Appendix A. Frobenius structures and unipotent monodromy at infinity by Hélène Esnault and Michael Groechenig

We fix an irreducible affine base scheme S which is of finite type over a universally Japanese ring. For the purpose of this appendix, S will either be $\operatorname{Spec}\mathbb{C}$, $\operatorname{Spec}\mathbb{F}_q$ or $\operatorname{Spec} R$, where R is a finite type algebra. Let us denote by $\bar{X}_S$ a smooth and projective S-scheme with a relatively very ample line bundle $\mathcal{O}_{\bar{X}_S}(1)$. Let $X_S \subset \bar{X}_S$ be an open subscheme such that $\bar{X}_S \setminus X_S$ is a strict normal crossings divisor (snc) $D_S = \bigcup_{\mu=1}^{c} D_S^{\mu}$. The sheaf of degree n Kähler differentials with log-poles along D will be denoted by $\Omega^n_{\bar{X}_S/S}\langle D\rangle$. For $\mu = 1,\ldots,c$ we write $\operatorname{res}_\mu\colon \Omega^1_{\bar{X}_S/S}\langle D\rangle \to \mathcal{O}_{D_S^\mu/S}$ for the residue map.

Definition A.1. (a) A *log-dR local system* on $\bar{X}_S$ is a pair (E_S,∇_S) where E_S is a vector bundle of rank r on $\bar{X}_S$ and

$$\nabla\colon E_S \to E_S \otimes \Omega^1_{\bar{X}_S/S}\langle D\rangle$$

is a flat logarithmic connection such that $\operatorname{res}_\mu(\nabla) \in H^0(D_S^\mu, \mathsf{End}(E_S|_{D_S^\mu}))$ is *nilpotent* for all $\mu = 1,\ldots,c$.

(b) We say that (E_S,∇_S) is *strongly cohomologically rigid*, if

$$\mathbb{H}^1\big(\bar{X}_S, [\mathsf{End}(E_S) \xrightarrow{\mathsf{End}(\nabla_S)} \mathsf{End}(E_S)\otimes\Omega^1\langle D\rangle \xrightarrow{\mathsf{End}(\nabla_S)} \cdots]\big) = 0. \tag{1}$$

(c) A *log-Higgs bundle* on $\bar{X}_S$ is a pair (V_S,θ_S), where V_S is a vector bundle of rank r on $\bar{X}_S$ and θ_S is an $\mathcal{O}$-linear morphism $V \to V\otimes\Omega^1_{\bar{X}}\langle D\rangle$ satisfying $\theta_S\wedge\theta_S = 0$.

(d) A log-Higgs bundle (V_S,θ_S) is called *strongly cohomologically rigid*, if

$$\mathbb{H}^1\big(\bar{X}_S, [\mathsf{End}(V_S) \xrightarrow{\mathsf{End}(\theta_S)} \mathsf{End}(V_S)\otimes\Omega^1\langle D\rangle \xrightarrow{\mathsf{End}(\theta_S)} \cdots]\big) = 0. \tag{2}$$

Remark A.2. (a) If $S = \operatorname{Spec}\mathbb{C}$, the underlying vector bundle E of a log-dR local system has vanishing Chern classes. This follows from the formula for the Atiyah class of E given in [EV86, Proposition B.1]. In addition, the left-hand side of (1) computes $H^1(X(\mathbb{C}), \mathsf{End}(E_{\mathbb{C},\mathrm{an}})^{\mathsf{End}(\nabla_{\mathbb{C}})})$. Indeed, as $\operatorname{res}_\mu(\nabla_{\mathbb{C}})$ is nilpotent for $\mu = 1,\ldots,c$, so is $\operatorname{res}_\mu(\mathsf{End}(\nabla_{\mathbb{C}}))$, thus $\mathsf{End}(E_{\mathbb{C}})$ is Deligne's extension the cohomology of which computes analytically Rj_* where $j : X_{\mathbb{C},\mathrm{an}} \to \bar{X}_{\mathbb{C},\mathrm{an}}$, see [Del70, II, Proposition 3.13, Corollaire 3.14].

(b) The notion of *strong cohomological rigidity* is more restrictive than the one of *cohomological rigidity* used in [Kat96, EG18]. A cohomological rigid local system, in the traditional sense, does not have any non-trivial infinitesimal deformations which leave the monodromies at infinity invariant. A strongly cohomologically rigid log-dR local system does not have any non-trivial infinitesimal deformations, independently of any constraints at the boundary.

Definition A.3 (Arithmetic models)**.** Let $(\bar{X}_{\mathbb{C}}, D_{\mathbb{C}}, \mathcal{O}_{\bar{X}}(1))$ be a triple consisting of a smooth projective complex variety $\bar{X}_{\mathbb{C}}$, an snc divisor $D_{\mathbb{C}}$, and a very ample line bundle $\mathcal{O}_{\bar{X}_{\mathbb{C}}}(1)$.

(a) An *arithmetic model* for $(\bar{X}_{\mathbb{C}}, D_{\mathbb{C}}, \mathcal{O}_{\bar{X}_{\mathbb{C}}}(1))$ is given by an affine scheme S where $\Gamma(S,\mathcal{O}_S)$ is a finite type subring $R\subset\mathbb{C}$, a smooth projective S-scheme $\bar{X}_S$ together with an snc divisor D_S such that

$$\bar{X}_{\mathbb{C}} = \bar{X}_S \times_S \operatorname{Spec}\mathbb{C} \text{ and } D_{\mathbb{C}} = D_S \times_S \operatorname{Spec}\mathbb{C},$$

and a relatively very ample line bundle $\mathcal{O}_{\bar{X}_S}(1)$ pulling back to $\mathcal{O}_{\bar{X}_{\mathbb{C}}}(1)$.

(b) Let $\{(E^i_{\mathbb{C}}, \nabla^i_{\mathbb{C}})\}_{i\in I}$ be a family of log-dR local systems on $X_{\mathbb{C}}$. An arithmetic model for $(\bar{X}_{\mathbb{C}}, D_{\mathbb{C}}, \mathcal{O}_{\bar{X}_{\mathbb{C}}}, \{E^i_{\mathbb{C}}, \nabla^i_{\mathbb{C}}\}_{i\in I})$ is given by an arithmetic model for $(\bar{X}_{\mathbb{C}}, D_{\mathbb{C}}, \mathcal{O}_{\bar{X}_{\mathbb{C}}}(1))$ as in (a), and log-dR local systems $\{(E^i_S, \nabla^i_S)\}_{i\in I}$ on X/S satisfying

$$(E^i_{\mathbb{C}}, \nabla^i_{\mathbb{C}}) = (E^i_S, \nabla^i_S)|_{\bar{X}_{\mathbb{C}}} \text{ for all } i \in I.$$

Theorem A.4. *Suppose that every stable log-dR local system $(E_{\mathbb{C}}, \nabla_{\mathbb{C}})$ of rank r on $(\bar{X}_{\mathbb{C}}, D_{\mathbb{C}})$ is strongly cohomologically rigid. Then, there exists a finite type subalgebra $R \subset \mathbb{C}$ and a model of $(\bar{X}_{\mathbb{C}}, X_{\mathbb{C}}, D_{\mathbb{C}})$ over $S = \operatorname{Spec} R$ such that every stable log-dR local system of rank r on $(\bar{X}_{\mathbb{C}}, D_{\mathbb{C}})$ has an S-model (E_S, ∇_S) such that for every finite field k and every morphism $R \to W(k)$ the formal flat connection*

$$(\widehat{E}_W, \widehat{\nabla}_W)$$

is endowed with the structure of a torsionfree Fontaine-Lafaille module *on $X_W = \bar{X}_W \setminus D_W$.*

Remark A.5. In [EG20] we prove a stronger result for the case where $D_{\mathbb{C}} = \emptyset$. The assumptions of *loc. cit.* are less stringent, as they apply more generally to arbitrary rigid dR local system, i.e. isolated points of the moduli space $\mathcal{M}_{dR}$. The additional assumptions above allow one to simplify the argument significantly.

A.1. **Construction of a suitable arithmetic model.** Moduli spaces of logarithmic flat connections on complex varieties were constructed by Nitsure in [Nit93]. Using Langer's boundedness (see [Lan14]), this construction was extended to more general base schemes ([Lan14, Theorem 1.1]):

Theorem A.6 (Langer)**.** *For a fixed polynomial P there exists a quasi-projective S-scheme $\mathcal{M}_{dR}(\bar{X}_S, D_S)$ of stable flat logarithmic connections on $\bar{X}_S$ with Hilbert polynomial P.*

More generally, Langer constructs moduli spaces for semistable Λ-modules, where Λ is a ring of operators in the sense of [Sim94]. It is explained on p. 87 of *loc. cit.* that flat logarithmic connections are special case of the general theory of Λ-modules. We are interested in moduli spaces of flat logarithmic connections with *vanishing Chern classes* (see Remark A.2). The corresponding Hilbert polynomial satisfies

$$P_0(n) = \int r \cdot \mathrm{td}_{\bar{X}_{\mathbb{C}}} \mathrm{ch}(\mathcal{O}_{\bar{X}_{\mathbb{C}}}(n)) \text{ for all } n \in \mathbb{N}.$$

Corollary A.7. *There exists a closed subscheme $\mathcal{M}_{log\text{-}dR}(\bar{X}_S, D_S) \subset \mathcal{M}_{dR}(\bar{X}_S, D_S)$, which is the moduli space of stable log-dR local systems with Hilbert polynomial P_0.*

Proof. There is an étale covering $\left(U_i \to \mathcal{M}_{dR}(\bar{X}_S, D_S)\right)_{i\in I}$ such that we have a universal family $(\mathcal{E}_{U_i}, \nabla_{U_i})$ on $U_i \times_S \bar{X}_S$. By stability, such a universal log-dR U_i-family is well-defined up to tensoring by a line bundle on U_i. By construction, the characteristic polynomial $\chi_{i,\mu}(T)$ of $\mathrm{res}_\mu(\nabla_{U_i})$ is a section of a locally free sheaf on U_i. We let $Z_i \hookrightarrow U_i$ be the closed immersion corresponding to the vanishing locus of $(\chi_{i,\mu}(T) - T^r), \mu = 1, \ldots, c$. This closed immersion is independent of the choice of a U_i-universal family, since tensoring by a line bundle on U_i leaves $\chi_{i,\mu}$ invariant. We may thus apply faithfully flat descent theory to glue those closed immersions to a closed embedding

$$Z \hookrightarrow \mathcal{M}_{dR}(\bar{X}_S, D_S).$$

The scheme Z is the sought-for moduli space $\mathcal{M}_{log\text{-}dR}(\bar{X}_S, D_S)$. □

We record the following consequence of non-abelian Hodge theory for later reference.

Theorem A.8. *For every strongly cohomologically rigid log-dR local system $(E_{\mathbb{C}}, \nabla_{\mathbb{C}})$ on $\bar{X}_{\mathbb{C}}$ there exists an F-filtration $\cdots \subset F^i \subset F^{i-1} \subset \cdots F^0 = E$ satisfying Griffiths transversality $\nabla\colon F^i \to F^{i-1} \otimes_{\mathcal{O}_X} \Omega^1_X\langle D\rangle$ and with the associated graded sheaves $\mathrm{gr}^i_F E = F^i/F^{i+1}$ being locally free. The associated Higgs bundle is denoted by*

$$\big(\mathrm{gr}_F E, \mathrm{KS}\big),$$

where KS *stands for* Kodaira-Spencer *and is defined by the linear maps*

$$\mathrm{gr}_F \nabla\colon \mathrm{gr}^i_F E \to \mathrm{gr}^{i-1}_F E \otimes_{\mathcal{O}_X} \Omega^1_X\langle D\rangle.$$

Proof. Mochizuki proved in [Mo06, Theorem 10.5] that every log-dR local system on $\bar{X}_{\mathbb{C}}$ can be complex analytically deformed to a polarised variation of Hodge structures, which implies the existence of the requisite F-filtration on the rigid $(E_{\mathbb{C}}, \nabla_{\mathbb{C}})$. In *loc. cit.*, this is stated in terms of Betti local systems on $X_{\mathbb{C}} = \bar{X}_{\mathbb{C}} \setminus D_{\mathbb{C}}$. This is an equivalent perspective, by virtue of the Riemann-Hilbert correspondence which is complex analytic. Due to strong cohomological rigidity, $(E_{\mathbb{C}}, \nabla_{\mathbb{C}})$ cannot be deformed in a non-trivial manner. We conclude that $(E_{\mathbb{C}}, \nabla_{\mathbb{C}})$ underlies a polarised variation of Hodge structures. ☐

Remark A.9. Stability of the log-Higgs bundle $\left(\bigoplus_j \mathrm{gr}^{ij}_F E^i_{\mathbb{C}}, \mathrm{KS}(\nabla^i_{\mathbb{C}})\right)$ is implied by Mochizuki's parabolic Simpson correspondence [Mo06]. We remark that the parabolic structure is trivial in the case at hand, since we assume that the monodromies around the divisor at infinity are unipotent and therefore in this case, parabolic stability amounts to stability in the usual sense of log-Higgs bundles. See [Sim90, p. 722] where the triviality of the parabolic structure is justified for the curve case. The argument given there generalises directly to higher dimensional varieties.

Subsequently, for every strongly cohomologically rigid log-dR local system $(E_{\mathbb{C}}, \nabla_{\mathbb{C}})$ on $\bar{X}_{\mathbb{C}}$ we fix the F-filtration constructed in Theorem A.8.

Proposition A.10. *We keep the assumptions of Theorem A.4. There exists an arithmetic model $(S, \bar{X}_S, D_S, \mathcal{O}_{X_S}(1))$ of $(X_{\mathbb{C}}, D_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(1))$ such that*

(a) *all rank r log-dR local systems $(E^i_{\mathbb{C}}, \nabla^i_{\mathbb{C}})_{i\in I}$ have a locally free model $(E^i_S, \nabla^i_S)_{i\in I}$ over S,*
(b) *the models $(E^i_S, \nabla^i_S)_{i\in I}$ are also strongly cohomologically rigid,*
(c) *the filtrations $(F^{ij}_{\mathbb{C}} \subset E^i_{\mathbb{C}})$ are defined over S such that the S-relative filtrations $F^{ij}_S \subset E^i_S$ satisfy the* Griffiths-transversality condition*,*
(d) *for every $i \in I$ the associated graded*

$$\left(\bigoplus_j \mathrm{gr}^{ij}_F E^i_S, \mathrm{KS}(\nabla^i_S)\right)$$

is a stable *logarithmic Higgs bundle which is also locally free.*

Furthermore, if $s\colon \mathrm{Spec}\,\bar{k} \to S$ is a geometric point of S, then

(e) *(E_s, ∇_s) is a log-dR local system on $\bar{X}_s = \bar{X}_S \times_S \mathrm{Spec}\,\bar{k}$, then there exists $i \in I$ such that $(E_s, \nabla_s) = (E^i_S, \nabla^i_S)|_{X_s}$,*
(f) *$p = \mathrm{char}(\bar{k}) > 2r + 2$, and*
(g) *$S \to \mathrm{Spec}\,\mathbb{Z}$ is smooth.*

Proof. The proof is analogous to the one of [EG20, Proposition 3.3] and will therefore only be sketched. Consider the set $\mathcal{R}$ of all finite type subrings $R \subset \mathbb{C}$. Since $\mathbb{C} = \bigcup_{R\in\mathcal{R}} R$ and $(\bar{X}, D_{\mathbb{C}})$ are defined in terms of finitely many homogenous equations, there exists $\widetilde{R} \in \mathcal{R}$ such that $(\bar{X}_{\mathbb{C}}, D_{\mathbb{C}})$

are obtained by base change from a pair of projective schemes $(\bar{X}_{\widetilde{S}}, D_{\widetilde{S}}) \subset \mathbb{P}^N_{\widetilde{S}}$, where we write $\widetilde{S}$ for $\operatorname{Spec} \widetilde{R}$. We may assume that $D_{\widetilde{S}}$ is an snc divisor and that $\bar{X}_{\widetilde{S}}$ is smooth.

We now consider the moduli space $\mathcal{M}_{log\text{-}dR}(\bar{X}_{\widetilde{S}}/\widetilde{S})$. Since

$$\mathcal{M}_{log\text{-}dR}(\bar{X}_{\mathbb{C}}/\mathbb{C}) \simeq \mathcal{M}_{log\text{-}dR}(\bar{X}_{\widetilde{S}}/\widetilde{S}) \times_{\widetilde{S}} \operatorname{Spec} \mathbb{C}$$

is finite and flat over $\operatorname{Spec} \mathbb{C}$, there exists a finite type algebra $\tilde{R} \subset R$, such that the base change (we denote $\operatorname{Spec} R$ by S)

$$\mathcal{M}_{log\text{-}dR}(\bar{X}_S/S) = \mathcal{M}_{log\text{-}dR}(\bar{X}_{\widetilde{S}}/\widetilde{S}) \times_{\widetilde{S}} S$$

is finite and flat over S.

Since there are only finitely many log-dR local systems $(E^i_{\mathbb{C}}, \nabla^i_{\mathbb{C}})_{i\in I}$ over $\mathbb{C}$, we may assume that they have stable and locally free models $(E^i_S, \nabla^i_S)_{i\in I}$ over S. This amounts to property (a) above. By further enlarging R we obtain strong cohomological rigidity (property (b)), and properties (c,d) about the F-filtrations and the associated graded log-Higgs bundles.

The S-models above give rise to sections

$$[(E^i_S, \nabla^i_S)]_{i\in I} \colon S \colon \mathcal{M}_{log\text{-}dR}(\bar{X}_{\widetilde{S}}/\widetilde{S}). \tag{3}$$

Since the structural morphism $\mathcal{M}_{log\text{-}dR}(\bar{X}_{\widetilde{S}}/\widetilde{S}) \to S$ is finite and flat, we infer that the sections of (3) are jointly surjective. This implies (e). By inverting $(2r+2)!$ we can achieve (f). And, property (g) can be arranged by passing to the maximal open subset of S which is smooth over $\operatorname{Spec} \mathbb{Z}$. $\square$

A.2. **Applications of the Higgs-de Rham flow.** In this subsection, we apply the *logarithmic Higgs-de Rham flow* from [LSYZ19] (the smooth and proper case is due to [LSZ13]).

We fix an arithmetic model as in Proposition A.10. Let $\bar{k}$ be an algebraic closure of a finite field and let $s\colon \operatorname{Spec} \bar{k} \to S$ be a geometric point of S.

Definition A.11 ([LSZ13, LSYZ19])**.** An f-periodic Higgs-de Rham flow on X_s is a tuple

$$(E_0, \nabla_0, F_0, \phi_0, E_1, \nabla_1, F_1, \dots, E_{f-1}, \nabla_{f-1}, F_{f-1}, \phi_{f-1}),$$

where for all $i \in \mathbb{Z}/f\mathbb{Z}$ we have a log-dR local system (E_i, ∇_i, F_i) with nilpotent p-curvature of level $\leq p-1$, a Griffiths-transversal filtration F_i, and an isomorphism $\phi_i \colon C_1^{-1}(\mathrm{gr}_F E_i, \mathrm{KS}_i) \simeq (E_{i+1}, \nabla_{i+1})$.

We denote the *set* of isomorphism classes of stable rank r logarithmic Higgs bundles on X_s with Hilbert polynomial P_0 by $M_{Dol}(s)$. Likewise, we write $M_{dR}(s)$ for the set of isomorphism classes of stable rank r log-dR local systems on X_s with Hilbert polynomial P_0. For the purpose of this subsection, it will not matter that those sets are $\bar{k}$-rational points of moduli spaces, which could be constructed with Langer's methods (see Theorem A.6).

We informally refer to the following diagram as the *Higgs-de Rham flow*:

$$M_{Dol}(s) \underset{\mathrm{gr}}{\overset{C^{-1}}{\rightleftarrows}} M_{dR}(s).$$

The dashed arrows represent merely correspondences, rather than actual maps. The reason is that $\mathrm{gr}(E,\nabla)$ could be not stable, and C^{-1} can only be defined if the p-curvature is nilpotent of level $\leq p-1$ and the residues at infinity are nilpotent.

Using this viewpoint, one calls an element $[(E,\nabla)]$ of $M_{dR}(s)$ periodic, if there exists $f \in \mathbb{N}$ with

$$[(E,\nabla)] = (C^{-1} \circ \mathrm{gr})^f([(E,\nabla)]).$$

We let $R_{Dol}(s) \subset M_{Dol}(s)$ denote the subset of stable rank r log Higgs bundles with nilpotent Higgs field θ and nilpotent $\mathrm{res}_\mu \theta$ for all $\mu = 1, \dots, c$ of level $\leq p-1$. We denote by $R_{dR}(s) \subset M_{dR}(s)$ the subset of stable log-dR local systems with nilpotent residues or level $\leq p-1$. Restricting the Higgs-de Rham flow to these subsets has the added advantage of turning the correspondences above into maps of sets:

$$R_{Dol}(s) \underset{\mathrm{gr}}{\overset{C^{-1}}{\rightleftarrows}} R_{dR}(s). \tag{4}$$

It is not immediately obvious that the above maps are well-defined, since one has to justify that strong cohomological rigidity and stability is preserved by gr and C^{-1}.

Lemma A.12. *The maps in* (4) *are well-defined.*

Proof. Proposition A.10(c) allows us to fix for every $(E_s, \nabla_s) \in R_{dR}(s)$ an F-filtration. It follows from Proposition A.10(d) that $\mathrm{gr}(E_s, \nabla_s) = (\mathrm{gr}_F E_s, \mathrm{KS})$ is stable. This shows that $\mathrm{gr}\colon R_{dR}(s) \to R_{Dol}(s)$ is a well-defined map, which a priori depends on the chosen filtration (but see the end of the proof of Lemma A.15). Arguing as in [Lan14, Corollary 5.10] one shows that C^{-1} preserves stability. □

Lemma A.13. *Every element of $R_{Dol}(s)$ is strongly cohomologically rigid.*

Proof. There is an equivalence of categories (see [LSYZ19, Theorem 6.1])

$$C^{-1}\colon \mathsf{Higgs}_{p-1}(\bar{X}_s, D_s) \cong \mathsf{MIC}_{p-1}(\bar{X}_s, D_s),$$

where the left-hand side denotes a subcategory of logarithmic Higgs bundles (V, θ) satisfying several technical assumptions, and similarly, the right-hand side denotes a subcategory of log-dR local systems with nilpotent p-Higgs bundles which are required to satisfy various assumptions. We refer the reader to [LSYZ19, Section 6] for more details. This is an equivalence of categories, and therefore

$$\mathrm{Ext}(C^{-1}(V_s, \theta_s), C^{-1}(V_s, \theta_s)) = \mathrm{Ext}((V_s, \theta_s), (V_s, \theta_s)) = 0.$$

Here, we implicitly use Proposition A.10(e) to guarantee that all self-extensions of (V_s, θ_s) (respectively $C^{-1}(V_s, \theta_s)$) belong to $\mathsf{Higgs}_{p-1}(\bar{X}_s, D_s)$ (respectively $\mathsf{MIC}_{p-1}(\bar{X}_s, D_s)$). Indeed, since $p > 2r+2$ by Proposition A.10(f), the Higgs field of such a self-extension is automatically nilpotent of level $\leq p-1$. Thus, C^{-1} preserves strong cohomological rigidity. The same assertion holds for its inverse functor C.

We conclude the proof of the lemma by applying assertion (e) of Proposition A.10, according to which every log-dR local system with Hilbert polynomial P_0 on $\bar{X}_s$ is strongly cohomologically rigid. Therefore, for every $[(V,\theta)] \in R_{Dol}(s)$ we have that $C^{-1}(V,\theta)$ is strongly cohomologically rigid. This implies that $(V,\theta) = \mathrm{gr} \circ C^{-1}(V,\theta)$ is strongly cohomologically rigid. □

Lemma A.14. *The set $R_{dR}(s)$ is finite.*

Proof. Let (E, ∇) be a strongly cohomologically rigid log-dR local system which is stable and has Hilbert polynomial P_0. The hypercohomology group (1) computes the tangent space of $\mathcal{M}_{dR}(\bar{X}, D)$ in $[(E,\nabla)]$. By the vanishing assumption, the point $[(E,\nabla)]$ is isolated. We conclude the proof by recalling that the number of isolated points of a Noetherian scheme is finite. □

Lemma A.15. *The maps* gr *and C^{-1} are bijections.*

Proof. It suffices to prove that gr and C^{-1} are injective. Indeed, it then follows from Lemma A.14, they must be of equal cardinality if both maps are injective. The pigeonhole principle is used to conclude that gr and C^{-1} are bijections.

Since C^{-1} is defined using an equivalence of categories (see [LSYZ19, Theorem 6.1]), it is clear that $C^{-1}\colon R_{Dol}(s) \to R_{dR}(s)$ is injective.

The associated graded gr is injective for different reasons. In particular, we will use strong cohomological rigidity to prove this. The Artin-Rees construction applied to the F-filtration on (E, ∇) yields a $\mathbb{G}_m$-equivariant $\mathbb{A}^1_s$-family of vector bundles $(\mathcal{V}, \nabla_t)$, endowed with a log-t-connection ∇_t, where $t\colon \mathbb{A}^1_s \to \mathbb{A}^1_s$ denotes the identity map. Furthermore, we have

$$(\mathcal{V}, \nabla_t)|_{t=0} \simeq (\mathrm{gr}_F E, \mathrm{KS}).$$

Recall from Proposition A.10(d) that the right-hand side is a strongly cohomologically rigid log-Higgs bundle. There is therefore a unique way to lift it to a t-connection over $\operatorname{Spec} \bar{k}[t]/(t^2)$, and likewise for $\operatorname{Spec} \bar{k}[t]/(t^n)$. We infer from the Grothendieck existence theorem that there is a unique way to lift it to a t-connection on $\operatorname{Spec} \bar{k}[[t]]$. This implies that there cannot be a pair of distinct elements

$$(E^1_s, \nabla^1_s), (E^2_s, \nabla^2_s) \in R_{dR}(s) \text{ such that } (\mathrm{gr}_F E^1_s, \mathrm{KS}) \simeq (\mathrm{gr}_F E^2_s, \mathrm{KS}).$$

Otherwise, we would have

$$(E^1_s, \nabla^1_s) \otimes \bar{k}((t)) \simeq (E^2_s, \nabla^2_s) \otimes \bar{k}((t)),$$

which implies the existence of an isomorphism over $\bar{k}$ (by stability). This concludes the proof of injectivity, and furthermore proves that the map gr doesn't depend on the chosen F-filtration. $\square$

Proposition A.16. *The p-curvature of $[(E, \nabla)] \in R_{dR}(s)$ is nilpotent.*

Proof. By virtue of definition of C^{-1}, every log-dR local system in the image of C^{-1} has nilpotent p-curvature. According to Lemma A.15 the map C^{-1} is bijective. This concludes the proof. $\square$

Proposition A.17. *Every $[(E, \nabla)] \in R_{dR}(s)$ is* periodic*.*

Proof. Let $\sigma = C^{-1} \circ \mathrm{gr}$. By definition, it is a permutation of the finite set $R_{dR}(s)$. Let f' be the order of σ. We then have that $\sigma^{f'}([(E, \nabla)]) = [(E, \nabla)]$, and thus $[(E, \nabla)]$ is f-periodic for some $f | f'$. $\square$

A.3. **Higgs-de Rham flow over truncated Witt rings.** As before, we denote by $\bar{k}$ the algebraic closure of a finite field of characteristic p, and let $s\colon \operatorname{Spec} \bar{k} \to S$ be a $\bar{k}$-point of S. Furthermore, we write $W = W(\bar{k})$ for the associated Witt ring, and K for its fraction field. Hensel's lemma and Proposition A.10(g) implies that s can be extended to a morphism

$$s_W\colon \operatorname{Spec} W \to S.$$

For $n \in \mathbb{N}$ we denote by W_n the ring of n-th Witt vectors and by $\bar{X}_n$ the base change $\bar{X}_S \times_S W_n$.

We define $\mathcal{H}(\bar{X}_n/W_n)$ to be the category of tuples $(V, \theta, \bar{E}, \bar{\nabla}, \bar{F}, \phi)$, where (V, θ) is a graded log-Higgs bundle on $\bar{X}_n$ of level $\leq p-1$, $(\bar{E}, \bar{\nabla}, \bar{F})$ is a log-dR local system on $\bar{X}_{n-1}$ with a Griffiths-transversal filtration $\bar{F}$ of level $\leq p-2$, and $\phi\colon gr_{\bar{F}}(\bar{E}, \bar{\nabla}) \simeq (V, \theta) \times_{W_n} W_{n-1}$ is an isomorphism of graded log-Higgs bundles.

Similarly, we denote by $\mathsf{MIC}(\bar{X}_n/W_n)$ the category of quasi-coherent sheaves with W_n-linear flat connections on $\bar{X}_n$. There is a functor

$$C_n^{-1}\colon \mathcal{H}(\bar{X}_n/W_n) \to \mathsf{MIC}(\bar{X}_n/W_n)$$

which extends the logarithmic inverse Cartier transform. In the proper non-logarithmic case this is due to [LSZ13, Theorem 4.1]. Closely related results were obtained by Xu in [Xu19]. The logarithmic version is covered in [LSYZ19, Section 5] immediately before the proof of Proposition 5.2.

Let $(E_{W_n}, \nabla_{W_n}, F_n)$ be an W_n-linear log-dR local system endowed with an F-filtration. We denote by $\overline{\mathrm{gr}}(E, \nabla, F)$ the tuple $(\mathrm{gr}_F(E), \mathrm{KS}, (E, \nabla, F)_{W_{n-1}}, \mathrm{id})$.

Definition A.18 (Lan–Sheng–Zuo & Lan–Shen–Yang–Zuo)**.** An f-periodic log-dR local system on $\bar{X}_n/W_n$ is a tuple

$$(E^0_{W_n}, \nabla^0_{W_n}, F^0_{W_n}, \phi_0, E^1_{W_n}, \nabla^1_{W_n}, F^1_{W_n}, \ldots, E^{f-1}_{W_n}, \nabla^{f-1}_{W_n}, F^{f-1}_{W_n}, \phi_{f-1}),$$

where for all i we have that (E^i, ∇^i, F^i) is a log-dR local system on $\bar{X}_{W_n}$ (nilpotent of level $\leq p-2$ on the special fibre) with a Griffiths-transversal filtration $F^i_{W_n}$, such that for all integers n we have that $\overline{\mathrm{gr}}_F(E^i_{W_n}, \nabla^i_{W_n})$ belongs to $\mathcal{H}(\bar{X}_n/W_n)$ and $\phi_i\colon C_1^{-1}(\mathrm{gr}_F E^i_{W_n}, \mathrm{KS}_i) \simeq (E^{i+1}_{W_n}, \nabla^{i+1}_{W_n})$.

By taking the inverse limit with respect to n, we obtain a notion of periodicity relative to W. Using [LSZ13, p.3, Theorem 3.2, Variant 2], [LSYZ19, Theorem 1.1] together with [Fal88, Theorem 2.6*,p.43 i)] one obtains:

Theorem A.19 (Lan–Sheng–Zuo & Lan–Sheng–Yang–Zuo)**.** *A 1-periodic log-dR local system on $\bar{X}_W/W$ gives rise to a torsion-free Fontaine–Lafaille module on $X_W = \bar{X}_W \setminus D_W$. Furthermore, we can associate to an f-periodic log-dR local system on $\bar{X}_W/W$ a crystalline étale local system of free $W(\mathbb{F}_{p^f})$-modules on X_K. This is a fully faithful functor.*

We remark that Faltings only treats the case $f = 1$, in which he constructs a fully faithful functor from Fontaine-Lafaille modules to étale local systems of $\mathbb{Z}_p$-modules. The general case can be reduced to this one using a categorical construction, as explained in [LSZ13, Variant 2]. It is clear that this formal procedure preserves fully faithfulness of the functor. In combination with the above, the following result concludes the proof of Theorem A.4.

Theorem A.20. *Every element $[(E_s, \nabla_s)] \in R_{dR}(s)$ can be lifted to a periodic Higgs-de Rham flow over W_n on $\bar{X}_n$.*

Proof. Recall from Lemma A.14 that there is a finite number of non-isomorphic log-dR local systems $(E^i_s, \nabla^i_s)_{i\in I}$ in $R_{dR}(s)$. For every $i \in I$ there exists an extension to an S-family of log-dR local systems (E^i_S, ∇^i_S) (see Proposition A.10(a,b)). By pulling back along $s_W\colon\ \mathrm{Spec}\, W \to S$ we therefore obtain a lift to a W-family (E^i_W, ∇^i_W), and hence also a W_n-lift $(E^i_{W_n}, \nabla^i_{W_n})$.

Since (E^i_s, ∇^i_s) is strongly cohomologically rigid, deformation theory implies that such a W-lift is unique up to isomorphism.

We have seen in Proposition A.17 that every (E^i_s, ∇^i_s) is periodic over s (this corresponds to the case $n = 0$) since the map

$$\sigma = C^{-1} \circ \mathrm{gr}\colon R_{dR}(s) \to R_{dR}(s)$$

is a permutation (Lemma A.15). The W_n-relative log-dR local system $(E^i_{W_n}, \nabla^i_{W_n})$ is endowed with an F-filtration by Proposition A.10(c). We can therefore evaluate $\overline{\mathrm{gr}}(E^i_{W_n}, \nabla^i_{W_n})$.

Since the functor C_n^{-1} extends C^{-1} on the special fibre, we see that we have

$$(C_n^{-1} \circ \overline{\mathrm{gr}})(E^i_{W_n}, \nabla^i_{W_n}) \simeq (E^{\sigma(i)}_{W_n}, \nabla^i_{W_n}).$$

In particular, for (E^i_s, ∇^i_s) being f-periodic, we have $\sigma^f(i) = i$, and thus

$$(C_n^{-1} \circ \overline{\mathrm{gr}})(E^i_{W_n}, \nabla^i_{W_n}) \simeq (E^i_{W_n}, \nabla^i_{W_n}).$$

This equation establishes periodicity of $(E^i_{W_n}, \nabla^i_{W_n})$ relative to W_n. $\square$

We will now state a Betti version of Theorem A.4. For this purpose, let us recall that the Betti moduli space $\mathcal{M}_B(X)$ of irreducible rank r complex local systems of is zero-dimensional, since every such local system is assumed to be strongly cohomologically rigid. Furthermore, $\mathcal{M}_B(X)$ is defined over $\mathbb{Q}$, and therefore the irreducible rank r local systems $\rho_1, \dots, \rho_{n_r}$ are defined over a number field F. By finite generation of $\pi_1^{\mathrm{top}}(X)$ we have that the representations $\rho_1, \dots, \rho_{n_r}$ can be defined over $\mathcal{O}_F[M^{-1}]$, for a sufficiently big positive integer M.

Remark A.21. As by Remark A.2 (a), $H^1(X(\mathbb{C}), \mathsf{End}(E_{\mathbb{C},\mathrm{an}})^{\mathsf{End}(\nabla_{\mathbb{C}})})$ computes the left-hand side of (1), so is equal to zero, we can apply [EG18, Theorem 1.1] to conclude that Simpson's integrality conjecture holds. Therefore, M can be chosen to be 1. In fact, under the assumption that all log-dR bundles in a given rank are rigid, the proof of *loc. cit.* applies without verifying this vanishing assumption. We do not use this remark in the sequel.

For every prime $p \nmid M$, and every choice of an embedding $\mathcal{O}_F \to W(\bar{\mathbb{F}}_p)$ we can therefore consider the induced $W(\bar{\mathbb{F}}_p)$-representations

$$\rho_1^{W(\bar{\mathbb{F}}_p)}, \dots, \rho_{n_r}^{W(\bar{\mathbb{F}}_p)} \colon \pi_1^{\mathrm{top}}(X_{\mathbb{C}}) \to \mathrm{GL}_r(W(\bar{\mathbb{F}}_p)).$$

The étale fundamental group $\pi_1(X_{\mathbb{C}})$ is the profinite completion of $\pi_1^{\mathrm{top}}(X_{\mathbb{C}})$. Thus, we obtain continuous representations

$$\rho_1^{W(\bar{\mathbb{F}}_p)}, \dots, \rho_{n_r}^{W(\bar{\mathbb{F}}_p)} \colon \pi_1(X_{\mathbb{C}}) \to \mathrm{GL}_r(W(\bar{\mathbb{F}}_p)).$$

Theorem A.22. *Let $(\bar{X}_{\mathbb{C}}, X_{\mathbb{C}}, D_{\mathbb{C}})$ and $(\bar{X}_S, X_S, D_S)$ be as in Theorem A.4. Suppose that p is a prime, which belongs to the image of $S \to \operatorname{Spec}\mathbb{Z}$. Let k be a finite field of characteristic p and fix a morphism $\operatorname{Spec} W(k) \to S$. Then, the representations $\{\rho_i^{W(\bar{\mathbb{F}}_p)}\}_{i=1,\dots,n_r}$ descend to crystalline representations $\{\rho_i^{\mathrm{cris}}\}_{i=1,\dots,n_r} : \pi_1(X_K) \to GL_r(W(\bar{\mathbb{F}}_p))$, where $K = \mathrm{Frac}(W(k))$.*

Proof. By combining Theorem A.19 and Theorem A.4, we obtain crystalline representations

$$\{\pi_i\}_{i=1,\dots,n_r} : \pi_1(X_K) \to GL_r(W(\bar{\mathbb{F}}_p))$$

associated to the corresponding log-dR systems $(E_i, \nabla_i)_{i=1,\dots,r}$ on $\bar{X}_K$ Restricting these representations further to the geometric fundamental group $\pi_1(X_{\bar{K}})$, we obtain

$$(\pi_i^{\mathrm{geom}})_{i=1,\dots,n_r} \colon \pi_1(X_{\bar{K}}) \to GL_r(W(\bar{\mathbb{F}}_p)).$$

Claim A.23. *The geometric representations $(\pi_i^{\mathrm{geom}})_{i=1,\dots,n_r}$ are irreducible.*

Proof of the claim. Assume by contradiction that there exists π_i^{geom} which is reducible. Then, the residual representation $\pi_i^{\mathrm{geom}} \otimes \bar{\mathbb{F}}_p : \pi_1(X_{\bar{K}}) \to GL_r(\bar{\mathbb{F}}_p)$ is reducible as well. The continuous representation $\pi_i \otimes \bar{\mathbb{F}}_p : \pi_1(X_K) \to GL_r(\bar{\mathbb{F}}_p)$ factors through the finite group $GL_r(\mathbb{F}_q)$ for a p-power q. By Proposition A.10 (g) we may assume that X_K has a rational point, which yields a section $\mathrm{Gal}(\bar{K}/K) \to \pi_1(X_K)$. The kernel of the restriction $\pi_i|_{\mathrm{Gal}(\bar{K}/K)}$ yields a finite extension K'/K such that $\pi_i \otimes \bar{\mathbb{F}}_p|_{\pi_1(X_{K'})}$ is reducible.

Let α be a subrepresentation of $\pi_i \otimes \bar{\mathbb{F}}_p|_{\pi_1(X_{K'})}$. We will now use work by Sun–Yang–Zuo. It develops a version of the Higgs-de Rham flow over ramified extensions of W. Theorem 5.15 in [SYZ22] implies that the subrepresentation α gives rise to a sub-log-dR local system of (E_i, ∇), which is furthermore periodic and thus of slope 0. This contradicts stability. Note that *loc. cit.* deals with the more general setting of twisted Higgs-de Rham flows and projective representations.

When applying their result we may therefore assume that the twisting line bundle $\mathcal{L}$ is trivial, since our representations are not projective. $\square$

Claim A.24. *The geometric representations π_i^{geom} are pairwise non-isomorphic.*

Proof of the claim. As before, it suffices to show that the residual $\bar{\mathbb{F}}_p$-representations are pairwise non-isomorphic. We will use the same strategy as before. An isomorphism between $\pi_i^{\mathrm{geom}} \otimes \bar{\mathbb{F}}_p$ and $\pi_j^{\mathrm{geom}} \otimes \bar{\mathbb{F}}_p$ therefore implies the existence of a finite extension K'/K such that there exists an isomorphism

$$\pi_i^{\mathrm{geom}} \otimes \bar{\mathbb{F}}_p \simeq \pi_j^{\mathrm{geom}} \otimes \bar{\mathbb{F}}_p \colon \pi_1(X_{K'}) \to GL_r(\bar{\mathbb{F}}_p).$$

According to [SYZ22, Theorem 5.15] (which relies on [SYZ22, Theorem 5.8(ii)]), the functor from periodic Higgs-de Rham flows to $\bar{\mathbb{F}}_p$-linear representations of $\pi_1(X_{K'})$ is fully faithful. Thus, we obtain an isomorphism of the associated Higgs-de Rham flows, and in particular we have that $(E_i, \nabla_i)_{K'} \simeq (E_j, \nabla_j)_{K'}$. This implies $i = j$ since the associated complex log-dR local systems $(E_i, \nabla_i)_{\mathbb{C}}$ and $(E_j, \nabla_j)_{\mathbb{C}}$ are non-isomorphic, and hence concludes the proof. $\square$

Applying these two claims we see that the geometric representations

$$\pi_i|_{\pi_1(X_{\bar{K}})} : \pi_1(X_{\bar{K}}) \to GL_r(W(\bar{\mathbb{F}}_p))$$

for $i = 1, \ldots, n_r$ remain irreducible since the set $\{\pi_i|_{\pi_1(X_{\bar{K}})}\}_{i=1,\ldots,n_r}$ defines n_r pairwise non-isomorphic $W(\bar{\mathbb{F}})_p$-local systems on $X_{\bar{K}}$, which by the pigeonhole principle has to be the set of $W(\bar{\mathbb{F}}_p)$-local systems defined by $\{\rho_i^{W(\bar{\mathbb{F}}_p)}\}_{i=1,\ldots,n_r}$, and thus each single one of them descends to a crystalline representation. $\square$

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