ARITHMETIC SUBSPACES OF MODULI SPACES OF RANK ONE LOCAL SYSTEMS

HÉLÈNE ESNAULT AND MORITZ KERZ

Abstract. We show that closed subsets of the character variety of a complex variety with negatively weighted first homology, which are p-adically integral and stabilized by an arithmetic Galois action, are motivic.

1. Introduction

Let $F \subset \mathbb{C}$ be a subfield of the field of complex numbers, $\overline{F}$ be its algebraic closure in $\mathbb{C}$, X be a separated geometrically connected scheme of finite type over F. Let Π be the abelianized fundamental group $\pi_1^{\mathrm{ab}}(X(\mathbb{C})) = H_1(X(\mathbb{C}), \mathbb{Z})$. The $\mathbb{Q}$-vector space $\Pi_{\mathbb{Q}} = \Pi \otimes_{\mathbb{Z}} \mathbb{Q}$ is endowed with a mixed Hodge structure with weights in $\{-2, -1, 0\}$. The weights of $\Pi_{\mathbb{Q}}$ are in $\{-2, -1\}$ if and only if for any open dense subscheme $U \subset X$ the map $\pi_1^{\mathrm{ab}}(U(\mathbb{C}))_{\mathbb{Q}} \to \Pi_{\mathbb{Q}}$ is surjective, see Lemma 6.1. For example this occurs if X is irreducible and geometrically unibranch.

For any ring A, the rank one A-linear local systems on $X(\mathbb{C})$ are in bijection with $\mathrm{Hom}(\Pi, A^{\times})$. The functor

$$B \mapsto \mathrm{Hom}(\Pi, B^{\times})$$

on the category of A-algebras is corepresentable by the group ring $A[\Pi]$, so the *character variety*

$$\mathrm{Char}_A^{\Pi} = \mathrm{Spec}(A[\Pi])$$

is the fine moduli space of rank one A-linear local systems on $X(\mathbb{C})$ and is a multiplicative affine group scheme over A.

In this note we study closed subsets of $\mathrm{Char}_{\overline{\mathbb{Q}}_p}^{\Pi}$ which are integral and stabilized by an arithmetic Galois group in a suitable sense. In fact we show that those subsets are necessarily of a very special form, namely they consist of torsion translated motivic subtori. By a *motivic subtorus* of Char_A^{Π} we mean a subgroup scheme, which roughly speaking is the character variety of a 1-motive. More precisely, it is a subgroup scheme of the form $\mathrm{Char}_A^{\Pi'} \hookrightarrow \mathrm{Char}_A^{\Pi}$, where Π' is a torsion free quotient of Π such that $\Pi'_{\mathbb{Q}}$ underlies a quotient mixed Hodge structure of $\Pi_{\mathbb{Q}}$, see Proposition 7.3 for equivalent definitions.

Our work is motivated by Simpson's classical article [Sim93], in which he studies closed algebraic subsets of $\mathrm{Char}_{\mathbb{C}}^{\Pi}$ which remain algebraic after applying the Riemann-Hilbert correspondence. One of his central results reads as follows. When X is smooth projective, there is also a fine moduli

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space $\mathrm{Pic}^{\nabla}(X)$ of rank one integrable connections on X. The Riemann-Hilbert correspondence yields an isomorphism of complex Lie groups

$$\mathrm{RH} : \mathrm{Char}^{\Pi}_{\mathbb{C}}(\mathbb{C}) \simeq \mathrm{Pic}^{\nabla}(X)(\mathbb{C})$$

which is not an algebraic isomorphism if the dimension is at least one. In [Sim93, Thm. 3.1 (c)] he proves that if $S \subset \mathrm{Char}^{\Pi}_{\mathbb{C}}$ is a Zariski closed irreducible subset such that $\mathrm{RH}(S(\mathbb{C}))$ is again Zariski closed, then S is a motivic subtorus translated by a character χ. When $F = \overline{\mathbb{Q}} \subset \mathbb{C}$, S and $\mathrm{RH}(S)$ are defined over F, then χ can be chosen to be a torsion character [Sim93, Thm.3.3].

In order to formulate our main result, Theorem 1.2 below, we have to introduce a Galois action related to the character variety. Let π be the profinite completion of Π.

Assumption 1.1. From now on we assume that F is finitely generated.

Let $G = \mathrm{Aut}(\overline{F}/F)$ be its Galois group. As π is equal to $\pi_1^{\text{ét,ab}}(X_{\overline{F}})$, G acts on π. We consider the group $\mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_p^{\times})$ of continuous homomorphisms, which by definition is the colimit of $\mathrm{Hom}_{\mathrm{cont}}(\pi, E^{\times})$ over all finite extensions $E \supset \mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$. The homomorphism $\Pi \to \pi$ induces a composite map

$$\begin{aligned} \varphi : \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_p^{\times}) = \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Z}}_p^{\times}) = \mathrm{Hom}(\Pi, \overline{\mathbb{Z}}_p^{\times}) \\ \hookrightarrow \mathrm{Hom}(\Pi, \overline{\mathbb{Q}}_p^{\times}) = \mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p}(\overline{\mathbb{Q}}_p) \to \mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p}, \end{aligned}$$

where $\overline{\mathbb{Z}}_p$ is the ring of integers of $\overline{\mathbb{Q}}_p$. Our main theorem is a p-adic arithmetic equivalent of Simpson's result recalled above, which is expressed using the map φ.

Theorem 1.2. *Assume that the weights of* $\Pi_{\mathbb{Q}}$ *are in* $\{-2, -1\}$. *Let* $S \subset \mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p}$ *be a Zariski closed subset such that* $\varphi^{-1}(S)$ *is stabilized by an open subgroup of* G. *Then*

1) *every irreducible component* S' *of* S *such that* $\varphi^{-1}(S')$ *is non-empty is a subtorus* T *of* $\mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p}$ *translated by a torsion character;*
2) *each such* T *is a motivic subtorus.*

The condition on weights in Theorem 1.2 is optimal in the sense that the weight zero part of $\Pi_{\mathbb{Q}}$ describes the homology of the dual complex of the configuration of irreducible components of X. For example if X is a rational double point curve defined over $\mathbb{Q}$, with normalization $X' \to X$ such that the two points above the node are still rational, then $\mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_{p'}^{\times}) = \overline{\mathbb{Z}}_p^{\times}$ on which G acts as the identity, and

$$\varphi : \overline{\mathbb{Z}}_p^{\times} \to \mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p} = \mathbb{G}_{m,\overline{\mathbb{Q}}_p}.$$

So taking S to be the image by φ of a non-torsion element in $\overline{\mathbb{Z}}_p^{\times}$ yields a counter-example.

Note that by the characterization of motivic subtori in Proposition 7.3, the converse of Theorem 1.2 also holds, i.e. for a closed subset $S \subset \mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p}$

satisfying 1) and 2) of the theorem there exists an open subgroup of G stabilizing $\varphi^{-1}(S)$.

Theorem 1.2 immediately implies the following independence of p result. Let p' be a prime number and let $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\simeq} \overline{\mathbb{Q}}_{p'}$ be an isomorphism of fields. It induces an isomorphism of schemes $\iota : \mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p} \xrightarrow{\simeq} \mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_{p'}}$, so in particular, if $S \hookrightarrow \mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p}$ is a closed subset, $\iota(S) \hookrightarrow \mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_{p'}}$ is a closed subset as well. In the same way as φ one defines

$$\varphi' : \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_{p'}^{\times}) \to \mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_{p'}}.$$

Corollary 1.3. *With the assumptions as in Theorem 1.2, let $S \subset \mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p}$ be a closed subset such that $\varphi^{-1}(S)$ is stabilized by an open subgroup of G and let S' be an irreducible component of S such that $\varphi^{-1}(S')$ is non-empty. Then $(\varphi')^{-1}(\iota(S'))$ is non-empty and stabilized by an open subgroup of G.*

We do not even know conjecturally what a complete analog of Theorem 1.2 could be for local systems of rank greater one. However, we formulate a conjectural analog of Corollary 1.3 for higher rank in Section 9.

We now give a summary of the methods used in the proof of Theorem 1.2. For simplicity, we assume that X is smooth and proper over F and that S is irreducible with $\varphi^{-1}(S) \neq \varnothing$. We first discuss 1), which is proved in Section 6. Bogomolov in [Bog80] shows the existence of an element of $\sigma \in G$ which acts as a homothety of infinite order on the pro-p-completion $\pi^{(p)}$ of π. We first use this element to show that torsion points are Zariski dense on S. To this aim, we show that $\varphi^{-1}(S)$ is 'p-adically conical' around every torsion point of $\mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_p^{\times})$, see Proposition 4.3.

Using again σ, we further show, using p-adic analysis, that $\varphi^{-1}(S)$ is linear around smooth torsion points. By a classical argument, this implies that S is a subtorus translated by a torsion point, see Section 5. In order to prove 2) in Section 7, i.e. motivicity of this subtorus, we use Faltings' theorem (Tate conjecture) on homomorphisms of abelian varieties over F applied to the Albanese variety of X.

We give an alternative shorter proof of a weaker version of Theorem 1.2 in Section 10 in which $\mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p}$ is replaced by Char^{Π}_{K}, where K is a number field. It is based on known cases of de Jong's conjecture and the Mordell–Lang conjecture for tori.

One of the main applications of Simpsons' result or of Theorem 1.2 is to cohomological jumping loci. It is a classical question, initiated by [GL87] for cohomology of coherent sheaves, how for a fixed bounded constructible complex $\mathcal{F} \in D^b_c(X(\mathbb{C}), \mathbb{C})$ the dimensions

$$h^i(\mathcal{F} \otimes L) = \dim H^i(X(\mathbb{C}), \mathcal{F} \otimes L)$$

vary with $L \in \mathrm{Char}^{\Pi}_{\mathbb{C}}(\mathbb{C})$. It is not hard to see that for all $j \in \mathbb{Z}$ the jumping loci

$$\Sigma^i(\mathcal{F}, j) = \{L \in \mathrm{Char}^{\Pi}_{\mathbb{C}}(\mathbb{C}) \mid h^i(\mathcal{F} \otimes L) > j\}$$

are Zariski closed in $\mathrm{Char}^{\Pi}_{\mathbb{C}}(\mathbb{C})$, see Section 8.

Recall that if K is a number field and $\mathcal{F} \in D^b_c(X(\mathbb{C}), K)$, then for all but finitely many places $K \hookrightarrow \overline{\mathbb{Q}}_p$ of K, $\mathcal{F}$ induces an object $\mathcal{F}_{\text{ét}}$ in $D^b_c(X_{\mathbb{C},\text{ét}}, \overline{\mathbb{Q}}_p)$.

Definition 1.4. For a prime number p we say that $\mathcal{F}_{\text{ét}} \in D^b_c(X_{\mathbb{C},\text{ét}}, \overline{\mathbb{Q}}_p)$ is *arithmetic* if there is a finitely generated field extension $F' \supset F$ such that for all $\sigma \in \mathrm{Aut}(\mathbb{C}/F')$ the complex $\sigma(\mathcal{F}_{\text{ét}})$ is isomorphic to $\mathcal{F}_{\text{ét}}$ inside $D^b_c(X_{\mathbb{C},\text{ét}}, \overline{\mathbb{Q}}_p)$.

We say that $\mathcal{F} \in D^b_c(X(\mathbb{C}), \mathbb{C})$ is *arithmetic* if there exists a number field K such that $\mathcal{F}$ descends to an object of $D^b_c(X(\mathbb{C}), K)$ and such that for infinitely many embeddings $K \hookrightarrow \overline{\mathbb{Q}}_p$, $\mathcal{F}$ induces an arithmetic object $\mathcal{F}_{\text{ét}}$ in $D^b_c(X_{\mathbb{C},\text{ét}}, \overline{\mathbb{Q}}_p)$.

Remark 1.5. Note that any perverse semi-simple object $\mathcal{F} \in D^b_c(X(\mathbb{C}), \mathbb{C})$ which is of geometric origin in the sense of [BBD82, 6.2.4-6.2.5] is arithmetic, in fact, $\mathcal{F}_{\text{ét}} \in D^b_c(X_{\mathbb{C},\text{ét}}, \overline{\mathbb{Q}}_p)$ is then arithmetic for all but finitely many embeddings $K \hookrightarrow \overline{\mathbb{Q}}_p$.

As a direct application of Theorem 1.2 we derive in Section 8 the following corollary.

Theorem 1.6. *Assume that the weights of* $\Pi_{\mathbb{Q}}$ *are in* $\{-2, -1\}$ *and that* $\mathcal{F} \in D^b_c(X(\mathbb{C}), \mathbb{C})$ *is arithmetic. For* $i, j \in \mathbb{Z}$, *each irreducible component of* $\Sigma^i(\mathcal{F}, j)$ *is a motivic subtorus* $T \subset \mathrm{Char}^{\Pi}_{\mathbb{C}}$ *translated by a torsion character* χ.

For the same reason as in Theorem 1.2, the condition on the weights in Theorem 1.6 is sharp. Even for a constant sheaf $\mathcal{F}$, Theorem 1.6 was not known in this generality. For such $\mathcal{F}$ it was known for X smooth projective by Simpson [Sim93, Sections 6-7] and for X smooth by Budur–Wang [BW17, Thm. 1.3.1]. We refer to Schnell [Schn15, Thm. 2.2] for a detailed discussion of the problem in terms of D-modules on abelian varieties. All these works rely on complex analysis and the Riemann–Hilbert correspondence. There is also an approach to a coherent version of the theorem [PR04, Prop. 1.2] using mod p reduction of the variety.

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2. Some p-adic analysis

In our local p-adic arguments below we need some p-adic analysis summarized in this section. In the sequel, $\overline{\mathbb{Q}}_p$ is an algebraic closure of $\mathbb{Q}_p$ and $\overline{\mathbb{Z}}_p$ is its ring of integers. We endow $\overline{\mathbb{Q}}_p$ with the p-adic absolute value $|\cdot| : \overline{\mathbb{Q}}_p \to \mathbb{R}$ normalized by $|p| = 1/p$ and $\overline{\mathbb{Q}}_p^d$ with the norm $|(x_1, \dots, x_d)| = \sup_i |x_i|$.

For an integer $d \geq 0$ and for $\rho \in |\overline{\mathbb{Q}}_p^\times|$, we define the *closed polydisc*

$$P^d(\rho) = \{x \in \overline{\mathbb{Q}}_p^d, |x| \leq \rho\}$$

around 0 of radius ρ and we let

$$P^d(\rho; y) = P^d(\rho) + y$$

be the corresponding polydisc around $y \in \overline{\mathbb{Q}}_p^d$. We get an isomorphism

$$P^d(\rho; y) = \prod_{i=1}^d P^1(\rho; y_i) \text{ with } y = (y_1, \dots, y_d).$$

Definition 2.1. A function $f : P^d(\rho; y) \to E$ is called *globally analytic* (or *rigid analytic*) if it can be expressed as a convergent power series with coefficients in a finite extension $E \subset \overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, i.e. in coordinates as above there exist $a_J \in E$ for $J \in (\mathbb{Z}_{\geq 0})^d$ such that

$$f(x_1, \dots, x_d) = \sum_{J=(j_1,\dots,j_d)} a_J (x-y)^J, \;\; (x-y)^J := (x_1 - y_1)^{j_1} \cdots (x_d - y_d)^{j_d},$$

converges for all $x \in P^d(\rho; y)$.

The convergence of the power series in Definition 2.1 means that $|a_J|\,\rho^{|J|} \to 0$ as $|J| \to \infty$, so the $\overline{\mathbb{Q}}_p$-algebra of globally analytic functions $A(P^d(\rho; y))$ on $P^d(\rho; y)$ is isomorphic to colimit of Tate algebras

$$\overline{\mathbb{Q}}_p\langle T_1, \dots, T_d\rangle = \operatorname*{colim}_E E\langle T_1, \dots, T_d\rangle$$

where E runs through all finite extensions E of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$ and where we set $T_i = (x_i - y_i)/z$ with $z \in \overline{\mathbb{Q}}_p$, $|z| = \rho$. In the following proposition we summarize some properties of this Tate algebra over $\overline{\mathbb{Q}}_p$, which are proved in the same way as over a complete p-adic field E, see [FvdP04, Ch. 3].

Proposition 2.2. *We have:*

1) *$\overline{\mathbb{Q}}_p\langle T_1, \dots, T_d\rangle$ is a noetherian ring of dimension d;*
2) *all maximal ideals of $\overline{\mathbb{Q}}_p\langle T_1, \dots, T_d\rangle$ have residue field $\overline{\mathbb{Q}}_p$;*
3) *$\overline{\mathbb{Q}}_p\langle T_1, \dots, T_d\rangle$ is Jacobson, i.e. every prime ideal of $\overline{\mathbb{Q}}_p\langle T_1, \dots, T_d\rangle$ is an intersection of maximal ideals;*
4) *the canonical map $\overline{\mathbb{Q}}_p[T_1, \dots, T_d] \to \overline{\mathbb{Q}}_p\langle T_1, \dots, T_d\rangle$ is flat.*

Proof. Only the Noether property 1) is not immediately deduced from the corresponding classical results about Tate algebras over complete fields, see [FvdP04, Ch. 3]. The standard proof that a non-zero ideal $I \subset \overline{\mathbb{Q}}_p\langle T_1, \dots, T_d\rangle$ is finitely generated goes through. Weierstrass preparation produces after applying an automorphism a T_d-distinguished element $f \in I$. Then I is generated by $I \cap \overline{\mathbb{Q}}_p\langle T_1, \dots, T_{d-1}\rangle[T_d]$ and we can argue by induction on d. □

Definition 2.3. A subset $S \subset P^d(\rho; y)$ is *closed analytic* if it is the zero locus of finitely many globally analytic functions on $P^d(\rho; y)$.

A closed analytic $S \subset P^d(\rho; y)$ corresponds to an ideal $I \subset A(P^d(\rho; y))$ which equals its own nilradical. We denote by

$$A(S) = A(P^d(\rho; y))/I$$

the *ring of globally analytic functions on* S which by definition are restrictions of globally analytic function on $P^d(\rho; y)$. An *irreducible component* of S is by definition the zero locus of a minimal prime ideal of $A(S)$, so S is the finite union of its irreducible components by Proposition 2.2 1).

A map $f : P^{d'}(\rho'; z) \to S$ is called *globally analytic* if the d composed maps

$$P^{d'}(\rho'; z) \xrightarrow{f} S \hookrightarrow P^d(\rho; y) \to P^1(\rho; y_i)$$

are. For a point $x \in S$ we define the *stalk of analytic functions* $A_{S,x}$ at x in the obvious way. It is a noetherian local ring, faithfully flat over the localization $A(S)_x$ of $A(S)$ at x as the completion at x is. The *tangent space* T_xS of S at x is defined as $\mathrm{Hom}_{\overline{\mathbb{Q}}_p}(\mathfrak{m}_{S,x}/\mathfrak{m}_{S,x}^2, \overline{\mathbb{Q}}_p)$, where $\mathfrak{m}_{S,x}$ is the maximal ideal of the local ring $A_{S,x}$. A point $x \in S$ is called *smooth* if the ring $A_{S,x}$ (or equivalently the ring $A(S)_x$) is regular, see [FvdP04, Prop. 4.6.1].

From Proposition 2.2 1) we immediately deduce the classical:

Lemma 2.4 (Strassmann)**.** *If $d = 1$ the closed analytic subsets are finite or the whole $P^1(\rho; y)$.*

In the following we assume that we are given a direct sum decomposition $\overline{\mathbb{Q}}_p^d = \overline{\mathbb{Q}}_p^{d_1} \oplus \overline{\mathbb{Q}}_p^{d_2}$ of $\overline{\mathbb{Q}}_p$-vector spaces. We write $P(\rho)$ for $P^d(\rho)$, $P_1(\rho)$ for $P^{d_1}(\rho)$ and $P_2(\rho)$ for $P^{d_2}(\rho)$. The linear projections are denoted

$$r : P(\rho) \to P_1(\rho),\ q : P(\rho) \to P_2(\rho). \tag{1}$$

We consider the action

$$\overline{\mathbb{Z}}_p \times P(\rho) \to P(\rho),\ (\alpha, (x_1, x_2)) \mapsto \alpha \cdot (x_1, x_2) = (\alpha x_1, \alpha^2 x_2). \tag{2}$$

Definition 2.5. Let $\rho \in |\overline{\mathbb{Q}}_p^\times|$.

1) A subset $S \subset P(\rho)$ is said to be *conic* if $\overline{\mathbb{Z}}_p \cdot S = S$.
2) A subset $S \subset P(\rho)$ is said to be *linear* if there exist $\overline{\mathbb{Q}}_p$-linear subspaces $V_1 \subset \overline{\mathbb{Q}}_p^{d_1}$ and $V_2 \subset \overline{\mathbb{Q}}_p^{d_2}$ such $S = (V_1 \oplus V_2) \cap P(\rho)$. In particular linear implies conic and closed analytic.

Fix $\alpha \in \overline{\mathbb{Z}}_p^\times$ which is not a root of unity and consider the automorphism

$$\sigma : P(\rho) \to P(\rho),\ \sigma(x_1, x_2) = (\alpha x_1, \alpha^2 x_2).$$

Lemma 2.6. *Let $S \subset P(\rho)$ be a closed analytic subset with $\sigma(S) \subset S$. Then S is conic.*

Proof. Consider a globally analytic function $f : P(\rho) \to \overline{\mathbb{Q}}_p$ vanishing on S. For $x \in S$ we consider the globally analytic function $g : \overline{\mathbb{Z}}_p \to \overline{\mathbb{Q}}_p$, $g(\beta) = f(\beta \cdot x)$. By assumption, the set $\{\alpha^n \mid n \geq 0\}$ is infinite and g vanishes on it. Lemma 2.4 implies that g vanishes identically. Thus f vanishes on $\overline{\mathbb{Z}}_p \cdot x$. As S is the zero locus of all such f by assumption, we get $\overline{\mathbb{Z}}_p \cdot x \subset S$. □

Lemma 2.7. *If $S, S' \subset P(\rho)$ are conic closed analytic subsets such that there exists $\rho' \in (0, \rho)$ with $S \cap P(\rho') = S' \cap P(\rho')$ then $S = S'$.*

Proof. Let $f : P(\rho) \to \overline{\mathbb{Q}}_p$ be globally analytic and assume that f vanishes on S'. We want to show that f vanishes on S. For $x \in S$ we consider the globally analytic function $g : \overline{\mathbb{Z}}_p \to \overline{\mathbb{Q}}_p$, $g(\beta) = f(\beta \cdot x)$. Then for $|\beta| < \rho'/\rho$ we have $g(\beta) = 0$. So g has infinitely many zeros and must vanish identically by Lemma 2.4. In particular $0 = g(1) = f(x)$. □

Proposition 2.8. *Consider a closed analytic subset $S \subset P(\rho)$ such that $\sigma(S) \subset S$ and such that S is smooth at the point 0. Assume that there exists a closed analytic subset $S_2 \subset P_2(\rho)$ such that $q(S) \subset S_2$ and such that $d_0 q : T_0(S) \to T_0(S_2)$ is surjective. Then S is linear.*

Proof. In order to simplify the notation we canonically identify the tangent space $T_0(P(\rho))$ with $\overline{\mathbb{Q}}_p^d$. From Lemma 2.6 we deduce that S is conic. By the eigenspace decomposition of $d_0\sigma : T_0 S \to T_0 S$ we see that $T_0(S) = V_1 \oplus V_2$ with $V_1 = \overline{\mathbb{Q}}_p^{d_1} \cap T_0(S)$ and $V_2 = \overline{\mathbb{Q}}_p^{d_2} \cap T_0(S)$.

It is sufficient to show that

$$S \subset T_0(S) \cap P(\rho). \tag{3}$$

Indeed, by dimension reasons this inclusion has to induce an isomorphism on the stalks of analytic functions around 0. From this we conclude that there exists $\rho' \in (0, \rho]$ such that $S \cap P(\rho') = T_0(S) \cap P(\rho')$, so the proposition follows from Lemma 2.7.

In order to show (3) it is sufficient to show $r(x) \in V_1$ and $q(x) \in V_2$ for any $x \in S$. Fixing $x \in S$, by Lemma 2.6 we have an analytic map $g_x : \overline{\mathbb{Z}}_p \to S$, $\beta \mapsto \beta \cdot x$ with $\mathrm{im}(d_0 g_x) = \overline{\mathbb{Q}}_p \cdot (r(x) \oplus 0)$. So we obtain $\overline{\mathbb{Q}}_p \cdot (r(x) \oplus 0) \subset T_0(S) = V_1 \oplus V_2$, thus $r(x) \in V_1$.

By definition and Lemma 2.6, we have $q(\overline{\mathbb{Z}}_p \cdot x) \subset q(S) \subset S_2$. Using Lemma 2.4 one deduces that $\beta q(x) \in S_2$ for all $\beta \in \overline{\mathbb{Z}}_p$. So we get an analytic map $h_{q(x)} : \overline{\mathbb{Z}}_p \to S_2$, $\beta \mapsto \beta q(x)$ with $\mathrm{im}(d_0 h_{q(x)}) = \overline{\mathbb{Q}}_p q(x)$. Here the action of $\overline{\mathbb{Z}}_p$ on V_2 is just the (unweighted) linear action. So we obtain $\overline{\mathbb{Q}}_p q(x) \subset T_0(S_2)$. By the surjectivity assumption, $T_0(S_2) = V_2$. Thus $q(x) \in V_2$. This finishes the proof. □

Remark 2.9. The results of this section carry over to the case where the power 2 in the definition of a conic set etc. is replaced by any positive natural number.

3. The p-adic exponential map

In this section we recall some properties of the p-adic exponential and logarithm functions in our setting, see [Cas86, Ch. 12] for proofs.

For a topologically finitely generated pro-finite abelian group π and a prime number p we denote by $\pi^{(p)}$ its pro-p completion and by $\pi^{(p)}/\mathrm{tor}$ the quotient modulo the torsion subgroup. Let $\pi^{(p)}_{\mathbb{Q}_p}$ be the finite dimensional $\mathbb{Q}_p$-vector space $\pi^{(p)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

The set of *continuous characters*

$$H_p^\pi = \mathrm{Hom}_{\mathrm{cont}}(\pi^{(p)}/\mathrm{tor}, \overline{\mathbb{Q}}_p^\times),$$

is after the choice of coordinates $\pi^{(p)}/\mathrm{tor} = \oplus_1^d \mathbb{Z}_p \cdot \gamma_i$, isomorphic to the open polydisc

$$\{x \in \overline{\mathbb{Q}}_p^d \mid |x| < 1\},$$

by sending $\chi \in H_p^\pi$ to $(\chi(\gamma_i) - 1)_{i=1}^d$ in $\overline{\mathbb{Q}}_p^d$.

For $\rho \in |\overline{\mathbb{Q}}_p^\times|$ in the open interval $(0,1)$, we denote by $H_p^\pi(\rho)$ the subgroup of characters $\chi \in H_p^\pi$ with $|\chi(\gamma)-1| \leq \rho$ for all $\gamma \in \pi$. Clearly, in coordinates $H_p^\pi(\rho)$ is identified with the closed polydisc $P^d(\rho;1)$.

Similarly, we consider

$$T_p^\pi = \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_p)$$

and the subset $T_p^\pi(\rho) \subset T_p^\pi$ consisting of those $\chi \in T_\rho^\pi$ with $|\chi(\gamma)| \leq \rho$ for all $\gamma \in \pi$. In coordinates $T_p^\pi(\rho)$ is identified with the closed polydisc $P^d(\rho)$ by sending $\chi \in T_p^\pi(\rho)$ to the image $(\chi(\gamma_i))_{i=1}^d$ in $\overline{\mathbb{Q}}_p^d$. There is a canonical isomorphism between T_p^π and the *analytic tangent space* $T_1 H_p^\pi(\rho)$ at the character 1 defined above.

Set $\rho_\circ = p^{-\frac{1}{p-1}}$. For $\rho \in (0,\rho_\circ) \cap |\overline{\mathbb{Q}}_p^\times|$ the *p-adic exponential homomorphism* is a globally analytic isomorphism

$$\exp : T_p^\pi(\rho) \xrightarrow{\cong} H_p^\pi(\rho)$$

This exponential map is explicitly given by

$$\chi \mapsto \sum_{j=0}^\infty \frac{\chi^j}{j!}$$

and its inverse

$$\log : H_p^\pi(\rho) \xrightarrow{\cong} T_p^\pi(\rho)$$

it the usual globally analytic *p-adic logarithm*.

Data 3.1. For the rest of this section we assume given the following data:

- a closed subgroup $\pi_\natural \subset \pi$;
- a continuous automorphism $\sigma : \pi \xrightarrow{\simeq} \pi$ which induces a $\mathbb{Q}_p$-linear semi-simple map on $\pi^{(p)}_{\mathbb{Q}_p}$ stabilizing $\pi^{(p)}_{\natural,\mathbb{Q}_p}$, which is multiplication by α^{-2} on $\pi^{(p)}_{\natural,\mathbb{Q}_p}$ and by α^{-1} on $(\pi/\pi_\natural)^{(p)}_{\mathbb{Q}_p}$, where $\alpha \in \mathbb{Z}_p^\times$ is not a root of unity.

We denote the canonical projection $H_p^\pi \to H_p^{\pi_\natural}$ by q defined by the restriction of characters. We choose a $\mathbb{Q}_p$-basis $e_1', \ldots, e_{d_1+d_2}'$ of $\pi^{(p)}_{\mathbb{Q}_p}$ consisting of eigenvectors of σ with $\sigma(e_i') = \alpha^{-1} e_i'$ for $i \leq d_1$ and with $\sigma(e_i') = \alpha^{-2} e_i'$ for $i > d_1$. We can furthermore choose the eigenvectors such that

$$\pi^{(p)}/\mathrm{tor} \subset \pi' := \mathbb{Z}_p e_1' + \cdots + \mathbb{Z}_p e_{d_1+d_2}'.$$

We get an isomorphism $T_p^\pi \cong \overline{\mathbb{Q}}_p^{d_1} \oplus \overline{\mathbb{Q}}_p^{d_2}$. Choose an integer $w \geq 0$ such that $p^w \pi' \subset \pi^{(p)}/\mathrm{tor}$. On $T_p^{\pi'}$ the linear map σ acts as in Section 2 and there are σ-equivariant inclusions $T_p^\pi(\rho/p^w) \subset T_p^{\pi'}(\rho) \subset T_p^\pi(\rho)$. So via the exponential isomorphism we can deduce from Lemma 2.6 and Proposition 2.8 the following result.

Proposition 3.2. *For some $\rho \in (0, \rho_\circ) \cap |\overline{\mathbb{Q}}_p^\times|$ let $S \subset H_p^\pi(\rho)$ be a closed analytic subset which is stabilized by σ such that $S \cap H_p^\pi(\rho/p^w)$ is non-empty*

1) *Then the character 1 is in S.*
2) *If moreover 1 is a smooth point of S and if there exists a closed analytic subset $S_\natural \subset H_p^{\pi_\natural}(\rho)$ such that $q(S) \subset S_\natural$ and such that $d_1 q : T_1 S \to T_1 S_\natural$ is surjective, then $S \cap H_p^\pi(\rho/p^w)$ is a subgroup of $H_p^\pi(\rho/p^w)$.*

4. Torsion points in character varieties

We start with recalling some simple facts about global character varieties. Let Π be a finitely generated abelian group and let A be commutative ring. One defines the functor $\{A-\text{algebras}\} \to \mathrm{Ab}$ to the category of abelian groups which assigns to B the set of characters $\Pi \to B^\times$. It is corepresented by the group ring $A[\Pi]$. So we define the *character variety* as the commutative group scheme

$$\mathrm{Char}_A^\Pi = \mathrm{Spec}(A[\Pi]).$$

Recall that $\pi^{(p)}$ denotes the pro-p completion of a pro-finite group π.

Data 4.1. In this section and the next section we assume that we are given the following data:

- an inclusion of finitely generated abelian groups $\Pi_\natural \subset \Pi$, the pro-finite completion of which we denote by $\pi_\natural \subset \pi$;
- a prime number p;
- a continuous automorphism $\sigma : \pi \xrightarrow{\simeq} \pi$ which induces a $\mathbb{Q}_p$-linear semi-simple map on $\pi^{(p)}_{\mathbb{Q}_p}$ stabilizing $\pi^{(p)}_{\natural,\mathbb{Q}_p}$, which is multiplication by α^{-2} on $\pi^{(p)}_{\natural,\mathbb{Q}_p}$ and by α^{-1} on $(\pi/\pi_\natural)^{(p)}_{\mathbb{Q}_p}$, where $\alpha \in \mathbb{Z}_p^\times$ is not a root of unity.

We denote the projection $\mathrm{Char}^\Pi_{\mathbb{Q}_p} \to \mathrm{Char}^{\Pi_\natural}_{\mathbb{Q}_p}$ by q.

For a character $\chi \in \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_p^\times) = \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Z}}_p^\times)$, we let

$$\bar\chi \in \mathrm{Hom}(\pi, \overline{\mathbb{F}}_p^\times)$$

denote its associated *residue character*. For a character $\xi \in \mathrm{Hom}(\pi, \overline{\mathbb{F}}_p^\times)$ we let

$$[\xi] \in \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_p^\times)$$

denote its *Teichmüller lift*. It is defined by post-composing ξ with the Teichmüller lift $\overline{\mathbb{F}}_p^\times \hookrightarrow W(\overline{\mathbb{F}}_p)^\times$ followed by the inclusion $W(\overline{\mathbb{F}}_p)^\times \hookrightarrow \overline{\mathbb{Q}}_p^\times$.

We consider the composite map

$$\begin{aligned}\varphi : \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_p^\times) = \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Z}}_p^\times) = \mathrm{Hom}(\Pi, \overline{\mathbb{Z}}_p^\times) = \mathrm{Char}^\Pi_{\overline{\mathbb{Z}}_p}(\overline{\mathbb{Z}}_p) \\ \hookrightarrow \mathrm{Hom}(\Pi, \overline{\mathbb{Q}}_p^\times) = \mathrm{Char}^\Pi_{\overline{\mathbb{Q}}_p}(\overline{\mathbb{Q}}_p) \to \mathrm{Char}^\Pi_{\overline{\mathbb{Q}}_p}.\end{aligned}$$

Note the homomorphisms $\pi \to \pi^{(p)} \to \pi^{(p)}/\mathrm{tor}$ induce an injective homomorphism

$$H_p^\pi \hookrightarrow \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_p^\times).$$

In the following lemma we collect properties of φ that we need.

Lemma 4.2. *Let $\rho \in |\overline{\mathbb{Q}}_p^\times|$ be in the interval $(0,1)$ and let $S \subset \mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p}$ be a Zariski closed subset.*

1) *The preimage $D = \varphi^{-1}(S) \cap H_p^\pi(\rho)$ is a closed analytic subset of $H_p^\pi(\rho)$ and there is an identification of tangent spaces $T_1 D \xrightarrow{\simeq} T_1 S$ if 1 is in S.*
2) *If σ stabilizes $\varphi^{-1}(S) \cap H_p^\pi(\rho)$ then there exists an integer $m > 0$ such that σ^m stabilizes $\varphi^{-1}(S') \cap H_p^\pi(\rho)$ for each irreducible component S' of S.*
3) *If S is irreducible and if $D = \varphi^{-1}(S) \cap H_p^\pi(\rho)$ is non-empty then $\varphi(D)$ is dense in S.*
4) *If S is irreducible and if $\varphi^{-1}(S) \cap H_p^\pi(\rho)$ is a subgroup of $H_p^\pi(\rho)$ then S is a subtorus of $\mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p}$.*

Proof. By Proposition 2.2 4) the map $\varphi|_{H_p^\pi(\rho)}$ corresponds to a flat ring homomorphism $\varphi^* : \mathcal{O}(\mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p}) \to A(H_p^\pi(\rho))$ from the algebraic functions on $\mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p}$ to the globally analytic functions on $H_p^\pi(\rho)$. This ring homomorphism φ^* induces an isomorphism on the completions of local rings

$$\mathcal{O}(\mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p})^\wedge_{(\varphi^*)^{-1}(\mathfrak{m})} \xrightarrow{\simeq} A(H_p^\pi(\rho))^\wedge_{\mathfrak{m}}$$

for any maximal ideal $\mathfrak{m} \subset A(H_p^\pi(\rho))$. In particular, it induces an isomorphism of tangent spaces as in 1). If S is the zero set of the finitely many algebraic funtions $f_i \subset \mathcal{O}(\mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p})$, then $\varphi^{-1}(S) \cap H_p^\pi(\rho)$ is the zero set of the analytic functions $\varphi^*(f_i)$ and therefore defines a closed analytic subset. This proves 1).

The closed subset S is defined by an ideal I which is the intersection of the finitely many minimal prime ideals $\mathfrak{p}_i \supset I$ defining its irreducible components. Any prime ideal $\mathfrak{q} \subset A(H_p^\pi(\rho))$ which is minimal containing $\varphi^*(\mathfrak{p}_i)$ is also minimal over $\varphi^*(I)$. This follows from going-down for the flat map φ^*. Expressed geometrically this means that for an irreducible component S' of S, $\varphi^{-1}(S') \cap H_p^\pi(\rho)$ consists of a finite union of irreducible components of $\varphi^{-1}(S) \cap H_p^\pi(\rho)$. As σ is an isomorphism, it permutes the finitely many irreducible components of $\varphi^{-1}(S) \cap H_p^\pi(\rho)$. So in 2) we can choose $m > 0$ such that σ^m stabilizes all these irreducible components. This finishes the proof of 2).

The conditions of 3) imply that $I = (\varphi^*)^{-1}(\varphi^*(I) A(H_p^\pi(\rho)))$. The closure of $\varphi(D)$ is defined by the ideal

$$\cap_{\mathfrak{m}} (\varphi^*)^{-1}(\mathfrak{m}) = (\varphi^*)^{-1}(\cap_{\mathfrak{m}} \mathfrak{m}) \tag{4}$$

where $\mathfrak{m}$ runs through the maximal ideals of $A(H_p^\pi(\rho))$ containing $\varphi^*(I)$. By Proposition 2.2 4) we see that $\cap_{\mathfrak{m}} \mathfrak{m}$ is the nilradical of the ideal of $A(H_p^\pi(\rho))$ generated by $\varphi^*(I)$, so the ideal in the equation (4) is equal to I, in other words the closure of $\varphi(D)$ is S. This proves 3).

In order to show 4) it is sufficient to show that $S(\overline{\mathbb{Q}}_p)$ is a subgroup of $\mathrm{Char}^{\Pi}_{\mathbb{Q}_p}(\overline{\mathbb{Q}}_p)$. Consider the commutative diagram

$$\begin{array}{ccc} D \times D & \longrightarrow & D \\ {\scriptstyle \varphi\times\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ S(\overline{\mathbb{Q}}_p) \times S(\overline{\mathbb{Q}}_p) & \longrightarrow & \mathrm{Char}^{\Pi}_{\mathbb{Q}_p}(\overline{\mathbb{Q}}_p) \end{array}$$

in which the horizontal maps are the group operations $(a,b) \mapsto a-b$ and where $D = \varphi^{-1}(S) \cap H_p^{\pi}(\rho)$. The image of the left vertical map is Zariski dense by part (iii). The image of the right vertical map is contained in $S(\overline{\mathbb{Q}}_p)$, so the same is true for the image of the lower horizontal map. $\square$

For a character $\xi \in \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{F}}_p^{\times})$, we let $\mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_p^{\times})^{\xi}$ denote the set of characters $\chi \in \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_p^{\times})$ with $\bar{\chi} = \xi$.

Proposition 4.3. *Let $\xi \in \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{F}}_p^{\times})$ be fixed by σ. Let $S \subset \mathrm{Char}^{\Pi}_{\mathbb{Q}_p}$ be a closed subset such that $\varphi^{-1}(S) \cap \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_p^{\times})^{\xi}$ is stabilized by σ. Let $S' \subset S$ be an irreducible component such that $\varphi^{-1}(S') \cap \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_p^{\times})^{\xi}$ is non-empty. Then there exists a torsion point $\chi \in S'$ with $\bar{\chi} = \xi$.*

Proof. As the Teichmüller lift $[\xi]$ of ξ is fixed by σ, and has the same order as ξ, translating by $[\xi]$ is σ-equivariant on $\mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_p^{\times})$ and preserves the torsion points. So we can replace S by $[\xi^{-1}] \cdot S$ in the following and thereby assume that $\xi = 1$.

For $n > 0$ we consider the Zariski closed subsets $[p^n](S') \subset [p^n](S)$ of $\mathrm{Char}^{\Pi}_{\mathbb{Q}_p}$. We note that $\varphi^{-1}([p^n](S)) = [p^n](\varphi^{-1}(S))$. This implies that $\varphi^{-1}([p^n](S)) \cap H_p^{\pi}$ is stabilized by σ.

Choose a character $\psi \in \varphi^{-1}(S') \cap H_p^{\pi}$. We fix $0 < \rho' < \rho_\circ = p^{-\frac{1}{p-1}}$ and choose n large such that $[p^n](\psi)$ lies in $H_p^{\pi}(\rho'/p^w)$, where w is as in Proposition 3.2. By Lemma 4.2 there exists $m > 0$ such that σ^m stabilizes $D = \varphi^{-1}([p^n](S')) \cap H_p^{\pi}(\rho')$. From Proposition 3.2 1) we deduce that D contains 1, therefore $[p^n](S')$ contains 1 and S' contains a torsion point. $\square$

Corollary 4.4. *Let $S \subset \mathrm{Char}^{\Pi}_{\mathbb{Q}_p}$ be a closed subset such that $\varphi^{-1}(S)$ is stabilized by σ. Then the torsion points are dense on each irreducible component S' of S for which $\varphi^{-1}(S')$ is non-empty.*

Proof. Let $S' \subset S$ be an irreducible component of S for which $\varphi^{-1}(S')$ is non-empty and let $U \subset S'$ be a non-empty open subset. The closure $\mathcal{S}'$ of S' in $\mathrm{Char}^{\Pi}_{\mathbb{Z}_p}$ has the property that $\dim(\mathcal{S}'_{\overline{\mathbb{F}}_p}) = \dim(S') \geq 0$. The closure $\mathcal{V}$ of $S' \setminus U$ in $\mathrm{Char}^{\Pi}_{\mathbb{Z}_p}$ has the property that $\dim(\mathcal{V}_{\overline{\mathbb{F}}_p}) \leq \dim(S') - 1$. So we can find $\xi \in \mathcal{S}'(\overline{\mathbb{F}}_p) \setminus \mathcal{V}(\overline{\mathbb{F}}_p)$. The point ξ is the moduli point of a character in $\mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{F}}_p^{\times})$. So there is an $m > 0$ such that σ^m fixes ξ. We may assume without loss of generality that $m = 1$. Note that by the choice of ξ, all characters $\chi \in S'(\overline{\mathbb{Z}}_p)$ with $\bar{\chi} = \xi$ automatically lie in $U(\overline{\mathbb{Q}}_p)$.

Now S, S' and ξ satisfy the assumptions of Proposition 4.3 and the χ constructed there is a torsion point on U. $\square$

Remark 4.5. As observed by M. Groechenig, Corollary 4.4 together with the Mordell–Lang conjecture for tori, shown by M. Laurent in [Lau84, Introduction] using Diophantine approximation, see Proposition 10.2, immediately implies Theorem 5.2. However, in Section 5 we explain how to deduce the latter purely in terms of elementary p-adic analysis.

Corollary 4.6. *With the notation of Propositon 4.3, $\chi \in \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_p^\times)$ is fixed by a power of σ if and only if it is torsion.*

5. Criterion for a closed subset of $\mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p}$ to be a subtorus

The notation is as in Section 4.

Proposition 5.1. *Let $S \subset \mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p}$ be an irreducible closed subset such that $\varphi^{-1}(S) \cap H_p^\pi(\rho)$ is stabilized by σ for some $\rho \in (0,1) \cap |\overline{\mathbb{Q}}_p^\times|$. Assume that 1 is a regular point of S and that there exists a closed subset $S_\natural \subset \mathrm{Char}^{\Pi_\natural}_{\overline{\mathbb{Q}}_p}$ such that $q(S) \subset S_\natural$ and such that $d_1 q : T_1 S \to T_1 S_\natural$ is surjective. Then S is a subtorus of $\mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p}$.*

Proof. We can assume without loss of generality that $\rho < \rho_\circ$, see Section 3. Write D for $\varphi^{-1}(S) \cap H_p^\pi(\rho)$ and $D_\natural$ for $\varphi^{-1}(S_\natural) \cap H_p^{\pi_\natural}(\rho)$. By Lemma 4.2 1) we have a canonical isomorphism of tangent spaces $T_1 D \cong T_1 S$ and $T_1 D_\natural \cong T_1 S_\natural$, so Proposition 3.2 2) implies that $D \cap H_p^\pi(\rho/p^w)$ is a subgroup of $H_p^\pi(\rho/p^w)$ for some integer $w \geq 0$. We deduce from Lemma 4.2 4) that S is a subtorus. $\square$

Theorem 5.2. *Let $S \subset \mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p}$ be a closed subset such that $\varphi^{-1}(S)$ is stabilized by σ. Then each irreducible component S' of S with $\varphi^{-1}(S')$ non-empty is a subtorus T translated by a torsion character.*

Proof. By Chevalley's theorem [EGAIV, 1.8.4], the image $q(S')$ is constructible in $\mathrm{Char}^{\Pi_\natural}_{\overline{\mathbb{Q}}_p}$, so we can find an open subset $U_\natural \subset \mathrm{Char}^{\Pi_\natural}_{\overline{\mathbb{Q}}_p}$ such that $q(S') \cap U_\natural$ is non-empty and closed in $U_\natural$. We can furthermore assume that $q(S') \cap U_\natural$ is regular when endowed with the reduced subscheme structure. We let $S'_\natural$ be the closure of $q(S') \cap U_\natural$ in $\mathrm{Char}^{\Pi_\natural}_{\overline{\mathbb{Q}}_p}$. Let $U \subset S' \cap q^{-1}(U_\natural)$ be a non-empty open subset such that $q|_U : U \to q(S') \cap U_\natural$ is smooth, when endowed with the reduced subscheme structure.

By Corollary 4.4 there exists a torsion point $\chi \in U$. After replacing σ by some power we can assume that σ fixes χ, so translation by χ is σ-equivariant and we can replace S by $\chi^{-1} \cdot S$ etc. Now S' contains 1 and by Lemma 4.2 2) for some $\rho \in (0,1)$ there exists $m > 0$ such that σ^m stabilizes $\varphi^{-1}(S') \cap H_p^\pi(\rho)$. After replacing σ by σ^m our sets S' and $S'_\natural$ satisfy the assumptions of Proposition 5.1, so S' is a subtorus. $\square$

6. Subspaces of moduli spaces of rank one local systems

Let X be a reduced, separated scheme of finite type over a subfield F of the field of complex numbers. We denote by $\overline{F}$ the algebraic closure of F in $\mathbb{C}$. Let $G = \mathrm{Aut}(\overline{F}/F)$ be the Galois group of F. Let Π be the abelian fundamental group $\pi_1^{\mathrm{ab}}(X(\mathbb{C}))$. The group Π is finitely generated and its pro-finite completion π is isomorphic to the étale fundamental group $\pi_1^{\text{ét},\mathrm{ab}}(X_{\overline{F}})$ on which G acts continuously.

We define a weight filtration $W_{-2}\Pi \subset W_{-1}\Pi \subset \Pi$, as follows. Choose a regular, open and dense subscheme $q : U \to X$ and an open embedding $U \to \bar{U}$ into a proper smooth variety $\bar{U}$ over F such that $\bar{U} \setminus U$ is a simple normal crossings divisor. Then

$$W_{-2}\Pi = q_* \ker(\pi_1^{\mathrm{ab}}(U(\mathbb{C})) \to \pi_1^{\mathrm{ab}}(\bar{U}(\mathbb{C})))$$

and

$$W_{-1}\Pi = \mathrm{im}(q_*).$$

Lemma 6.1. *The weight filtration on $\Pi_{\mathbb{Q}}$ defined in* [Del74] *is the same as the one defined above. It is independent of the choice of q and $U \to \bar{U}$ integrally. In particular, the weights lie in $\{-2,-1\}$ if X is irreducible and geometrically unibranch.*

Proof. Dually, the mixed Hodge structure on $H^1(X(\mathbb{C}),\mathbb{Q})$ is defined in ([Del74, Prop.8.2.2]) by descent $H^1(X(\mathbb{C}),\mathbb{Q}) = H^1(X_\bullet(\mathbb{C}),\mathbb{Q})$ where $X_\bullet \to X$ is a simplicial resolution, X_0. Thus one has an exact sequence

$$\mathrm{Ker}(H^0(X_1(\mathbb{C}),\mathbb{Q}) \to H^0(X_2(\mathbb{C}),\mathbb{Q}))/\mathrm{Im}H^0(X_\bullet(\mathbb{C}),\mathbb{Q}) \to \\ H^1(X(\mathbb{C}),\mathbb{Q}) \to H^1(X_0(\mathbb{C}),\mathbb{Q})$$

of mixed Hodge structures. By [Del74, Thm.8.2.4], the weights on the left are 0, the ones on the right are in $\{1,2\}$. In particular the weights of $H^1(X(\mathbb{C}),\mathbb{Q})$ lie in $\{1,2\}$ if and only if the map $H^1(X(\mathbb{C}),\mathbb{Q}) \to H^1(X_0(\mathbb{C}),\mathbb{Q})$ coming from the desingularization $X_0 \to X$ is injective, or equivalently, as the restriction q factors through $U \to X_0$ and $H^1(X_0(\mathbb{C}),\mathbb{Q}) \to H^1(U(\mathbb{C}),\mathbb{Q})$ is injective, if and only if the restriction $q^* : H^1(X(\mathbb{C}),\mathbb{Q}) \to H^1(U(\mathbb{C}),\mathbb{Q})$ is injective. Finally, the weight 1 part of $H^1(X(\mathbb{C}),\mathbb{Q})$ is then its subquotient $q^*H^1(X(\mathbb{C}),\mathbb{Q}) \cap H^1(\bar{U}(\mathbb{C}),\mathbb{Q}) \subset H^1(U(\mathbb{C}),\mathbb{Q})$ and the weight 2 part is the quotient $\mathrm{Im}H^1(X(\mathbb{C}),\mathbb{Q}) \subset H^1(U(\mathbb{C}),\mathbb{Q})/H^1(\bar{U}(\mathbb{C}),\mathbb{Q})$. This proves the first part.

Since $\pi_1(\bar{U}(\mathbb{C}))$, and thus $\pi_1^{\mathrm{ab}}(\bar{U}(\mathbb{C}))$, is a birational invariant among smooth varieties, to prove the independence integrally for two opens U, V we may assume that $V \subset U \subset \bar{U}$ so that $\bar{U} \setminus V$ is a normal crossings divisor. Then $\mathrm{Ker}(\pi_1^{\mathrm{ab}}(V(\mathbb{C})) \to \pi_1^{\mathrm{ab}}(\bar{U}(\mathbb{C}))$ is the subgroup spanned by the local monodromies around the components of $\bar{U} \setminus V$, which is the union of the components of $\bar{U} \setminus U$ with the Zariski closure in $\bar{U}$ of the components of $U \setminus V$. But the subgroup in $\pi_1^{\mathrm{ab}}(V(\mathbb{C}))$ spanned by the latter ones is $\mathrm{Ker}(\pi_1^{\mathrm{ab}}(V(\mathbb{C})) \to \pi_1^{\mathrm{ab}}(U(\mathbb{C}))$. As the map $V \to X$ factors through U, this shows the independency of $W_{-2}\Pi$. Furthermore, as $\pi_1^{\mathrm{ab}}(V(\mathbb{C})) \to \pi_1^{\mathrm{ab}}(U(\mathbb{C}))$ is surjective, this shows the independency of $W_{-2}\Pi$. Finally for the last part, we quote [dJStack, Lem. 0BQI]. This finishes the proof. □

In the following we write $\Pi_\natural$ for $W_{-2}\Pi$.

For the rest of this section assume that F is finitely generated and that $\Pi_\mathbb{Q}$ has weights in $\{-2,-1\}$. By [Del80, Thm. 1, Thm. 2], G acts purely of weight -2 on the pro-finite completion $\pi_\natural$ of $\Pi_\natural$ and purely of weight -1 on $\pi/\pi_\natural$. As a consequence of this weight filtration and using the Hodge-Tate property for the Galois representation on $\pi^{(p)}_{\mathbb{Q}_p}$ one obtains:

Proposition 6.2 (Bogomolov, Litt)**.** *If $\Pi_\mathbb{Q}$ has weights in $\{-2,-1\}$ then for any prime number p and any $\alpha \in \mathbb{Z}_p^\times$ sufficiently close to 1 there exists an element $\sigma \in G$ which induces a semi-simple map on $\pi^{(p)}_{\mathbb{Q}_p}$ which is multiplication by α^{-2} on $\pi^{(p)}_{\natural,\mathbb{Q}_p}$ and by α^{-1} on $(\pi/\pi_\natural)^{(p)}_{\mathbb{Q}_p}$.*

The proposition is shown in [Bog80, Cor. 1] for X smooth and proper, and in [Lit18, Lem. 2.10] for X smooth, which implies it in general, as $\pi^{(p)}_{\mathbb{Q}_p}$ is a G-equivariant quotient of $\pi_1^{\text{ét,ab}}(X^{\text{reg}}_{\overline{F}})^{(p)}_{\mathbb{Q}_p}$.

Proof of Theorem 1.2 1). If we choose $\alpha \in \mathbb{Z}_p^\times$ in Proposition 6.2 not to be a root of unity, the resulting automorphism $\sigma : \pi \xrightarrow{\simeq} \pi$ satisfies the conditions of Data 4.1, so part 1) of Theorem 1.2 is a direct consequence of Theorem 5.2. ☐

7. On subgroup schemes of semi-abelian varieties

Let $F \subset \mathbb{C}$ be a finitely generated field. Let $\overline{F}$ be its algebraic closure in $\mathbb{C}$, and $G = \mathrm{Gal}(\overline{F}/F)$ be its Galois group. Let M/F be a semi-abelian variety. By definition, M is an extension of an abelian variety A by a torus T. Let $V^M = H_1(M(\mathbb{C}),\mathbb{Q})$ and let $V_p^M = H^1_{\text{ét}}(M_{\overline{F}},\mathbb{Q}_p)^\vee$ be the dual of p-adic étale cohomology, endowed with its canonical G-action. There is a canonical comparison isomorphism

$$\psi^M : V^M \otimes_\mathbb{Q} \mathbb{Q}_p \xrightarrow{\simeq} V_p^M .$$

Proposition 7.1. *For a $\mathbb{Q}$-linear subspace $U \subset V^M$ such that $\psi^M(U_{\mathbb{Q}_p}) \subset V_p^M$ is stabilized by G, there exists a unique semi-abelian subvariety $N \hookrightarrow M$ defined over F such that $U = V^N$.*

Before starting the proof, let us state a small lemma.

Lemma 7.2. *Let V be a finite dimensional vector space over a field K of characteristic 0, and let $K \subset K'$ be a field extension. Let $C \subset \mathrm{End}_K(V)$ be a K-subalgebra and let $U \subset V$ be a K-linear subspace. If there exists an idempotent $e' \in C_{K'}$ with $e'(V_{K'}) = U_{K'}$ then there exists an $e \in C$ with $e(V) = U$.*

Proof. Using a spreading argument one reduces to a finite field extension $K \subset K'$. Then $e = \mathrm{tr}_{K'/K}(e') \in C$ is the requested element. ☐

Proof of Proposition 7.1. Uniqueness is clear as the tangent space of N at the neutral element, which is identified with $U_\mathbb{C}$, uniquely characterizes N.

Recall that Faltings [Fal86, Thm. 1] showed that V_p^A is a semi-simple G-representation and that for semi-abelian varieties M, M' we have an isomorphism

$$V_p : \mathrm{Hom}_k(M, M')_{\mathbb{Q}_p} \xrightarrow{\simeq} \mathrm{Hom}_G(V_p^M, V_p^{M'}). \tag{5}$$

In fact Faltings showed the latter for abelian varieties only, but it can be extended to semi-abelian varieties using the Mordell-Weil theorem, see [Jan95, Thm. 4.3].

We show the existence of N in three steps.

Case 1: $M = T$ torus.
In this case we have $V^M = \mathrm{Hom}(X(T_{\overline{F}}), \mathbb{Q})$, where $X(T_{\overline{F}})$ is the group of characters, so U induces by duality a free abelian quotient Y of $X(T_{\overline{F}})$. As the induced p-adic quotient $X(T_{\overline{F}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to Y \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is stabilized by G, so is the quotient $X(T) \to Y$. We set $N_{\overline{F}} = \mathrm{Spec}(\overline{F}[Y])$ and observe that the G-action descends this torus to the requested torus $N/\overline{F}$.

Case 2: $M = A$ abelian variety.
Note that the Betti realization

$$V : \mathrm{End}_k(A)_{\mathbb{Q}} \to \mathrm{End}(V^A)$$

identifies $\mathrm{End}_k(A)_{\mathbb{Q}}$ with a semi-simple subalgebra of $\mathrm{End}(V^A)$.

Falting's results recalled above imply that there exists an idempotent $e' \in \mathrm{End}_k(A)_{\mathbb{Q}_p}$ such $V_p(e')(V_p^A) = \psi^A(U_{\mathbb{Q}_p})$. By Lemma 7.2 there is an $e \in \mathrm{End}_k(A)_{\mathbb{Q}}$ with $V(e)(V^A) = U$. Write $e = f/n$ with $f \in \mathrm{End}_F(A)$ and $n \in \mathbb{N}$. We define N to be $\mathrm{im}(f) \subset A$.

Case 3: M semi-abelian variety with torus part T and abelian quotient A.
We construct a semi-abelian quotient variety M' of M such that N is the connected component of $\ker(M \to M')$. By Case 1 the subspace $U \cap V^T$ comes from a subtorus $T^\flat$ of T. Replacing M by $M/T^\flat$ we may assume without loss of generality that $U \cap V^T = 0$, so we get an isomorphism $\tau : U \xrightarrow{\simeq} \tau(U) \subset V^A$, which is G-invariant after tensoring with $\mathbb{Q}_p$. By Case 2 there is an abelian subvariety $t : A'' \hookrightarrow A$ such that $V(t)(V^{A''}) = \tau(U)$. In view of the isomorphism (5) there exists a map $s \in \mathrm{Hom}_F(A'', M)_{\mathbb{Q}_p}$ with étale realization $\tau_{\mathbb{Q}_p}^{-1} \circ V_p(t)$. We obtain a commutative diagram

$$\begin{tikzcd} & & A'' \arrow[dl, dashed, "s"'] \arrow[d, "t"] \\ T \arrow[r] & M \arrow[r] & A \end{tikzcd}$$

where the dotted arrow means that s has $\mathbb{Q}_p$-coefficients. As the realization of s is induced by the $\mathbb{Q}$-linear map $\tau^{-1} \circ V(t)$, so is $s \in \mathrm{Hom}_F(A'', M)_{\mathbb{Q}}$. So $M' = \mathrm{coker}(s)$ is the requested semi-abelian quotient of M.

$\square$

Theorem 1.2 2) follows immediately from the following proposition. Let Y be a reduced, separated scheme of finite type over $\mathbb{C}$ and define $\Pi = \pi_1^{\mathrm{ab}}(Y(\mathbb{C}))$.

Proposition 7.3. *If $\Pi_{\mathbb{Q}}$ has weights in $\{-2,-1\}$ and $T \subset \mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p}$ is a subtorus the following are equivalent:*

1) *T is a motivic subtorus,*
2) *there exists a morphism $\psi_T : Y^{\mathrm{reg}} \to B$ to a complex semi-abelian variety B such that*
$$\tau^*(T) = \mathrm{im}\left(\psi_T^* : \mathrm{Char}^{\Gamma}_{\overline{\mathbb{Q}}_p} \to \mathrm{Char}^{\Pi^\circ}_{\overline{\mathbb{Q}}_p}\right) \subset \mathrm{Char}^{\Pi^\circ}_{\overline{\mathbb{Q}}_p}$$
where $\Gamma = \pi_1^{\mathrm{ab}}(B(\mathbb{C}))$ and $\tau : \Pi^\circ = \pi_1^{\mathrm{ab}}(Y^{\mathrm{reg}}(\mathbb{C})) \to \Pi$ is induced by the open embedding $Y^{\mathrm{reg}} \hookrightarrow Y$ of the regular locus.
3) *$Y = X \otimes_F \mathbb{C}$ for a scheme X of finite type over a finitely generated field $F \subset \mathbb{C}$ such that $\varphi^{-1}(T)$ is stabilized by $\mathrm{Gal}(\overline{F}/F)$, where φ is as defined in Section 4.*

Proof. Applying Lemma 6.1 and noticing that if U is regular, the weights of $H_1(U(\mathbb{C}),\mathbb{Q})$ are in $\{-2,-1\}$, we may assume that Y is regular and connected.

We prove 1) implies 2). A quotient Hodge structure of a polarized Hodge structure inherits the polarization and by [Del74, Const. 10.1.3], we know that polarizable integral Hodge structures of weight $\{-2,-1\}$ correspond to complex semi-abelian varieties.

We consider Serre's Albanese morphism $\mathrm{alb} : Y \to M$ to a complex semi-abelian variety, defined once we have chosen a rational point $x_0 \in Y(\mathbb{C})$ such that $\mathrm{alb}(x_0) = 0$, see [Ser58, Thm. 4]. We know that $\mathrm{alb} : H_1(Y(\mathbb{C}),\mathbb{Q}) \to H_1(M(\mathbb{C}),\mathbb{Q})$ is an isomorphism [Ser58, Ann. II]. By 1) our torus T is induced by a torsion free quotient of Π compatible with the Hodge structure, so it corresponds to a quotient semi-abelian variety B of M and we define ψ_T as the composite map $Y \xrightarrow{\mathrm{alb}} M \to B$.

We prove 2) implies 3). The morphism ψ_T is defined over a finitely generated subfield $F \subset \mathbb{C}$. Therefore $\varphi^{-1}(T)$ is stabilized by $\mathrm{Gal}(\overline{F}/F)$.

We prove 3) implies 1). Let M be the Albanese of Y as above, which also descends to F. The torus T induces a $\mathbb{Q}$-linear subspace $U \subset V^M \simeq H_1(Y(\mathbb{C}),\mathbb{Q})$. As $\varphi^{-1}(T)$ is stabilized by $\mathrm{Gal}(\overline{F}/F)$ so is $\psi^M(U_{\mathbb{Q}_p})$. From Proposition 7.1 we deduce that U is a Hodge substructure of V^M, so the torus T is motivic. □

From Theorem 1.2 one also obtains the following compatiblility result with automorphisms of $\mathbb{C}$. The notation is as in Theorem 1.2.

Remark 7.4. For a finitely generated subfield $F \subset \mathbb{C}$ and for $\sigma \in \mathrm{Aut}(\mathbb{C})$ let $F^\sigma = \sigma(F)$. For a scheme X separated and of finite type over F consider the F^σ-scheme $X^\sigma = X \otimes_\sigma F^\sigma$ with abelian fundamental group $\Pi^\sigma = \pi_1^{\mathrm{ab}}(X^\sigma(\mathbb{C}))$, the pro-finite completion of which we denote by π^σ. Then σ induces a continuous isomorphism $\pi \xrightarrow{\simeq} \pi^\sigma$. Let $\mathcal{S} \subset \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_p^\times)$ be a subset stabilized by an open subgroup of G such that
$$\varphi^{-1}(S) = \mathcal{S} \text{ where } S = \overline{\varphi(\mathcal{S})}.$$
Then $\mathcal{S}^\sigma = \sigma(\mathcal{S}) \subset \mathrm{Hom}_{\mathrm{cont}}(\pi^\sigma, \overline{\mathbb{Q}}_p^\times)$ satisfies
$$(\varphi^\sigma)^{-1}(S^\sigma) = \mathcal{S}^\sigma \text{ where } S^\sigma = \overline{\varphi(\mathcal{S}^\sigma)}$$

and $\varphi^\sigma : \mathrm{Hom}_{\mathrm{cont}}(\pi^\sigma, \overline{\mathbb{Q}}_p^\times) \to \mathrm{Char}^{\Pi^\sigma}_{\overline{\mathbb{Q}}_p}$ is the map analog to φ. Indeed, up to translation by a torsion character, the irreducible components of S^σ are the subtori of $\mathrm{Char}^{\Pi^\sigma}_{\overline{\mathbb{Q}}_p}$ induced by the maps $\psi_T \otimes_\sigma \mathbb{C} : X^{\mathrm{reg}} \otimes_\sigma \mathbb{C} \to B \otimes_\sigma \mathbb{C}$ of complex varieties, where ψ_T is defined in Theorem 1.2 2).

8. Application to jumping loci

Let the notation be as in Theorem 1.6. Denoting by

$$\mathcal{L} : \Pi \to \mathbb{C}[\Pi]^\times$$

the canonical $\mathbb{C}[\Pi]$-valued character, $R\Gamma(X(\mathbb{C}), \mathcal{F} \otimes \mathcal{L})$ is quasi-isomorphic to a bounded above complex P of finitely generated free $\mathbb{C}[\Pi]$-modules, see [Bor84, Lemma V.5, 10.13]. Moreover, for $L \in \mathrm{Char}^{\Pi}_{\mathbb{C}}(\mathbb{C})$ we get a quasi-isomorphism $R\Gamma(X(\mathbb{C}), \mathcal{F} \otimes L) \simeq P \otimes_{\mathbb{C}[\Pi],L} \mathbb{C}$. So by [EGAIII, Thm. 7.6.9] the sets $\Sigma^i(\mathcal{F}, j)$ are Zariski closed in $\mathrm{Char}^{\Pi}_{\mathbb{C}}(\mathbb{C})$ for any $i, j \geq 0$.

Proof of Theorem 1.6. As $\mathcal{F}$ descends to $D^b_c(X(\mathbb{C}), K)$ for some number field $K \subset \mathbb{C}$ see Definition 1.4) we see that $\Sigma^i(\mathcal{F}, j)$ is defined by a Zariski closed subset in Char^{Π}_K, which we denote by the same symbol. Note that for all but finitely many embeddings $K \hookrightarrow \overline{\mathbb{Q}}_p$ the closure of each irreducible component of $S = \Sigma^i(\mathcal{F}, j)_{\overline{\mathbb{Q}}_p}$ in $\mathrm{Char}^{\Pi}_{\overline{\mathbb{Z}}_p}$ has non-empty special fibre. Choose an embedding $K \hookrightarrow \overline{\mathbb{Q}}_p$ with this property such that furthermore $\mathcal{F}$ induces an arithmetic $\mathcal{F}_{\text{ét}}$ in $D^b_c(X_{\mathbb{C},\text{ét}}, \overline{\mathbb{Q}}_p)$. After replacing F by a finitely generated extension we can assume that each $\sigma \in \mathrm{Aut}(\mathbb{C}/F)$ fixes $\mathcal{F}_{\text{ét}}$ up to quasi-isomorphism.

Clearly, $\varphi^{-1}(S)$ consists of those $L_{\text{ét}} \in \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_p^\times)$, such that

$$\dim H^i(X_{\mathbb{C},\text{ét}}, \mathcal{F}_{\text{ét}} \otimes L_{\text{ét}}) > j$$

and each $\sigma \in \mathrm{Aut}(\mathbb{C}/F)$ stabilizes this subset. So by Theorem 1.2 we observe that each irreducible component of S is a torsion translated motivic subtorus of $\mathrm{Char}^{\Pi}_{\overline{\mathbb{Q}}_p}$, so the same is true for $\Sigma^i(\mathcal{F}, j)$ over $\mathbb{C}$. □

9. Some remarks

Let X be a separated scheme, which is geometrically connected and of finite type over a finitely generated field $F \subset \mathbb{C}$. Let $G = \mathrm{Aut}(\overline{F}/F)$ be the Galois group of F, where $\overline{F}$ is the algebraic closure of F in $\mathbb{C}$. Let $x_0 \in X(F)$ be a fixed F-rational point. Let Π be the fundamental group $\pi_1(X(\mathbb{C}), x_0)$ and let π be its pro-finite completion, which is isomorphic to $\pi_1^{\text{ét}}(X_{\overline{F}}, x_0)$ and which is endowed with a continuous action of G.

Let K be a field. Let $\mathrm{M}^{\mathrm{irr}}_K$ be the moduli space of isomorphism classes of K-linear irreducible local systems of rank r on $X(\mathbb{C})$. For a prime number p we obtain a map

$$\varphi : \mathrm{Hom}_{\mathrm{cont}}(\pi, \mathrm{GL}_r(\overline{\mathbb{Q}}_p))^{\mathrm{irr}} \to \mathrm{M}^{\mathrm{irr}}_{\overline{\mathbb{Q}}_p}$$

from the irreducible continuous representations to the closed points of the moduli space. Through its action on π we have an induced action of G on the domain of φ. For p' another prime number we obtain an analogous map φ'. Let $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\simeq} \overline{\mathbb{Q}}_{p'}$ be an isomorphism of fields.

As a potential generalization of Corollary 1.3 we suggest the following problem.

Question 9.1. Let S be a closed subset of $\mathrm{M}^{\mathrm{irr}}_{\overline{\mathbb{Q}}_p}$ such that $\varphi^{-1}(S)$ is stabilized by an open subgroup of G. Let S' be an irreducible component of S such that $\varphi^{-1}(S')$ is non-empty.

1) Is the set of arithmetic points $s \in S'$ dense on S'? Recall that a closed point $s \in \mathrm{M}^{\mathrm{irr}}_{\overline{\mathbb{Q}}_p}$ is called *arithmetic* if $\varphi^{-1}(s)$ is non-empty and stabilized by an open subgroup of G.
2) Is $(\varphi')^{-1}(\iota(S'))$
   a) non-empty;
   b) stabilized by an open subgroup of G?
3) In case X is smooth and projective over F, is $\mathrm{RH}(S'(\mathbb{C}))$ a Zariski closed subset of the moduli space of irreducible rank r integrable connections, where RH is the Riemann-Hilbert correspondence?

Question 9.1 1) was suggested to us by M. Groechenig. One can show (see [EG18, Section 3]) that if S consists of a single isolated point, then a positive answer to Question 9.1 2) implies Simpson's integrality conjecture [Sim90, Conj. 5]. We also observe that the converse of 1), which would be a André-Oort type question, has a negative answer: it is not the case that the Zariski closure of a set of torsion points is the union of motivic subtori translated by torsion points. For example, let X be an elliptic curve defined over $F = \mathbb{Q}$, so the two-dimensional Hodge structure H_1 is irreducible. Then Char^{Π}_{A} is a two-dimensional torus, which contains one-dimensional subtori, which are thus non-motivic. However the set of torsion points on such subtori is dense.

10. An alternative approach

In this section we sketch an alternative proof of a weaker version of Theorem 1.2, which is however sufficient for our application to jumping loci in Theorem 1.6. This alternative proof is short and does not use p-adic analysis, instead it is based on class field theory, more precisely the rank one case of de Jong's conjecture [deJ01], and the torus case of the Mordell-Lang conjecture [Lau84]. A similar technique is used by Drinfeld in [Dr01].

Let the notation be as in Section 6, i.e. $F \subset \mathbb{C}$ is a finitely generated field, X/F is a reduced separated scheme of finite type, $\Pi = \pi_1^{\mathrm{ab}}(X(\mathbb{C}))$, π is the pro-finite completion of Π and $\pi^{(p)}$ denotes its pro-p completion.

Let K be a number field. For each embedding $\iota : K \hookrightarrow \overline{\mathbb{Q}}_p$ we define a map

$$\varphi_\iota : \mathrm{Hom}_{\mathrm{cont}}(\pi, \overline{\mathbb{Q}}_p^{\times}) \to \mathrm{Char}^{\Pi}_{K}$$

as in Section 4.

Let $S \subset \mathrm{Char}^{\Pi}_{K}$ be a closed subset. In the following we sketch an alternative argument for the following corollary of Theorem 1.2.

Corollary 10.1. *Assume that $\Pi_{\mathbb{Q}}$ has weights in $\{-2, -1\}$ and that for infinitely many embeddings $\iota : K \hookrightarrow \overline{\mathbb{Q}}_p$ there exist open subgroups of G which stabilize $\varphi_\iota^{-1}(S)$. Then $S_{\overline{K}} \subset \mathrm{Char}^{\Pi}_{\overline{K}}$ is a finite union of subtori translated by torsion points.*

Proof. By what is explained in Section 6 we can assume without loss of generality that X is smooth over F. We will show that the torsion points are dense in S. Then the corollary follows from the following version of the Mordell–Lang conjecture, proved by M. Laurent [Lau84, Thm. 2].

Proposition 10.2. *Let T be a split torus over K and let $S \subset T$ be a closed subset such that $T_{\mathrm{tor}} \cap S$ is dense in S. Then $S_{\overline{K}}$ is a finite union of subtori of $T_{\overline{K}}$ translated by torsion points.*

Let $V \subsetneq S$ be a closed subset. We want to find a torsion point on $S \setminus V$. Let $\mathcal{S}$ be the closure of S in $\mathrm{Char}^{\Pi}_{\mathcal{O}_K}$ and $\mathcal{V}$ be that of V. Choose a closed point $s \in \mathcal{S} \setminus \mathcal{V}$ which satisfies the following two conditions

1) $\mathcal{S}$ is smooth over $\mathbb{Z}$ at s and
2) a $\iota : K \hookrightarrow \overline{\mathbb{Q}}_p$ which induces the place of K corresponding to the image of s in $\mathrm{Spec}(\mathcal{O}_K)$ has the property that $\varphi_\iota^{-1}(S)$ is stabilized by an open subgroup of G.

Then the completion R of $\mathcal{O}_{\mathcal{S},s}$ is canonically a quotient of the Iwasawa algebra $W(k(s))[\![\pi^{(p)}/\mathrm{tor}]\!]$ by an ideal stabilized by the action of an open subgroup of G on $\pi^{(p)}$. Here $W(k(s))$ is the ring of Witt vectors of the residue field $k(s)$ of s. We may assume without loss of generality that this open subgroup is equal to G. By smoothness in 1) we see that $\mathcal{O}_{\mathcal{S},s}$ is isomorphic to a formal power series ring $W(k(s))[\![T_1, \ldots, T_d]\!]$.

By [Del77, Th.Fin. 1.9] there exists a connected normal scheme $\mathcal{W}$ of finite type over $\mathbb{Z}[1/p]$ with $k(\mathcal{W}) = F$ and a smooth scheme $f : \mathcal{X} \to \mathcal{W}$ with generic fibre X such that $R^1 f_* \mathbb{Z}/p\mathbb{Z}$ is locally constant and compatible with base change. Then the action of G on $\pi^{(p)}$ factors through $\pi_1(\mathcal{W})$.

Choose a closed point $w \in \mathcal{W}$ and let $\mathrm{Fr}_w \in \pi_1(\mathcal{W})$ be the corresponding Frobenius. Class field theory implies the finiteness of the Frobenius coinvariants $\pi^{(p)}_{\mathrm{Fr}_w}$ of the maximal p-adic quotient of the abelian étale fundamental group $\pi^{(p)} \cong \pi_1^{\mathrm{ab}}(\mathcal{X}_{\bar{w}})^{(p)}$, see [Del80, Thm. 1.3.1]. De Jong's observed in [deJ01, Sec. 3] that this finiteness implies that the (non-zero) quotient ring

$$\bar{R} = R/(\mathrm{Fr}_w(x) - x | x \in R) = R/(T_1 - \mathrm{Fr}_w(T_1), \ldots, T_d - \mathrm{Fr}_w(T_d))$$

is finite and flat over $\mathbb{Z}_p$. Indeed, otherwise $\dim(\bar{R}/(p))$ would be > 0 and the canonical character

$$\pi^{(p)}_{\mathrm{Fr}_w} \to (\bar{R}/(p))^\times$$

would have infinite image, contradicting the above observation. A minimal prime ideal of $\bar{R}$ finally gives rise to a torsion point of $S \setminus V$. $\square$

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Freie Universität Berlin, Arnimallee 3, 14195, Berlin, Germany
Email address: `esnault@math.fu-berlin.de`

Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
Email address: `moritz.kerz@mathematik.uni-regensburg.de`