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Cohomological Dimension in Pro-p Towers

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We give a proof without use of perfectoid geometry of the following vanishing theorem of Scholze: for $X \subset \mathbb{P}^n$ a projective scheme of dimension d over an algebraically closed characteristic 0 field, and X_r the inverse image of X via the map that assigns $(x_0^{p^r} : \cdots : x_n^{p^r})$ to the homogeneous coordinates $(x_0 : \ldots : x_n)$, the induced map $H^i(X, \mathbb{F}_p) \to H^i(X_r, \mathbb{F}_p)$ on étale cohomology dies for $i > d$ and r large. Our proof holds in characteristic $\ell \neq p$ as well.

1 Introduction

If X is a proper scheme of finite type of dimension d defined over an algebraically closed field k of characteristic $p > 0$, Artin–Schreier theory implies that the cohomological dimension of étale cohomology of X with $\mathbb{F}_p$-coefficients is at most d, that is, $H^i(X, \mathbb{F}_p) = 0$ for $i > d$. If k has characteristic not equal to p, the cohomological dimension of étale cohomology of X with $\mathbb{F}_p$-coefficients is $2d$ if X is proper and, by Artin's vanishing theorem, at most d if X is affine. However, when X is projective, Peter Scholze showed that there is a specific tower of p-power degree covers of X that makes its cohomological dimension at most d in the limit.

Let $X \subset \mathbb{P}^n$ be a projective scheme of dimension d. We choose coordinates $(x_0 : \ldots : x_n)$ on $\mathbb{P}^n$. With this choice of coordinates, we define the covers

$$\Phi_r^n : \mathbb{P}^n \to \mathbb{P}^n,\ (x_0 : \ldots : x_n) \mapsto (x_0^{p^r} : \ldots : x_n^{p^r}).$$

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We define X_r as the inverse image of X by Φ_r^n.

Theorem 1.1 (Scholze, [4], Theorem 17.3). If k is an algebraically closed field of characteristic 0, for $i > d$, one has

$$\varinjlim_r H^i(X_r, \mathbb{F}_p) = 0.$$

Scholze obtains the theorem as a corollary of his theory of perfectoid spaces. He does not detail the proof in *loc. cit.*, but his argument is documented in [5]. By smooth base change, we may assume that $k = \bar{\mathbb{Q}}_p$. By the comparison theorem [5, Thm. IV.2.1], $\varinjlim_r H^r(X_r, \mathbb{F}_p) \otimes \mathcal{O}_C/p$ is "almost" equal to $H^i(\mathcal{X}, \mathcal{O}^+_{\mathcal{X}}/p)$ where $\mathcal{X}$ is a perfectoid space he constructs, associated to $\varprojlim_r X_r$, and $C = \hat{\bar{\mathbb{Q}}}_p$. By [3, Thm. 4.5], the spectral space $\mathcal{X}$ has cohomological dimension at most the Krull dimension of X.

The aim of this short note is to give an elementary proof, as was asked for over $\mathbb{C}$ in [4, Section 17]. It turns out that the proof holds in characteristic not equal to p as well. One obtains the following theorem.

Theorem 1.2. If k is an algebraically closed field of characteristic not equal to p, for $i > d$, one has

$$\varinjlim_r H^i(X_r, \mathbb{F}_p) = 0.$$

The ingredients are constructibility and base change properties for relative étale cohomology with compact supports, functoriality, and some easy facts of representation theory of a cyclic group of p-power order.

2 General Reduction

As all cohomologies considered are étale cohomology with coefficients in $\mathbb{F}_p$, we drop $\mathbb{F}_p$ from the notation so it does not create confusion. As étale cohomology only depends on the underlying reduced structure, we may assume that X is reduced.

We first observe that Theorem 1.2 is true for all X if and only if it is true for all X that are irreducible. We argue by induction on the dimension d of X and its number s of components. If X has dimension 0, its cohomological dimension is 0, and there is nothing to prove. If X has only one component, there is nothing to prove by assumption. If X has $s \geq 2$ components, then it is the union of X_1 and X_2, where X_2 is irreducible and is not contained in X_1, and X_1 has $(s-1)$ components. Then $X_1 \cap X_2$ had dimension $\le (d-1)$. The Mayer–Vietoris exact sequence

$$\ldots \to H^{i-1}(X_1 \cap X_2) \to H^i(X_1 \cup X_2) \to H^i(X_1) \oplus H^i(X_2) \to H^i(X_1 \cap X_2) \to \ldots$$

shows that $H^i(X_1) = H^i(X_2) = H^{i-1}(X_1 \cap X_2) = 0$ for $i > d$ and implies $H^i(X_1 \cup X_2) = 0$ for $i > d$. But $H^i(X_2) = 0$ for $i > d$ by assumption, $H^i(X_1) = 0$ for $i > d$ by induction on the number of components, and $H^{i-1}(X_1 \cap X_2) = 0$ for $i > d$ by induction on d.

Let $H_a \subset \mathbb{P}^n$ denote the hyperplane defined by $x_a = 0$, and $Y_a = H_a \cap X$. If there is one a such that the dimension of Y_a is d, then $X \cap Y_a = X$ as X is irreducible, and one replaces in the statement and the proof $\mathbb{P}^n$ by $H_a = \mathbb{P}^{n-1}$. So we may assume that $Y = \bigcup Y_a$ is a divisor on X.

Notations 2.1. Assuming $Y = \bigcup Y_a$ is a divisor on X, let $U \subset X \setminus Y$ be open and dense and $Z = X \setminus U \supset Y$ be the boundary closed subscheme. We let U_r, resp. Y_r, resp. Z_r denote the pull-back of U, resp. Y, resp. Z along Φ^n_r.

Then $U_r = X_r \setminus Z_r$ is open dense, and Z_r is closed in X_r of smaller dimension. The morphism Φ^n_r restricted to U_r is proper and étale, thus the direct system $\varinjlim_r H^i_c(U_r)$ of étale cohomology with compact supports and coefficients $\mathbb{F}_p$ is defined.

From the excision sequence

$$\ldots \to H^{i-1}(Z_r) \to H^i_c(U_r) \to H^i(X_r) \to H^i(Z_r) \to \ldots$$

and induction on the dimension, one deduces that the theorem is true if and only if $\varinjlim_r H^i_c(U_r) = 0$ for $i > d$.

On the other hand, for $d = n$ then $X = \mathbb{P}^n$, $H^{2i}(\mathbb{P}^n) = \mathbb{F}_p \cdot [L]$, where L is linear of codimension i. Thus $\Phi^{n*}_1[L] = \mathbb{F}_p \cdot p^i[L]$ that is equal to 0 as soon as $i > 0$.

So throughout the rest of the note, we make the general assumption:

Assumption 2.2. X is an irreducible, reduced projective variety of dimension d with $0 < d < n$ over an algebraically closed field k of characteristic not equal to p.

With Notations 2.1, we want to draw the conclusion

$$\varinjlim_r H^i_c(U_r) = 0 \text{ for all } i > d.$$

3 Local Systems

3.1 Geometry preparation

We define the torus $\mathbb{T} = \mathbb{P}^n \setminus \cup_{i=0}^n H_i$. It has coordinates $(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$. We denote by $\phi_r^n : \mathbb{G}_m^n = (\Phi_r^n)^{-1}(\mathbb{G}_m^n) \to \mathbb{G}_m^n$ the restriction of Φ_r^n to the torus. By analogy with the notation $X_r = (\Phi_r^n)^{-1}(X)$, we write $\phi_r^n : (\mathbb{G}_m^n)_r \to \mathbb{G}_m^n$. We also use the same notation $U_r = (\Phi_r^n)^{-1}(U)$ for the open U. The projection $q : \mathbb{T} \to \mathbb{G}_m$ to any of the factors has the property that $q \circ \phi_r^n$ factors through ϕ_r^1. For $U \subset \mathbb{T} \cap X$ open dense, there is a projection $q : U \to \mathbb{G}_m$ to one of the factors that is dominant. As U is irreducible, all the fibers of q have dimension $\leq (d-1)$.

The composite map $U_r \xrightarrow{\phi_r^n} U \xrightarrow{q} \mathbb{G}_m$ factors through $\phi_r^1 : (\mathbb{G}_m)_r \to \mathbb{G}_m$, defining $q_r : U_r \to (\mathbb{G}_m)_r$. Concretely, if $q(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) = \frac{x_1}{x_0}$ (say), then $q_r(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) = \frac{x_1}{x_0}$.

Thus one has a commutative diagram

$$\begin{array}{ccccc} & & \overset{\phi_r^n}{\curvearrowright} & & \\ U_r & \longrightarrow & U'_r & \longrightarrow & U \\ & \searrow^{q_r} & \downarrow{\scriptstyle q'_r} & & \downarrow{\scriptstyle q} \\ & & (\mathbb{G}_m)_r & \xrightarrow[\phi_r^1]{} & \mathbb{G}_m \end{array}$$

where $U'_r = U \times_{\mathbb{G}_m, \phi_r^1} (\mathbb{G}_m)_r$, $q'_r = (\phi_r^1)^* q$.

3.2 Constructibility

Recall that $R^j q_! \mathbb{F}_p$ is constructible, see [2, Thm. 5.3.5]. As ϕ_r^1 is proper, for any $j \in \mathbb{N}$, $\phi_r^{1*}(R^j q_! \mathbb{F}_p)$ is constructible as well, and one has a morphism

$$\phi_r^{1*}(R^j q_! \mathbb{F}_p) \to R^j q_{r!} \mathbb{F}_p$$

of constructible sheaves, which for any $i \in \mathbb{N}$, induces an $\mathbb{F}_p$-linear map

$$\phi_r^{n*} : H_c^i(\mathbb{G}_m, R^j q_! \mathbb{F}_p) \to H_c^i((\mathbb{G}_m)_r, \phi_r^{1*}(R^j q_! \mathbb{F}_p)) \to H_c^i((\mathbb{G}_m)_r, R^j q_{r!} \mathbb{F}_p) = H_c^i(\mathbb{G}_m, \phi_{r*}^1 R^j q_{r!} \mathbb{F}_p).$$

Here the left map is defined by adjunction.

We pose the:

Induction hypothesis on d': given a subscheme $X \subset \mathbb{P}^n$ of dimension d' and a Zariski open subscheme $U \subset \mathbb{T} \cap X$ that is dense in X, there is a natural number r_0, such that for all $r \geq r_0$, the map $\phi_r^{n*} : H_c^i(U) \to H_c^i(U_r)$ vanishes for all $i > d'$.

The induction hypothesis is trivially verified for $d' = 0$. In the sequel, we assume it is verified for $d' \leq d - 1$.

Lemma 3.1. With the assumption 2.2 on X and (U, q) as in 3.1, for $j > d - 1$, there is an $r_1 \in \mathbb{N}$ such that for all $r \geq r_1$,

$$\phi_r^{n*} : R^j q_! \mathbb{F}_p \to \phi_{r*}^1 R^j q_{r!} \mathbb{F}_p$$

vanishes.

Proof. By [2, Thm. 5.2.8], $R^j q_! \mathbb{F}_p$ verifies base change with stalks $(R^j q_! \mathbb{F}_p)_x = H_c^j(q^{-1}(x))$ on geometric points $x \in \mathbb{G}_m$. Thus we can apply the induction hypothesis on the dimension of the fibers $q^{-1}(x)$. As $(\phi_r^n)^{-1}(q^{-1}(x)) = q_r^{-1}((\phi_r^1)^{-1}(x))$, it follows that the map

$$\phi_r^{n*} : H_c^j(q^{-1}(x)) \to H_c^j(q_r^{-1}(\phi_r^1)^{-1}(x))$$

vanishes by induction for $r \geq r(x)$ large enough depending on x. Taking x to be the geometric generic point $\mathrm{Spec}(\overline{k(\mathbb{G}_m)})$ defines $r = r(x)$. If $\mathcal{U} \subset (\mathbb{G}_m)_r$ is a dense open on which $R^j q_{r!} \mathbb{F}_p$ is a local system, which is lying in the smooth locus $\mathcal{U}^0$ of ϕ_r^1, then $\cap_g g^* \mathcal{U}$, for g in the Galois group $\mathbb{Z}/p^r$ of ϕ_r^1, is Galois invariant and dense in $(\mathbb{G}_m)_r$, thus of the shape $(\mathbb{G}_m^0)_r$ for some dense open $\mathbb{G}_m^0 \subset \mathbb{G}_m$. Then for all closed points $x \in \mathbb{G}_m^0$, we may take r constant equal to $r(x)$. We take r_1 greater or equal to r and to the finitely many $r(x)$ for x closed in $\mathbb{G}_m \setminus \mathbb{G}_m^0$. This finishes the proof. ■

3.3 Representation theory

With the assumption 2.2 on X and (U, q) as in 3.1, we fix some j and consider a dense open $\mathbb{G}_m^0 \subset \mathbb{G}_m$ over which $R^j q_! \mathbb{F}_p$ is a local system. As $\mathbb{G}_m \setminus \mathbb{G}_m^0$ is 0-dimensional, the excision map $H_c^2(\mathbb{G}_m^0, R^j q_! \mathbb{F}_p) \to H_c^2(\mathbb{G}_m, R^j q_! \mathbb{F}_p)$ is an isomorphism.

Proposition 3.2. There is an $r_2 \in \mathbb{N}$ such that for all $j \in \mathbb{N}$, all $r \geq r_2$,

$$\phi_r^{1*} : H_c^2(\mathbb{G}_m, R^j q_! \mathbb{F}_p) \to H_c^2((\mathbb{G}_m)_r, R^j q'_{r!} \mathbb{F}_p)$$

vanishes.

Proof. On $\mathbb{G}_m^0$, we denote by $\mathcal{V}$ the local system of $\mathbb{F}_p$-vector spaces dual to $R^j q_! \mathbb{F}_p$. By classical duality, the cup-product $H^2_c(\mathbb{G}_m^0, \mathcal{V}^\vee) \times H^0(\mathbb{G}_m^0, \mathcal{V}) \to H^2_c(\mathbb{G}_m, \mathbb{F}_p)$ is a perfect duality. On the other hand, on $(\mathbb{G}_m^0)_r$, base change again implies

$$\phi_r^{1*} R^j q_! \mathbb{F}_p = R^j q'_{r!} \mathbb{F}_p,$$

thus $\phi_r^{1*}\mathcal{V}$ is the local system dual to $R^j q'_{r!}\mathbb{F}_p$. Thus

$$\phi_r^{1*} : H^2_c(\mathbb{G}_m, R^j q_! \mathbb{F}_p) \to H^2_c((\mathbb{G}_m)_r, R^j q'_{r!} \mathbb{F}_p)$$

is dual to the trace map

$$\mathrm{Tr}(\phi_r^1) : H^0((\mathbb{G}_m^0)_r, \phi_r^{1*}\mathcal{V}) \to H^0(\mathbb{G}_m^0, \mathcal{V})$$

from which we show now that it vanishes for r large. As $\mathcal{V}$ is a local system, the dimension of $H^0((\mathbb{G}_m^0)_r, \phi_r^{1*}\mathcal{V})$ as an $\mathbb{F}_p$-vector space is bounded above by the rank of $R^j q_* \mathbb{F}_p$ and thus does not depend on r. For N a natural number, in $GL(N, \mathbb{F}_p)$ the order of a p-power torsion element is bounded by a constant depending on N and p. Thus for r large, the representation ρ of the Galois group $\mathbb{Z}/p^r$ of ϕ_r^1 on $H^0((\mathbb{G}_m^0)_r, \phi_r^{1*}\mathcal{V})$ cannot be faithful. Thus ρ factors as $\bar{\rho}$ through $\mathbb{Z}/p^s$ for some $s < r$. This implies that for any $v \in H^0((\mathbb{G}_m^0)_r, \phi_r^{1*}\mathcal{V})$

$$\mathrm{Tr}(\phi_r^1)(v) = \sum_{i=0}^{p^r-1} \rho(i)(v) = \sum_{\bar{i} \in \mathbb{Z}/p^s} \sum_{j=0}^{p^{r-s}-1} \rho(i + jp^s)(v) = p^{r-s} \left(\sum_{\bar{i} \in \mathbb{Z}/p^s} \bar{\rho}(\bar{i})(v) \right) = 0.$$

In the formula, $i \in \mathbb{Z}/p^r$ and maps to $\bar{i} \in \mathbb{Z}/p^s$. This finishes the proof. ∎

Corollary 3.3. With r_2 as in Proposition 3.2, for all $r \geq r_2$,

$$\phi_r^{n*} : H^2_c(\mathbb{G}_m, R^j q_! \mathbb{F}_p) \to H^2_c((\mathbb{G}_m)_r, R^j q_{r!} \mathbb{F}_p)$$

vanishes.

Proof. From the factorization q'_r of q_r over q, one obtains a factorization

$$\phi_r^{n*} : H^2_c(\mathbb{G}_m, R^j q_! \mathbb{F}_p) \xrightarrow{\phi_r^{1*}} H^2_c(\mathbb{G}_m, R^j q'_{r!} \mathbb{F}_p) = H^2_c(\mathbb{G}_m, \phi_r^{1*} R^j q_! \mathbb{F}_p) \to H^2_c((\mathbb{G}_m)_r, R^j q_{r!} \mathbb{F}_p)$$

where the first map is the one considered in Proposition 3.2 and the second one comes by functoriality $R^j q'_{r!}\mathbb{F}_p \to R^j q_{r!}\mathbb{F}_p$. This finishes the proof. ■

Remark 3.4. We could have taken U to be $\mathbb{T}$ in the Induction Hypothesis. In the proof, we have to consider the inverse image by q of some dense open in $\mathbb{G}_m$ that is then a dense open in $\mathbb{T}$. For this reason, we just kept a neutral letter U.

4 Proof of Theorem 1.2

We argue by induction on d, starting with $d = 0$. We use the notations of the previous sections, make the assumption 2.2 on X, and take (U, q) as in 3.1.

We consider the Leray spectral sequence for q and $H^i_c(U)$ for $i > d$. One first has the corner map

$$H^i_c(U) \to E_2^{0i}(q) = H^0_c(\mathbb{G}_m, R^i q_! \mathbb{F}_p).$$

As $i > d > d - 1$ we apply Lemma 3.1. Thus there is an r_1 such that for all $r \geq r_1$, the image of $H^i_c(U)$ in $H^i_c(U_r)$ lies in a subquotient of

$$E_2^{1,i-1}(q_r) = H^1_c((\mathbb{G}_m)_r, R^{i-1} q_{r!} \mathbb{F}_p).$$

As $i-1 > d-1$ the same argument shows that there is an $r'_1 \geq r_1$ such that for all $r \geq r'_1$, the image of $H^i_c(U)$ in $H^i_c(U_r)$ lies in the image of

$$E_2^{2,i-2}(q_r) = H^2_c((\mathbb{G}_m)_r, R^{i-2} q_{r!} \mathbb{F}_p).$$

If $i > d + 1$ then $i - 2 > d - 1$, and one applies again the same argument that finishes the proof. Or else, one applies the following argument. If $i \geq d + 1$, then $i - 2 \geq d - 1$; Corollary 3.3 implies that there is an $r_2 \geq r'_1$ such that for all $r \geq r_2$, the image of $H^i_c(U)$ in $H^i_c(U_r)$ is 0.

As in the whole argument, it does not matter whether we start the proof for U or for U_{r_0} for some given natural number r_0; we, in fact, proved $\varinjlim_r H^i_c(U_r) = 0$ for $i > d$. This finishes the proof of Theorem 1.2.

5 Remarks

1) If $Z \subset \mathbb{P}^n$ is any locally closed subscheme, with compactification $\bar{Z} \subset \mathbb{P}^n$, applying again the excision exact sequence

$$\ldots \to H^i_c(Z) \to H^i(\bar{Z}) \to H^i(\bar{Z} \setminus Z) \to H^{i+1}_c(Z) \to \ldots,$$

one sees that Theorem 1.1 implies (and in fact is equivalent to)

$$\varinjlim_r H^i_c(Z_r) = 0 \text{ for all } i > d,$$

where $Z_r = (\Phi^n_r)^{-1}(Z)$.

2) Theorem 1.2 can be expressed by writing $H^i(X, \mathbb{F}_p)$ as $H^i(\mathbb{P}^n, \mathcal{F})$, where $\mathcal{F}$ is the constructible sheaf $i_* \mathbb{F}_{p,X}$, where $i : X \hookrightarrow \mathbb{P}^n$ is the closed embedding, and where $\mathbb{F}_{p,X}$ is the constant étale sheaf $\mathbb{F}_p$ on X, and writing $H^i(X_r, \mathbb{F}_p)$ as $H^i(\mathbb{P}^n, (\Phi^n_r)^* \mathcal{F})$. More generally, we can take in Theorem 1.2 $\mathcal{F}$ to be any constructible sheaf.

Theorem 5.1. If k is an algebraically closed field of characteristic not equal to p, and $\mathcal{F}$ is a constructible sheaf of finite dimensional $\mathbb{F}_p$-vector spaces on $\mathbb{P}^n$ with support of dimension d, then for $i > d$, one has

$$\varinjlim_r H^i((\mathbb{P}^n)_r, (\Phi^n_r)^* \mathcal{F}) = 0.$$

The proof is exactly the same, and we do not write the details.

3) It may happen that even if X is smooth, one needs $r \geq 2$ in Theorem 1.2. For example if $p = 2$, and X is a smooth conic in $\mathbb{P}^2$ such that the H_i, $i = 0, 1, 2$ are tangent to X, then X_1 splits entirely into the union of four lines. So the minimum r that kills the whole cohomology $H^i(X), i > d$ is perhaps an intriguing geometric invariant of the triple

$$\left(X, \mathbb{P}^n, (x_0 : \ldots : x_n)\right).$$

4) Beilinson remarked that Theorem 1.2 has some relation to [1, Thm. 4.5.1]. In [1], *loc. cit.* the formulation is with $\bar{\mathbb{Q}}_p$-coefficients, but they immediately go down to $\mathbb{F}_p$ in the proof. They do not prove vanishing, and they are not in $\mathbb{P}^n$, but they bound the growth of the dimension of the cohomology in étale towers in function of the degree of the covers.

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