RIGID NON-COHOMOLOGICALLY RIGID LOCAL SYSTEMS

JOHAN DE JONG, HÉLÈNE ESNAULT, AND MICHAEL GROECHENIG

Abstract. For any even natural number $r \geq 2$, we construct an irreducible rigid non-cohomologically rigid complex local system of rank r on a smooth projective variety depending on r. For $r = 2$, we construct an irreducible rigid non-cohomogically rigid local system of rank 2 on a quasi-projective variety which becomes cohomologically rigid after fixing the conjugacy classes of the monodromies at infinity.

1. Introduction

In [EG18, Theorem 1.1] it is proven that on a smooth complex quasi-projective variety X, irreducible cohomologically rigid local systems with a fixed torsion determinant and fixed quasi-unipotent conjugacy classes of the monodromies at infinity are integral. This answers positively a conjecture of Simpson [Sim92, p.9] (formulated in the projective case) under the cohomological rigidity condition, which simply means that the Zariski tangent space of the considered moduli space at the point corresponding to the local system is 0. This property is used in a crucial way in the proof *loc. cit.* However, Simpson's conjecture concerns irreducible rigid local systems, that is those for which the corresponding moduli point is isolated, but perhaps fat. In dimension 1, Katz proved that all irreducible rigid local systems are cohomologically rigid [Kat96, Cor. 1.2.5]. The irreducible local systems on a Shimura variety of real rank ≥ 2 are also cohomologically rigid, see [EG21, Remark A.2]. In this short note we prove that

i) for any even natural number $r \geq 2$, there exists a smooth complex projective variety depending on r with an irreducible rigid non-cohomologically rigid complex local system of rank r on it,

ii) and there exists a smooth complex quasi-projective variety with an irreducible rigid non-cohomogically rigid local system of rank 2 which becomes cohomologically rigid after fixing the conjugacy classes of the monodromies at infinity.

The examples are an adaptation to algebraic geometry of [LM85, (2.10.4)] in which Lubotzky and Magid define the semi-direct product Γ_2 coming from the standard representation of the symmetric group in 3 variables and show that the so defined rank 2 representation Λ_2 of Γ_2 is rigid, see *loc.cit* (2.10.2), and has dim $H^1(\Gamma_2, \mathcal{E}nd(\Lambda_0)) = 1$.

In our initial construction, the monodromy was finite, in particular the Zariski closure of the monodromy group is equal to itself, the determinant is finite, and the monodromies at infinity are finite as well. Alexander Petrov noticed that we could apply Künneth formula to the exterior product of our examples with a standard cohomologically rigid local system with infinite monodromy to obtain examples as in (i) and (ii) (with different ranks) with infinite monodromy, see Remark 4.3.

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2. The standard representation of the symmetric group

We fix once and for all an even natural number $r \geq 2$. For S_{r+1} the symmetric group on $(r+1)$ elements, we define the standard representation

$$\Lambda_r = \mathrm{Ker}(\mathbb{Z}^{r+1} \xrightarrow{\Sigma} \mathbb{Z}),\ \underline{a} = (a_1, \ldots, a_{r+1}) \mapsto \sum_1^{r+1} a_i = \Sigma(\underline{a})$$

$$\rho_r : S_{r+1} \to GL(\Lambda_r)$$

by permuting the elements. This rank r representation has among others two properties:

- ($\star$) Λ_r is absolutely irreducible, i.e. $\Lambda_e \otimes_{\mathbb{Z}} \mathbb{C}$ is irreducible;
- ($\star\star$) $\Lambda_r \otimes_{\mathbb{Z}} \mathbb{C}$ is a direct summand of $\mathcal{E}nd^0(\Lambda_r \otimes_{\mathbb{Z}} \mathbb{C})$.

The property ($\star$) is very classical and is straightforward to check: the basis vectors $(e_i - e_{r+1})_{i=1,\ldots,r}$ of Λ_0 are permuted by the permutations $(i, r+1)_{i=1,\ldots,r}$,where $(e_i)_{i=1,\ldots,r+1}$ is the standard basis of $\mathbb{Z}^{r+1}$. The property ($\star\star$) is verified for $r=2$ by the character table in [LM85, (2.10.4)] and in [BDO15, Ex.1.2] in general. Here $\mathcal{E}nd^0$ denotes the trace-free endomorphisms. This defines the semi-direct product

$$\Gamma_r := \Lambda_r \rtimes S_{r+1}$$
$$0 \to \Lambda_r \to \Gamma_r \to S_{r+1} \to 1.$$

3. A rigid non-cohomologically rigid complex system on a smooth projective variety

Let E be an elliptic curve. We define the abelian variety

$$A_r = E \otimes_{\mathbb{Z}} \mathbb{Z}^r = \underbrace{E \times \ldots \times E}_{r-\text{times}}$$

of dimension r. Via the basis $(\epsilon_i = e_i - e_{r+1})_{i=1,\ldots,r}$ of Section 2, we define S_{r+1} as the subgroup of $\mathrm{Aut}(A_r)$ defined by

$$S_{r+1} = \mathbb{I}_E \otimes S_{r+1} \xrightarrow{\mathbb{I}\otimes\rho_r \text{ injective}} \mathrm{Aut}(E) \otimes_{\mathbb{Z}} GL(\Lambda_r) \subset \mathrm{Aut}(A_r).$$

For example, setting $\sigma = (12\ldots(r+1))$

$$(1)\quad \sigma\big(x_1, \ldots, x_r\big) = \big(-(x_1 + x_2 + \ldots + x_r), x_1, x_2, \ldots, x_{r-1}\big) \quad \text{for } (x_1, \ldots, x_r) \in E \otimes_{\mathbb{Z}} \mathbb{Z}^r.$$

In particular, the origin 0_A of the abelian variety is a fixpoint for this action.

We now do Serre's construction as in [Ser58, Sect. 15]. Let P_r be a smooth projective simply connected variety over $\mathbb{C}$ on which S_{r+1} acts without fixpoint ([Ser58, Prop. 15]). We define

$$Y_r = A_r \times_{\mathbb{C}} P_r,\ X_r = Y_r/S_{r+1}$$

where S_{r+1} acts diagonally. This yields the S_{r+1}-Galois cover

$$\pi_r : Y_r \to X_r$$

and the associated Galois exact sequence

$$0 \to \pi_1(Y_r, y_r) = H_1(A_r) = H_1(E) \otimes_{\mathbb{Z}} \Lambda_0 \to \pi_1(X_r, x_r) \to S_{r+1} \to 1$$

where $y_r = (0_A, p_r)$ is a $\mathbb{C}$-point of Y_r and $x_r = \pi_r(y_r)$. The Galois exact sequence is the same as the homotopy exact sequence of the smooth projective morphism

$$f_r : X_r \to P_r/S_{r+1}$$

with fibres isomorphic to A and with section

$$P_r/S_{r+1} \xrightarrow{\cong} (0_A \times_{\mathbb{C}} P_r)/S_{r+1} \subset X.$$

Thus

$$\pi_1(X_r, x_r) = \pi_1(Y_r, y_r) \rtimes S_{r+1}$$

where the action of S_{r+1} on $\pi_1(Y_r, y_r) = H_1(E) \otimes_{\mathbb{Z}} \mathbb{Z}^r$ is equal t to $\mathbb{I}_{H_1(E)} \otimes_{\mathbb{Z}} \rho_r$.

We denote by $\mathbb{L}_r$ the local system defined by the composite

$$\pi_1(X_r, x_r) \to S_{r+1} \xrightarrow{\rho_r} GL(\Lambda_r).$$

We denote by $M_B(X_r, 2, \det(\mathbb{L}_r))$ the Betti moduli space of rank r irreducible local systems on X_r with fixed determinant $\det(\mathbb{L}_r)$, by $M_B(X_r, r)$ the Betti moduli space of rank r irreducible local systems on X_r ([EG18, Prop. 2.1]) and by

$$\phi_r : M_B(X_r, r, \det(\mathbb{L}_r)) \to M_B(X_r, r)$$

the closed immersion induced by omitting the restriction on the determinant. We have

$$\mathbb{L}_r \in M_B(X_r, r, \det(\mathbb{L}_r)).$$

Proposition 3.1. *When r is even, $\mathbb{L}_r$ is rigid in $M_B(X_r, r, \det(\mathbb{L}_r))$ and $\phi_r(\mathbb{L}_r)$ is rigid in $M_B(X_r, r)$.*

Proof. It suffices to prove this for $\phi_r(\mathbb{L}_r) \in M_B(X_r, r)$ since ϕ is a closed immersion. Let T be an irreducible complex affine curve, together with a morphism $\tau : T \to M_B(X_r, r)$ and a complex point $t_0 \in T$ such that $\tau(t_0) = \mathbb{L}_r$. We fix a complex point $t \in T$, and denote by $\mathbb{L}_r(t)$ the irreducible complex local system corresponding to $\tau(t)$. As $H_1(A_r)$ is normal in $\pi_1(X, x)$, Clifford theory implies that $\mathbb{L}_r(t)|_{H_1(A_r)}$ is a sum of r characters $\chi_{t,1} \oplus \ldots \oplus \chi_{t,r}$, with $\chi_{t,i} \in \mathrm{Hom}(H_1(A_r), \mathbb{C}^*)$. As $\mathbb{L}_r(t)|_{H_1(A_r)}$ is S_{r+1}-invariant, the subset $\{\chi_{t,1}, \ldots, \chi_{t,r}\} \in \mathrm{Hom}(H_1(A_r), \mathbb{C}^*)$ consists of a union of S_{r+1}-orbits.

Claim 3.2. If r is even, the characters $\chi_{t,i}$ for $i = 1, \ldots, r$ are torsion characters, that is they lie in

$$\mathrm{Hom}(H_1(E) \otimes_{\mathbb{Z}} \mathbb{Z}^r, 2\pi i\mathbb{Q}/2\pi i\mathbb{Z}) \subset \mathrm{Hom}(H_1(A_r), \mathbb{C}^*).$$

Proof. Each single orbit of S_{r+1} in the set $\{\chi_{t,1}, \dots, \chi_{t,r}\}$ has length $s \le r < r+1$ and defines a quotient $S_{r+1} \to S_s$, where $s = 1$ or 2 if $r = 2$. For $r \ge 4$, the only non-trivial normal subgroup of S_{r+1} is the alternate group. Thus the image of S_{r+1} in S_s has order 1 or 2 in all cases. This implies that σ, which has odd order, must map to the identity in S_s, that is

$$\sigma(\chi_{t,i}) = \chi_{t,i} \text{ for } i = 1, \dots, r. \tag{2}$$

Pick $0 \neq \gamma \in H_1(E)$ and some $i \in \{1, \dots, r\}$. Write for $(x_1, \dots x_r) \in \mathbb{Z}^r$

$$\chi_{t,i}(\gamma \otimes x_1, \dots \gamma \otimes x_r) = \exp(2\pi i(a^1 x_1 + \dots + a^r x_r))$$

for some $a^1, \dots, a^r \in \mathbb{C}$. Then (1) and (2) yield

$$\exp(2\pi i(a^1 x_1 + \dots + a^r x_r)) = \exp(2\pi i(-a^1(x_1 + \dots + x_r) + a^2 x_1 + \dots + a^r x_{r-1})).$$

As this is true for all $x_j \in \mathbb{Z}$ we derive

$$a^j \in \frac{1}{(r+1)}\mathbb{Z} \text{ for all } j = 1, \dots, r,$$

thus $\chi_{t,i}$ has order dividing $(r+1)$. This finishes the proof. $\square$

The scheme map $T \to \mathrm{Hom}(H_1(A_r), \mathbb{C}^*)/S_{r+1}$ which sends t to $(\chi_{t,1}, \dots, \chi_{t,r})$ up to ordering, has by Claim 3.2 values in $(\mathrm{Hom}(H_1(A_r), 2\pi i\mathbb{Q}/2\pi i\mathbb{Z}))^r/S_r$. This is a countable subset of complex points, thus has to be finite. As T is connected, it is constant, thus equal to its value at $t = t_0$, thus is the image of the trivial character of rank r. Thus $\mathbb{L}_r(t)$ factors through S_{r+1}. Since there are only finitely many complex points in $M_B(X, 2)$ the monodromy representation of which factors through S_{r+1}, $\mathbb{L}_r$ is rigid. This concludes the proof for $\phi(\mathbb{L}_r)$. For $\mathbb{L}_r$, we assume that τ factors through $\tau_0 : T \to M_B(X, r, \det(\mathbb{L}_r))$ and we argue with $\mathbb{L}_r(t) \in M_B(X, r, \det(\mathbb{L}_r))$ corresponding to $\tau_0(t)$ in place of $\mathbb{L}(t)$, which does not change the argument. This concludes the proof. $\square$

Proposition 3.3. *Under the assumption of Proposition 3.1, $\mathbb{L}_r$ is not cohomologically rigid in $M_B(X_r, r, \det(\mathbb{L}_r))$, and $\phi(\mathbb{L}_r)$ is not cohomologically rigid in $M_B(X_r, r)$.*

Proof. The Hochschild-Serre spectral sequence for $\pi_1(X_r, x_r)$ reads

$$H^1(X_r, \mathcal{E}nd(^0\mathbb{L}_r)) = H^1(H_1(A_r), \mathcal{E}nd^0(\mathbb{L}_r))^{S_{r+1}} = \mathrm{Hom}_{S_{r+1}}(H_1(A_r), \mathcal{E}nd^0(\mathbb{L}_r)) = \mathrm{Hom}_{S_{r+1}}(\underbrace{\Lambda_r \oplus \dots \oplus \Lambda_r}_{r-\text{times}}, \mathcal{E}nd^0(\mathbb{L}_r)) \neq 0.$$

The non-vanishing on the right comes from Property $(\star\star)$. The Zariski tangent space to $M_B(X_r, r, \det(\mathbb{L}_r))$ at $\mathbb{L}_r$ is precisely $H^1(X_r, \mathcal{E}nd^0(\mathbb{L}_r))$, which is a direct summand $H^1(X_r, \mathcal{E}nd(\mathbb{L}_r))$, which in turn is the Zariski tangent space to $M_B(X_r, r)$ at $\phi(\mathbb{L}_r)$. This concludes the proof. $\square$

Remark 3.4. In Proposition 3.3, $\mathbb{L}_r$ by definition is of the shape $f_r^*\mathbb{L}'_r$, for an irreducible local system $\mathbb{L}'_r$ on P_r/S_r, which is uniquely defined. But

$$H^1(P_r/S_r, \mathcal{E}nd^0(\mathbb{L}'_r)) = H^1(S_r, \mathcal{E}nd^0(\mathbb{L}'_r)) = 0$$

as S_r is finite, thus $\mathbb{L}'_r$ is cohomologically rigid on P_r/S_r. Given an irreducible local system $\mathbb{L}$ defined by the representation $\rho : \pi_1(X, x) \to GL_r(\mathbb{C})$ on X smooth projective, we could dream of the existence of a factorization $\rho : \pi_1(X, x) \to \pi_1(Y, y) \xrightarrow{\rho'} GL_r(\mathbb{C})$ where Y is smooth projective, defining $\mathbb{L}'$ on Y, with the property that $\mathbb{L}'$ is cohomologically rigid. Even without requesting $\pi_1(X, x) \to \pi_1(Y, y)$ to come from geometry, we have no access to it. Together with [EG18, Theorem 1.1] it would prove Simpson's integrality conjecture in general on X smooth projective.

4. The quasi-projective case

We now assume $r = 2$. We replace A_2 in the previous section with the 2-dimensional torus

$$T = \mathbb{G}_m^3/(\text{diagonal } \mathbb{G}_m)$$

which is the Jacobian of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. We take the same P_2 as in Section 3 and define $V = T \times_{\mathbb{C}} P_2$, $U = V/S_3$ with the S_3-Galois cover $q : V \to U$. This yields the Galois exact sequence

$$0 \to \pi_1(V, v) = \pi_1(T) = \mathbb{Z}^2 \to \pi_1(U, u) \to S_3 \to 1$$

based at a point $(1, 1, 1) \times p = v$ with $u = q(v)$, which is identified with the homotopy exact sequence of $V \to P_2/S_3$ with fibre T. This fibration has a section $\{1, 1, 1\} \times P/S_3$, thus

$$\pi_1(U, x) = \Gamma_2.$$

Thus ρ_2 defines a local system $\mathbb{M}$ on U. Again we introduce the moduli spaces $M_B(U, 2, \det(\mathbb{M}))$ and $M_B(U, 2)$ with the forgetful map

$$\varphi : M_B(U, 2, \det(\mathbb{M})) \to M_B(U, 2).$$

Lemma 4.1. $\mathbb{M}$ *is rigid and not cohomologically rigid in* $M_B(U, 2, \det(\mathbb{M}))$, $\varphi(\mathbb{M})$ *is rigid and not cohomologically rigid in* $M_B(U, 2)$.

Proof. For $\varphi(\mathbb{M})$ the rigidity is proved in [LM85, Lemma 2.11] and is in fact a consequence of $M_B(U, 2)$ being 0-dimensional ([LM85, (2.10.1)]), which also shows that $\mathbb{M}$ is rigid. The cohomological statement $H^1(\Gamma_2, \mathcal{E}nd^0(\mathbb{M})) = \mathbb{C}$ is proved in [LM85, 2.10.4], which is a special case of Property $(\star\star)$. Hence, $\mathbb{M}$ and $\varphi(\mathbb{M})$ are not cohomologically rigid. $\square$

We now fix the monodromies at infinity to be those of $\mathbb{M}$. As the global monodromy of $\mathbb{M}$ is finite, so are the monodromies at ∞, in particular they are quasi-unipotent. We define $M_B(U, 2, \det(\mathbb{M}), \mathrm{mon}(\mathbb{M}))$ as in [EG18, Section 2]. By [EG18, Remark 2.4], the Zariski tangent space at $\mathbb{M}$ of $M_B(U, 2, \det(\mathbb{M}), \mathrm{mon}(\mathbb{M}))$ is equal to

$$\mathrm{Ker}\big(H^1(U, \mathcal{E}nd^0(\mathbb{M})) \xrightarrow{\text{restriction}} \oplus_\gamma H^1(\langle\gamma\rangle, \mathcal{E}nd^0(\mathbb{M}))\big), \tag{3}$$

where the γ are small loops around the components at ∞ of a smooth projective compactification $U \hookrightarrow \bar{U}$ with strict normal crossings at infinity, and $\langle\gamma\rangle$ is the free commutative group spanned by it. We denote by $\mathbb{M}_0$ the local system $\mathbb{M}$ viewed in $M_B(U, 2, \det(\mathbb{M}), \mathrm{mon}(\mathbb{M}))$.

Lemma 4.2. $\mathbb{M}_0$ *is cohomologically rigid.*

Proof. By (3), it is enough to find one loop γ at infinity such that the restriction map

$$H^1(\pi_1(U,x), \mathcal{E}nd^0(\mathbb{M})) \to H^1(\langle\gamma\rangle, \mathcal{E}nd^0(\mathbb{M}))$$

is injective. The action of S_3 on on $\mathbb{P}^2 \setminus T$, which is a union of the 3 coordinate lines, permutes them. This defines a S_3-equivariant compactification

$$V = T \times_{\mathbb{C}} P \hookrightarrow \bar{V} = \mathbb{P}^2 \times_{\mathbb{C}} P,$$

thus a compactification

$$j : U \hookrightarrow \bar{U} = \bar{V}/S_3$$

where S_3 acts on $\bar{V}$ diagonally. This yields the commutative diagram

$$\begin{tikzcd}
P_2/S_3 \arrow[r] \arrow[dr, "="] & U \arrow[r] \arrow[d] & \bar{U} \arrow[d] \\
& P_2/S_3 \arrow[r, "="] & P/S_3
\end{tikzcd}$$

with fibres $T \hookrightarrow \mathbb{P}^2$ above the image $\bar{p}$ of $p \in P_2$ in P_2/S_3. We take γ in the fibre $\mathbb{P}^2$ which winds around one of the irreducible components of $\mathbb{P}^2 \setminus T$. Thus, it maps to 1 in $\pi_1(P_2/S_3, \bar{p}) = S_3$, that is $\gamma \in \mathbb{Z}^2 = \pi_1(T) \subset \pi_1(U,x)$. Thus $\mathbb{M}$ restricted to $\langle\gamma\rangle$ is trivial. The restriction map

$$H^1(\pi_1(U,u), \mathcal{E}nd^0(\mathbb{M})) \xrightarrow{\text{rest}} H^1(\langle\gamma\rangle, \mathcal{E}nd(^0\mathbb{M}))$$

is then identified with

$$\mathrm{Hom}_{S_3}(\mathbb{M}, \mathcal{E}nd^0(\mathbb{M})) \to \mathrm{Hom}(\mathbb{Z}, \mathcal{E}nd^0(\mathbb{M}))$$

via the inclusion $\mathbb{Z}\langle\gamma\rangle \to \mathbb{Z}^2$ of $\mathbb{Z}$-modules. Therefore, the map rest is injective. This concludes the proof. □

Remark 4.3. This remark is due to Alexander Petrov. Let S be a smooth complex projective variety, $\mathbb{L}_S$ be an irreducible rigid local system on S. We denote by $\mathbb{L}'_r$ the exterior product $\mathbb{L}_r \boxtimes \mathbb{L}_S$ on $X_r \times_{\mathbb{C}} S$ in Proposition 3.1, by $\mathbb{L}'$, resp. $\mathbb{L}'_0$ the exterior product $\mathbb{L} \boxtimes \mathbb{L}_S$ on $U \times_{\mathbb{C}} S$ viewed in $M_B(U \times_{\mathbb{C}} S, 2 \cdot \mathrm{rank}(\mathbb{L}_S), \det(L'))$, resp. $M_B(U \times_{\mathbb{C}} S, 2 \cdot \mathrm{rank}(\mathbb{L}_S), \det(L')), \mathrm{mon}(\mathbb{L}'))$ in Lemma 4.1, resp. Lemma 4.2. Then

Claim 4.4. Proposition 3.3 remains true with $\mathbb{L}_r$ replaced with $\mathbb{L}'_r$ and Lemma 4.1 and Lemma 4.2 remain true with $\mathbb{L}$ replaced with $\mathbb{L}'$.

Proof. We applied Künneth formula to H^1 of $\mathcal{E}nd(\mathbb{L}'_r) = \mathcal{E}nd(\mathbb{L}_r) \boxtimes \mathcal{E}nd(\mathbb{L}_S)$. This yields the cohomological part of the statement. Similary for $\mathbb{L}'$. As for rigidity: In a T-deformation $(\mathbb{L}'_r)_t$ of $\mathbb{L}'_r$ as in Proposition 3.1, the restriction of $(\mathbb{L}'_r)_t$ to $x \times S$ for any complex point $x \in X_r$ is isomorphic to the restriction of $(\mathbb{L}'_r)_{t_0}$ to $x \times S$ by rigidity of $\mathbb{L}_S$, and similarly the restriction of $(\mathbb{L}')_t$ to $X_r \times s$ for any complex point $s \in S$ is isomorphic to the restriction of $(\mathbb{L}')_{t_0}$ to $X_r \times s$ by rigidity of $\mathbb{L}_r$. The Künneth property for $\pi_1(X_r \times_{\mathbb{C}} S)$ implies that $(\mathbb{L}'_r)_t \cong (\mathbb{L}'_r)_{t_0}$. Similary for $\mathbb{L}'$.
□

In particular, we can take S to be a positive projective Shimura variety of real rank ≥ 2, on which all the irreducible local systems are rigid, even cohomologically rigid, and take $\mathbb{L}_S$ to be a rank ≥ 2 irreducible local system. In this way, we obtain examples as in (i) and (ii) (with different ranks) with infinite monodromy.

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Mathematics Hall, 2990 Broadway, New York, NY 10027, USA
Email address: `dejong@math.columbia.edu`

Freie Universität Berlin, Arnimallee 3, 14195, Berlin, Germany
Email address: `esnault@math.fu-berlin.de`

Mathematics, University of Toronto, Bahen Centre, Toronto, Ontario Canada
Email address: `michael.groechenig@utoronto.ca`