A NOTE ON FIERCE RAMIFICATION

HÉLÈNE ESNAULT, LARS KINDLER, AND VASUDEVAN SRINIVAS

Abstract. We show that bounding ramification at infinity bounds fierce ramification. This answers positively a question of Deligne posed to the first named author.

1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$ and let ℓ be a prime different from p. One fixes an algebraic closure $\bar{\mathbb{F}}_\ell$ of $\mathbb{F}_\ell$. Let $\bar{C}$ be a nonsingular connected projective curve over k, with a dense Zariski open subset $C \hookrightarrow \bar{C}$, and a geometric point $\bar{c} \in C$. To any continuous representation $\rho : \pi_1(C, \bar{c}) \to \mathrm{GL}_r(\bar{\mathbb{F}}_\ell)$, thus with values $\mathrm{GL}_r(\mathbb{F}_{\ell^n})$ for some non-zero natural number n, one associates its Swan conductor $Sw(\rho)$ in the group $Z_0(\bar{C})$ of zero-cycles. It is an effective divisor supported on $\bar{C} \setminus C$, which measures the wild ramification of ρ ([Ser67, III.20], [Lau81, 1.1.2], [KR14, 4.4].)

The definition of the Swan conductor is local: if K is the function field of C, then for each closed point $x \in \bar{C} \setminus C$, consider the corresponding complete discrete valuation field obtained from K by completion at x, and the associated local Galois representation ρ_x into $\mathrm{GL}_r(\bar{\mathbb{F}}_\ell)$ (well-defined up to conjugation). The non-negative integer invariant assigned to it, see e.g. [KR14, 4.82,4.84], is the coefficient of x in the zero cycle $Sw(\rho)$. We will use the term "Swan conductor" to mean either the local or global invariant, depending on the context.

When X is a normal connected variety of finite type over k, a modulus condition for continuous representations $\rho : \pi_1(X, \bar{x}) \to GL_r(\bar{\mathbb{F}}_\ell)$ is defined in [EK12, Defn. 3.6]: if $X \hookrightarrow \bar{X}$ is a normal compactification of X, and Δ is a Cartier divisor supported in $\bar{X} \setminus X$, we say that $\rho : \pi_1(X, \bar{x}) \to GL_r(\bar{\mathbb{F}}_\ell)$ has *ramification bounded by* Δ, if for any morphism $\varphi : \bar{C} \to \bar{X}$ from a connected nonsingular projective curve $\bar{C}$, such that $\varphi^{-1}(X) = C$ is nonempty, the induced representation $\rho \circ \varphi_* : \pi_1(C, \bar{c}) \to GL_r(\bar{\mathbb{F}}_\ell)$ verifies

$$Sw(\rho \circ \varphi_*) \preceq \varphi^* \Delta$$

with respect to the order on $Z_0(\bar{C})$. As $\bar{X}$ is normal, the intersection of its smooth locus $\bar{X}_{\mathrm{reg}}$ with $\bar{X} \setminus X$ is dense, so $\Delta \cap \bar{X}_{\mathrm{reg}}$ as a divisor is a sum $\sum_i m_i \Delta_i$, where Δ_i is an irreducible divisor and $m_i \in \mathbb{N}$ are the called the multiplicities of Δ. Let N be a natural number. We say that ρ *has ramification bounded by N if it has ramification bounded by Δ for an effective divisor supported in $\bar{X} \setminus X$ with multiplicities $m_i \leq N$ for all i.* Similarly we say that ρ *has ramification bounded by N along a divisorial discrete valuation v* if there exists some normal compactification $X \hookrightarrow \bar{X}$ as above, with a boundary component divisor D_0 corresponding to v, such that ρ has ramification bounded by an effective Cartier divisor Δ, where the coefficient of D_0 in Δ is $\leq N$.

A natural question is whether, for a fixed effective Cartier divisor Δ as above, or for a chosen divisorial valuation v, the class of representations $\rho : \pi_1(X, \bar{x}) \to GL_r(\bar{\mathbb{F}}_\ell)$ with ramification bounded by Δ (or with ramification bounded by N along v) has other "finiteness properties" with respect to wild ramification.

One such is the notion of fierce ramification along an irreducible component D_0 of $\bar{X} \setminus X$. Let $\pi : Y \to X$ be the Galois cover of Galois group $\mathrm{Im}(\rho)$ determined by the quotient $\pi_1(X, \bar{x}) \to \mathrm{Im}(\rho)$, and $\bar{\pi} : \bar{Y} \to \bar{X}$ be the normalization of $\bar{X}$ in the field of functions of Y. So the smooth locus $\bar{Y}_{\mathrm{reg}} \subseteq \bar{Y}$ has complement of codimension at least 2. Let E_0 be an irreducible component of $\bar{\pi}^{-1}(D_0) \cap \bar{Y}_{\mathrm{reg}}$. Then the *fierce ramification index* of D_0 is the purely inseparable degree of the function field extension $k(D_0) \subseteq k(E_0)$. It depends on the local system defined by ρ and D_0, not on the choices of $\bar{x}$ and E_0. Indeed, $\mathrm{Im}(\rho)$ acts transitively on the set $\{E'_0\}$ of components of $\bar{\pi}^{-1}(D_0) \cap \bar{Y}_{\mathrm{reg}}$, on the set $\{k(D_0) \hookrightarrow k(E'_0)\}$ of extensions of $k(D_0)$ preserving the separable closures and the purely inseparable ones. Changing $\bar{x}$ conjugates the representation. The conjugation sends $\{k(D_0) \hookrightarrow k(E'_0)\}$ defined for $\bar{x}$ to the corresponding set defined for the other base point, preserving the separable closures and the purely inseparable ones. We say that $(\rho, X \hookrightarrow \bar{X})$ has *fierce ramification bounded by a natural number M* if for all D_0, the fierce ramification index of D_0 is at most M.

Similarly we have the notion of *fierce ramification index along a divisorial valuation v*, which equals the fierce ramification index along D_0 for any normal compactification $X \hookrightarrow \bar{X}$ with a boundary component D_0 associated to the divisorial valuation v. This notion depends only on the discrete valuation, since it can be defined using the extension of discrete valuation rings associated to $E_0 \to D_0$.

The aim of this note is to prove that bounding the ramification along a divisorial valuation v also bounds the fierce ramification index.

Theorem 1. *Let $(X,\ \ell,\ v,\ r)$ be as above. Let N be a natural number. Then there is a natural number M such that for all continuous representations $\rho : \pi_1(X, \bar{x}) \to \mathrm{GL}_r(\bar{\mathbb{F}}_\ell)$ of ramification bounded by N along v, the fierce ramification of $(\rho, X \hookrightarrow \bar{X})$ along v is bounded by M.*

The theorem answer positively a question posed by Pierre Deligne in [Del16].

The proof consists of two parts. In Section 2, we first make a local analysis of ramification at a point on a nonsingular curve, in relation to a bound on the Swan conductor at that point. This is formulated in terms of a boundedness assertion for representations of the corresponding local Galois group (see Proposition 2). To globalize the argument and perform the proof of Theorem 1, we notably use a version of the Bertini theorem controlling that the inverse image of a curve by a ramified Galois cover remains unibranch [Zha95].

Acknowledgements: The first author thanks Pierre Deligne for his letter

[Del16] sent in relation to [Esn17]. This note, which answers one of the two questions in his letter, has been circulating among experts for two years. The second question, asking whether fixing k algebraically closed, and $(X \hookrightarrow \bar{X}, \ell, r, \Delta)$ one can find a curve $C \hookrightarrow X$ such that the restriction of any irreducible representation of rank $\le r$ and ramification bounded by Δ remains irreducible, remains unanswered. By Deligne's theorem [EK12, Thm. 1.1] and the standard Lefschetz theorem over finite fields (see [Esnl17] and references in there), this is true over $k = \bar{\mathbb{F}}_p$ for the representations which descend to $\mathbb{F}_{p^m}$ for a fix natural number m.

The first named author is supported by the Einstein program. The third author was supported by an Einstein Visiting Fellowship, and by a J. C. Bose Fellowship of the Department of Science and Technology, India. The authors also thank the referee for a careful reading of the paper.

2. Local arguments

Let k be an algebraically closed field of characteristic $p > 0$ and let K be a discretely valued, complete field of characteristic p with residue field k. Fix an algebraic closure $\bar{K}$ and write $G_K := \mathrm{Gal}(\bar{K}/K)$.

Proposition 2. *Fix a prime $\ell \neq p$, and positive natural numbers r, N. There exists a number $M(\ell, r, N)$ with the following property. For any continuous representation $\rho : G_K \to \mathrm{GL}_r(\mathbb{F}_{\ell^n}) \subseteq \mathrm{GL}_r(\bar{\mathbb{F}}_\ell)$ with $Sw(\rho)$ bounded by N, there exists a finite Galois extension L_ρ of K of degree $\le M(\ell, r, N)$, such that $\rho|_{L_\rho} := \rho|_{G_{L_\rho}}$ is tamely ramified.*

Proof. Given ρ as in the statement, we write $\bar{I} := \rho(G_K)$, and we let $\bar{P}$ denote the (unique) p-Sylow subgroup of $\bar{I}$. The Galois theory of discretely valued fields shows that there is a short exact sequence

$$1 \to \bar{P} \to \bar{I} \to \mathbb{Z}/M \to 1, \tag{1}$$

where $(p, M) = 1$. Writing $M = \ell^n M'$ with $(\ell, M') = 1$, we have $\mathbb{Z}/M \cong \mathbb{Z}/\ell^n \times \mathbb{Z}/M'$.

Claim 1. *The exponent n is bounded by a constant only depending on r.*

Proof. In $\bar{I}$ there exists an element σ of order precisely ℓ^n (e.g. by the theorem of Schur-Zassenhaus). Considering σ as an element of $\mathrm{GL}_r(\bar{\mathbb{F}}_\ell)$, we can write $\sigma = 1 + x$, with x a nilpotent matrix such that $x^{\ell^{n-1}} \neq 0$ but $x^{\ell^n} = 0$. This shows that $\ell^{n-1} \le r$. ∎

Claim 2. *The order of $\bar{P}$ is bounded by a constant only depending on r, N.*

Proof. Let $\tilde{\bar{I}} \subseteq \bar{I}$ be the preimage of $\mathbb{Z}/M' \subseteq \mathbb{Z}/M$. We proceed in several steps.

(a) As $(M', p) = 1$, $\bar{P} \subseteq \tilde{\bar{I}}$ and $\bar{P}$ is the unique p-Sylow subgroup of $\tilde{\bar{I}}$.

(b) As $(M', \ell) = 1$, we see that $(\ell, |\tilde{\bar{I}}|) = 1$. Recall Jordan's theorem (see [CJ1878], or [MC08] for a modern source) according to which there is a constant $J(r)$, the *Jordan constant*, depending only on r, such that, given a subgroup $\mathrm{GL}_r(\bar{\mathbb{F}}_\ell)$ of order prime to ℓ, it contains an abelian normal subgroup of index at most $J(r)$ (the constant is the same as for subgroups of $\mathrm{GL}_r(\mathbb{C})$). It yields a normal abelian subgroup $A \lhd \tilde{\tilde{I}}$, such that $B := \tilde{\tilde{I}}/A$ is of order bounded by $J(r)$. Consider the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & A \cap \bar{P} \arrow[r] \arrow[d, hook] & \bar{P} \arrow[r] \arrow[d, hook] & \bar{P}/A \cap \bar{P} \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & A \arrow[r] & \tilde{\tilde{I}} \arrow[r] & B \arrow[r] & 1
\end{tikzcd}$$

To prove the claim, it suffices to show that the order of $A \cap \bar{P}$ is bounded by a constant only depending on N, r.

(c) Translating the group theoretic picture via Galois theory, we obtain the following diagram of field extensions, where the arrows are labeled by the corresponding Galois groups.

$$\begin{tikzcd}
L \\
L' \arrow[u, "A"] \\
\tilde{L} \arrow[u, "B"] \arrow[uu, bend right, "\tilde{\tilde{I}}"'] \\
K \arrow[u, "\mathbb{Z}/\ell^n"'] \arrow[uuu, bend left, "\bar{I}"]
\end{tikzcd}$$

(d) If $\rho_A : \rho^{-1}(A) \to \mathrm{GL}_r(\bar{\mathbb{F}}_\ell)$ is the induced representation, then there exists a bound N' depending only on N, r, such that $Sw(\rho_A)$ is bounded by N'.

To see this, apply Lemma 5 below to L'/K and ρ_A. This yields the desired estimate with $N' \leq \ell^r J(r) N$.

(e) For every element $\sigma \in A \cap \bar{P}$, we have $\sigma^{p^{N'}} = 1$. Indeed, if $x_1, \ldots, x_t \in \mathbb{Z}_{>0}$ are the jumps of the ramification filtration of $A = \mathrm{Gal}(L/L')$ in ascending order, and if $x_0 = 0$, then

$$\mathrm{Swan}(\rho_A) = \sum_{i=1}^{t} x_i \underbrace{\dim(V^{A^{(x_i)}}/V^{A^{(x_{i-1})}})}_{\geq 1} \leq N',$$

where V is the r-dimensional $\bar{\mathbb{F}}_\ell$-vector space on which A acts. It follows that $t \leq N'$, as the jumps x_i are integers, according to the theorem of Hasse-Arf. But the associated graded pieces of the filtration $A^{(x)}$ are products of copies of $\mathbb{Z}/p$. It follows that every element of $\bar{P} \cap A$ is killed by $p^{N'}$.

(f) Finally, as $p \neq \ell$, the elements of the finite abelian subgroup $A \cap \bar{P} \subseteq \mathrm{GL}_r(\bar{\mathbb{F}}_\ell)$ are simultaneously diagonizable matrices with eigenvalues $p^{N'}$-th roots of unity. It follows that the cardinality of $A \cap \bar{P}$ is bounded by $p^{rN'}$, which is a constant depending only on N and r, as claimed.

■

Remark 3. Our argument yields the upper bound $p^{rN'}J(r)$ for the order of $\bar{P}$. Since $N' \leq \ell^n J(r)N \leq \ell r J(r)N$ (since $\ell^{n-1} \leq r$), we get an upper bound for the order of $\bar{P}$ to be $p^{\ell r^2 J(r)N} J(r)$.

We finish the proof of Proposition 2. The short exact sequence (1) induces a homomorphism of groups $\alpha : \mathbb{Z}/M \to \mathrm{Aut}(\bar{P})$, given by conjugation. Write $H := \ker(\alpha)$ and let $\bar{I}^\alpha \subseteq \bar{I}$ be the preimage of H. By construction $\bar{I}^\alpha \cong \bar{P} \times H$, with both $\bar{P}$ and H characteristic subgroups of $\bar{I}^\alpha$. We obtain an injective map $H \hookrightarrow \bar{I}$, such that the composition $H \hookrightarrow \bar{I} \twoheadrightarrow \mathbb{Z}/M$ is the identity on H, and H is normal in $\bar{I}$. This shows that there is a short exact sequence

$$1 \to \bar{P} \to \bar{I}/H \to (\mathbb{Z}/M)/H \to 1.$$

The orders of the two outer terms are bounded by constants depending only on r, N (because of Claim 2 and because $(\mathbb{Z}/M)/H \subseteq \mathrm{Aut}(\bar{P})$). It follows that the order of $\bar{I}/H$ is bounded by such a constant.

Let L_ρ be the Galois extension of K corresponding to $\rho^{-1}(H)$. Then the restriction of ρ to L_ρ is tame, as it takes values in the prime-to-p-group H. On the other hand, by construction we have $[L_\rho : K] = [\bar{I} : H]$. This finishes the proof.

■

Remark 4. Our argument gives an upper bound M for the index of the subgroup H to be the order of the automorphism group of $\bar{P}$. Thus, if $M_0 = p^{\ell r^2 J(r)N} J(r)$ is our bound for the order of $\bar{P}$, a crude upper bound for M is the order of the permutation group, that is, $M_0!$

Lemma 5. *Let $K \subseteq K'$ be a finite Galois extension of complete discretely valued fields with algebraically closed residue fields of characteristic $p > 0$. If $\rho : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_r(\mathbb{F}_{\ell^n})$ is a continuous representation with $Sw(\rho)$ bounded by N, then $Sw(\rho_{K'})$ is bounded by $[K' : K]N$.*

Proof. Write $G := \rho(\mathrm{Gal}(\bar{K}/K))$ and $H := \rho(\mathrm{Gal}(\bar{K}/K'))$. Then there is a finite extension L/K, such that $G = \mathrm{Gal}(L/K)$ and $H = \mathrm{Gal}(L/K')$. Since the rank r is fixed, and since the Swan conductor is additive, we may assume that ρ is irreducible. In this case, the existence of the "break decomposition" implies that there exists a unique $u \in \mathbb{Z}_{>0}$ such that $G_u \neq G_{u+1}$. As $H_u = G_u \cap H$, u is also the only index where the ramification filtration of H jumps (we assume $H \neq 1$). One has

$$Sw(\rho) = \varphi_{L/K}(u) \cdot r,$$

and

$$Sw(\rho|_{K'}) = \varphi_{L/K'}(u) \cdot \dim(V^{H_u}/V^{H_{u-1}}) \leq \varphi_{L/K'}(u) \cdot r,$$

where φ denotes the Herbrand function. We know that

$$\varphi_{L/K} = \varphi_{K'/K} \circ \varphi_{L/K'},$$

and for any $v \in \mathbb{Z}_{>0}$, one has

$$\varphi_{K'/K}(v) = \frac{1}{[K':K]}(|\operatorname{Gal}(K'/K)_1| + \ldots + |\operatorname{Gal}(K'/K)_v|) \geq \frac{v}{[K':K]}.$$

It follows that

$$\frac{1}{[K':K]}\operatorname{Swan}(\rho|_{K'}) \leq \frac{1}{[K':K]}\varphi_{L/K'}(u)\cdot r \leq \operatorname{Swan}(\rho) \leq N,$$

as we wanted to prove. ■

3. Global arguments

Let k be an algebraically closed field, $\bar{X} \subseteq \mathbb{P}^n_k$ a normal projective k-variety and let $X \subseteq \bar{X}$ be an open subscheme such that $(\bar{X} \setminus X)$ is a divisor. Let D_0 be an irreducible component of $\bar{X}_{\mathrm{reg}} \setminus X$, with generic point ξ. This corresponds to a divisorial valuation v with discrete valuation ring $\mathcal{O}_{\bar{X},\eta}$. We fix $\rho : \pi_1(X, \bar{x}) \to \mathrm{GL}_r(\bar{\mathbb{F}}_\ell)$, a continuous representation with finite image $\bar{I}$ and with ramification bounded by an effective Cartier divisor Δ which has multiplicty at most N along D_0.

We then introduce some notations.

- Let $\pi : Y \to X$ be finite Galois covering with group $\bar{I}$ and let $\bar{\pi} : \bar{Y} \to \bar{X}$ be the normalization of $\bar{X}$ in Y. Denote by $\eta_1, \ldots, \eta_t$ the codimension 1 points of $\bar{Y}$ lying over ξ.
- As $k(X) \subseteq k(Y)$ is Galois, the ramification index of the extension of discrete valuation rings $\mathcal{O}_{\bar{X},\xi} \subseteq \mathcal{O}_{\bar{Y},\eta_i}$ is independent of i; we denote it by e.
- Let f be the degree of the residue extensions $k(\xi) \subseteq k(\eta_i)$, and $f^{\mathrm{sep}}, f^{\mathrm{insep}}$ the separable, resp. inseparable degree. Note that f is independent of i, and so are f^{sep}, f^{insep}: indeed, the integral closure B of $\mathcal{O}_{\bar{X},\xi}$ in $k(Y)$ is a Dedekind domain and if $\mathfrak{P}$ is a prime ideal lying over the maximal ideal $\mathfrak{m}_\xi$, then any Galois automorphism $\sigma \in \bar{I}$ induces an isomorphism of $k(\xi)$-extensions $B/\mathfrak{P} \xrightarrow{\cong} B/\sigma(\mathfrak{P})$, and $\bar{I}$ acts transitively on the set of prime ideals of B lying over $\mathfrak{m}_\xi$.

Proposition 6. *With $\bar{\pi} : \bar{Y} \to \bar{X}$ as above, a general complete intersection curve $\bar{C} \subseteq \bar{X}$ has the following properties:*

(a) $\bar{C}$ is smooth and intersects $(\bar{X} \setminus X)_{\mathrm{red}}$ transversely at smooth points.
(b) Write $C := \bar{C} \cap X$, $D = Y \times_X C$, then $D \to C$ is Galois étale with group $\bar{I}$.
(c) Write $\bar{D} := \bar{Y} \times_{\bar{X}} \bar{C}$. The normalization map $\nu : \bar{D}^{\mathrm{norm}} \to \bar{D}$ is a universal homeomorphism.
(d) For $x \in \bar{C} \cap D_0$, there are precisely tf^{sep} points in $\bar{D}$ (and hence in $\bar{D}^{\mathrm{norm}}$) mapping to x.

Proof. That $\bar{C}$ is nonsingular projective and intersects $\bar{X} \setminus X$ only at nonsingular points, and that the intersection is transverse, follows easily from Bertini's theorem. This proves (a). For (c), we use [Zha95, Thm. 1.2] where it is shown that the inverse image scheme $\bar{D}$ of a general complete intersection curve $\bar{C} \subseteq \bar{X}$ is *geometrically unibranch* at all points. In addition, it is connected, since $\bar{Y}$ is normal (see the proof of [Zha95, Cor. 1.7], for example). Since $\bar{Y}$ is Cohen-Macaulay outside a closed subset of codimension ≥ 3, we may assume $\bar{D}$ is contained in the Cohen-Macaulay locus, and so $\bar{D}$ (which is a complete intersection) is also Cohen-Macaulay, and hence reduced. Thus $\bar{D}$ is irreducible, and the normalization morphism $\bar{D}^{\mathrm{norm}} \to \bar{D}$ is bijective, and a universal homeomorphism. This proves (b). For (d), note that for every $i \in \{1, \dots, t\}$, the finite map

$$(\{\bar{\eta_i}\})_{\mathrm{red}} \to (\bar{X} \setminus X)_{\mathrm{red}}$$

has degree $f = f^{\mathrm{sep}} f^{\mathrm{insep}}$, and can be factored in a homeomorphism of degree f^{insep} followed by a generically étale morphism of degree f^{sep}. We may now assume $\bar{C}$ is chosen so that its intersection with D_0 is in the open subset over which the morphism f^{sep} is étale, which again follows from Bertini's theorem. ∎

Definition 7. A curve $\bar{C}$ as in Proposition 6 is said to be *in good position relative to* D_0. If $\bar{C}$ is in good position relative to each irreducible component of $\bar{X} \setminus X$, we say that C is *in good position relative to* $\bar{X} \setminus X$.

Theorem 8. *We keep the assumptions and notations from above. Let $\bar{C} \subseteq \bar{X}$ be a curve which is in good position relative to D_0. Fix a closed point $x \in \bar{C} \cap D_0$ and a closed point $y \in \bar{D}^{\mathrm{norm}}$ mapping to x. Denote by $\bar{I}^C_{y/x} \subseteq \bar{I}$ be the associated decomposition group. Then*

$$|\bar{I}^C_{y/x}| = e f^{\mathrm{insep}}.$$

Proof. Let $\nu : \bar{D}^{\mathrm{norm}} \to \bar{D}$ be the normalization morphism. We have the following diagram

$$\begin{tikzcd} \bar{D}^{\mathrm{norm}} \arrow[r, "\nu"] \arrow[dr, "h"'] & \bar{D} \arrow[r] \arrow[d, "\bar{\pi}_C"] & \bar{Y} \arrow[d, "\bar{\pi}"] & \ni \eta_1, \dots, \eta_t \\ & \bar{C} \arrow[r] & \bar{X} & \ni \xi, \end{tikzcd}$$

and we utilize the notations introduced at the beginning of the section. Note that $\bar{I}$ acts on $\bar{D}$ and that the normalization $\nu : \bar{D}^{\mathrm{norm}} \to \bar{D}$ is an $\bar{I}$-equivariant homeomorphism.

For $x \in \bar{C} \setminus C$, the properties from Proposition 6 imply that $|\bar{\pi}_C^{-1}(x)| = r f^{\mathrm{sep}}$. On the other hand, for $y \in \bar{D}^{\mathrm{norm}}$ mapping to x we have

$$|\bar{\pi}_C^{-1}(x)| = [\bar{I} : \mathrm{Stab}_{\bar{I}}(\nu(y))],$$

and

$$\bar{I}^C_{y/x} = \mathrm{Stab}_{\bar{I}}(y) = \mathrm{Stab}_{\bar{I}}(\nu(y)).$$

It follows that
$$|\bar{I}^C_{y/x}| = \frac{|\bar{I}|}{tf^{\text{sep}}} = ef^{\text{insep}}$$
which is what we wanted to prove. ∎

Recall that a by [KS10, Thm. 1.1], a Galois cover is tamely ramified if and only if it is in restriction to all curves. We make the small observation, implicit in *loc.cit.*, that there are test curves for all the generic codimension *one* points at infinity. Explicitly:

Corollary 9. *The covering $\pi : Y \to X$ is tamely ramified along the codimension one points of $\bar{X}$ with support in $\bar{X} \setminus X$ if and only if $\pi_C : D = Y \times_X C \to C$ is tamely ramified for a curve $\bar{C}$ which is in good position relative to $\bar{X} \setminus X$.*

Proof. If $\pi_C : D \to C$ is tamely ramified, then for all $x \in \bar{C} \setminus C$, and all $y \in \bar{D}^{\text{norm}}$ mapping to x, the order of $|\bar{I}^C_{y/x}|$ is prime to p. The theorem implies that, for each boundary component D_0, $f^{\text{insep}} = 1$ and $(e, p) = 1$, so π is tamely ramified with respect $\bar{X} \setminus X$. ∎

Proof of Theorem 1. Let $\bar{C} \subseteq \bar{X}$ be a curve as in Proposition 6, i.e. which is in good position relative to D_0. Fix $x \in \bar{C} \setminus C$, and $y \in \bar{D}^{\text{norm}}$ mapping to x. As $\bar{C}$ intersects $(\bar{X} \setminus X)_{\text{red}}$ transversely, the ramification of $\rho|_C$ is bounded by $N \cdot (\bar{C} \setminus C)_{\text{red}}$.

Claim 2 in the proof of Proposition 2 showed that the order of the p-Sylow subgroup of $\bar{I}^C_{y/x}$ is bounded by a constant only depending on N, r, ℓ (an explicit bound is given by the constant M_0). As f^{insep} is a p-power, Theorem 8 implies that f^{insep} is bounded by the same constant. ∎

4. Some comments

Using the notation from above, for any $x \in \bar{C} \cap D_0$, and any $y \in \bar{D}^{\text{norm}}$ lying on $\overline{\{\eta_1\}}$ mapping to x, we have
$$\bar{I}^C_{y/x} \subseteq \ker\left(\text{Stab}_{\bar{I}}(\eta_1) \twoheadrightarrow \text{Aut}(k(\eta_1)/k(\xi))\right). \tag{$\star$}$$
The order of $\text{Stab}_{\bar{I}}(\eta_1)$ is ef, the order of $\text{Aut}(k(\eta_1)/k(\xi))$ is f^{sep}, so the kernel has order ef^{insep}. It follows that ($\star$) is an equality, and hence that the decomposition groups $\bar{I}^C_{y/x}$ only depend on the component $\overline{\{\eta_i\}}$ on which y lies (recall that $\eta_1, \ldots, \eta_t$ are the generic points of $\bar{Y} \setminus Y$).

We can also interpret $\ker\left(\text{Stab}_{\bar{I}}(\eta_1) \twoheadrightarrow \text{Aut}(k(\eta_1)/k(\xi))\right)$ as the inertia group of $\mathcal{O}^h_{\bar{X},\xi} \subseteq \mathcal{O}^h_{\bar{Y},\eta_1}$. As such, it has a unique p-Sylow subgroup, which according to the discussion above, has to coincide with the p-Sylow subgroup of $\bar{I}^C_{y/x}$ for any $y \in \bar{D}^{\text{norm}}$ lying on the closure of η_1.

Then this raises the following interesting side question. On the p-Sylow subgroup of $\bar{I}^C_{y/x}$ we have two filtrations: the upper numbering ramification filtration associated to $\mathcal{O}_{\bar{C},x} \subseteq \mathcal{O}_{\bar{D}^{\text{norm}},y}$, for some $y \in \bar{D}^{\text{norm}}$ lying on $\overline{\{\eta_1\}}$, and the Abbes-Saito ramification filtration on the inertia group of $\mathcal{O}^h_{\bar{X},\xi} \subseteq \mathcal{O}^h_{\bar{Y},\eta_1}$. Do they coincide?

References

[Zha95] Zhang, Bin, *Théorèmes du type Bertini en caractéristique positive*, Arch. Math. (Basel) **64** (1995), no. 3, 209–215.

[MC08] Michael J. Collins, *Modular analogues of Jordan's theorem for finite linear groups*, J. Reine Angew. Math. **624** (2008), 143–171.

[Del16] Deligne, P., *Email to Hélène Esnault*, dated Feb. 29th, 2016.

[EK12] H. Esnault, M. Kerz, *A finiteness theorem for Galois representations of function fields over finite fields (after Deligne)*, Acta Mathematica Vietnamica **37** 4 (2012), 531–562.

[Esn17] Esnault, H., *A remark on Deligne's finiteness theorem*, Int. Math. Res. Not. **16** (2017), 4962–4970.

[Esnl17] Esnault, H., *Survey on some aspects of Lefschetz theorems in algebraic geometry*, Revista Matemática Complutense **30** (2) (2017), 217–232.

[CJ1878] C. Jordan, *Mémoire sur les équations différentielles linéaires à intégrale algébrique*, J. Reine Angew. Math. **84** (1878), 89–215.

[KS10] Kerz, M., Schmidt, A.: *On different notions of tameness in arithmetic geometry*, Math. Ann. **346** (3) (2010), 641- 668.

[KR14] Kindler, L., Rülling, K.: *Introductory course on ℓ-adic sheaves and the ramification theory on curves*, https://arxiv.org/abs/1409.6899

[Lau81] Laumon, G.: *Semi-continuité du conducteur de Swan (d'après P. Deligne)*, in The Euler-Poincaré characteristic, Astérisque, **83** (1981), 173–219, Soc. Math. France.

[Ser67] Serre, J.-P.: *Représentations linéaires des groupes finis*, Hermann, Paris 1967.

Freie Universität Berlin, Arnimallee 3, 14195, Berlin, Germany
E-mail address: esnault@math.fu-berlin.de

E-mail address: lars.kindler@posteo.de

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai-400005, India
E-mail address: srinivas@math.tifr.res.in