INTEGRALITY OF THE BETTI MODULI SPACE

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ABSTRACT. If in a given rank r, there is an irreducible complex local system with torsion determinant and quasi-unipotent monodromies at infinity on a smooth quasi-projective variety, then for every prime number ℓ, there is an absolutely irreducible ℓ-adic local system of the same rank, with the same determinant and monodromies at infinity, up to semi-simplification. A finitely presented group is said to be weakly integral with respect to a torsion character and a rank r if once there is an irreducible rank r complex linear representation, then for any ℓ, there is an absolutely irreducible one of rank r and determinant this given character, which is defined over $\bar{\mathbb{Z}}_\ell$. We prove that this property is a new obstruction for a finitely presented group to be the fundamental group of a smooth qusi-projective complex variety. The proofs rely on the arithmetic Langlands program via the existence of Deligne's companions (L. Lafforgue, Drinfeld) and the geometric Langlands program via de Jong's conjecture (Gaitsgory for $\ell \geq 3$). We also define weakly arithmetic complex local systems and show they are Zariski dense in the Betti moduli. Finally we show that our method gives an arithmetic proof of the Corlette-T. Mochizuki theorem, proved using tame pure imaginary harmonic metrics, which shows the pull-back by a morphism between two smooth complex algebraic varieties of a semi-simple complex local system is semi-simple.

1. INTRODUCTION

Let X be a smooth quasi-projective variety defined over the field of complex numbers. Let r be a positive natural number. Let $\mathcal{L}$ be a rank 1 complex local system on X of finite order. We fix a smooth projective compactification $X \hookrightarrow \bar{X}$ with boundary divisor $\bar{X} \setminus X$ being a strict normal crossings divisor, called in the sequel "good compactification". For each irreducible component D_i of $\bar{X} \setminus X$, we fix a quasi-unipotent conjugacy class $T_i \subset \mathrm{GL}_r(\mathbb{C})$. For any conjugacy class $T \subset \mathrm{GL}_r(K)$, where K is an algebraically closed field of characteristic 0, we denote by $T^{ss} \subset \mathrm{GL}_r(K)$ the conjugacy class of its semi-simplification.

Theorem 1.1 (Weak integrality property with respect to $(r, \mathcal{L}, T_i)$). *Assume there is an irreducible topological rank r complex local system $\mathbb{L}_\mathbb{C}$ with determinant $\mathcal{L}$ and monodromies in T_i at infinity. Then for any prime number ℓ, there is an ℓ-adic local system $\mathbb{L}_\ell$ which*

1) *has rank r and is irreducible over $\bar{\mathbb{Q}}_\ell$,*
2) *has determinant $\mathcal{L}$,*
3) *has monodromies $T_{i,\ell}$ at infinity such that $T_i^{ss} = T_{i,\ell}^{ss}$.*

For example, if X is smooth projective and $\mathbb{L}_{\mathbb{C}}$ is an irreducible rank r local system with trivial determinant, then Theorem 1.1 says that for all ℓ, there is an irreducible rank r ℓ-adic local system with trivial determinant.

Forgetting the conditions at infinity, if we fix a finitely presented group Γ, a natural number $r \geq 1$, and a rank 1 torsion complex character $\chi : \Gamma \to \mathbb{C}^{\times}$, we pose the

Definition 1.2. Γ *has the weak integrality property with respect to* (r, χ) *if, assuming there is an irreducible representation* $\rho : \Gamma \to \mathrm{GL}_r(\mathbb{C})$ *with determinant* χ, *then for any prime number* ℓ, *there is a representation* $\rho_\ell : \Gamma \to \mathrm{GL}_r(\bar{\mathbb{Z}}_\ell)$ *with determinant* χ *which is irreducible over* $\bar{\mathbb{Q}}_\ell$.

This property depends only on the isomorphism class of Γ. In [BBV22], the authors study the $\mathrm{SL}(2, \mathbb{C})$-character variety of irreducible representations of a residually finite group Γ_0 defined in [DS05, Theorem 4], with 2 generators $\{a, b\}$ and one relation $b^2 = a^2 b a^{-2}$. They in particular show that it is 0-dimensional, defined over $\mathbb{Q}$, with only two conjugate points, which correspond to irreducible dense complex representations, which are *not integral* at the prime 2. See Section 7.4. We conclude that Γ_0 does not have the weak integrality property with respect to $(2, \mathbb{I})$ where $\mathbb{I}$ is the trivial character.

Theorem 1.3 (Obstruction)**.** *If* X *is a smooth complex quasi-projective variety, then* $\Gamma = \pi_1(X(\mathbb{C}))$ *satisfies the weak integrality property for any pair* (r, χ). *In particular, the group* Γ_0 *above cannot be the topological fundamental group of a smooth complex quasi-projective variety.*

Our obstruction for an abstract finitely presented group Γ to be the topological fundamental group of a smooth complex quasi-projective seems to be of a new kind. We do not have to specify a finite set of conjugacy classes in Γ which would be the local monodromies at infinity. See Section 4.1 for the proof of Theorem 1.3.

Ultimately, as we shall see below, the proof of Theorem 1.1 relies on the arithmetic Langlands program as proven by L. Lafforgue [Laf02], on the existence of ℓ-adic companions shown by him in dimension 1 and Drinfeld [Dri12] in higher dimension, and on de Jong's conjecture [dJ01], proved for $\ell \geq 3$ by Gaitsgory [Gai07], using the geometric Langlands program.

We now explain in which framework Theorem 1.1 is located and proven. Let S be an affine scheme of finite type over $\mathbb{Z}$ with $\mathcal{O}(S) \subset \mathbb{C}$, such that a good compactification $X \hookrightarrow \bar{X}$ and a given complex point $x \in X$ have a model over S. This means that we have a relative good compactification $X_S \hookrightarrow \bar{X}_S$ over S such that $\bar{X}_S \setminus X_S$ is a relative normal crossings divisor, we have an S-point x_S of X_S whose base change to $\mathrm{Spec}(\mathbb{C})$ recovers $x \in X \hookrightarrow \bar{X}$. We also assume the orders of $\mathcal{L}$ and of the eigenvalues of the T_i are invertible on S. For any closed point $s \in |S|$ of residue field $\mathbb{F}_q$ of characteristic $p > 0$, with a $\bar{\mathbb{F}}_p$-point $\bar{s}$ above it, we denote by

$$sp_{\mathbb{C}, \bar{s}} : \pi_1(X_{\mathbb{C}}, x_{\mathbb{C}}) \to \pi_1^t(X_{\bar{s}}, x_{\bar{s}})$$

the continuous surjective specialization homomorphism to the tame fundamental group [SGA1, Exposé XIII 2.10, Corollaire 2.12]. Precomposing with the profinite completion homomorphism

$$\pi_1(X(\mathbb{C}), x(\mathbb{C})) \to \pi_1(X_{\mathbb{C}}, x_{\mathbb{C}})$$

from the topological fundamental group to the étale one over $\mathbb{C}$ yields the homomorphism

$$sp^{\text{top}}_{\mathbb{C},\bar{s}} : \pi_1(X(\mathbb{C}), x(\mathbb{C})) \to \pi_1^t(X_{\bar{s}}, x_{\bar{s}}).$$

This enables us to consider the topological pull-back $(sp^{\text{top}}_{\mathbb{C},\bar{s}})^*(\mathbb{L}_{\ell,\bar{s}})$ of any tame ℓ-adic local system $\mathbb{L}_{\ell,\bar{s}}$ on $X_{\bar{s}}$, in particular of those $\mathbb{L}_{\ell,\bar{s}}$ which are arithmetic, that is defined over $X_{\mathbb{F}_{q'}}$ for a finite extension $\mathbb{F}_q \to \mathbb{F}_{q'} \subset \bar{\mathbb{F}}_p$, as in [EG18, Section 3]. The topological pull-back $(sp^{\text{top}}_{\mathbb{C},\bar{s}})^*(\mathbb{L}_{\ell,\bar{s}})$ is defined over $\bar{\mathbb{Z}}_\ell$. We prove

Theorem 1.4. *If there is one (resp. infinitely many pairwise non-isomorphic) irreducible topological rank r complex local system (resp. systems) $\mathbb{L}_{\mathbb{C}}$ with determinant $\mathcal{L}$ and monodromies in T_i at infinity, then there is a non-empty open subscheme $S^\circ \subset S$ such that for any two closed points $s, s' \in |S|$ of residual characteristics $p \neq p'$ it holds:*

1) *for any prime number $\ell \neq p$ there is one (resp. infinitely many pairwise non-isomorphic) arithmetic local system (resp. systems) $\mathbb{L}_{\ell,\bar{s}}$ on $X_{\bar{s}}$;*
2) *which has (resp. have) determinant $\mathcal{L}$, with quasi-unipotent monodromies $T_{i,\ell,\bar{s}}$ at infinity such that $T_i^{ss} = T_{i,\ell,\bar{s}}^{ss}$;*
3) *which is (are) irreducible over $\bar{\mathbb{Q}}_\ell$;*
4) *for $\ell = p$ there is one (resp. infinitely many pairwise non-isomorphic) arithmetic local system (resp. systems) $\mathbb{L}_{p,\bar{s}'}$ on $X_{\bar{s}'}$ with 2),3) where ℓ is replaced by p;*
5) *for any prime number ℓ, the topological pull-backs $(sp^{\text{top}}_{\mathbb{C},\bar{s}})^*(\mathbb{L}_{\ell,\bar{s}})$ (which in the resp. case are pairwise non-isomorphic) have properties 2) and 3) as topological local systems.*

Theorem 1.4 5) for the non-resp. case immediately implies Theorem 1.1 by considering the topological pull-backs $(sp^{\text{top}}_{\mathbb{C},\bar{s}})^*(\mathbb{L}_{\ell,\bar{s}})$.

In Theorem 1.4 we can in addition single out specific extra properties for $\mathbb{L}_{\mathbb{C}}$ and request that the $(sp^{\text{top}}_{\mathbb{C},\bar{s}})^*(\mathbb{L}_{\ell,\bar{s}})$ keep the same properties.

Theorem 1.5.   i) *If in Theorem 1.4, we assume that $\mathbb{L}_{\mathbb{C}}$ is cohomologically rigid, then we can choose $\mathbb{L}_{\ell,\bar{s}}$ such that the topological pull-back $(sp^{\text{top}}_{\mathbb{C},\bar{s}})^*(\mathbb{L}_{\ell,\bar{s}})$ is cohomologically rigid.*

ii) *If in Theorem 1.4 we assume that the Zariski closure of the monodromy of $\mathbb{L}_{\mathbb{C}}$ contains $\mathrm{SL}_r(\mathbb{C})$, then we can choose $\mathbb{L}_{\ell,\bar{s}}$ such that the topological pull-back $(sp^{\text{top}}_{\mathbb{C},\bar{s}})^*(\mathbb{L}_{\ell,\bar{s}})$ has the same property.*

Theorem 1.5 i) follows from [EG18, Theorem 1.1] which is Simpson's integrality conjecture for cohomologically rigid local systems. Its method of proof, developed with M. Groechenig, is the starting point of this article. We comment on this in Remark 6.1. Theorem 1.5 ii) answers positively a question by A. Landesman asked to us after the second named author lectured on Theorem 1.4 in Harvard in October 2022.

We now describe the method of proof of the non-resp. case of Theorem 1.4, see Section 2. We make use of the fact that the Betti moduli space $M_B(X, r, \mathcal{L}, T_i)$ parametrizing irreducible local systems of rank r with prescribed torsion determinant $\mathcal{L}$ and quasi-unipotent monodromies at infinity in T_i is of finite type over a number ring $\mathcal{O}_K$, see Section 2.1. We denote by $M_B(X, r, \mathcal{L}, T_i^{ss})$ the disjoint union

of the finitely many $M_B(X, r, \mathcal{L}, T_i)$ so the semi-simplification of T_i is T_i^{ss} for all i. The existence of an irreducible rank r complex local system $\mathbb{L}_{\mathbb{C}}$ is equivalent to $M_B(X, r, \mathcal{L}, T_i)$ being dominant over $\mathcal{O}_K$. By generic smoothness, the underlying reduced scheme $M_B(X, r, \mathcal{L}, T_i)_{red} \subset M_B(X, r, \mathcal{L}, T_i)$ is smooth over $\mathcal{O}_K$ on some non-trivial open subscheme. We pick a closed point z in this locus, with residue field $\mathbb{F}_{\ell^m}$ for $\ell \geq 3$. This defines an $\mathbb{F}_{\ell^m}$-local system on $X_{\mathbb{C}}$. By Grothendieck's theory of the specialization of the étale fundamental group of $X_{\mathbb{C}}$ to the tame one in characteristic $p > 0$, this $\mathbb{F}_{\ell^m}$-local system descends to the mod p reduction $X_{\bar{\mathbb{F}}_p}$ for p large prime to ℓ, and as the monodromy is finite, to $X_{\mathbb{F}_q}$ where $q = p^s$ for some $s \in \mathbb{N}_{>0}$. The completion of $M_B(X, r, \mathcal{L}, T_i)$ at z is identified with Mazur's deformation space of z keeping the same conditions $(\mathcal{L}, T_i)$. We can then apply de Jong's conjecture to the effect that z lifts to an ℓ-adic local system $\mathbb{L}_{\ell, \bar{\mathbb{F}}_p}$ which is artihmetic and is still absolutely irreducible. We now apply the existence of companions for any $\ell' \neq p$. This yields arithmetic ℓ'-adic local systems $\mathbb{L}_{\ell', \bar{\mathbb{F}}_p}$ on $X_{\bar{\mathbb{F}}_p}$ with the same determinant and monodromies at infinity, modulo semi-simplification. Pulling-back those to $X(\mathbb{C})$ yields a point in $M_B(X, r, \mathcal{L}, T_i^{ss})(\bar{\mathbb{Z}}_{\ell'})$. This proves the theorem for $\ell \neq p$. At p we just choose a different specialization to $X_{\bar{\mathbb{F}}_{p'}}$ for $p \neq p'$.

In the resp. case we do the same replacing $M_B(X, r, \mathcal{L}, T_i)$ by one component, dominant over $\mathcal{O}_K$, which over $\mathbb{C}$ contains infinitely many of them.

For Theorem 1.5 ii), we use in addition to Theorem 1.4 the fact that the existence of $\mathbb{L}_{\mathbb{C}}$ with Zariski dense monodromy implies that the locus in $M_B(X, r, \mathcal{L}, T_i)(\mathbb{C})$ of complex points with Zariski dense monodromy is Zariski dense, see Section 6.

The method of proof described above invites us to define the notion of a *weakly arithmetic complex local system* $\mathbb{L}_{\mathbb{C}}$: there is an identification of $\mathbb{C}$ with $\bar{\mathbb{Q}}_\ell$ such that the resulting topological local system $\mathbb{L}_{\bar{\mathbb{Q}}_\ell}$ defined over $\bar{\mathbb{Q}}_\ell$ is in fact ℓ-adic and descends to an arithmetic ℓ-adic local system on some reduction $X_{\bar{\mathbb{F}}_p}$ mod p (see Definition 3.1). The method of proof described above enables us to show that the weakly arithmetic complex local systems with a fixed determinant $\mathcal{L}$ are dense in the Betti moduli space parametrizing irreducible complex local systems $M_B(X, r, \mathcal{L})$ of rank r and determinant $\mathcal{L}$. In fact the proof does not use the companions in characteristic $p > 0$, instead it uses the invariance of this locus by complex conjugation over $\mathbb{C}$, see Theorem 3.5.

To summarize, in Section 2 we prove Theorem 1.1 and Theorem 1.4, non-resp. case. In Section 3 we define the notion of weakly arithmetic complex local systems and prove their density in the Betti moduli. In Section 4 we prove Theorem 1.3. In Section 5 we prove the resp. case of Theorem 1.4. In Section 6 we prove Theorem 1.5 ii). Finally in Section 7 we make some comments, formulate some questions, and, as a curiosity, we give a proof of the theorem by Corlette and T. Mochizuki that a morphism between normal complex varieties respects semi-simplicity of local systems. The original proof uses the harmonic theory. Our proof uses de Jong's conjecture (and not the companions).

Emmanuel Breuillard for kindly writing down for us the example documented in Section 7.4, which shows that our weak integrality property for a finitely presented group is indeed an obstruction for this group to come from algebraic geometry. We warmly thank the referee for a thorough, precise and helpful report which helped us to improve the presentation of our article and correct a mistake in the proof of Theorem 7.3.

2. PROOF OF THEOREMS 1.1 AND 1.4, NON-RESP. CASE

2.1. **The Betti moduli space.** Let $(X \hookrightarrow \bar{X}, r, D_i, T_i, \mathcal{L})$ be the notation used in Theorem 1.1. The datum $(X, r, \mathcal{L}, T_i)$ defines a number field K and its ring of integers $\mathcal{O}_K$ over which $\mathcal{L}$ and the eigenvalues of the T_i are defined. There is an algebraic stack $\mathcal{M}(X, r, \mathcal{L}, T_i)$ of finite type over $\mathrm{Spec}\,(\mathcal{O}_K)$ parametrizing irreducible local systems on X of rank r, with determinant $\mathcal{L}$, and monodromies at infinity in T_i. See [Dri01, 2.1] where it is denoted by Irr_r^X, and in [EG18, Section 2] where the determinant and the conditions at infinity are taken into account, it is denoted by $\underline{M}$. See also the footnote [1]. We denote by $M_B(X, r, \mathcal{L}, T_i)$ the associated coarse moduli space, and call it the Betti moduli space. In the proof of the following lemma we will see that it exists and is a separated scheme of finite type over $\mathrm{Spec}\,(\mathcal{O}_K)$.

Lemma 2.1. *The morphism* $\mathcal{M}(X, r, \mathcal{L}, T_i) \to M_B(X, r, \mathcal{L}, T_i)$ *exhibits the source as a* $\mathbb{G}_m$*-gerbe over the target.*

Proof. Most of the steps in this proof are justified in [WE18] and [EG18]; we only add arguments for the parts which are not shown there. Choose generators $\gamma_1, \ldots, \gamma_n$ for the topological fundamental group $\Gamma = \pi_1(X(\mathbb{C}), x(\mathbb{C}))$. Then there is a locally closed (closed if the conjugacy classes T_i are semi-simple) subscheme $M^\square \subset (\mathrm{GL}_{r, \mathcal{O}_K})^n$ with the following property: for any $\mathcal{O}_K$-algebra R, the R-points of $M^\square$ correspond bijectively to homomorphisms $\rho : \Gamma \to \mathrm{GL}_r(R)$ whose associated local system on X defines an R-point of the stack $\mathcal{M}(X, r, \mathcal{L}, T_i)$. The correspondence sends ρ to $(\rho(\gamma_1), \ldots, \rho(\gamma_n))$. We obtain a morphism $M^\square \to \mathcal{M}(X, r, \mathcal{L}, T_i)$. There is an action of the group scheme $G = \mathrm{GL}_{r, \mathcal{O}_K}$ on $M^\square$ over $\mathrm{Spec}\,(\mathcal{O}_K)$, and we have

$$\mathcal{M}(X, r, \mathcal{L}, T_i) = [M^\square / G]$$

as algebraic stacks. Since $M^\square$ is of finite type over $\mathrm{Spec}\,(\mathcal{O}_K)$ the same is true for $\mathcal{M}(X, r, \mathcal{L}, T_i)$. The action of G on $M^\square$ factors through an action of the group scheme $\bar{G} = \mathrm{PGL}_{r, \mathcal{O}_K}$ on $M^\square$ over $\mathrm{Spec}\,(\mathcal{O}_K)$. Since for every algebraically closed field k, every representation $\rho : \Gamma \to \mathrm{GL}_r(k)$ corresponding to a k-point of $M^\square$ is by definition irreducible, we see that the action of $\bar{G}$ on $M^\square$ is scheme theoretically free. Hence the quotient

$$M_B(X, r, \mathcal{L}, T_i) = [M^\square / \bar{G}]$$

is an algebraic space by [SP, Tag 06PH]. Since $M^\square$ is of finite type over $\mathrm{Spec}\,(\mathcal{O}_K)$ the same is true for $M_B(X, r, \mathcal{L}, T_i)$. For any scheme T endowed with an action of $\bar{G}$ the morphism $[T/G] \to [T/\bar{G}]$ is a $\mathbb{G}_m$-gerbe. Although in the rest of the article we never use anything beyond the facts already proven (at the cost of working with algebraic spaces in addition to schemes), below we briefly indicate why

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[1] There is a typo defining the irreducibility condition, which on p. 4282 should be tested not only on geometric generic points but on all geometric points.

$M_B(X, r, \mathcal{L}, T_i)$ is separated and a scheme. This is standard but we have not been able to find a reference in the literature.

To show that $M_B(X, r, \mathcal{L}, T_i)$ is separated is equivalent to proving that the action of $\bar{G}$ on $M^\square$ is closed, i.e., that the morphism $\Psi : \bar{G} \times M^\square \to M^\square \times M^\square$ is a closed immersion. Freeness of the action means that Ψ is a monomorphism. By [SP, Tag 04XV] it suffices to show that Ψ is universally closed. To check this in turn by [SP, Tag 04XV] it suffices to check the existence part of the valuative criterion for discrete valuation rings. Unwinding the definitions this boils down to the following: given a discrete valuation ring R with uniformizer π, residue field k, and fraction field L, given two homomorphisms $\rho_1, \rho_2 : \Gamma \to \mathrm{GL}_r(R)$ such that $\bar{\rho}_1 : \Gamma \to \mathrm{GL}_r(k)$ is irreducible, if ρ_1 and ρ_2 are isomorphic as representations over L, then ρ_1 and ρ_2 are isomorphic as representations over R. This follows from the fact that all Γ-ρ_1-invariant lattices in L^r are of the form $\pi^n R^r$, using irreducibility and integrality of ρ_i.

By [SP, Tag 03XX] if we can construct a quasi-finite morphism $M_B(X, r, \mathcal{L}, T_i) \to N$ to a scheme N, then $M_B(X, r, \mathcal{L}, T_i)$ is a scheme. Let $\Omega \subset \Gamma$ be a finite subset. Given $\gamma \in \Omega$ we can associate to a representation ρ of Γ the characteristic polynomial of $\rho(\gamma)$. This defines a G-invariant morphism $M^\square \to \prod_{\gamma \in \Omega} \mathbb{A}^r_{\mathcal{O}_K}$ and hence a morphism

$$M_B(X, r, \mathcal{L}, T_i) \to \prod_{\gamma \in \Omega} \mathbb{A}^r_{\mathcal{O}_K}$$

We claim that if Ω is large enough, this morphism is quasi-finite onto its image and the set of field value points of its fibres is either empty or consists of one point. This then finishes the proof. Namely, by the Brauer-Nesbit theorem the isomorphism class of an irreducible representation ρ over an algebraically closed field k is determined by its character (see for example [Lam91, Theorem 7.20]). On the other hand, since Γ is finitely generated, the pseudocharacter $\gamma \mapsto \det(T - \rho(\gamma))$ for any representation ρ of fixed rank r is determined by the values on finitely many elements of Γ for example by [Che14, Proposition 2.38].   □

2.2. **Proof of Theorem 1.4, non-resp. case.** Let $(S, X_S \hookrightarrow \bar{X}_S, x_S)$ be as in the introduction. The existence of a point $\mathbb{L}_\mathbb{C}$ in $M_B(X, r, \mathcal{L}, T_i)(\mathbb{C})$ in the assumption of Theorem 1.4 tells us that the structure morphism

$$\epsilon : M_B(X, r, \mathcal{L}, T_i) \to \mathrm{Spec}(\mathcal{O}_K)$$

is dominant. By generic smoothness, there is a non-empty open subscheme $M^\circ \subset M_B(X, r, \mathcal{L}, T_i)_{red}$ of the reduced scheme such that

$$\epsilon|_{M^\circ} : M^\circ \to \mathrm{Spec}(\mathcal{O}_K)$$

is smooth, dominant, and has values in $\mathrm{Spec}(\mathcal{O}_K)^\circ$ where $\mathrm{Spec}(\mathcal{O}_K)^\circ \to \mathrm{Spec}(\mathbb{Z})$ is smooth over its image. Let $z \in |M^\circ|$ be a closed point, so of residue field $\mathbb{F}_{\ell^m}$ for a prime number $\ell \geq 3$ and some $m \in \mathbb{N}_{>0}$. By Lemma 2.1 and the vanishing of the Brauer group of a finite field, we see that z corresponds to an absolutely irreducible local $\mathbb{F}_{\ell^m}$-system $\mathbb{L}_z$ over X.

We define $S^\circ \subset S$ to be the non-empty open subscheme which is the complement of closed points of residual characteristic dividing the order of $\mathrm{GL}_r(\mathbb{F}_{\ell^m})$. We claim

S° satisfies the assertions of Theorem 1.4. Pick $s \in |S^\circ|$ of characteristic p and consider the diagram

$$sp^{\text{top}}_{\mathbb{C},\bar{s}} : \pi_1(X(\mathbb{C}), x(\mathbb{C})) \to \pi_1(X_{\mathbb{C}}, x_{\mathbb{C}}) \xrightarrow{sp_{\mathbb{C},\bar{s}}} \pi_1^t(X_{\bar{s}}, x_{\bar{s}})$$

from the introduction. Since $sp_{\mathbb{C},\bar{s}}$ is an isomorphism on prime to p quotients, we see that $\mathbb{L}_z$ gives rise to an absolutely irreducible local $\mathbb{F}_{\ell^m}$-system $\mathbb{L}_{z,\bar{s}}$ over $X_{\bar{s}}$. Its determinant is $\mathcal{L}$ and its monodromies at infinity are in T_i by the compatibility of $sp_{\mathbb{C},\bar{s}}$ with the local fundamental groups, see [Del73, Section 1.1.10].

Let $D_{z,\bar{s}} = \text{Spf}(R_{z,\bar{s}})$ where $R_{z,\bar{s}}$ is Mazur's formal deformation ring of the rank r representation $\rho_{z,\bar{s}}$ of $\pi_1^t(X_{\bar{s}}, x_{\bar{s}})$ over $\mathbb{F}_{\ell^m}$ corresponding to $\mathbb{L}_{z,\bar{s}}$ ([Maz89, Proposition 1]). Consider the formal closed subscheme $D_{z,\bar{s}}(r, \mathcal{L}, T_i) \subset D_{z,\bar{s}}$ corresponding to deformations where the universal deformation has determinant $\mathcal{L}$ and monodromies at infinity in T_i (it is indeed a closed condition.) By construction we obtain a morphism

$$D_{z,\bar{s}}(r, \mathcal{L}, T_i) \to \mathcal{M}(X, r, \mathcal{L}, T_i) \to M_B(X, r, \mathcal{L}, T_i)$$

Thus we obtain

$$D_{z,\bar{s}}(r, \mathcal{L}, T_i) \xrightarrow{\iota} M_B(X, r, \mathcal{L}, T_i)_z^\wedge$$

where $(-)_z^\wedge$ indicates the formal completion at z.

Proposition 2.2. *The morphism ι is an isomorphism.*

Proof. Let R be an Artinian local ring whose residue field is identified with the residue field $\mathbb{F}_{\ell^m}$ of z. To construct the inverse to ι we will show that morphisms

$$m : \text{Spec}\,(R) \to M_B(X, r, \mathcal{L}, T_i)$$

which send the closed point to z, are in one to one correspondence with deformations of $\rho_{z,\bar{s}}$ in $D_{z,\bar{s}}(r, \mathcal{L}, T_i)$. There are two steps. First, by Lemma 2.1 the morphism m lifts to a morphism $m : \text{Spec}\,(R) \to \mathcal{M}(X, r, \mathcal{L}, T_i)$ into the stack (as the Brauer group of R is trivial) and moreover the isomorphism class of the lift is well defined. This lift defines a local R-system $\mathbb{L}_m$ on X such that $\mathbb{L}_R \otimes_R \mathbb{F}_{\ell^m} = \mathbb{L}_z$. Second, by exactly the same arguments as above, this descends to a local R-system $\mathbb{L}_{m,\bar{s}}$ on $X_{\bar{s}}$ (unique up to isomorphism) with determinant $\mathcal{L}$ and monodromies in T_i at infinity. The corresponding continuous representation $\rho_{R,\bar{s}} : \pi_1^t(X_{\bar{s}}, x_{\bar{s}}) \to \text{GL}_r(R)$ is the desired deformation. $\square$

Corollary 2.3. *The reduced deformation space $D_{z,\bar{s}}(r, \mathcal{L}, T_i)_{red}$ is smooth over $\text{Spf}(W(\mathbb{F}_{\ell^m}))$.*

Proof. This follows as by the choices made above the morphisms

$$M^\circ \to \text{Spec}(\mathcal{O}_K) \text{ and } \text{Spec}(\mathcal{O}_K)^\circ \to \text{Spec}(\mathbb{Z})$$

are smooth and the scheme M° is an open subscheme of the $M_B(X, r, \mathcal{L}, T_i)_{red}$. $\square$

Recall that $D_{z,\bar{s}}(r, \mathcal{L}, T_i)$ is a deformation space for the residual representation $\rho_{z,\bar{s}}$ of $\pi_1^t(X_{\bar{s}}, x_{\bar{s}})$. We have the homotopy exact sequence

$$1 \to \pi_1^t(X_{\bar{s}}, x_{\bar{s}}) \to \pi_1^t(X_s, x_s) \to \text{Gal}(\bar{s}/s) \to 1$$

of profinite groups. Denote $\Phi \in \text{Gal}(\bar{s}/s)$ the Frobenius of s. By the outer action of $\text{Gal}(\bar{s}/s)$ on $\pi_1^t(X_{\bar{s}}, x_{\bar{s}})$, we see that Φ acts on the set of isomorphism classes of representations of $\pi_1^t(X_{\bar{s}}, x_{\bar{s}})$. Since $\rho_{z,\bar{s}}$ has finite image, a power Φ^n for some

$n \geq 1$ of Φ stabilizes it. Thus there is an action of Φ^n on $D_{z,\bar{s}}$. On the other hand, $\Phi(\mathcal{L})$ is another torsion local system of the same order. Since there are finitely many of them, a power Φ^{nm} for some $m \geq 1$ stabilizes $\mathcal{L}$. By [SGA7.2, XIV.1.1.10] (see also [EK22, Lemma 7.1]), the action of Φ on the conjugacy class t_i of the monodromies at infinity is via the cyclotomic character. So Φ acts on the set of conjugacy classes of quasi-unipotent matrices in $GL_r(\bar{\mathbb{Q}}_\ell)$ with eigenvalues being powers of the eigenvalues of the T_i. Thus for some $t \geq 1$, Φ^{nmt} stabilizes $(D_{z,\bar{s}}, \mathcal{L}, T_i)$ and thus the formal closed subscheme $D_{z,\bar{s}}(r, \mathcal{L}, T_i) \hookrightarrow D_{z,\bar{s}}(r)$. We abuse notation setting $n = nmt$. So, Φ^n acts on $D_{z,\bar{s}}(r, \mathcal{L}, T_i)$.

We now apply de Jong's conjecture [dJ01, Conjecture 2.3], proved by Gaitsgory [Gai07] for $\ell \geq 3$, in the way Drinfeld did in [Dri01, Lemma 2.8]: Corollary 2.3 implies that there is a $\bar{\mathbb{Z}}_\ell$-point of $D_{z,\bar{s}}(r, \mathcal{L}, T_i)_{red}$ which is invariant under Φ^n. This corresponds to an irreducible ℓ-adic local system $\mathbb{L}_{z,\bar{s},\ell}$ on $X_{\bar{s}}$, which descends to a Weil sheaf, thus by [Del80, Proposition 1.3.14] to an arithmetic étale (we just say arithmetic in the sequel) local system with determinant $\mathcal{L}$ and monodromies T_i at infinity. This yields Theorem 1.4 1), 2), 3) 5) for this one ℓ.

We now use the method of [EG18, Section 3]. For any $\ell' \neq p$ and any algebraic isomorphism $\sigma : \bar{\mathbb{Q}}_\ell \cong \bar{\mathbb{Q}}_{\ell'}$, we denote $\mathbb{L}^\sigma_{z,\bar{s},\ell}$ the restriction to $X_{\bar{s}}$ of the companion of the arithmetic descent of $\mathbb{L}_{z,\bar{s},\ell}$. It is an ℓ'-adic local system on $X_{\bar{s}}$ which is irreducible over $\bar{\mathbb{Q}}_{\ell'}$ ([EG18, Proof of Theorem 1.1]), has determinant $\mathcal{L}$ by compatibility of companions and exterior powers, and monodromies at infinity $T_{z,\bar{s},\ell',i}$ such that $T_i^{ss} = T_{z,\bar{s},\ell',i}^{ss}$ by [Del72, Théorème 9.8]. We set $\mathbb{L}_{\ell,\bar{s}} = \mathbb{L}_{z,\bar{s},\ell}$ and $\mathbb{L}_{\ell',\bar{s}} = \mathbb{L}^\sigma_{z,\bar{s},\ell}$ for all $\ell' \neq p, \ell$. For $\ell = p$ we just choose another closed point s' in S° of residual characteristic p' not equal to p, redo the same construction ane set $\mathbb{L}_{p,\bar{s}'} = \mathbb{L}_{z,p,\bar{s}'}$. This finishes the proof.

2.3. **Proof of Theorem 1.1.** For any $\ell \neq p$ we set $\mathbb{L}_\ell = (sp^{top}_{\mathbb{C},\bar{s}})^* \mathbb{L}_{\ell,\bar{s}}$ and for $\ell = p$ we set $\mathbb{L}_p = (sp^{top}_{\mathbb{C},\bar{s}'})^* \mathbb{L}_{p,\bar{s}'}$. This finishes the proof.

3. Density of weakly arithmetic local systems

In this section we use the notation of Section 2. Our aim is to define a notion of weakly arithmetic local systems and to show density of them in the Betti moduli space.

3.1. **Definitions.** If $\tau : K_1 \to K_2$ is a homomorphism of fields, and $\mathbb{L}_1$ is a topological local system defined over K_1 by the homomorphism $\rho : \pi_1(X(\mathbb{C}), x(\mathbb{C})) \to GL_r(K_1)$, we denote by $\mathbb{L}^\tau_{K_1}$ the local system defined over K_2 obtained by post-composing ρ by the homomorphism $GL_r(K_1) \to GL_r(K_2)$ defined by τ.

Choose a finitely generated subfield $F \subset \mathbb{C}$ such that X and x descend to X_F and x_F over F. For example F could be the function field of the affine finite type scheme S over which X has a model. Denote

$$sp^{top}_{\mathbb{C},F} : \pi_1(X(\mathbb{C}), x(\mathbb{C})) \to \pi_1(X_F, x_F)$$

the comparison morphism. Recall that $M_B(X, r, \mathcal{L}, T_i)$ is defined over the number ring $\mathcal{O}_K \subset \mathbb{C}$.

Definition 3.1. *With notation as above:*

1) *A point $\mathbb{L}_{\mathbb{C}} \in M_B(X, r)(\mathbb{C})$ is said to be arithmetic if there exist a finite extension F'/F, an étale ℓ-adic local system $\mathbb{L}_{\ell,F'}$ on $X_{F'}$, and a field isomorphism[2] $\tau : \bar{\mathbb{Q}}_\ell \to \mathbb{C}$ such that $\mathbb{L}_{\mathbb{C}}$ and $\left((sp_{\mathbb{C},F'}^{\mathrm{top}})^{-1}(\mathbb{L}_{\ell,F'})\right)^\tau$ are isomorphic.*

2) *A point $\mathbb{L}_{\mathbb{C}} \in M_B(X, r, \mathcal{L}, T_i)(\mathbb{C})$ is said to be weakly arithmetic if there exist*

   i) *a prime number ℓ and a field isomorphism $\tau : \bar{\mathbb{Q}}_\ell \to \mathbb{C}$ (so τ defines an $\mathcal{O}_K$-algebra structure on $\bar{\mathbb{Q}}_\ell$ via $\mathcal{O}_K \subset \mathbb{C}$);*

   ii) *a finite type scheme S over $\mathrm{Spec}(\mathcal{O}_K)$ such that $(X \hookrightarrow \bar{X}, x, \mathcal{L}, T_i)$ has a model over S (see introduction);*

   iii) *a closed point $s \in |S|$ of residual characteristic different from ℓ;*

   iv) *a tame arithmetic ℓ-adic local system $\mathbb{L}_{\ell,\bar{s}}$ on $X_{\bar{s}}$ with determinant $\mathcal{L}$ and monodromy at infinity in T_i;*

   such that $\mathbb{L}_{\mathbb{C}}$ and $\left((sp_{\mathbb{C},\bar{s}}^{\mathrm{top}})^{-1}(\mathbb{L}_{\ell,\bar{s}})\right)^\tau$ are isomorphic.

We can express the definition in an imprecise way by saying the following. As $\pi_1(X(\mathbb{C}), x(\mathbb{C}))$ is finitely generated, the topological local system $\mathbb{L}_{\mathbb{C}}$ is defined over a ring A of finite type over $\mathbb{Z}$, so $\mathbb{L}_{\mathbb{C}} = \mathbb{L}_A \otimes_A \mathbb{C}$. For almost all prime numbers ℓ, there is a non-zero homomorphism $A \to \bar{\mathbb{Z}}_\ell$. For any such, $\mathbb{L}_A \otimes_A \bar{\mathbb{Z}}_\ell$ defines an ℓ-adic local system. Then $\mathbb{L}_{\mathbb{C}}$ is weakly arithmetic if there is such a non-zero $A \to \bar{\mathbb{Z}}_\ell$ such that $\mathbb{L}_A \otimes_A \bar{\mathbb{Z}}_\ell$ comes via the specialization homomorphism from a tame arithmetic ℓ-adic local system on X_s for some s of large characteristic.

Remark 3.2. The notion of an arithmetic local system does not depend on the choice of F. As we could not find a reference, we give a short argument. Let $F_1, F_2 \subset \mathbb{C}$ be two finitely generated subfields over which X and x can be defined. Then the compositum is a third one. So we many assume that $F_1 \subset F_2 \subset \mathbb{C}$. We have the commutative diagram of homotopy exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X_{\bar{F}_2}, x_{\bar{F}_2}) & \longrightarrow & \pi_1(X_{F_2}, x_{\bar{F}_2}) & \longrightarrow & \mathrm{Gal}(\bar{F}_2/F_2) & \longrightarrow & 1 \\
& & \downarrow{\cong} & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(X_{\bar{F}_1}, x_{\bar{F}_1}) & \longrightarrow & \pi_1(X_{F_1}, x_{\bar{F}_1}) & \longrightarrow & \mathrm{Gal}(\bar{F}_1/F_1) & \longrightarrow & 1
\end{array}$$

Since F_2/F_1 is a finitely generated extension, the right vertical map is open. Let $\mathbb{L}$ be a geometrically irreducible arithmetic ℓ-adic local system over $X_{\bar{F}_2}$ with underlying representation ρ. Thinking of X as being defined over F_2, by replacing F_2 by a finite extension, we may assume that ρ is defined over F_2. Then $K := \mathrm{Ker}(\mathrm{Gal}(\bar{F}_2/F_2) \to \mathrm{Gal}(\bar{F}_1/F_1))$ lifts to $\pi_1(X_{F_2}, x_{\bar{F}_2})$ and stabilizes $\mathbb{L}|_{X_{\bar{F}_2}}$, which is irreducible. Thus $\rho(K) \subset \bar{\mathbb{Q}}_\ell^\times \subset \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$ and is torsion as the determinant is torsion. This defines a finite (abelian) extension of F_2. Replacing F_2 by it, ρ factors through $\pi_1(X_{F_2}, x_{\bar{F}_2})/K$ which is equal to $\pi_1(X_{F'_1}, x_{\bar{F}_1})$ where $F_1 \subset F'_1$ is the finite Galois extension defined by the image of the right vertical map which is open.

Remark 3.3. An arithmetic local system is weakly arithmetic. Namely, suppose that $\mathbb{L}_{\mathbb{C}}$ is arithmetic. Let $(S, X_S \hookrightarrow \bar{X}_S, x_S)$ be as in the introduction. Denote by F the function field of S. Then $\mathbb{L}_{\mathbb{C}}$ comes from an étale ℓ-adic local system $\mathbb{L}_{\ell,F'}$ on $X_{F'}$ for F'/F finite. After replacing S by a finite cover, we may assume $F' = F$.

---

[2]It would be equivalent to say "homomorphism of fields" here.

A standard argument shows that after replacing S by a suitable Zariski open, we may assume that $\mathbb{L}_{\ell,F}$ comes from an étale ℓ-adic local system $\mathbb{L}_{\ell,S}$ on X_S (see e.g. [Pet20, Proposition 6.1] where the argument is performed for S the spectrum of a number ring, and references therein). Then, for a closed point $s \in S$ we see that $\mathbb{L}_{\mathbb{C}}$ comes from $\mathbb{L}_{\ell,\bar{s}} := \mathbb{L}_{\ell,S}|_{X_{\bar{s}}}$ as desired. We omit the details.

Notations 3.4. 1) Fixing $(X, r, \mathcal{L}, T_i)$ as in Section 2, we denote by

$$W(X, r, \mathcal{L}, T_i) \subset M_B(X, r, \mathcal{L}, T_i)(\mathbb{C})$$

the locus of weakly arithmetic local systems.
2) Fixing $(X, r, \mathcal{L})$, we denote by

$$W(X, r, \mathcal{L}) = \cup_{\{T_i\}} W(X, r, \mathcal{L}, T_i) \subset M_B(X, r, \mathcal{L})(\mathbb{C})$$

the locus of all weakly arithmetic local systems of rank r and determinant $\mathcal{L}$.

3.2. **Density.**

Theorem 3.5. *1) Fixing* $(X, r, \mathcal{L}, T_i)$, $W(X, r, \mathcal{L}, T_i)$ *is dense in* $M_B(X, r, \mathcal{L}, T_i)(\mathbb{C})$.
2) Fixing $(X, r, \mathcal{L})$, $W(X, r, \mathcal{L})$ *is dense in* $M_B(X, r, \mathcal{L})(\mathbb{C})$.

Proof. Ad 1): Let $T_{\mathbb{C}}$ be the Zariski closure of $W(X, r, \mathcal{L}, T_i)$ in $M_B(X, r, \mathcal{L}, T_i)(\mathbb{C})$ and $T_{\mathcal{O}_K}$ be the Zariski closure of $T_{\mathbb{C}}$ in $M_B(X, r, \mathcal{L}, T_i)$. As $T_{\mathbb{C}}$ is invariant under the group $\mathrm{Aut}_{\mathcal{O}_K}(\mathbb{C})$ of field automorphisms of $\mathbb{C}$ over $\mathcal{O}_K$, $T_{\mathcal{O}_K}$ is the Zariski closure of $W(X, r, \mathcal{L}, T_i)$ in $M_B(X, r, \mathcal{L}, T_i)$.

If $T_{\mathbb{C}} \neq M_B(X, r, \mathcal{L}, T_i)(\mathbb{C})$, one chooses a closed point $z \in M_B(X, r, \mathcal{L}, T_i)$ in Section 2.2 outside of $T_{\mathcal{O}_K}$. Then $(sp_{\mathbb{C},\bar{s}}^{\mathrm{top}})^{-1}(\mathbb{L}_{\ell,\bar{s},z})$ constructed in Section 2.2, with $\mathbb{L}_{\ell,\bar{s},z}$ arithmetic on $X_{\bar{s}}$, does not lie in $T_{\mathcal{O}_K}(\bar{\mathbb{Q}}_\ell)$. By invariance, for any τ, $((sp_{\mathbb{C},\bar{s}}^{\mathrm{top}})^{-1}(\mathbb{L}_{\ell,\bar{s},z}))^\tau$ does not lie on $T(\mathbb{C})$, a contradiction to the definition of weak arithmeticity.

Ad 2): By [EK23, Theorem 1.3], for a torsion rank 1 local system $\mathcal{L}$ given,

$$\cup_{T_i} M_B(X, r, \mathcal{L}, T_i)$$

is dense in $M_B(X, r, \mathcal{L})(\mathbb{C})$. Combined with 1), this yields 2).

$\square$

Remark 3.6. In [EK22, Weak Conjecture] it is predicted that arithmetic ℓ-adic local systems on a smooth quasi-projective variety $X_{\mathbb{F}}$ defined over a finite field $\mathbb{F}$ of characteristic different from ℓ are Zariski dense in a Mazur, or more generally, in a Chenevier deformation space. In [EK23, Conjecture 1.1] it is predicted that arithmetic local systems in $M_B(X, r, \mathcal{L})(\mathbb{C})$ are Zariski dense, where $\mathcal{L}$ is a torsion rank one local system. [3] This is not correct, see [LL22, Theorem 8.1.2] in which Landesman-Litt prove that over a very general genus ≥ 3 curve, for r small, arithmetic local systems have finite monodromy, and [LL22a, Corollary 1.2.10] for the consequence on non-density. Theorem 3.5 yields a replacement for this. However, there may be uncountably many weakly arithmetic local systems due to the free choice of τ.

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[3] In fact in [EK23] is expressed for local systems of geometric origin.

4. Proof of Theorem 1.3

4.1. **Proof.** [Proof of Theorem 1.3.] We have to prove that $\Gamma = \pi_1(X(\mathbb{C}), x(\mathbb{C}))$ has the weak integrality property (Definition 1.2) with respect to any (r, χ) where $r \geq 1$ is a natural number and χ is a character of Γ.

Let $\rho : \Gamma \to \mathrm{GL}_r(\mathbb{C})$ be as in Definition 1.2. Then the density of weakly arithmetic local systems in Theorem 3.5 shows that we may assume ρ corresponds to a weakly arithmetic local system $\mathbb{L}_{\mathbb{C}}$. Since such a system has quasi-unipotent local monodromies at infinity by Grothendieck's theorem [ST68, Appendix], we conclude that $\mathbb{L}_{\mathbb{C}}$ is as in Theorem 1.1 for some choice of quasi-unipotent conjugacy classes T_i. Thus by Theorem 1.1, we find a representation $\rho_\ell : \Gamma \to \mathrm{GL}_r(\bar{\mathbb{Z}}_\ell)$ with determinant χ which is irreducible over $\bar{\mathbb{Q}}_\ell$.

Next, the discussion in the introduction shows that the group Γ_0 presented after Definition 1.2 does not have the weak integrality property for $r = 2$ and χ being the trivial character. Thus the weak integrality property for Γ is a non-trivial obstruction for it to be of the shape $\pi_1(X(\mathbb{C}), x(\mathbb{C}))$. The proof of Theorem 1.3 is finished.

4.2. **Comments.** 1) In [Kli19, Theorem 1.1], the main integrality theorem [EG20, Theorem 1.1] for cohomologically rigid local systems is used as *the* criterion to decide that certain p-adic lattices are not the fundamental group of a smooth projective variety (or are not Kähler groups). Recall that Theorem 1.1 of the present article relies in part on the proof of [EG20, Theorem 1.1], but for the other part on de Jong's conjecture. The obstruction we obtain in Theorem 1.3 is of a new kind, it enables one consider all quasi-projective varieties at once, with all boundary conditions and not only smooth projective varieties.

2) In [dJEG22, Section 4] examples of local systems $\mathbb{L}_{\mathbb{C}}$ are constructed with the following property: they lie in $M_B(X, r, \mathcal{L}, T_i)(\mathbb{C})$ for some $\mathcal{L}$ and T_i, are in this Betti moduli with boundary conditions cohomologically rigid, and viewed in $M_B(X, r, \mathcal{L})(\mathbb{C})$, they are still rigid but no longer cohomologically rigid. So in a way this is good to keep track of the conditions at infinity.

5. Proof of Theorem 1.4, resp. case

The goal of this section is to prove the resp. case of Theorem 1.4. That is we have to show that we can construct infinitely many ℓ-adic local systems as Section 2 if we start with infinitely many $\mathbb{L}_{\mathbb{C}}$. The existence of infinitely many $\mathbb{L}_{\mathbb{C}}$ implies there is an irreducible component of $M_B(X, r, \mathcal{L}, T_i)$ the base change of which to K has dimension > 0. Thus, after replacing K by a finite extension over which all components are defined, we may assume that there is an irreducible component $Z \subset M_B(X, r, \mathcal{L}, T_i)$ such that Z_K is geometrically irreducible and of dimension > 0. Then Z is an integral scheme and the morphism $Z \to \mathrm{Spec}\,(\mathcal{O}_K)$ is dominant and of relative dimension ≥ 1. Let $Z^\circ \subset Z$ be a nonempty open subscheme which is smooth over $\mathrm{Spec}\,(\mathcal{O}_K)$ which maps into the open $\mathrm{Spec}\,(\mathcal{O}_K)^\circ$ of absolutely unramified points. We also may and do assume that Z° does not meet any irreducible component of $M_B(X, r, \mathcal{L}, T_i)$ except Z. Finally, we may further shrink $\mathrm{Spec}\,(\mathcal{O}_K)^\circ$ such that all fibres of $Z^\circ \to \mathrm{Spec}\,(\mathcal{O}_K)^\circ$ are geometrically irreducible.

We redo the argument of the proof 2.2 with Z° replacing M°. Namely, we pick $z \in |Z^\circ|$ closed with residue field $\mathbb{F}_{\ell^m}$. This determines an $\mathbb{F}_{\ell^m}$ local system $\mathbb{L}_z$ on X. Denote $t \in \mathrm{Spec}\,(\mathcal{O}_K)$ be the image of z and recall that Z_t° is smooth and geometrically irreducible of dimension ≥ 1. We let $S^\circ \subset S$ be the open where the residue characteristics are prime to $|\mathrm{GL}_r(\mathbb{F}_{\ell^m})|$. Next, let $s \in |S^\circ|$ of residue characteristic p. Since, for any finite field $\mathbb{F}$, $|\mathrm{GL}_r(\mathbb{F})|$ is a polynomial in $|\mathbb{F}|$, it follows that $|\mathrm{GL}_r(\mathbb{F}_{\ell^{m p^N}})|$ is prime to p for all $N \geq 0$. By the Lang-Weil estimates

$$\mathcal{Z} = \bigcup_{N \geq 0} Z_t^\circ(\mathbb{F}_{\ell^{m p^N}})$$

is infinite. Thus we may choose an infinite sequence $z_\alpha \in Z_t^\circ$ of closed points such that $|\mathrm{GL}_r(\kappa(z_\alpha))|$ is prime to p. Here $\kappa(z_\alpha)$ denotes the residue field. Hence, all the local systems $\mathbb{L}_{z_\alpha}$ give rise to local systems $\mathbb{L}_{z_\alpha,\bar{s}}$ on $X_{\bar{s}}$. Redoing the construction of Section 2.2 we find irreducible ℓ-adic systems $\mathbb{L}_{z_\alpha,\bar{s},\ell}$ on $X_{\bar{s}}$, which have arithmetic descent, with determinant $\mathcal{L}$ and monodromies T_i at infinity. These systems are pairwise non-isomorphic, as their mod ℓ reductions $\mathbb{L}_{z_\alpha,\bar{s}}$ are pairwise non-isomorphic and absolutely irreducible. For a prime $\ell' \neq p$ we fix an isomorphism $\sigma : \bar{\mathbb{Q}}_\ell \cong \bar{\mathbb{Q}}_{\ell'}$. Then the companions $\mathbb{L}^\sigma_{z_\alpha,\bar{s},\ell}$ constructed in Section 2.2 are likewise pairwise non-isomorphic. Indeed, assume $\mathbb{L}^\sigma_{z_\alpha,\bar{s},\ell}$ and $\mathbb{L}^\sigma_{z_\beta,\bar{s},\ell}$ are isomorphic. Let us choose a point $s' \to s$ such that the residue field extension $\kappa(s) \hookrightarrow \kappa(s')(\hookrightarrow \kappa(\bar{s}))$ is finite, so the arithmetic descents $\mathbb{L}^\sigma_{z_\alpha,s',\ell}$ and $\mathbb{L}^\sigma_{z_\beta,s',\ell}$ are defined. So they differ by a character of $\kappa(s')$. Thus $\mathbb{L}_{z_\alpha,\bar{s},\ell}$ and $\mathbb{L}_{z_\beta,\bar{s},\ell}$ descend to $X_{s'}$ over which they differ by a character of $\kappa(s')$. So they are isomorphic. Furthermore, they have the same determinant $\mathcal{L}$ and the same semi-simplification of the monodromies at infinity.

In the same manner we may deal with the case $\ell' = p$; we omit the details. This finishes the proof.

6. Proof of Theorem 1.5 i) ii)

The aim of this section is to prove Theorem 1.5 i) ii). The first part i), which comes from one part of the proof of the integrality for cohomologically rigid local systems, is entirely contained in [EG18] and is repeated here for the reader's convenience. We also insert a comment on Theorem 1.5 i), see Remark 6.1. The second part ii) is new. It says that the monodromy of the ℓ-adic local systems constructed in Theorem 1.1 is large if the initial topological local system $\mathbb{L}_\mathbb{C}$ has large monodromy.

6.1. **Part i).** If in Theorem 1.4, we assume that $\mathbb{L}_\mathbb{C}$ is cohomologically rigid, then the topological local system $(sp^{\mathrm{top}}_{\mathbb{C},\bar{s}})^*(\mathbb{L}_{\ell,\bar{s}})$ of the statement of Theorem 1.5 i) is constructed in this article exactly as in [EG18]. We give here a brief account: $\mathbb{L}_\mathbb{C}$ is defined over a ring of finite type A, so $\mathbb{L}_C = \mathbb{L}_A \otimes_A \mathbb{C}$, and we first take a non-zero homomorphism $A \to \bar{\mathbb{Z}}_{\ell_0}$, where ℓ_0 is a prime number. This defines an ℓ_0-adic local system $\mathbb{L}_{\ell_0}$ which descends, as all ℓ-adic local systems do, to $X_{\bar{s}}$ for $p = \mathrm{char}(s)$ large. Call it $\mathbb{L}_{\ell,0\bar{s}}$. By the very definition, $(sp^{\mathrm{top}}_{\mathbb{C},\bar{s}})^*(\mathbb{L}_{\ell_0,\bar{s}})$ has an A-model which via the embedding $A \subset \mathbb{C}$ gives back $\mathbb{L}_\mathbb{C}$. Cohomological rigidity implies that $\mathbb{L}_{\ell_0,\bar{s}}$ descends to $\mathbb{L}_{\ell_0,s}$ so we do not need de Jong's conjecture to have this arithmetic descent. Choose a field isomorphism $\sigma : \bar{\mathbb{Q}}_{\ell_0} \xrightarrow{\cong} \bar{\mathbb{Q}}_\ell$ for some prime ℓ prime to p. Cohomological rigidity also implies that the companions $\mathbb{L}^\sigma_{\ell_0,s} =: \mathbb{L}_{\ell,s}$ has the property that $(sp^{\mathrm{top}}_{\mathbb{C},\bar{s}})^*(\mathbb{L}_{\ell,\bar{s}})$ is cohomologically rigid. So for σ given and

each such $\mathbb{L}_{\mathbb{C}}$, we produce a topological local system $(sp_{\mathbb{C},\bar{s}}^{\mathrm{top}})^*(\mathbb{L}_{\ell,\bar{s}})$ which comes from an étale local system, is cohomologically rigid (with all the extra conditions preserved, determinant, monodromies at infinity). This finishes the proof.

Remark 6.1. The statement and the proof of Theorem 1.5 i) do not explain the relation between the initial $\mathbb{L}_{\mathbb{C}}$ and $(sp_{\mathbb{C},\bar{s}}^{\mathrm{top}})^*(\mathbb{L}_{\ell,\bar{s}})$. To do this, in [EG18] we argued that two non-isomorphic such $\mathbb{L}_{\mathbb{C}}$ yield two non-isomorphic $(sp_{\mathbb{C},\bar{s}}^{\mathrm{top}})^*(\mathbb{L}_{\ell,\bar{s}})$. So by finiteness of the set of such $\mathbb{L}_{\mathbb{C}}$, they are integral. As rigidity (alone) implies that A could have been taken from the beginning to be the localization at finitely many places of a number ring, we conclude that A can be taken to be a number ring. But we can not conclude that $\mathbb{L}_A \otimes_A \bar{\mathbb{Q}}_\ell$ is the system $(sp_{\mathbb{C},\bar{s}}^{\mathrm{top}})^*(\mathbb{L}_{\ell,\bar{s}})$ constructed earlier.

6.2. **Part ii).** Consider the set

$$Z_K \subset M_B(X, r, \mathcal{L}, T_i) \times_{\mathrm{Spec}\,(\mathcal{O}_K)} \mathrm{Spec}\,(K)$$

of points parametrizing local systems such that the Zariski closure of the monodromy does not contain SL_r. This set is closed, because the complement W_K is open by [AB94, Theorem 8.2]. Let $Z \subset M_B(X, r, \mathcal{L}, T_i)$ be the Zariski closure of Z_K. Then

$$W = M_B(X, r, \mathcal{L}, T_i) \setminus Z$$

is an open subscheme with $W_K = W \times_{\mathrm{Spec}\,(\mathcal{O}_K)} \mathrm{Spec}\,(K)$, i.e., all of the points of W in characteristic zero correspond to local systems such that the Zariski closure of the monodromy contains SL_r. So it is non-empty by assumption. Then in the arguments of 2.2 if we pick $z \in W$ we see that our arithmetic ℓ-adic local system $\mathbb{L}_{z,\bar{s},\ell}$ pulls back to a topological local system on X which corresponds to a point of W_K and hence has Zariski closure of the monodromy containing SL_r.

Having constructed one arithmetic ℓ-adic local system of rank r on $X_{\bar{s}}$ with large monodromy (in the sense that the Zariski closure of the image of monodromy contains SL_r) with determinant $\mathcal{L}$ and monodromies in T_i we can take the companions and show they have large monodromy too. For this one can use known facts on compatible systems of Galois representations, see for example [D'Ad20, Theorem 1.2.1] (where we disregard the crystalline statement) which guarantees that the companion of an ℓ-adic local system with large monodromy also has large monodromy.

Remark 6.2. We can argue as above and as in Section 5 to prove a variant of Theorem 1.5 ii) for infinite collections. We omit the detailed formulation and proof.

7. Remarks and Questions

7.1. **The crystalline version.** It is expected that there are ℓ to p companions in the sense of Deligne, see [Del80, Conjecture 1.2.10] for the original formulation and [AE19, Definition 1.4 5)] for a precise formulation. If so, Theorem 1.4 implies what is as of today a conjecture.

Conjecture 7.1. *If there is one (resp. infinitely many pairwise non-isomorphic) irreducible topological rank r complex local system (resp. systems) $\mathbb{L}_{\mathbb{C}}$ with torsion determinant $\mathcal{L}$ and quasi-unipotent monodromies in T_i at infinity, then there is a non-empty open subscheme $S^\circ \subset S$ such that for any closed point $s \in |S|$ it holds: there is one (resp. infinitely many pairwise non-isomorphic) Frobenius invariant iscrystal $M_{s'}$ on $X_{s'}$ (resp. infinitely many pairwise non-isomorphic Frobenius invariant isocrystals M_{s_α}, each defined on some X_{s_α}), where $s' \to s$ and $s_\alpha \to s$*

are closed points, with determinant $\mathcal{L}$ (as an F-isocrystal) and residues modulo $\mathbb{Z}$ being along $D_{i,\bar{s}}$ the log of the eigenvalues of T_i.

7.2. **Specific subloci.** Recall that Simpson's integrality conjecture [Sim92, p.9] is proven only for cohomologically rigid local systems [EG18, Theorem 1.1] while there are rigid non-cohomologically rigid local systems [dJEG22].

Question 7.2. Given a locally closed subset $W \subset M_B(X, r, \mathcal{L}, T_i^{ss})$ does Theorem 1.4 hold with the following modifications

  a) $\mathbb{L}_\mathbb{C}$ is assumed to lie in $W(\mathbb{C})$;
  b) the resulting $(sp_{\mathbb{C},\bar{s}}^{\text{top}})^*(\mathbb{L}_{\ell,\bar{s}}) \otimes_{\bar{\mathbb{Z}}_\ell} \bar{\mathbb{Q}}_\ell$ should lie in $W(\bar{\mathbb{Q}}_\ell)$?

The answer will not be positive for all W; we should only consider suitably natural loci in the moduli space. For example, Theorem 1.5 i) says that the answer is "yes" if W is the union of the smooth isolated points of $M_{dR}(X, r, \mathcal{L}, T_i^{ss})(\mathbb{C})$, i.e. the underlying $\mathbb{L}_\mathbb{C}$ are cohomologically rigid. Theorem 1.5 ii) says the answer is "yes" if we take W to be the locus of points in $M_{dR}(X, r, \mathcal{L}, T_i^{ss})(\mathbb{C})$ such that the Zariski closure of monodromy contains SL_r.

On the other hand, Simpson conjecture says the answer is "yes" if W is the union of the isolated points of $M_{dR}(X, r, \mathcal{L}, T_i^{ss})(\mathbb{C})$, i.e., the underlying $\mathbb{L}_\mathbb{C}$ are rigid. As a tentative generalization of Simpson's conjecture, we ask if the answer is "yes" when W is the union of the irreducible components of $M_B(X, r, \mathcal{L}, T_i^{ss})(\mathbb{C})$ of a given dimension.

7.3. **Pullbacks of semisimple local systems.** Using our methods we can reprove the following theorem. Its only known proof, as far as we understand, uses the existence of a tame pure imaginary harmonic metric.

Theorem 7.3. [Moc07, Theorem 25.30] [4] *Let $f : Y \to X$ be a morphism of normal quasi-projective complex varieties. The pullback by f of a semi-simple complex local system $\mathbb{L}_\mathbb{C}$ is semi-simple.*

We sketch a proof of the theorem using our techniques. The subtle part is that "being semi-simple" is neither a closed nor an open condition in moduli of representations, but it is constructible and this is enough for our arguments. We briefly indicate the proof strategy.

Proof. Let us reduce to the case where X and Y are smooth. Let $g : X' \to X$ be a resolution of singularities. Note that $g^*\mathbb{L}_\mathbb{C}$ is semi-simple as $\pi_1(X'(\mathbb{C})) \to \pi_1(X(\mathbb{C}))$ is surjective due to the fact that X is normal. Since $X' \times_X Y \xrightarrow{\text{projection}} Y$ is surjective, we can find a generically finite morphism of varieties $h : Y' \to Y$ with Y' smooth and a morphism $f' : Y' \to X'$ such that $g \circ f' = f \circ h$. Here is the corresponding picture

$$
\begin{array}{ccc}
Y' & \xrightarrow{\ f'\ } & X' \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\
Y & \xrightarrow{\ f\ } & X
\end{array}
$$

After replacing Y' by a generically finite cover, we may asume the function field extension $\mathbb{C}(Y')/\mathbb{C}(Y)$ is Galois. Then there exists an open $Y^\circ \subset Y$ which is

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[4]There is a typo in *loc.cit.* where the condition of normality is dropped. The reduction to the smooth case is explained in the proof of Theorem 7.3 .

smooth and such that $(Y')^\circ = h^{-1}(Y^\circ) \to Y^\circ$ is finite étale and Galois. Since Y is normal the map $\pi_1(Y^\circ(\mathbb{C})) \to \pi_1(Y(\mathbb{C}))$ is surjective and hence it suffices to show that $f^*\mathbb{L}_\mathbb{C}|_{Y^\circ}$ is semi-simple. Since $(Y')^\circ \to Y^\circ$ is Galois it is enough to show that $h^*f^*\mathbb{L}_\mathbb{C}|_{(Y')^\circ}$ is semi-simple. Since $h^*f^*\mathbb{L}_\mathbb{C} = (f')^*g^*\mathbb{L}_\mathbb{C}$ we see that it suffices to prove the result for $f'|_{(Y')^\circ}$.

From now on we assume X and Y smooth. To obtain a contradiction, we assume that $\mathbb{L}_\mathbb{C}$ is irreducible and that we are given a subsystem $\mathbb{L}' \subset f^*\mathbb{L}_\mathbb{C}$ of rank $0 < r' < r$ which does not split off. Consider the morphisms of Betti moduli spaces

$$(\star) \quad M_B^{split}(f, r', r) \longrightarrow M_B(f, r', r)$$

Here the moduli space $M_B(f, r', r)$ parametrizes pairs $(\mathbb{L}, \mathbb{L}' \subset f^*\mathbb{L})$ consisting of an irreducible local system $\mathbb{L}$ over X together with a rank r' subsystem $\mathbb{L}'$ over Y. And the moduli space $M_B^{split}(f, r', r)$ parametrizes triples $(\mathbb{L}, \mathbb{L}' \subset f^*\mathbb{L}, \tau)$ where $\tau : f^*\mathbb{L} \to \mathbb{L}'$ is a splitting of the inclusion map.

The moduli schemes in $(\star)$ are finite type schemes over $\mathbb{Z}$; this follows from arguments similar to those in Subsection 2.1. Hence, by Chevalley's theorem the image of $(\star)$ is a constructible set. Let $W \subset M_B(f, r', r)$ be the complement of the image; this is also a constructible subset. The assumption that we have $\mathbb{L}_\mathbb{C}$ and $\mathbb{L}' \subset f^*\mathbb{L}_\mathbb{C}$ tells us that W has a characteristic zero point. Let $W' \subset W$ be a subset which is an irreducible, locally closed subset of $M_B(f, r', r)$ containing a generic point of W of characteristic zero. After replacing W' by an open subset, we may and do assume that W' does not meet the closure of $W \setminus W'$. We view W' as a reduced, irreducible, locally closed subscheme of $M_B(f, r', r)$. As in Subsection 2.2 we choose a closed point $z \in W'$ in the smooth locus of $W' \to \mathrm{Spec}(\mathbb{Z})$. Say $\kappa(z) = \mathbb{F}_{\ell^m}$ for a prime number $\ell \geq 3$ and some $m \in \mathbb{N}_{>0}$.

Next, as in the introduction, we choose a model for f, i.e., we choose an integral affine scheme S of finite type over S, a morphism $Y_S \to X_S$ of smooth schemes over S such that Y_S and X_S have a good compactifications over S, and such that there is a dominant morphism $\mathrm{Spec}(\mathbb{C}) \to S$ such that the base change of $Y_S \to X_S$ by this morphism is isomorphic to f. Consider an open S° and a closed point $s \in S^\circ$ as in Subsection 2.2. We obtain an absolutely irreducible local $\mathbb{F}_{\ell^m}$-system $\mathbb{L}_{z,\bar{s}}$ over $X_{\bar{s}}$ which now comes endowed with a subsystem $\mathbb{L}'_{z,\bar{s}} \subset f^*_{\bar{s}}\mathbb{L}_{z,\bar{s}}$ of rank r' over $Y_{\bar{s}}$. Next, we consider the deformation space

$$D_{z,\bar{s}}(f, r', r)$$

classifying deformations of $\mathbb{L}_{z,\bar{s}}$ endowed with a rank r' subsystem of the pullback to $Y_{\bar{s}}$ deforming $\mathbb{L}'_{z,\bar{s}}$. The analogue of Proposition 2.2 holds in this situation: the morphism

$$D_{z,\bar{s}}(f, r', r) \xrightarrow{\iota} M_B(f, r', r)^\wedge_z$$

to the formal completion of the moduli space is an isomorphism. (But this time we do not know or claim formal smoothness for this deformation space or its reduced structure.) Since the local system $\mathbb{L}_{z,\bar{s}}$ and the subsystem $\mathbb{L}'_{z,\bar{s}} \subset f^*_{\bar{s}}\mathbb{L}_{z,\bar{s}}$ can be defined over a finite extension of $\kappa(s)$, we obtain an action of Φ^n on $D_{z,\bar{s}}(f, r', r)$; please compare with the discussion following Corollary 2.3.

To get a fixed point for the action of Φ^n, we restrict to deformations lying in W' as follows. Denote

$$W'_{z,\bar{s}} \subset D_{z,\bar{s}}(f, r', r)$$

the inverse image of the closed formal subscheme $(W')^\wedge_z \subset M_B(f, r', r)^\wedge_z$ by the isomorphism ι. Now we claim that $W'_{z,\bar{s}}$ is invariant under the action of Φ^n. Namely, $W'_{z,\bar{s}} \cong (W')^\wedge_z$ is formally smooth over $W(\mathbb{F}_{\ell^m})$ by our choice of z. Hence in order to see that it is stabilized by Φ^n it suffices to see that its set of $\bar{\mathbb{Z}}_\ell$-points is stabilized. However, $\bar{\mathbb{Z}}_\ell$-points of $(W')^\wedge_z$ are those $\bar{\mathbb{Z}}_\ell$-points of $D_{z,\bar{s}}(f, r', r)$ such that the corresponding $\bar{\mathbb{Q}}_\ell$ pair $(\mathbb{L}, \mathbb{L}' \subset f^*\mathbb{L})$ does not split[5]. Pulling back by the Frobenius automorphism does not change this property. So $W'_{z,\bar{s}}$ is invariant under the action of Φ^n.

We conclude as before that we obtain a $\bar{\mathbb{Z}}_\ell$-point of $W'_{z,\bar{s}}$ fixed by Φ^n. This point corresponds to a pair $(\mathbb{M}, \mathbb{M}' \subset f^*_{\bar{s}}\mathbb{M})$ consisting of a $\bar{\mathbb{Z}}_\ell$-local system on $X_{\bar{s}}$ and a subsystem of rank r' over $Y_{\bar{s}}$. Since our fixed point lies in W', by definition of W', the local system $\mathbb{M}$ is irreducible and the inclusion $\mathbb{M}' \subset f^*_{\bar{s}}\mathbb{M}$ does not split! Being fixed by Φ^n exactly signifies that $\mathbb{M}$ descends to a Weil sheaf ([Del80, Definition 1.1.10]) $\mathbb{M}^\circ$ on X_κ for a finite extension $\kappa/\kappa(s)$ and that the inclusion $\mathbb{M}' \subset f^*_{\bar{s}}\mathbb{M}$ comes from an inclusion $(\mathbb{M}')^\circ \subset f^*_\kappa\mathbb{M}^\circ)$ of Weil sheaves. We will see that this leads to a contradiction.

Namely, by [Del80, Proposition 1.3.14], we may assume that $\mathbb{M}^\circ$ is arithmetic ("par torsion"). While there is no condition on the determinant and the monodromies at infinity in the definition of moduli in $(\star)$, we now conclude the determinant of $\mathbb{M}$ is torsion (and the monodromies at infinity are quasi-unipotent). This means we can further assume $\mathbb{M}^\circ$ has finite order determinant (again "par torsion"). By [Laf02, Théorème VII.6] in dimension 1, and [EK12, Theorem 4.4], or [Del12, Section 0.7] in higher dimension, $\mathbb{M}$ is pure of weight 0, and so is its pullback to $Y_{\bar{s}}$. Thus [Del80, Lemme 3.4.3] implies that $f^*\mathbb{M}|_{Y_{\bar{\kappa}}}$ is semi-simple. This is the desired contradiction. $\square$

7.4. **The example of Becker-Breuillard-Varjú.** In [BBV22], the authors study the $\mathrm{SL}(2, \mathbb{C})$-character variety $\mathrm{Ch}(\Gamma_0, 2, \mathbb{I})$ of irreducible representations of the residually finite group Γ_0 defined in [DS05, Theorem 4], with two generators $\{a, b\}$ and one relation $b^2 = a^2 b a^{-2}$. They prove that $\mathrm{Ch}(\Gamma_0, 2, \mathbb{I})(\mathbb{C})$ consists of two points. The first one $\mathbb{L}_1$ has representation

$$\rho_1(a) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \rho_1(b) = \begin{pmatrix} j & 0 \\ 0 & j^2 \end{pmatrix},$$

where j is a primitive 3-rd root of unity. It is defined over $\mathbb{Q}(j)$. $\mathbb{L}_2$ is Gaois conjugate to $\mathbb{L}_1$. The authors compute

$$\rho(ab) = \frac{j}{\sqrt{2}} \begin{pmatrix} 1 & j \\ -1 & j \end{pmatrix}.$$

As $\mathrm{Trace}(\rho(ab)) = -\frac{1}{\sqrt{2}}$, $\mathbb{L}_1$ is not integral at $\ell = 2$, so $\mathbb{L}_2$ is not integral at $\ell = 2$ either. Furthermore, $\rho_1(a)$ does not preserve the eigenvalues of $\rho_1(b)$, so $\mathbb{L}_1$ and thus $\mathbb{L}_2$ are irreducible with dense monodromy in $\mathrm{SL}_2(\mathbb{C})$. They also compute that $H^1(\Gamma_0, \mathcal{E}nd^0(\mathbb{L}_i)) = 0$. That is, in $\mathrm{Ch}(\Gamma_0, 2, \mathbb{I})$, those two points are cohomologically rigid.

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[5]Here we need to use the careful choice of the component W' and the fact that no other component of the closure of W passes through z.

References

[AE19] Abe, T., Esnault, H.: *A Lefschetz theorem for overconvergent isocrystals with Frobenius structure*, Annales de l'École Normale Supérieure, **52** (4) (2019), 1243–1264.

[AB94] A'Campo, N., Burger, M.: *Réseaux arithmétiques et commensurateur d'après G. A. Margulis*, Invent. Math. **116** (1994), no. 1-3, 1–25.

[BBV22] Becker, O., Breuillard, E., Varjú, P.: *Random character varieties*, 2022, in progress.

[Che14] Chenevier, G.: *The p-adic analytic space of pseudocharacters of a profinite group and pseudorepresentations over arbitrary rings*, Automorphic forms and Galois representations. Vol. 1, 221–285, London Math. Soc. Lecture Note Ser., **414**, Cambridge Univ. Press, Cambridge, 2014.

[D'Ad20] D'Addezio, M.: *The monodromy groups of lisse sheaves and overconvergent F-isocrystals*, Selecta Math. (N.S.) **26** (2020), no. 3, Paper No. 45, 41 pp.

[dJ01] de Jong, J.: *A conjecture on the arithmetic fundamental group*, Israel J. of Math. **121** (2001), 61–84.

[dJEG22] de Jong, J., Esnault, H., Groechenig, M.: *Rigid non-cohomologically rigid local systems*, preprint 2022, 6 pages, https://arxiv.org/pdf/2206.03590.pdf v2, to appear in Algebraic Geometry and Physics.

[Del72] Deligne, P.:*Les constantes des équations fonctionnelles des fonctions L*, Proc. Antwerpen Conference, vol. 2, Lecture Notes in Mathematics, **349**, pp. 501–597, Springer Verlag, Berlin.

[Del73] Deligne, P.: *Comparaison avec la théorie transcendente*, in: SGA 7 II, Exp. No. XIV, pp. 116–164, Lecture Notes in Mathematics, vol. **340**, (1973).

[Del80] Deligne, P.: *La conjecture de Weil II*, Publ. math. I.H.É.S. **52** (1980), 137–252.

[Del12] Deligne, P.: *Finitude de l'extension de $\mathbb{Q}$ engendrée par des traces de Frobenius, en caractéristique finie,* Volume dedicated to the memory of I. M. Gelfand, Moscow Math. J. **12** (2012) no. 3.

[Dri01] Drinfeld, V.: *On a conjecture of Kashiwara*, Math. Res. Lett. **8** (2001), no. 5-6, 713–728.

[Dri12] Drinfeld, V.: *On a conjecture of Deligne*, Volume dedicated to the memory of I. M. Gelfand, Moscow Math. J. **12** (2012) no. 3.

[DS05] Druţu, C., Sapir, M.: *Non-linear residually finite groups*, J. of Algebra **285** (2005), 174–178.

[EG18] Esnault, H., Groechenig, M.: *Cohomologically rigid connections and integrality*, Selecta Mathematica **24** (5) (2018), 4279—4292.

[EG20] Esnault, H., Groechenig, M.: *Rigid connections and F-isocrystals*, Acta Mathematica **225** (1) (2020), 103—58.

[EK12] Esnault, H., Kerz, M.: *A finiteness theorem for Galois representations of function fields over finite fields (after Deligne).* Acta Mathematica Vietnamica **37** 4 (2012), 531–562.

[EK22] Esnault, H., Kerz, M.: *Density of Arithmetic Representations of Function Fields*, 18 pages, Epiga **6** (2022).

[EK23] Esnault, H., Kerz, M.: *Local systems with quasi-unipotent monodromy at infinity are dense*, preprint 2021, 9 pages

[Gai07] Gaitsgory, D.: *On de Jong's conjecture*, Israel J. Math. **157** (2007), 155–191.

[Kli19] Klingler, B.: *p-adic lattives are not Kähler groups*, Epiga **3** (2019), Article 6.

[Laf02] Lafforgue, L.: *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. math. **147** (2002), no. 1, 1–241.

[Lam91] Lam, T.Y.: *A first course in noncommutative rings*, Graduate Texts in Mathematics, **131**. Springer-Verlag, New York, 1991. xvi+397 pp. ISBN: 0-387-97523-3

[LL22a] Landesman, A., Litt, D.: *Geometric local systems on very general curves and isomonodromy*, https://arxiv.org/pdf/2202.00039.pdf v2 .

[LL22] Landesman, A., Litt, D.: *Canonical representations of surface groups*, https://arxiv.org/pdf/2205.15352.pdf v1.

[Maz89] Mazur, B.: *Deforming Galois Representations*, In: Galois groups over $\mathbb{Q}$ (Proc. Workshop, Berkeley/CA (USA), 1987), pp. 385–437, Mathematical Sciences Research Institute Publications, vol. **16**, Springer-Verlag, New York, 1989.

[Moc07]   Mochizuki, T.: *Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D-modules. II.* Mem. Amer. Math. Soc. **185** (2007), no. 870, xii+565 pp.

[MF82]   Mumford, D., Fogarty, J.: *Geometric invariant theory*, Second Edition. Ergebnisse der Mathematik und ihrer Grenzgebiete, **34**. Springer-Verlag, Berlin, 1982. xii+220 pp.

[Pet20]   Petrov, A.: *Geometrically irreducible p-adic local systems are de Rham up to a twist*, to appear in Duke, `https://arxiv.org/abs/2012.13372`.

[ST68]   Serre, J.-P., Tate, J.: *Good Reduction of Abelian Varieties*, Annals of Math. **88** no 3 (1968), 492–517.

[Sim92]   Simpson, C.: *Higgs bundles and local systems*, Publ. math. I.H.É.S. **75** (1992), 5–95.

[WE18]   Wang-Erickson, C., *Algebraic families of Galois representations and potentially semi-stable pseudodeformation rings*, Math. Ann. **371** (2018), no. 3-4, 1615–1681.

[SP]   *The Stacks project*, `https://stacks.math.columbia.edu`.

[SGA1]   *Séminaire de Géométrie Algébrique Revêtements étales et groupe fondamental*, Lecture Notes in Mathematics **224**, Springer Verlag (1971).

[SGA7.2]   *Séminaire de Géométrie Algébrique: Groupes de monodromie en géométrie algébrique*, Lecture Notes in Mathematics **340**, Springer Verlag (1973).

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