SMALL REPORT ON SOME OF THE CONTRIBUTIONS OF STEVE ZUCKER DOCUMENTED IN [Z02], [Z03], [Z12], [Z13]

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[Z02]: The author proves the Hodge conjecture in codimension 2 for complex cubic fourfolds: rational classes in Betti cohomology H^4 of type $(2,2)$, which are called codimension 2 Hodge classes, are the cohomology classes of algebraic cycles of codimension 2. It was at the time the first example which was not derived from the well understood codimension 1 case. The proof uses "the method of normal functions, based on an outline presented by Phillip Griffiths" as the author says in the introduction to *loc. cit.*: a Hodge cycle produces a normal function with values in the intermediate Jacobian of the cubic threefolds which are the fibres of a Lefschetz pencil. Algebraicity in the intermediate Jacobian of cubic threefolds had been studied by Clemens-Griffiths (*Annals of maths.* (1972)). At the same time Clemens gave a purely algebraic proof of the theorem based on the observation that there is a correspondence between the cubic fourfold and the family of Fano varieties of lines in the fibres of a Lefschetz pencil, which consists of the tautological $\mathbb{P}^1$-bundle above the latter, to which then the problem is reduced. This is the content of Appendix A of *loc. cit.*, written by Zucker. Finally he writes in Appendix B an example of Mumford to the effect that the Hodge conjecture is not true in Kähler geometry. In addition, Murre (*Nederl. Akad. Wetensch. Proc. Ser* (1977)) observes that a cubic fourfold is unirational. This reduces the problem to the compatibility of the Hodge conjecture with blow-ups in dimension 4. Bloch observed that Murre's proof applies to quartic fourfolds as well.

[Z03]: Deligne in the 70s endowed the cohomology $H^i(S,\mathcal{V})$ of a polarised variation $\mathcal{V}$ of real Hodge structure of pure weight w on a smooth complex variety S, a notion due to Griffiths (*Am. J. of Maths.* 1968), with a mixed real Hodge structure of weight $\geq (i+w)$, which is pure of weight $(i+w)$ if S is projective. It is functorial in $\mathcal{V}$ and S. If $\mathcal{V}$ is a subquotient (which by Deligne's semi-simplicity theorem is necessarily a summand) of a local system $R^jf_*\mathbb{C}$, where $f: X \to S$ is a smooth projective morphism between smooth varieties, (in which case we say that $\mathcal{V}$ is geometric), then $H^i(S,\mathcal{V})$ is a subquotient of $H^{i+j}(X,\mathbb{C})$ as a mixed Hodge structure, or pure Hodge structure if X, S are projective. It is natural to ask how to recognise geometrically the pure weight $(i+w)$ sub $W_{i+j}H^i(S, \mathcal{V}) \subset H^i(S, \mathcal{V})$. The main theorem of the article is that if $j : S \hookrightarrow \bar{S}$ is a good compactification, and S has dimension 1, then $W_{i+j}H^i(S, \mathcal{V}) = H^i(\bar{S}, j_*\mathcal{V})$. In addition, if $\mathcal{V}$ is geometric, and $\bar{f} : \bar{X} \to \bar{S}$ is a good compactification of f, then $W_{i+j}H^i(S, \mathcal{V})$ is a subquotient of $H^{i+1}(\bar{X}, \mathbb{C})$ as a Hodge structure via the map $R^j\bar{f}_*\mathbb{C} \to j_*R^jf_*\mathbb{C}$. The proof is purely analytic, it uses L^2 and harmonic methods (thus applies in the Kähler case as well). In his introduction, the author thanks Deligne both for the formulation of the theorem and for the methods developed and used. This article has been of great importance. In modern terminology one could summarise the purity result by saying that if $\mathcal{V}$ is any local system on S, then $j_*\mathcal{V}$ is the intermediate extension. If in addition $\mathcal{V}$ is a polarisable variation of pure real Hodge structure, then its intermediate extension is a pure sheaf, and cohomology of a pure sheaf on a smooth projective variety is pure. The existence of intermediate extensions and their purity in any dimension has been at the core of the theory of perverse sheaves initiated by Goresky-MacPherson, generalised by Deligne (*Invent. math.* 1983), then Beilinson-Bernstein-Deligne-Gabber (*Astérisque* 1982).

[Z12] [Z13]: In view of [Z03], El Zein's cohomological mixed Hodge complexes (*CRAS* 1983), Schmid's $SL2$-orbit theorem (*Invent. math* 1973), and Steenbrink's definition of a limiting mixed Hodge structure (*Invent. math.* 1976), it is natural to ask whether there is a notion of a polarised variation of mixed Hodge structure $\mathcal{V}$ on a smooth curve S, such that $H^i(S, \mathcal{V})$ carries a mixed Hodge structure with the property that if $\mathcal{V} = R^jf_*\mathbb{C}$ comes from geometry via a not necessarily smooth or proper morphism $f : X \to S$, the mixed Hodge structure is compatible with the one on $H^\bullet(X, \mathbb{C})$. The problem is solved in [Z12] by the introduction of the relative weight filtration, a notion, as the authors explain in the acknowledgement, due to Deligne. It is characterised by a simple compatibility property. The rational local system $\mathcal{V}$ may be assumed to be locally unipotent, with associated nilpotent operator N, it is endowed with a Hodge filtration satisfying Griffiths' transversality, and a flat weight filtration W. The graded Gr^W_k are polarisable and pure. The condition is then that there is a new filtration M on the fibres at the punctures of Deligne's extension of $\mathcal{V}$, such that N^i equates $Gr^M_{k+i}Gr^W_k$ with $Gr^M_{k-i}Gr^W_k$. If the filtration W is trivial, this is Deligne's nilpotent operator. The authors show using the methods of their previous articles, without new harmonic method, that this condition is the right one to yield a mixed Hodge structure in $H^i(S, \mathcal{V})$. The compatibility with the mixed Hodge structure of $H^i(X, \mathbb{C})$ in the geometric case is handled in [Z13]. The definition has had an important impact on M. Saito's definition of the theory of mixd Hodge modules (*Publ. Res. Inst. Math. Sci.* 1990), and beyond on the motivic enhancements of those. If S is higher dimensional, Cattani-Kaplan (*Invent. math.* 1979) show the uniqueness of the right monodromy filtration, called monodromy-weight filtration, an important step to show the algebraicity of the Hodge locus in a higher dimensional variation (Cattani-Deligne-Kaplan *J. Amer. Math. Soc.* 1995), which very recently has been recovered and vastly generalised using o-minimality techniques.

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