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A remark on a non-vanishing theorem of P. Deligne and G. D. Mostow

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In [1], Proposition 2. 14, P. Deligne and G. D. Mostow consider meromorphic one-forms ω with values in a rank one local constant system L on the complement U of N points in $\mathbb{P}^1(\mathbb{C})$ $(N \geqq 3)$. If L has non trivial monodromies in the N points and if ω has no pole on U and if the sum of the multiplicities of the zeros of ω in U is smaller than $N-2$ they prove that ω defines a non vanishing cohomology class in $H^1(U, L)$. As a straightforward application of results and methods considered in [2], we prove in this note a higher dimensional analogue of this criterion.

We thank F. Loeser, who motivated this note by asking for possible generalizations of Deligne's and Mostow's result.

Theorem. *Let X be an n-dimensional compact complex manifold and $D = \sum D_i$ a normal crossing divisor on X. Let L be a local constant system of rank one on $U = X - D$, none of whose monodromies around the D_i's is one. Assume that*

$$\omega \in H^0(U, \Omega^n_U \otimes_{\mathbb{C}} L)$$

has a meromorphic extension to X.

If Z is the closure in X of the zero divisor of ω on U and if $\Omega^n_X(\log D) \otimes \mathcal{O}_X(-Z)$ is numerically effective and $\kappa(\Omega^n_X(\log D) \otimes \mathcal{O}_X(-Z)) = n$, then ω defines a non vanishing cohomology class in $H^n(U, L)$.

Remarks. 1) In fact we will prove a slightly more general statement:

Since ω is supposed to have a meromorphic extension to X we can extend $\mathcal{O}_U \otimes_{\mathbb{C}} L$ to an invertible sheaf $\mathcal{L}$ on X such that

$$\omega \in H^0(X, \Omega^n_X(\log D) \otimes \mathcal{L}) \subset H^0(U, \Omega^n_U \otimes_{\mathbb{C}} L).$$

Then, if we choose $\mathcal{L}$ as small as possible, the natural map

$$H^0(X, \Omega^n_X(\log D) \otimes \mathcal{L}) \to H^n(U, L)$$

is injective.

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2) Since X carries an invertible sheaf of maximal Iitaka-dimension, X is a Moišezon manifold.

3) Since $\Omega^n_X(\log D)(-Z)$ is numerically effective, the assumption

$$\text{“}\kappa(\Omega^n_X(\log D)(-Z)) = n\text{”}$$

is equivalent to

$$\text{“}c_1(\Omega^n_X(\log D)(-Z))^n > 0\text{”}$$

when X is projective.

4) The condition on the monodromies implies $j_! L = Rj_* L$ where $j\colon U \to X$ denotes the embedding ([2], 1. 6). If U is affine, one obtains $H^j(U, L) = 0$ for $j \neq n$ (see [2], 1. 5). Therefore, in this case all cohomology classes defined by holomorphic j-forms are zero for $j < n$.

Proof of the theorem. Set $\mathcal{M} = \Omega^n_X(\log D)(-Z)$. For some $a \geqq 1$ the sheaf $\mathcal{M}^a(-D)$ has a non trivial global section and for $b \geqq 1$ $\mathcal{N} = \mathcal{M}^{a+b}(-D)$ contains $\mathcal{M}^b$ and therefore $\kappa(\mathcal{N}) = n$.

As in the proof of (2. 11) in [2] one finds a birational morphism $\sigma\colon X' \to X$, with X' smooth and projective, an effective divisor B and some $N > 0$, such that $\mathcal{H} = \sigma^* \mathcal{N}^N \otimes \mathcal{O}_{X'}(-B)$ is ample and $B + \sigma^* D$ is a normal crossing divisor. Since $\sigma^* \mathcal{M}$ is numerically effective, $\mathcal{H} \otimes \sigma^* \mathcal{M}^N$ is ample as well, and (as in [2], 2. 12) we may change b to make sure that $M = (a+b) \cdot N$ is larger than the multiplicites of the components of $B + \sigma^* D$.

Replacing N by a multiple, we may assume in addition that $\mathcal{H}$ is very ample (loc. cit.). If H is the zero divisor of a general section of $\mathcal{H}$, $X' - H$ is affine and for $D' = B + H + \sigma^* D$ we have: $(\sigma^* \mathcal{M})^M = \mathcal{O}_{X'}(D')$, D' is a normal crossing divisor with multiplicities smaller than M and $X' - D'_{\text{red}}$ is affine. Hence we may apply [2], 2. 8, 2) to obtain

$$(*) \qquad H^q(X', \Omega^p_{X'}(\log D') \otimes \sigma^* \mathcal{M}^{-1}) = 0 \quad \text{for} \quad p + q \neq n.$$

As in remark 1) we choose the smallest invertible extension $\mathcal{L}$ of $\mathcal{O}_U \otimes_{\mathbb{C}} L$ to X such that

$$\omega \in H^0(X, \Omega^n_X(\log D) \otimes \mathcal{L}) \subset H^0(U, \Omega^n_U \otimes_{\mathbb{C}} L).$$

The form ω considered as a section of $\Omega^n_X(\log D) \otimes \mathcal{L}$ has no zeros along the components of D. By the assumption made ω comes from a section of $\Omega^n_X(\log D) \otimes \mathcal{L} \otimes \mathcal{O}_X(-Z)$ without zeros and ω defines isomorphisms

$$\mathcal{L} \simeq (\Omega^n_X(\log D) \otimes \mathcal{O}_X(-Z))^{-1} = \mathcal{M}^{-1}.$$

The condition on the monodromies of L implies ([2], 1. 2, d)) that

$$H^n(U, L) = \mathbb{H}^n(X, \Omega^{\bullet}_X(\log D) \otimes \mathcal{L}),$$

where $(\Omega_X^\bullet(\log D) \otimes \mathcal{L}, \nabla)$ is the logarithmic de Rham complex corresponding to L (see [2], 1. 3). Therefore the theorem follows from the injectivity of

$$H^0(X, \Omega_X^n(\log D) \otimes \mathcal{L}) \to \mathbb{H}^n(X, \Omega_X^\bullet(\log D) \otimes \mathcal{L}).$$

On the blown up manifold X' considered above, we can extend ∇ to a connection ∇' on $\sigma^* \mathcal{L}$ with logarithmic poles along $\sigma^* D$ or—extending it trivially along B and H—with logarithmic poles along D'. The map of complexes

$$\sigma^{-1}(\Omega_X^\bullet(\log D) \otimes \mathcal{L}) \to \Omega_{X'}^\bullet(\log D') \otimes \sigma^* \mathcal{L}$$

gives a commutative diagram

$$\begin{array}{ccc} H^0(X, \Omega_X^n(\log D) \otimes \mathcal{L}) & \xrightarrow{\alpha} & \mathbb{H}^n(X, \Omega_X^\bullet(\log D) \otimes \mathcal{L}) \\ \big\downarrow{\scriptstyle \sigma^*} & & \big\downarrow \\ H^0(X', \Omega_{X'}^n(\log D') \otimes \sigma^* \mathcal{L}) & \xrightarrow{\alpha'} & \mathbb{H}^n(X', \Omega_{X'}^\bullet(\log D') \otimes \sigma^* \mathcal{L}). \end{array}$$

The kernel of α' is the image of $\mathbb{H}^{n-1}(X', \Omega_{X'}^\bullet(\log D')^{\leq n-1} \otimes \sigma^* \mathcal{L})$ under $\mathbb{H}^{n-1}(\nabla')$. Since $E_1^{pq} = H^q(X', \Omega_{X'}^p(\log D') \otimes \sigma^* \mathcal{L})$ is zero for $p+q=n-1$ by $(*)$, the spectral sequence of hypercohomology implies $\mathbb{H}^{n-1}(X', \Omega_{X'}^\bullet(\log D')^{\leq n-1} \otimes \sigma^* \mathcal{L}) = 0$. Therefore both, α' and α are injective.

Remark. As explained in [2], 1. 4, the monodromy conditions for ∇' imply that $\mathbb{H}^n(X', \Omega_{X'}^\bullet(\log D') \otimes \sigma^* \mathcal{L})$ is nothing but $H^n(X' - D', L|_{X'-D'})$. Therefore we obtained in fact that the composition $H^0(X, \Omega_X^n(\log D) \otimes \mathcal{L}) \to H^n(U, L) \to H^n(X' - D', L|_{X'-D'})$ is injective.

References

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