

• From the definition:

On X proper, $F - \mathrm{Overconv}(X/K) = F - \mathrm{Conv}(X/K)$.

The various categories of isocrystals under consideration IV

• Over $\mathbb{F}_q$, $q = p^s$, define $F_{\mathbb{F}_q} = F^s - \mathrm{Overconv}(X/K)$, so $K = \mathrm{Frac} W(\mathbb{F}_q)$-linear; *abuse of notations* $F - \mathrm{Overconv}(X/K)$.

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- L-linearisation, for $K \subset L \subset \bar{\mathbb{Q}}_p$, $L \to \bar{\mathbb{Q}}_p$, defines the category $F - \mathrm{Overconv}(X/K)_{\bar{\mathbb{Q}}_p}$.

Analogy $\mathbb{C} \leftrightarrow \mathbb{F}_q, \ell$-adic, expected analogy $/\mathbb{F}_q$, p-adic

Irreducible objects in $F - \mathrm{Overconv}(X/K)_{\bar{\mathbb{Q}}_p}$ with finite determinant;

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are analog over $\mathbb{C}$ to irreducible $\mathbb{Q}$-variations of polarisable pure Hodge structures of pure weight definable over $\mathbb{Z}$.

Crystalline version (*petits camarades cristallins*) on curves

Theorem (Abe, Crystalline version of Lafforgue's theorem '13)

Let X be a smooth curve over $\mathbb{F}_q$. Then

1) *an irreducible overconvergent $\bar{\mathbb{Q}}_p$-F-isocrystal with finite determinant is ι-pure of weight 0;*
2) *an irreducible lisse $\bar{\mathbb{Q}}_\ell$- étale sheaf with finite determinant has an overconvergent $\bar{\mathbb{Q}}_p$-F–isocrystal companion and vice-versa.*

Crystalline version (*petits camarades cristallins*) on higher dimensional varieties

No wild Lefschetz theorem for F-overconvergent isocrystals

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1) one does not know whether irreducible F-isocrystals with finite determinant are ι-pure of weight 0 on X smooth quasi-projective variety over $\mathbb{F}_q$;

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1) one does not know whether irreducible F-isocrystals with finite determinant are ι-pure of weight 0 on X smooth quasi-projective variety over $\mathbb{F}_q$;
2) a fortiori, one does not have a number field capturing the EV of $F_{\mathbb{F}_{q(x)}}$ acting on the stalks at closed points $\mathrm{Spec}\ \mathbb{F}_{q(x)}$;

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3) nor does one have a crystalline version of Deligne's finiteness theorem.

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1) one does not know whether irreducible F-isocrystals with finite determinant are ι-pure of weight 0 on X smooth quasi-projective variety over $\mathbb{F}_q$;
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3) nor does one have a crystalline version of Deligne's finiteness theorem.

So: no higher dimensional generalisation of Drinfeld/Deligne.

Abe's Čebotarev theorem

Theorem (Abe '13)

On X quasi-projective smooth over $\mathbb{F}_q$, ι-pure (or mixed, $\iota : \bar{\mathbb{Q}}_p \cong \mathbb{C}$) semi-simple objects in $F - \mathrm{Overconv}(X/K)$ are determined by their local EV at closed points.

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Weaker analogies

Rather than considering analogies between *some* irreducible complex local systems ('motivic' ones) with *some* lisse $\bar{\mathbb{Q}}_\ell$- sheaves (irreducible with finite determinant) over $\mathbb{F}_q$, and with *some* overconvergent $\bar{\mathbb{Q}}_p$- F isocrystals (irreducible with finite determinant), one can raise the

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Question

what is the analog of complex local systems on X over $\mathbb{C}$ for X over a perfect field of characteristic $p > 0$?

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X smooth over a char. 0 field k;

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X_∞: $U \hookrightarrow T$ infinitesimal thickening of a Zariski open U; coverings from the Us.

$\{\text{finitely presented crystals on } X_\infty\} = \{(E, \nabla)\}$, E coherent sheaf and ∇ flat connection (thus E is locally free);

k-linear category (assume here $k =$ field of constants of X, i.e. X geometrically connected over k);

Malčev-Grothendieck's theorem

Theorem (Malčev '40-Grothendieck '70)

X smooth over $\mathbb{C}$; then $\pi_1^{\text{ét}}(X) = \{1\}$ implies there are no non-trivial crystals in the infinitesimal site (with regular singularities at infinity in case X is not projective).

Proof.

Use Riemann-Hilbert correspondence to translate to finite dimensional complex local systems.

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Proof.

Use Riemann-Hilbert correspondence to translate to finite dimensional complex local systems.

Then $\pi_1^{\text{top}}(X(\mathbb{C}))$ is an abstract group *of finite type*, so $\rho : \pi_1^{\text{top}}(X(\mathbb{C})) \to GL(r, \mathbb{C})$ factors through $\rho_A : \pi_1^{\text{top}}(X(\mathbb{C})) \to GL(r, A)$, $A/\mathbb{Z}$ of finite type, and $\rho = 1$ iff $\rho_A = 1$ iff $\rho_a : \pi_1^{\text{top}}(X(\mathbb{C})) \to GL(r, \kappa(a))$ $\forall$ closed point $a \in \mathrm{Spec}(A)$. □

Conservativity

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More modest question: analogs in char. $p > 0$ of this conservativity theorem? (*Terminology 'conservativity' borrowed from Ayoub's work*).

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As in char. 0: X smooth over a char. $p > 0$ field k; X_∞: $U \hookrightarrow T$ infinitesimal thickening of a Zariski open U; coverings from the Us.

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As in char. 0: X smooth over a char. $p > 0$ field k; X_∞: $U \hookrightarrow T$ infinitesimal thickening of a Zariski open U; coverings from the Us.

$$\{\text{finitely presented crystals on } X_\infty\} = \{\mathcal{O}_X - \text{coherent } \mathcal{D}_X - \text{modules}\}$$

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As in char. 0: X smooth over a char. $p > 0$ field k; X_∞: $U \hookrightarrow T$ infinitesimal thickening of a Zariski open U; coverings from the Us.

$\{\text{finitely presented crystals on } X_\infty\} = \{\mathcal{O}_X - \text{coherent } \mathcal{D}_X - \text{modules}\}$ $= \{F - \text{d.c.s.}\}$ (divided coherent sheaves) (Cartier isomorphism, Katz' theorem).

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k-linear category (assume here k = field of constants of X, i.e. X geometrically connected over k).

Gieseker's conjecture

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On X projective smooth over $k = \bar{k}$ of char. $p > 0$, $\pi_1^{\text{ét}}(X) = \{1\}$ implies that there are no non-trivial crystals in the infinitesimal site.

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Theorem (E-Mehta '10)

Conjecture has a positive answer.

Variants of Gieseker's conjecture

• X not proper: theory of *regular singular* crystals in the infinitesimal site developed by Kindler ('13), so that for those with finite monodromy, it coincides with the notion of *tame* quotient of $\pi_1^{\text{ét}}(X)$.

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• So far no extension of the conservativity theorem, except for the tame abelian quotient (Kindler '13) and for $X =$ smooth locus of a normal projective variety and $k = \bar{\mathbb{F}}_q$ (E-Srinivas '14, using an improvement of Grothendieck's LEF theorem by Bost, '14).

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• Hrushovsky's theorem then guarantees the existence of a Frobenius invariant vector bundle on a specialization of X over $\bar{\mathbb{F}}_p$, which yields a non-trivial finite étale cover of this one.

Various crystalline sites and (iso)crystal categories

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$\mathrm{Conv}(X/K) \hookrightarrow \mathrm{Crys}(X/W)_{\mathbb{Q}}$.

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Partial positive answer.

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In the sequel, we report on it, raising a few questions on the way.

Abelian case

$\pi_1^{\text{ét}}(X \otimes_k \bar{k}) = \{1\}$ implies $H^1(X, \mathcal{O}^\times_{X/W}) = 0$, thus rank 1 locally free crystals, and thus isocrystals, are trivial,

as well as $H^1(X/W) = 0$, thus successive extensions of trivial rank 1 crystals, thus isocrystals, are trivial.

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Thus the conjecture essentially predicts a relation between the "non-abelian" part of $\pi_1^{\text{ét}}(X)$ and irreducible isocrystals of higher rank.

At least when $p \geq 3$ it is so; for $p = 2$ those statements are less direct and follow from the whole proof.

Gauß-Manin F-convergent isocrystal

Theorem (E-Shiho '15)

Let $f : Y \to X$ be a smooth projective morphism over X smooth projective over k perfect. If $\pi_1^{\text{ét}}(X \otimes_k \bar{k}) = \{1\}$, then the F-convergent isocrystal $R^i f_*$ is trivial in $\mathrm{Conv}(X/K)$.

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Model of Proof

Assume f was an abelian scheme and $k = \mathbb{F}_q$. May assume X has a rational point x_0. Then (argument of *Faltings*): $\pi_1^{\text{ét}}(X)$ acts on $R^i f_* \mathbb{Q}_\ell$ via $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_q)$, thus by the Honda-Tate theorem, all geometric fibres of f are isogeneous. Thus for all closed points $x \in |X|$, $H^1(Y_x/\mathrm{Frac}\,W(k(x))) = H^1(Y_{x_0}/K) \otimes_K \mathrm{Frac}\,W(k(x))$, thus the isocrystal $R^i f_*$ is trivial in $\mathrm{Conv}(X/K)$.

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Apply base change to get it over k. Yields triviality of the semi-simplification of $R^i f_*$ in $\mathrm{Conv}(X/K)$ over k.

Thus $R^i f_* \in \mathrm{Conv}(X/K) \subset \mathrm{Crys}(X/W)_{\mathbb{Q}}$ is trivial, as we already saw that there are no non-trivial extensions. □

From now on, we discuss the general case.

Existence of lattices

Lemma

X smooth over k perfect, $\mathcal{E} \in \mathrm{Crys}(X/W)_{\mathbb{Q}}$, there is a p-torsion-free $E \in \mathrm{Crys}(X/W)$ with $E_{\mathbb{Q}} = \mathcal{E}$, called a lattice.

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Proof.

Given any $E \in \mathrm{Crys}(X/W)$, with $E_{\mathbb{Q}} = \mathcal{E}$, the surjective maps $E/\mathrm{Ker}(p^{n+1}) \twoheadrightarrow E/\mathrm{Ker}(p^n)$ stabilise, as one sees locally on finitely many open affines U, as then $\mathrm{Crys}(U/W) \cong MIC(\hat{U}_W/W)^{qn}$, the quasi-nilpotent flat connections on a formal lift. □

Locally free lattices

$\mathcal{E}$ is said to be *locally free* if it has a locally free lattice E, so $E_{\mathbb{Q}} = \mathcal{E}$, that is equivalently if E_X, the value of E on $X \hookrightarrow X$, viewed in $\mathrm{Coh}(X)$, is locally free.

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Question

Are all $\mathcal{E} \in \mathrm{Crys}(X/W)_{\mathbb{Q}}$ locally free?

A positive answer would ease the understanding of de Jong's conjecture.

Crystalline Chern classes of the value on X of lattices

Theorem (E-Shiho '15)

Let $E \in \mathrm{Crys}(X/W)$ *be a lattice.*

1) *If* E *is locally free, then* $0 = c_{i,\mathrm{crys}}(E_X) \in H^{2i}(X/W)$, $i \geq 1$.
2) *If* $E_{\mathbb{Q}} \in \mathrm{Conv}(X/K)$, *then* $0 = c_{i,\mathrm{crys}}(E_X) \in H^{2i}(X/K)$, $i \geq 1$.

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Modified splitting principle: on $X \subset D \to \mathbb{P}_W$ PD-hull, one considers the quotient $\Omega^\bullet_D \to \bar{\Omega}^\bullet_D$ of DGAs defined by $dx^{[n]} = x^{[n-1]}dx$. This defines the quotient $\Omega^\bullet_{\mathbb{P}(E_D)} \to \bar{\Omega}^\bullet_{\mathbb{P}(E_D)}$ of DGAs by moding out by the 'same' kernel, where E_D is the value of E on $X \hookrightarrow D$. Let $\pi : \mathbb{P}(E_D) \to D$ be the principal bundle.

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Proof of the vanishing of the crystalline Chern classes of the value on X of lattices II

Proof.

Equating $\mathcal{O}$ and $\overline{dR}$-cohomology: for X smooth, define $D_\bullet$ to be the simplicial scheme defined by D_n =PD-hull of the diagonal in $\mathbb{P}_W^{\times(n+1)}$. Then $H^i(X/W) = H^i_{dR}(D_\bullet)(:= H^i(D_\bullet, \bar{\Omega}^\bullet_{D_\bullet})) = H^i_{\mathrm{Zar}}(D_\bullet, \mathcal{O})$.

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Proof of the vanishing of the crystalline Chern classes of the value on X of lattices III

Proof.

2) Show, the class of E_X in $K_0(X)$, where E is a lattice of $\mathcal{E} \in \mathrm{Crys}(X/W)_{\mathbb{Q}}$, depends only on $\mathcal{E}$. Thus since $\mathcal{E} \in \mathrm{Conv}(X/K)$ is F^∞-divisible, $ch_{i,\mathrm{crys}}(E_X) \in H^{2i}(X/K)$ is $p^{i\infty}$-divisible, thus $= 0$, thus $0 = c_{i,\mathrm{crys}}(E_X) \in H^{2i}(X/K)$.

□

Lift of trivial crystals over k

Lemma

Assume $E \in \mathrm{Crys}(X/W)$ is a lattice, such that $\exists m \in \mathbb{N}_{\geq 0}$ such that $(F^m)^* E_X \in MIC(X/k)^{qn}$ is trivial. Then if $\pi_1^{\text{ét,ab}}(X \otimes_k \bar{k}) = \{1\}$, $E \in \mathrm{Crys}(X/W)$ is trivial.

Begin of Proof.

$F^* : \mathrm{Crys}(X/W)_{\mathbb{Q}} \to \mathrm{Crys}(X/W)_{\mathbb{Q}}$ is fully faithful, so may assume E_X trivial. □

Proof of lift II

Proof.

For D PD-hull of $X \subset \mathbb{P}_W$, with $D_n = D \otimes_W W_n$, has

$$\mathrm{Ker}\big(MIC(D_{n+m}) \to MIC(D_n)\big) \cong M(r \times r, H^1_{dR}(D_m))\ 1 \leq^{\forall} m \leq n. \qquad (\star)$$

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Applying $(\star)$ to $(n, m) = (1,1), (2,1), \ldots, (N-1, 1)$, we conclude that there is $b \in \mathbb{N}$ depending only on X such that $((F^b)^* E)_{D_N}$ is trivial. Replace E by $(F^b)^* E$, may assume E_{D_N} is trivial.
Applying $(\star)$ to $(n, m) = (2N, N)$ we conclude $E_{D_{N+1}}$ = image $E_{D_{2N}}$ via $M(r \times r, H^1(D_N)) \to M(r \times r, H^1(D_1))$, is trivial.

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One continues, etc. □

Trivializing a crystal

Theorem (E-Shiho '15)

Let X be smooth projective over $k = \bar{k}$ of char. $p > 0$. Let $\mathcal{E}$ be $\in \mathrm{Conv}(X/K)$ or be locally free in $\mathrm{Crys}(X/W)_{\mathbb{Q}}$. If $\pi_1^{\text{ét}}(X) = \{1\}$, $\mu_{\max}(\Omega^1_X) < N(r)$ for a certain positive number $N(r)$ discussed below, and the irreducible constituents of the Jordan-Hölder filtration of $\mathcal{E}$ have rank $\leq r$, then $\mathcal{E}$ is trivial.

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• The slope assumption enables one to assume semi-stability of $(F^a)^*E_X$ for a certain $a \geq 0$.

□

Table of Contents

Base change for étale cohomology for torsion coefficients of order prime to p

SGA 4,5, IV Thm.1.2.: Let A be an henselian discrete valuation ring (d.v.r.), with residue field k of characteristic $p > 0$. Let X/A be a scheme, $(n, p) = 1$. Then if X/A is proper, one has *base change*, that is the restriction homomorphism $H^i_{\text{ét}}(X, \mathbb{Z}/n) \xrightarrow{\text{rest}} H^i_{\text{ét}}(Y, \mathbb{Z}/n)$ is an isomorphism, where $Y = X \otimes_A k$.

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Answer for relative 0*-cycles.*

In the sequel, we report in it, raising a few questions on the way.

Why relative 0-cycles?

Examples

• Let X/A be a K3-surface, with $k = \bar{k}$ and A large enough so $NS(X_{\bar{K}})$ is defined over $K = \mathrm{Frac}(A)$. Then $NS(X_K) \to NS(Y)$ is an injection of torsion-free lattices of possibly different (Néron-Severi) ranks, e.g. assume Y is supersingular! Thus composite $\mathrm{Pic}(X)/n \xrightarrow{\text{rest. surj.}} \mathrm{Pic}(X_K)/n \xrightarrow{\text{sp}} \mathrm{Pic}(Y)/n$, which is the restriction homomorphism to the special fiber, can't be surjective.

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• So restriction neither surjective nor injective.

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For relative 0-cycles one has

Theorem (Sato-S.Saito '10)

Assume A excellent, henselian discrete valuation ring, with finite *or* separably closed *residue field k of characteristic $p > 0$. Assume X/A projective, irreducible strict normal crossings (s.n.c.) scheme (so X in particular is regular) of relative dimension d. Then the cycle map $c_X : CH_1(X)/n \to H^{2d}_{\text{ét}}(X, \mathbb{Z}/n(d))$ is an isomorphism.*

Formulation of the problem

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$$\begin{array}{ccc} CH_1(X)/n & \xrightarrow{\rho\ \cong} & \mathcal{C}(Y)/n \\ \downarrow{\scriptstyle c_X} & & \downarrow{\scriptstyle c_Y} \\ H^{2d}_{\text{ét}}(X, \mathbb{Z}/n(d)) & \xrightarrow[\text{rest}\ \cong]{\cong} & H^{2d}_{\text{ét}}(Y, \mathbb{Z}/n(d)). \end{array} \qquad (\star)$$

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In fact we mix the two viewpoints: we want to construct a restriction homomorphism $\rho : CH_1(X)/n \to H^{2d}_{\mathrm{mot}}(Y, \mathbb{Z}(d))/n$, which is then an isomorphism and lifts the base change isomorphism.

Recall

$$\begin{array}{ccc} CH_1(X)/n & \xrightarrow{\rho\ \cong} & \mathcal{C}(Y)/n \\ \downarrow c_X & & \downarrow c_Y \\ H^{2d}_{\text{ét}}(X, \mathbb{Z}/n(d)) & \underset{\text{rest}\ \cong}{\xrightarrow{\cong}} & H^{2d}_{\text{ét}}(Y, \mathbb{Z}/n(d)). \end{array} \qquad (\star)$$

Theorem

Theorem (Kerz-E-Wittenberg '15)

1) *Let Y be a strict normal crossings variety of dimension d defined over a perfect field k. Then there is a description of $H^{2d}_{\mathrm{mot}}(Y, \mathbb{Z}[\frac{1}{p}](d))$ as a quotient of $\mathbb{Z}[Y^{\mathrm{sm}}]$ by explicit relations, and $H^{2d}_{\mathrm{mot}}(Y, \mathbb{Z}[\frac{1}{p}](d)) = H^d_{\mathrm{Nis}}(Y, \mathcal{K}^M_d)[\frac{1}{p}]$.*
2) *Assume A excellent henselian d.v.r., with perfect char. $p > 0$ residue field, and X/A be a projective s.n.c. scheme. Then the following holds.*
   i) *If A has equal char. then $CH_1(X)/n = H^d_{\mathrm{Nis}}(X, \mathcal{K}^M_d/n)$ (Kerz' theorem), ρ is then defined via restriction on $\mathcal{K}^M_d$ and one has $(\star)$;*
   ii) *If k is finite or algebraically closed, one has $(\star)$;*
   iii) *If $((d-1)!, n) = 1$, in particular if $d = 2$, one has $(\star)$.*

On Proofs

Proof.

Ad 1): uses localisation in motivic cohomology on Y, then duality to relate $H^{2d}_{c,\mathrm{mot}}(Y^{\mathrm{sm}}, \mathbb{Z}/n(d))$ with Suslin homology $= \mathbb{Z}[Y^{\mathrm{sm}}]/(\mathcal{R}I)$, $\mathcal{R}I$ spanned by certain (C, g), $C \subset Y$ integral 1-dimensional subscheme not contained in Y^{sing}, g rational function which is a unit generically and equal to 1 along Y^{sing}.

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Result: $H^{2d}_{\mathrm{mot}}(Y, \mathbb{Z}/n(d)) = \mathbb{Z}[Y^{\mathrm{sm}}]/(\mathcal{R}I, \mathcal{R}II)$, with $\mathcal{R}II$ spanned by (C, g), C simple n.c. curve and g unit along Y^{sing}. □

On Proofs II

Proof.

Ad 2): ρ uniquely defined by writing $CH_1(X)$ as a quotient of $Z_1^g(X) \subset Z_1(X)$ spanned by A-flat 1-cycles which intersect Y in $Y^{\rm sm}$.

i) A equal char.: one uses Kerz' theorem showing $CH_1(X)/n = H^d(X_{\rm Nis}, \mathcal{K}^M_{X,d})$ to define ρ;

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In all cases, once ρ is defined, one uses geometry to show it is an isomorphism. □

Some consequences

Corollary

1) *If k is finite, then* $CH_0(X_K)/n$ *is finite (already a consequence of Sato-S.Saito);*
2) $A = k[[t]]$, k *p-adic field, then* $CH_0(X_K)/n$ *is finite.*

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Ad 1): This is the link to the first lectures: Class Field Theory plus the Kato conjecture enable one to show finiteness of $\mathcal{C}(Y)/n$. One then applies the theorem.

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Ad 2): One uses again the Kato conjecture (and a result of Forré). □

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• What about mod p, and what about replacing Y by its thickenings Y_m? Assuming Gersten conjecture for Milnor K-theory on X one has a restriction homomorphism $CH_1(X)/n \to \varprojlim_m H^d_{\mathrm{Nis}}(Y_m, \mathcal{K}^M_d/n)$ (possibly p divides n) and one could ask, when A is the ring of integers of a number field, whether the prosystem is constant. This is related to Colliot-Thélène's conjecture on the structure $CH_0(X_K)$, which should be of the shape $\mathbb{Z}$ (for the degree) + a finite group + a free lattice over $\mathbb{Z}_p$ + a divisible group.