CHERN CLASSES OF AUTOMORPHIC VECTOR BUNDLES, II

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Abstract. We prove that the ℓ-adic Chern classes of canonical extensions of automorphic vector bundles, over toroidal compactifications of Shimura varieties of Hodge type over \( \overline{\mathbb{Q}}_p \), descend to classes in the ℓ-adic cohomology of the minimal compactifications. These are invariant under the Galois group of the \( p \)-adic field above which the variety and the bundle are defined.

Introduction

Let \( S \) be a Shimura variety. It is defined over a number field [Del71, Cor. 5.5], called the reflex field \( E \), and carries a family of automorphic vector bundles \( \mathcal{E} \), defined (collectively) over the same number field [Har85, Thm. 4.8]. The Shimura variety has a minimal compactification \( S \hookrightarrow S^\text{min} \) which in dimension \( \geq 2 \) is singular. This is minimal in the sense that each of the toroidal compactifications \( S^\text{tor} \), subject to the extra choice of a fan, admits a surjective birational proper morphism \( S^\text{tor} \rightarrow S^\text{min} \), which is an isomorphism on \( S \) and is a desingularization of \( S^\text{min} \). The automorphic bundle \( \mathcal{E} \) admits a canonical extension \( \mathcal{E}^\text{can} \) on \( S^\text{tor} \) constructed by Mumford [Mum77, Thm. 3.1] and more generally in [Har89]: \( \mathcal{E}^\text{can} \) is locally free and defined over a finite extension of the reflex field. On \( X^\text{min} \), on the other hand, there is generally no locally free extension of \( \mathcal{E} \), except in dimension 1 when \( S^\text{tor} \rightarrow S^\text{min} \) is an isomorphism. In [EH17] we studied the continuous ℓ-adic Chern classes of \( \mathcal{E} \) in case \( S \) is proper. In particular, we proved that the higher Chern classes in continuous ℓ-adic cohomology die if \( \mathcal{E} \) is flat ([EH17, Thm. 0.2]). A key ingredient in the proof is the study of the action of Hecke algebra on continuous ℓ-adic cohomology, and the fact that Chern classes lie in the eigenspace for the volume character [EH17, Cor. 1.18]. When \( S \) is not proper, the Hecke algebra does not act on any cohomology of \( X^\text{tor} \). Thus we can not apply the methods of

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loc. cit. to prove the analogous theorem for the canonical extensions of flat automorphic bundles on $S^{\text{tor}}$.

On the other hand, the Hecke algebra does act on the $\ell$-adic cohomology of $X^{\text{min}}$. In addition, over the field of complex numbers $\mathbb{C}$, Goresky-Pardon [GP02, Thm.] used explicit estimates on differential forms to construct classes $c_n(E^{\text{GP}}) \in H^{2n}(X^{\text{min}}, \mathbb{C})$ which descend the Chern classes of $E^{\text{can}}$ in Betti cohomology $H^{2n}(X^{\text{tor}}, \mathbb{Q}(n))$. They do not come from classes in $H^{2n}(X^{\text{min}}, \mathbb{Q}(n))$; precisely this fact enabled Looijenga ([Loo17], Theorem 5.1) to construct mixed Tate extensions in the Hodge category on the Siegel modular variety (see also the result announced by Nair).

Our main theorem gives an $\ell$-adic version of the Goresky-Pardon construction.

**Theorem 0.1** (Theorem 3.7, Theorem 3.11). Let $X$ be a Shimura variety of abelian type. Let $\ell$ be a prime number, $E_v$ be completion of $E$ at a place $v$ dividing the prime $p$ different from $\ell$, and $E_v \hookrightarrow \overline{\mathbb{Q}}_p$ be an algebraic closure. Then the Chern classes of $E^{\text{can}} \in H^{2n}(X^{\text{tor}}, \mathbb{Q}(\ell)(n))$ descend to well defined classes $c_n(E) \in H^{2n}(X^{\text{min}}, \mathbb{Q}(\ell)(n))$. The $c_n(E)$ are contravariant for the change of level and of Shimura data, verify the Whitney product formula, lie in the eigenspace for the volume character of the Hecke algebra, and are invariant under the action of the Galois group of $E_v$ on $H^{2n}(X^{\text{min}}, \mathbb{Q}(\ell)(n))$. Their pull-back under $H^{2n}(X^{\text{min}}, \mathbb{Q}(\ell)(n)) \to H^{2n}(X^{\text{tor}}, \mathbb{Q}(\ell)(n)) = H^{2n}(X^{\text{tor}}, \mathbb{Q}(\ell)(n))$ are the Chern classes of $E^{\text{can}}$ which are invariant under the Galois group of $E$.

As the classes of $E^{\text{can}}$ are even defined in continuous $\ell$-adic cohomology of $X^{\text{tor}}$, where $E \hookrightarrow \overline{\mathbb{Q}}$ is an algebraic closure, they are in particular invariant under the Galois group of $E$. The classes we construct rely on $p$-adic geometry and have no reason a priori not to depend on the chosen $p$, let alone to lie in $H^{2n}(X^{\text{min}}, \mathbb{Q}(\ell)(n))^{G_E}$ via the isomorphism $H^{2n}(X^{\text{min}}, \mathbb{Q}(\ell)(n))^{G_E} \cong H^{2n}(X^{\text{tor}}, \mathbb{Q}(\ell)(n))$.

We now describe the method of proof. The main tool used is the existence of a perfectoid space $P^{\text{min}}$ above $S^{\text{min}}$ which has been constructed by Peter Scholze in [Sch15, Thm. 4.1.1] for Shimura varieties of Hodge type, together with a Hodge-Tate period map $\pi_{HT} : P^{\text{min}} \to \hat{X}$ (see [CS17, Thm. 2..3] for the final form) with values in the adic space of the compact dual variety defined over the completion of $\overline{\mathbb{Q}}_p$. More precisely as explained in [Sch15, 4. 1], one has to replace here $S^{\text{min}}$ by its image in the Siegel space with a sufficiently small level; we ignore
this difficulty in the introduction, although it complicates our exposition in several places (perhaps unnecessarily). This allows us to define \textit{vector bundles} \( \pi^\ast_{HT}(E) \) on \( P^\text{min} \) where \( E \) are equivariant vector bundles on \( \hat{X} \) defining the automorphic bundles. So while the bundles \( E \) do not extend to \( S^\text{min} \), they do on Scholze’s limit space \( P^\text{min} \). On the other hand, on the perfectoid space \( P^{\text{tor}} \) above \( S^{\text{tor}} \), one has the pull-back of the bundles \( \pi^\ast_{HT}(E) \) and the pull-back via the tower defining the perfectoid space of the canonical extensions \( E^\text{can} \). Theorem 2.20 asserts that the two on \( P^{\text{tor}} \) are the same. This is the key point, the proof of which relies on [PS16, Cor. 1.6],

Descent for cohomology with torsion coefficients enables one to construct the classes as claimed in Theorem 0.1, see Appendix A.

The classes \( c_n(E) \in H^{2n}(S^\text{min}_{\bar{Q}_p}, Q_\ell(n)) \) map in particular to well-defined classes in \( \ell \)-adic intersection cohomology \( IC^{2n}(S^\text{min}_{\bar{Q}_p}, Q_\ell(n)) \). On intersection cohomology of \( S^\text{min} \) over \( \bar{Q} \), we identify the eigenspace under the volume character with the cohomology of the compact dual \( \hat{X} \) of \( S \) (Proposition 3.12). If the classes in \( IC^{2n}(S^\text{min}_{\bar{Q}_p}, Q_\ell(n)) \), identified with \( IC^{2n}(S^\text{min}_{\bar{Q}_p}, Q_\ell(n)) \), descended to continuous \( \ell \)-adic intersection cohomology (to be defined), we could try to apply the method developed in [EH17] to show that the classes of \( E^\text{can} \) in continuous \( \ell \)-adic cohomology on \( S^{\text{tor}} \) over \( E \) die when \( E \) if flat. This would be in accordance with [EV02, Thm. 1.1] where it is shown that the higher Chern classes of \( E^\text{can} \) in the rational Chow groups on the Siegel modular variety vanish when \( E \) is flat.

While we were writing the present note, it was brought to our attention that in an unpublished preprint [Nai14, 0.4], Nair has independently mentioned the possibility of using Scholze’s Hodge-Tate morphism to construct Chern classes in the cohomology of minimal compactifications.

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1. Generalities on automorphic bundles

If $G$ is a reductive algebraic group over $\mathbb{Q}$, by an *admissible irreducible representation* of $G(\mathbb{A})$ we mean an irreducible admissible $(\mathfrak{g}, K) \times G(\mathbb{A}_f)$-module, where $\mathfrak{g}$ is the complexified Lie algebra of $G$, $K \subset G(\mathbb{R})$ is a connected subgroup generated by the center of $G(\mathbb{R})$ and a maximal compact connected subgroup, $G(\mathbb{A})$ ($G(\mathbb{A}_f)$) is the group of (finite) adèles of $G$. If $\pi$ is such a representation then we write

$$\pi \simeq \pi_\infty \otimes \pi_f$$

where $\pi_\infty$ is an irreducible admissible $(\mathfrak{g}, K)$-module and $\pi_f$ is an irreducible admissible representation of $G(\mathbb{A}_f)$.

Let $(G, X)$ be a Shimura datum: $G$ is a reductive algebraic group over $\mathbb{Q}$, and $X$ is a (finite union of) hermitian symmetric spaces, endowed with a transitive action of $G(\mathbb{R})$ satisfying a familiar list of axioms. The compact dual $\hat{X}$ of $X$ is a (projective) flag variety for $G$. Thus one can speak of the category $\text{Vect}_G(\hat{X})$ of $G$-equivariant vector bundles on $\hat{X}$; the choice of a base point $h \in X \subset \hat{X}$ determines an equivalence of categories $\text{Rep}(P_h) \simeq \text{Vect}_G(\hat{X})$, where $P_h \subset G$ is the stabilizer of $h$, a maximal parabolic subgroup of $G$, and for any algebraic group $H$, $\text{Rep}(H)$ denotes the tensor category of its representations. Let $K_h$ be the Levi quotient of $P_h$; the group $K_h(\mathbb{R})$ can be identified with a maximal connected subgroup of $G(\mathbb{R})$ that is compact modulo the center of $G$. If $K \subset G(\mathbb{A}_f)$ is an open compact subgroup, we let $K_S(G, X)$ denote the Shimura variety attached to $(G, X)$ at level $K$; it has a canonical model over the reflex field $E(G, X)$ which is a number field (see [Del71, Cor. 5.5]). We always assume that $K$ is *neat*; then $K_S(G, X)$ is a smooth quasi-projective variety.

Furthermore, we use the notation of [EH17], Sections 1.1–1.2, specifically the notation $\mathcal{E} \in \text{Vect}(\hat{X})$ for the vector bundle on the compact dual, $[\mathcal{E}]_K \in \text{Vect}(K_S(G, X))$ for the automorphic vector bundle associated to the underlying representation of the compact group $P_h \subset G(\mathbb{R})$, the stabilizer of a chosen point $h \in X$. For the purposes of constructing Chern classes, we need only consider semisimple representations of $P_h$, which necessarily factor through representations of $K_h$.

The action of the Hecke algebra $\mathcal{H}_K = \{T(g) : g \in G(\mathbb{A}_f)\}$ is recalled in Section 3. The Chern character

$$\text{ch}_{\hat{X}} : \text{Vect}_G(\hat{X}) \to CH(\hat{X})_\mathbb{Q} \to \oplus_i H^{2i}(\hat{X}, \mathbb{Q}(i))$$

induces isomorphisms

$$K_0(\text{Rep}(K_h)) \otimes K_0(\text{Rep}(G))_\mathbb{Q} \xrightarrow{\sim} CH(\hat{X})_\mathbb{Q} \xrightarrow{\sim} \oplus_i H^{2i}(\hat{X}, \mathbb{Q}(i))$$
and analogously in ℓ-adic cohomology. On the other hand, the construction of automorphic vector bundles gives rise to a homomorphism

\[(1.2) \quad c_K : \text{Vect}_G(\hat{X}) \to CH_K(S(G,X))_{\mathbb{Q},v}, \quad \mathcal{E} \mapsto c([\mathcal{E}]_K),\]

where \(v\) indicates the invariants under the volume character. If \(E \in \text{Vect}_G(\hat{X})\) is in the image of \(\text{Rep}(G)\) via the factorization

\[\text{Rep}(K_h) \to \text{Rep}(G) \to \text{Vect}_G(\hat{X}),\]

then \([\mathcal{E}]\) is endowed with a natural flat connection, so its higher Chern classes in \(\oplus_i H^{2i}(K_S(G,X),\mathbb{Q}(i))\) are equal to 0. We obtain a morphism

\[(1.3) \quad K_0(\text{Rep}(K_h)) \otimes_{K_0(\text{Rep}(G))} \mathbb{Q} \to \oplus_i H^{2i}(K_S(G,X),\mathbb{Q}(i))_v \subset \oplus_i H^{2i}(K_S(G,X),\mathbb{Q}(i)),\]

analogous to (1.1), where the subscript \(v\) denotes a certain eigenspace for the action of the unramified Hecke algebra.

2. Chern classes for compactified Shimura varieties

2.1. Toroidal and minimal compactifications, and canonical extensions of automorphic vector bundles. Henceforward we assume the Shimura variety \(K_S(G,X)\) is not projective; equivalently, the derived subgroup \(G^{\text{der}}\) of \(G\) is isotropic over \(\mathbb{Q}\). In this case the automorphic theory naturally gives information about Chern classes of canonical extensions on toroidal compactifications on the one hand; on the other hand, the \(v\)-eigenspace most naturally appears in intersection cohomology of the minimal compactification. This has been worked out in detail in the \(C^\infty\) and the \(L_2\) theory by Goresky and Pardon in [GP02]. In what follows, we let \(j_K : K_S(G,X)^{\text{min}} \hookrightarrow K_S(G,X)^{\text{min}}\) denote the minimal (Baily-Borel-Satake) compactification. The minimal compactification is canonical; thus if \(K' \subset K\), there is a unique morphism \(K'_S(G,X)^{\text{min}} \to K_S(G,X)^{\text{min}}\) extending the natural map of open Shimura varieties, and if \(g \in G(\mathbb{A}_f)\), then the Hecke correspondence \(T(g)\) extends canonically to a correspondence on \(K_S(G,X)^{\text{min}} \times K_S(G,X)^{\text{min}}\) where the product is taken over the reflex field. In particular, for any cohomology theory \(H\) as above, we have Hecke operators

\[T(g) \in \text{End}(H^*(K_S(G,X)^{\text{min}})).\]

The minimal compactification is always singular, except in dimension 1, and the automorphic vector bundles do not in general extend as bundles to \(K_S(G,X)^{\text{min}}\). Thus the classical theory does not automatically attach Chern classes to automorphic vector bundles on non-proper Shimura varieties in some cohomology theory \(H^{**}(K_S(G,X)^{\text{min}},*)\).
On the other hand, there is a large collection of toroidal compactifications $\mathcal{K}_S(G,X)_\Sigma \leftarrow \mathcal{K}_S(G,X)$, indexed by combinatorial data $\Sigma$ (see [AMRT75], III, Section 6, Main Theorem for details; the adelic construction is in [Har89, Pin90]). The set of $\Sigma$ is adapted to the level subgroup $K$. It is partially ordered by refinement: if $\Sigma'$ is a refinement of $\Sigma$, then there is a natural proper morphism $p_{\Sigma',\Sigma} : \mathcal{K}_S(G,X)_{\Sigma'} \rightarrow \mathcal{K}_S(G,X)_\Sigma$ extending the identity map on the open Shimura variety. Any two $\Sigma$ and $\Sigma'$ can be simultaneously refined by some $\Sigma''$. Further, the open embedding $\mathcal{K}_S(G,X)_\Sigma \leftarrow \mathcal{K}_S(G,X)$, is completed to a diagram

\[
\begin{array}{ccc}
\mathcal{K}_S(G,X)_\Sigma & \xrightarrow{\varphi_\Sigma} & \mathcal{K}_S(G,X)^{\text{min}} \\
\downarrow & & \downarrow \\
\mathcal{K}_S(G,X) & \xrightarrow{\jmath_K} & \mathcal{K}_S(G,X)_{\Sigma'}
\end{array}
\]

where $\varphi_\Sigma$ is a projective morphism, which is an isomorphism on $\mathcal{K}_S(G,X)$, thus is a desingularization of the minimal compactification, which is constructed as in [AMRT75], loc.cit..

Now assume that $K$ is a neat open compact subgroup of $G(A_f)$, in the sense of [Har89] or [Pin90], [0.6]. In that case, we can choose $\Sigma$ so that $\mathcal{K}_S(G,X)_\Sigma$ is smooth and projective. We do so unless we specify otherwise. Then for any refinement $\Sigma'$ of $\Sigma$, $\mathcal{K}_S(G,X)_{\Sigma'}$ is again smooth and projective. Mumford proved in [Mum77], Theorem 3.1 that, if $\mathcal{E} \in \text{Vect}_G^\text{ss}(\hat{X})$, the automorphic vector bundle $[\mathcal{E}]_K$ on $\mathcal{K}_S(G,X)$ admits a canonical extension $[\mathcal{E}]_K^{\text{can}}$ to $\mathcal{K}_S(G,X)_\Sigma$; we write $[\mathcal{E}]_K^{\Sigma}$ if we want to emphasize the toroidal data. The adelic construction is carried out in Section 4 of [Har89], where Mumford’s result was generalized to arbitrary $\mathcal{E} \in \text{Vect}_G(\hat{X})$ (recall that the upper index $^\text{ss}$ stands for semi-simple). In particular it was shown there that if $\mathcal{E} = (G \times W)/P_h$, where $W$ is the restriction to $P_h$ of a representation of $G$, the action of $P_h$ is diagonal, and we have identified $G/P_h$ with $\hat{X}$, then $[\mathcal{E}]_K$ is a vector bundle on $\mathcal{K}_S(G,X)$ with a flat connection, and its canonical extension $[\mathcal{E}]_K^{\text{can}}$ on $\mathcal{K}_S(G,X)_\Sigma$ is exactly Deligne’s canonical extension. In particular, the connection has logarithmic poles along $\mathcal{K}_S(G,X)_\Sigma \setminus \mathcal{K}_S(G,X)$. Moreover, if $\Sigma'$ is a refinement of $\Sigma$, then

\[
p_{\Sigma',\Sigma}^*[\mathcal{E}]_K^{\Sigma'} = [\mathcal{E}]_K^{\Sigma'}
\]

(see [Har90], Lemma 4.2.4), which in particular implies

\[
p_{\Sigma',\Sigma,*}[\mathcal{E}]_K^{\Sigma'} = Rp_{\Sigma',\Sigma,*}[\mathcal{E}]_K^{\Sigma'} = [\mathcal{E}]_K^{\Sigma}.
\]
Finally, for any fixed $\Sigma$,

$$\mathcal{E} \rightarrow [\mathcal{E}]_{K}^{\text{can}}$$

is a monoidal functor from $\text{Vect}_{G}(\hat{X})$ to the category of vector bundles on $K S(G, X)_{\Sigma}$.

Unfortunately, the $g \in G(\mathbb{A}_{f})$ generally permute the set of $\Sigma$ and thus the Hecke correspondences $T(g)$ in general do not extend to correspondences on a given $K S(G, X)_{\Sigma}$. Thus the arguments of [EH17], which are based on the study of the $v$-eigenspace for the Hecke operators in the cohomology of $K S(G, X)$, cannot be applied directly to prove vanishing of higher Chern classes of $[\mathcal{E}]_{K}$ in continuous $\ell$-adic cohomology of $K S(G, X)_{\Sigma}$ over its reflex field, when $[\mathcal{E}]_{K}$ is a flat automorphic vector bundle.

One can obtain information about the classes of $[\mathcal{E}]_{K}$ on the open Shimura variety, but these lose information. From the standpoint of automorphic forms, the natural target of the topological Chern classes of automorphic vector bundles should be the intersection cohomology of the minimal compactification. We fix cohomological notation: $H^{*}(Z, \mathbb{Q})$, resp. $IH^{*}(Z, \mathbb{Q})$ denotes Betti resp. intersection cohomology of a complex variety $Z$, while $H^{*}(Z, \mathbb{Q}_{\ell})$, resp. $IH^{*}(Z, \mathbb{Q}_{\ell})$ denote $\mathbb{Q}_{\ell}$-étale cohomology.

**Remark 2.2.** The action of Hecke correspondences on Betti intersection cohomology $IH(K S(G, X), \mathbb{Q})$ are defined analytically by reference to Zucker’s conjecture. For a purely geometric construction of the action of Hecke correspondences on $IH(K S(G, X), \mathbb{Q})$ and thus on $IH(K S(G, X), \mathbb{Q}_{\ell})$ see [GM03], (13.3) (the argument applies more generally to weighted cohomology as defined there).

The following statement generalizes Proposition 1.20 of [EH17] and is proved in the same way.

**Proposition 2.3.** There is are canonical isomorphism of algebras

$$H^{*}(\hat{X}, \mathbb{Q}) \cong IH^{*}(K S(G, X)^{\text{min}}, \mathbb{Q})$$

$$H^{*}(\hat{X}, \mathbb{Q}_{\ell}) \cong IH^{*}(K S(G, X)^{\text{min}}, \mathbb{Q}_{\ell}).$$

**Proof.** The second statement is deduced directly from the first one by the comparison isomorphism. As in the proof of the analogous fact in [EH17], it suffices to prove the corresponding statement over $\mathbb{C}$. By Zucker’s Conjecture [Loo88, SS90], $IH^{*}(K S(G, X)^{\text{min}}, \mathbb{C})$ is computed using square-integrable automorphic forms and Matsushima’s formula ([Bor83], 3.6, formula (1)). Say the space $\mathcal{A}_{(2)}(G)$ of square-integrable automorphic forms on $G(\mathbb{Q}) \backslash G(\mathbb{A})$ decomposes as the direct sum

$$\mathcal{A}_{(2)}(G) = \bigoplus_{\pi} m(\pi)\pi$$
where \( \pi \) runs over irreducible admissible representations of \( G(\mathbb{A}) \) and \( m(\pi) \) is a non-negative integer, which is positive for a countable set of \( \pi \). Then

\[
IH^i(KS(G, X)^{\min}, \mathbb{C}) \xrightarrow{\sim} \oplus_{\pi} m(\pi)H^i(g, K_h; \pi_{\infty}) \otimes \pi_f^K.
\]

This implies

\[
(2.4) \quad IH^i(KS(G, X), \mathbb{C})_v \xrightarrow{\sim} \oplus_{\pi} m(\pi)H^i(g, K_h; \pi_{\infty}) \otimes (\pi^K)_f,v,
\]

where \( (\pi^K)_f,v \) is the eigenspace in \( \pi^K_f \) for the volume character of \( \mathcal{H}_K \).

Write \( \pi_f = \bigotimes_q \pi_q \), where \( q \) runs over rational primes. Now if \( q \) is unramified for \( K \) then \( \pi^K_q = 0 \) unless \( \pi_q \) is spherical; but the only spherical representation of \( G(\mathbb{Q}_q) \), the spherical subspace which is an eigenspace for the (local) volume character, is the trivial representation of \( G(\mathbb{Q}_q) \).

Thus \( (\pi^K)_f,v = 0 \) implies that \( \pi_q \) is the trivial representation for all \( q \) that are unramified for \( K \). It then follows from weak approximation that \( \pi \) is in fact the trivial representation. Thus for all \( i \),

\[
(2.5) \quad IH^i(KS(G, X), \mathbb{C})_v \xrightarrow{\sim} H^i(g, K_h; \mathbb{C}).
\]

But this is equal to \( H^i(\hat{X}, \mathbb{C}) = IH^i(\hat{X}, \mathbb{C}) \) by a standard calculation; see [GP02, Rmk. 16.6]. The rest of the proof follows as in the proof of [EH17], Proposition 1.20. \( \square \)

The point of the preceding proposition is that the Chern classes of automorphic vector bundles on the non-proper Shimura variety \( KS(G, X) \) are represented by square-integrable automorphic forms, more precisely, by the differential forms on \( G(\mathbb{Q}) \backslash G(\mathbb{A}) \) that are invariant under the action of \( G(\mathbb{A}) \). In other words, the only \( \pi \) that contributes is the space of constant functions on \( G(\mathbb{Q}) \backslash G(\mathbb{A}) \), which are square integrable modulo the center of \( G(\mathbb{R}) \). Thus \( IH^{2*}(KS(G, X)^{\min}, \mathbb{Q}(\ast))_v \) can be viewed as \( L^2 \)-Chern classes of automorphic vector bundles. In addition \( IH^{2*+1}(KS(G, X)^{\min}, \mathbb{Q}(\ast))_v = 0 \). Although most automorphic vector bundles do not extend as bundles to the minimal compactification \( KS(G, X)^{\min} \), it was proved by Goresky and Pardon in [GP02], Main Theorem, that under the natural homomorphism

\[
(2.6) \quad H^*(KS(G, X)^{\min}, \mathbb{C})_v \to IH^*(KS(G, X)^{\min}, \mathbb{C}).
\]

the classes in \( IH^*(KS(G, X)^{\min}, \mathbb{C})_v \), lift canonically, as differential forms, to ordinary singular cohomology \( H^*(KS(G, X)^{\min}, \mathbb{C})_v \).

We use the notation of the diagram (2.1). Let \( K \) be a neat open compact subgroup of \( G(\mathbb{A}_f) \), as above. Let \([\mathcal{E}]_K \) be the automorphic
vector bundle on $K S(G, X)$ attached to the homogeneous vector bundle $E \in \text{Vect}_G(\hat{X})$. Let
\[ c_n([E]_K) \in H^{2n}(K S(G, X)_\Sigma, \mathbb{Q}(n)), \quad c_n([E]_K) \in H^{2n}(K S(G, X), \mathbb{Q}(n)) \]
be the Chern classes in Betti cohomology. The following theorem summarizes the main results of the article [GP02].

**Theorem 2.7.**

(i) There are canonical classes $\bar{c}_n([E]_K) \in H^{2n}(K S(G, X)^{\text{min}}, \mathbb{C})$ such that
\[ \varphi_\Sigma^*(\bar{c}_n([E]_K)) = c_n([E]_K) \in H^{2n}(K S(G, X)_\Sigma, \mathbb{Q}(n)). \]

(ii) In particular
\[ j_\Sigma^*(\bar{c}_n([E]_K)) = c_n([E]_K) \in H^{2n}(K S(G, X), \mathbb{Q}(n)). \]

(iii) The classes $\bar{c}_n([E])$ are represented by square integrable automorphic forms. More precisely, the image of $\bar{c}_n([E])$ under the morphism of (2.6) is contained in the subspace of intersection cohomology that corresponds, under the isomorphism (2.4), to the relative Lie algebra cohomology of $\bigoplus \pi$ with trivial archimedean component $\pi_\Sigma$.

(iv) Suppose $G^{\text{der}}(\mathbb{R}) = \prod_i G_i$ is a product of simple Lie groups, all of which are of the form $\text{Sp}(2n, \mathbb{R})$, $\text{SU}(p, q)$, $\text{SO}^*(2n)$, or $\text{SO}(p, 2)$ with either $p = 2$ or $p$ odd. The $\mathbb{Q}$-subalgebra
\[ H_{\text{Chern}}^{2*}(K S(G, X)^{\text{min}}, \mathbb{C}) \]
generated by the $\bar{c}_n([E])$, as $E$ varies over $\text{Vect}_G(\hat{X})$ is endowed with a naturally defined surjective homomorphism
\[ H_{\text{Chern}}^{2n}(K S(G, X)^{\text{min}}) \xrightarrow{h} H^{2*}(\hat{X}, \mathbb{Q}(*) \bigotimes_{\mathbb{C}^\times}) \]
such that for any $n$ the diagram
\[
\begin{array}{ccc}
H_{\text{Chern}}^{2n}(K S(G, X)^{\text{min}}) & \xrightarrow{\text{natural map}} & IH^{2n}(K S(G, X)^{\text{min}}, \mathbb{Q}(n)) \\
\downarrow h & & \uparrow \cong \\
H^{2n}(\hat{X}, \mathbb{Q}(n)) & \xrightarrow{=} & H^{2n}(\hat{X}, \mathbb{Q}(n))
\end{array}
\]
commutes. In other words, the isomorphism of Proposition 2.3 tensor $\mathbb{C}$ factors through
\[ H^{2n}(K S(G, X)^{\text{min}}, \mathbb{C}) \xrightarrow{\text{natural map}} IH^{2n}(K S(G, X)^{\text{min}}, \mathbb{C}). \]
In the next sections, we use Peter Scholze’s perfectoid geometry and his Hodge-Tate morphism to prove an analogue of the Goresky-Pardon theorem for \(\ell\)-adic cohomology of Shimura varieties of abelian type.

2.1.1. Even-dimensional quadrics. Theorem 2.7 excludes the case where \(G^\text{der}(\mathbb{R})\) contains a factor isomorphic to \(SO(2k - 2, 2)\) with \(k > 2\). Assuming \(G^\text{der}\) is \(\mathbb{Q}\)-simple, there is then a totally real field \(F\), with \([F : \mathbb{Q}] = d\), say, such that \(\hat{X}_C\) is isomorphic to a product \(Q^d_n\) of \(d\) smooth projective complex quadrics \(Q_n\), each of dimension \(n = 2k - 2\).

The reason for this exclusion is explained in §16.6 of [GP02]. Following §16.5 of [BH58], we consider the cohomology algebra \(A := H^*(Q_n, C)\) of a complex quadric \(Q_n = SO(n + 2)/SO(n) \times SO(2)\) of dimension \(n\), with \(C = \mathbb{C}\) or \(\mathbb{Q}_\ell\). Then \(A\) contains a subalgebra \(A^+\) isomorphic to \(C[c_1]/(c_1^{n+1})\), with \(c_1 \in H^2(Q_n, C)\) given by the Chern class of the line bundle corresponding to the standard representation of \(SO(2)\). (There is a misprint in [GP02]; the total dimension of \(A^+\) as \(C\)-vector space is \(n + 1\), not \(n\).) Moreover there is an isomorphism

\[
A = A^+[e]/(e^2 - c_1^n)
\]

with \(e\) the Euler class of the the vector bundle arising from the standard representation of \(SO(n)\). In [BH58], the class \(c_1\) is denoted \(x_1\) and the class \(e\) is denoted \(\prod_{i=2}^n x_i\); the equation \(e^2 = c_1^n\) then follows immediately for formula (6) of [BH58].

More generally, if \(\hat{X}_C\) is isomorphic to \(Q^d_n\) as above, we denote by \(c_{1,r} \in H^2(\hat{X}_C, C(1))\) the class of the line bundle defined above corresponding to the \(r\)-th factor of \(Q^d_n\), \(r = 1, \ldots, d\), and let \(e_r\) correspond to the Euler class in the \(r\)-th factor. The isomorphism of Proposition 2.3 is valid in all cases, and Goresky and Pardon showed that the image in \(IH^{2j}(K \cdot S(G, X)^\text{min}, \mathbb{Q}(j))\) of the classes \(c_{1,r}^j \in H^{2j}(\hat{X}_C, C(j))\) lift canonically to the cohomology of the minimal compactification. (The twist \(j\) here is unnecessary for \(C = \mathbb{C}\), we write it for the case \(C = \mathbb{Q}_\ell\).) However, when \(n = 2k - 2 > 2\), they were unable to show that the \(e_r\) lift. We can extend the statement of Theorem 2.7 with the following definition.

**Definition 2.9.** (i) Suppose \(G^\text{der}\) is \(\mathbb{Q}\)-simple and \(G^\text{der}(\mathbb{R}) \simeq SO(2k - 2, 2)^d\) for some integer \(d\), with \(k > 2\). Suppose \(C = \mathbb{C}, \mathbb{Q}_\ell\), or \(\mathbb{Q}\). Define

\[
H^*(\hat{X}, C)^+ \subset H^*(\hat{X}, C);
\]

\[
IH^*(K \cdot S(G, X)^\text{min}, \mathbb{Q}(n))^+ \subset IH^*(K \cdot S(G, X)^\text{min}, \mathbb{Q}(n))_v
\]

to be the subalgebras generated respectively by the classes \(c_{1,r}, r = 1, \ldots, d\), and their images under the isomorphism of Proposition 2.3.
We similarly define
\[(2.11) \quad [K_0(\text{Rep}(K_h)) \otimes_{K_0(\text{Rep}(G))} C]^+ \subset K_0(\text{Rep}(K_h)) \otimes_{K_0(\text{Rep}(G))} C\]
to be the inverse image of $H^*(\hat{X},C)^+$ under the isomorphism (1.1).

(ii) If $G$ satisfies condition (iv) of Theorem 2.7, we let
\[H^*(\hat{X},C)^+ = H^*(\hat{X},C)\]
\[IH^*(K_S(G,X)^{\text{min}},\mathbb{Q}(n))^+_v = IH^*(K_S(G,X)^{\text{min}},\mathbb{Q}(n))^+_v ;\]
\[[K_0(\text{Rep}(K_h)) \otimes_{K_0(\text{Rep}(G))} C]^+ = K_0(\text{Rep}(K_h)) \otimes_{K_0(\text{Rep}(G))} C.\]

(iii) In either case, we define
\[\text{Vect}_G(\hat{X})^+ = ch^{-1}(H^*(\hat{X},C)^+) \subset K_0(\text{Vect}_G(\hat{X})).\]

We continue to use the notation $E$ to denote (virtual) bundles in $\text{Vect}_G(\hat{X})^+$.

**Theorem 2.12.** Suppose $G$ satisfies either (i) or (ii) of Definition 2.9. Then the conclusions of Theorem 2.7 hold with $H^*(\hat{X},C)$, $\text{Vect}_G(\hat{X})$, and $IH^*(K_S(G,X)^{\text{min}},\mathbb{Q}(n))^+_v$ replaced by the versions with superscript $^+$, and for $E \in \text{Vect}_G(\hat{X})$.

The theorem, and its application in Proposition 3.12 below, naturally extend to groups $G$ such that $G^\text{der}$ is a product of groups of type (i) and (ii) in Definition 2.9. We omit the details.

2.2. Perfectoid Shimura varieties and the Hodge-Tate morphism. The results of the present section are entirely due to Scholze [Sch15], then to Caraiani-Scholze [CS17] and Pilloni-Stroh [PS16].

Fix a level subgroup $K \subset G(A_f)$. Let $G(A_f) \rightarrow G(\mathbb{Q}_p)$, $k \mapsto k_p$ be the projection. Denote by $K_p$ its projection to $G(\mathbb{Q}_p)$. For $r \geq 0$ we let $K_{p,r} \subset K_p$ be a decreasing family of subgroups of finite index, with $K_{p,r} \supset K_{p,r+1}$ for all $r$, and such that $\bigcap_r K_{p,r} = \{1\}$. Let $K_r = \{k \in K \mid k_p \in K_{p,r}\}$, and let $K^p = \cap_r K_r$. We identify $K^p$ with its projection to the prime-to-$p$ adèles $G(A_f^p)$; then $K^p$ is an open compact subgroup of $G(A_f^p)$ called a “tame level subgroup”.

We assume that the Shimura datum $(G,X)$ is of *Hodge type*. Thus, up to replacing $K$ by a subgroup of finite index, $K_S(G,X)$ admits an embedding of Shimura varieties in a Siegel modular variety of some level attached to the Shimura datum $(GSp(2g),X_{2g})$ for some $g$, where $X_{2g}$ is the union of the Siegel upper and lower half-spaces. We let $\hat{X}_{2g}$ denote the compact dual flag variety of $X_{2g}$. Let $C$ denote the completion of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Denote by $K_r S(G,X)$, resp. $K_r S(G,X)^{\text{min}}$ the adic space over $C$ attached to $K_r S(G,X)$, resp.
\[ K_r S(G, X)^{\text{min}} \] We assume that \( K^p \) is contained in the principal congruence subgroup of level \( N \) for some \( N \geq 3 \), in the sense explained in §4 of [Sch15]. By [Sch15, 4.1] there is a level \( K'_r \) for \((GSp(2g), X_{2g})\) such that \( K_r = K'_r \cap G(A_f) \), and the scheme theoretic image of \( K_r S(G, X)^{\text{min}} \) in \( K'_r S(GSp(2g), X_{2g})^{\text{min}} \) does not depend on the choice of \( K'_r \). We denote it by \( K_r S(G, X)^{\text{min}} \), and by \( K_r S(G, X)^{\text{min}} \) its associated \( C \)-adic space.

We restate one of the main theorems of Scholze’s article [Sch15].

**Theorem 2.13** ([Sch15], Theorem 4.1.1). There is a commutative diagram of morphisms of adic spaces

\[
\begin{array}{ccc}
K^p S(G, X) & \sim & \lim_{\leftarrow r} K_r S(G, X) \\
\downarrow j & & \downarrow \lim_{\leftarrow r} j_r \\
K^p S(G, X)^{\text{min}} & \sim & \lim_{\leftarrow r} K_r S(G, X)^{\text{min}}.
\end{array}
\]

Here, on the left side, \( j \) is an open embedding of perfectoid Shimura varieties over \( C \), and on the right side, \( \lim_{\leftarrow r} j_r \) is the projective limit of open embeddings of adic spaces over \( C \). Moreover, there is a \( G(\mathbb{Q}_p) \)-equivariant Hodge-Tate morphism

\[
\pi_{HT} : K^p S(G, X)^{\text{min}} \to \hat{X}_{2g}
\]

which is compatible with change of tame level subgroup \( K^p \).

Here we use the notation \( \hat{X}_{2g} \) for the adic space over \( C \) attached to the flag variety \( X_{2g} \). In loc. cit. the notation \( \mathcal{X}_{K^p} \) is used for the perfectoid Shimura variety \( K^p S(G, X)^{\text{min}} \). The notation \( \sim \) indicates that the right hand side of (2.14) is not exactly a projective limit in the category of adic spaces. For the precise definition, see [SW13, Defn. 2.4.1]. The target of the Hodge-Tate morphism was clarified in [CS17].

Let \( \hat{X} \subset \hat{X}_{2g} \) be the closed embedding of \( C \)-adic spaces corresponding to the closed embedding \( \hat{X} \subset \hat{X}_{2g} \) of the compact duals of \( X \) and \( X_{2g} \).

**Theorem 2.15** ([CS17], Theorem 2.1.3). The Hodge-Tate morphism \( \pi_{HT} \) in Theorem 2.13 factors through \( \hat{X} \subset \hat{X}_{2g} \), yielding a \( G(\mathbb{Q}_p) \)-equivariant Hodge-Tate morphism

\[
\pi_{HT} : K^p S(G, X)^{\text{min}} \to \hat{X}.
\]

**Proof.** The existence of \( \pi_{HT} \) is stated for the open perfectoid Shimura variety \( K^p S(G, X) \) with values in \( \hat{X} \). Since \( \hat{X} \) is closed in \( \hat{X}_{2g} \), the extension to the boundary then follows from Theorem 2.13 by continuity. \( \square \)
Assume for the moment that \((G, X) = (\text{Sp}(2g), X_{2g})\). Fix a neat compact open subgroup \(K_g \subset \text{Sp}(2g, \mathbb{A}_f)\), with \(K_p = \text{Sp}(2g, \mathbb{Z}_p)\) and write \(K_g \mathcal{A}_g\) instead of \(K_g \mathcal{S}(\text{Sp}(2g), X_{2g})\) for the Siegel modular variety of genus \(g\) and level \(K_g\), viewed as an adic space over \(\mathbb{C}\). Let \(K_{g, r} \subset K_g\) be the principal congruence subgroup of \(K_g\) of level \(p^r\).

Let \(K_{g, r} \mathcal{A}_g\) denote the corresponding toroidal compactification of \(K_g \mathcal{A}_g\) for some combinatorial datum \(\Sigma_g\) as above. Following Pilloni-Stroh in [PS16], §1.3, but with a change of notation, we let \(K_{g, r} \mathcal{A}_g\) denote the corresponding toroidal compactification of \(K_{g, r} \mathcal{A}_g\) for each \(r\), with the same \(\Sigma_g\). The \(K_{g, r} \mathcal{A}_g\) form a projective system of adic spaces over \(\mathbb{C}\). The authors construct a projective system of normal models \(K_{g, r} \mathcal{A}_g\) over \(\text{Spec} \mathcal{O}_C\) for each \(r\), with the same \(\Sigma_g\). The perfectoid space \(K_{g, r} \mathcal{A}_g\) which is the generic fiber of the projective limit of the \(K_{g, r} \mathcal{A}_g\), in the sense of [SW13], Section 2.2. See [PS16], Section A. 12, and Corollaire A. 19 for the statement that the projective limit, which they denote \(X(p^\infty)\), is indeed perfectoid.

For any \(r \geq 0\) there is then a natural map in the category of adic spaces over \(\mathbb{C}\)

\[
K_{g}^p \mathcal{S}_g^\text{tor} \rightarrow K_{g}^p \mathcal{S}_g^\text{min} \times_{K_{g, r} \mathcal{A}_g^\text{min}} K_{g, r} \mathcal{A}_g^\text{tor},
\]

where \(K_{g}^p \mathcal{S}_g^\text{tor}\) was denoted \(K_{g}^p \mathcal{S}(\text{Sp}(2g), X_{2g})^\text{min}\) in Theorem 2.13, and where the right hand side is the fiber product in the category of adic spaces over \(\mathbb{C}\).

Let \((G, X)\) be any Shimura datum of Hodge type, with a fixed symplectic embedding \(\tau : (G, X) \hookrightarrow (\text{Sp}(2g), X_{2g})\). Choose a combinatorial datum \(\Sigma\) for \(K \mathcal{S}(G, X)\) that is compatible with the fixed \(\Sigma_g\) chosen above, so that there is a morphism

\[
[\tau] : K \mathcal{S}(G, X) \rightarrow K_g \mathcal{A}_g,\Sigma_g
\]

which factors through its scheme theoretic image

\[
K \mathcal{S}(G, X) \Sigma := K \mathcal{S}(G, X)^\text{min} \times_{K_g \mathcal{A}_g^\text{min}} K_g \mathcal{A}_g,\Sigma_g
\]

in \(K_g \mathcal{A}_g,\Sigma_g\). We denote the corresponding \(C\)-adic spaces by \(K \mathcal{S}(G, X)\) and \(K \mathcal{S}(G, X)\).

As in Theorem 4.1.1 of [Sch15], diagram (2.14) of Theorem 2.13 extends to a morphism of minimal compactifications of the perfectoid Shimura varieties:

\[
K^p \mathcal{S}(G, X)^\text{min} \rightarrow K^p \mathcal{S}_g^\text{min}.
\]

By [Sch12], Proposition 6.18, the fiber product

\[
K^p \mathcal{S}(G, X)^\text{tor} := K^p \mathcal{S}(G, X)^\text{min} \times_{K^p \mathcal{S}_g^\text{min}} K^p \mathcal{S}_g^\text{tor}
\]
exists as a perfectoid space. It maps, compatibly in the level $K$, to $K S(G, X)_\Sigma$.

By [Har85], Section 3.4, there is a correspondence

$$K I(G, X) \xrightarrow{a} \hat{X}$$

where $b'$ is a family of $G$-torsors, functorial with respect to inclusions $K' \subset K$ and translation by elements $g \in G(A_f)$, and $a'$ is a $G$-equivariant morphism. For any $G$-equivariant vector bundle $E$ over $\hat{X}$, the pullback $a'^* (E)$ is $G$-equivariant, and thus descends to the automorphic vector bundle $[E]$ over $K S(G, X)$. The $G$-torsor $K I(G, X)$ is constructed as the moduli space $Isom^\otimes ([V_\rho], V)$ of trivializations of a flat automorphic vector bundle $[V_\rho]$ attached to a faithful representation $\rho : G \to GL(V)$; the superscript $\otimes$ indicates that the isomorphisms respect absolute Hodge cycles. Thus the construction requires a priori the existence of the automorphic vector bundle $[V_\rho]$. When $K S(G, X)$ is of Hodge type, one can take $\rho$ to be a symplectic embedding $\rho : G \to GSp(V)$, and define $[V_\rho]$ to be the pullback to $K S(G, X)$ of the (dual of the) relative de Rham $H^1$ of the universal abelian scheme over the Siegel modular variety attached to $GSp(V)$. Then the morphism $a$ is defined by transferring the Hodge filtration on $[V_\rho]$ to the constant vector bundle defined by $V$. That this construction is canonically independent of the choice of symplectic embedding $\rho$ is explained in [Har85, Remark 4.9.1]; see also [CS17, Lemma 2.3.4].

Choose a base point $h \in X \subset \hat{X}$ as in Section 1; we may as well assume $h$ to be a CM point. Recall that $P_h$ denotes the stabilizer of $h$ and $K_h$ its Levi quotient. (In [CS17] this group is denoted $M_\mu$.) Any faithful representation $\tau : K_h \to GL(W_h)$ defines by pullback to $P_h$ a $G$-equivariant vector bundle $W_h$ over $\hat{X}$, and thus an automorphic vector bundle $[W_h]$ over $K S(G, X)$ that varies functorially in $K$. We can then define a family (depending on $K$) of $K_h$-torsors $b : T \to K S(G, X)$ with a $K_h$-equivariant morphism $a : K I_h(G, X) \to \hat{X}$ as the moduli space of trivializations of $[W_h]$ as above. More precisely, letting $R_a P_h$ denote the unipotent radical of $P_h$, the natural morphism $\hat{X}_h := G / R_a P_h \to \hat{X}$ is canonically a $G$-equivariant $K_h$-torsor, whose pullback $a'^* \hat{X}_h$ descends to a $K_h$-torsor over $K S(G, X)$, over the reflex field of the CM point $h$, that is naturally identified with $T$ defined above. Moreover, the construction is canonically independent of the choice of base point. This is also constructed in Section 2.3, especially Lemma 2.3.5, of [CS17].
The pull-back of $T$ via the ringed space morphism $\hat{X} \to \hat{X}$ is denoted by $\mathcal{T}$. The $K_h$-torsor $b$ has a Mumford extension $T_{\text{can}} \to K_S(G, X)_\Sigma$ as a $G$-torsor (see [HZ94], Lemma 4.4.2 and pages 320-321, where the Mumford extension of the $G$-torsor $b'$ is constructed. One obtains the $K_h$-torsor over the toroidal compactification by the procedure described in the previous paragraph.) One denotes by $M_{\text{can}}$ the pull-back of $T_{\text{can}}$ via the ringed space morphism $K S(G, X)_\Sigma \to K S(G, X)_\Sigma$.

This defines for all $r \geq 0$, the commutative diagram of morphisms of adic spaces

\[(2.18)\]

where

\[K_p S(G, X)^{\text{tor}, r} = K_p S(G, X)^{\text{tor}} \times_{K_r S(G, X)_\Sigma} K_r S(G, X)_\Sigma\]

is the fiber product in the category of $C$-adic spaces and we use the notation $\nu(r)$ (for “normalization”) for the map induced by $[\tau]$, in order to avoid confusion. For $r' > r$, by the universal property of the fiber product, one has

\[K_p S(G, X)^{\text{tor}, r'} \to K_p S(G, X)^{\text{tor}, r}.\]

We define the $K_h$-torsor

\[M_p := \pi_{HT}^* \mathcal{T}\]

on $K_p S(G, X)^{\text{min}}$, which coincides by definition with the $K_h$-torsor defined in [CS17], Lemma 2.3.8. We define the $K_h$-torsor

\[M_{dR, \Sigma, r} := \pi_{K_r, K_p}^* M_{\text{can}}\]

on $K_p S(G, X)_\Sigma$. 

Proposition 2.19 ([CS17], Proposition 2.3.9). There is a canonical isomorphism

\[ j^* \mathcal{M}_p \sim \to j^*_{\Sigma, r} \mathcal{M}_{dR, \Sigma, r} \]

of $K_h$-torsors over $K^p S(G, X)$.

Although the article [CS17] is written for compact Shimura varieties, the argument developed there for this point is valid for any Shimura variety of Hodge type. Strictly speaking, as explained in [CS17], the torsors $\mathcal{M}_p$ and $\mathcal{M}_{dR, \Sigma}$ have natural extensions to $G$-torsors by pullback to torsors for opposite parabolics, followed by pushforward to $G$, so the comparison only applies to semisimple automorphic vector bundles.

The following theorem is essentially due to Pilloni and Stroh.

Theorem 2.20. The isomorphism of Proposition 2.19 extends to a canonical isomorphism

\[ \nu(r)^* q^* \mathcal{M}_p \sim \to \mathcal{M}_{dR, \Sigma, r} \]

of $K_h$-torsors over $K^p S(G, X)^{tor, r}$. As $K^p$ varies, these isomorphisms are equivariant under the action of $G(A_f^{p})$.

In [PS16] this is proved for the Siegel modular variety, although it is not stated in this form. In Section 4 we explain their result and show how to obtain Theorem 2.20 for general Shimura varieties of Hodge type. In Section 3 we admit Theorem 2.20 and derive from Theorem 2.20 a construction of $\ell$-adic Chern classes.

3. Construction of $\ell$-adic Chern classes

The constructions in the present section depend in an essential way on Theorem 2.20.

3.1. Construction using $\pi_{HT}$. As in [CS17], Theorem 2.1.3, part (2), the existence of the $K_h$-torsor $\mathcal{M}_p$ over $K^p S(G, X)^{min}$ implies that there is a functor

(3.1) $\text{Vect}_{G}^{ss}(\breve{X}) [\cong \text{Rep}_{K_h}] \to \{ \text{Vector bundles over } K^p S(G, X)^{min} \}$

$\mathcal{E} \mapsto [\mathcal{E}]_p$.

With the notation of (2.18), we define

(3.2) $[\mathcal{E}]_{p, \Sigma, r} = \nu(r)^* q^* [\mathcal{E}]_p$, $[\mathcal{E}]_{dR, \Sigma, r} = \pi_{K, K^p}^* [\mathcal{E}^{can}]$.

From Theorem 2.20 one immediately obtains the following corollary.
Corollary 3.3. There is a canonical isomorphism of tensor functors

\[
\begin{array}{ccc}
\mathcal{E} & \longrightarrow & \mathcal{E}_{p,\Sigma,r} \\
\downarrow & & \downarrow \\
\mathcal{E} & \longrightarrow & \mathcal{E}_{dR,\Sigma,r}
\end{array}
\]

extending the identity over \( K_S(G,X) \).

We return to this theme in the next paragraph. Recall briefly how the Hecke algebra acts on automorphic bundles on \( K_S(G,X) \). See [EH17], Section 1.2. Fix \( K \subset G(\mathbb{A}_f) \). Let \( g \in G(\mathbb{A}_f) \) and consider \( K_g = K \cap gKg^{-1} \subset K \). Right multiplication by \( g \) defines an isomorphism \( r_g : gKg^{-1}S(G,X) \sim \rightarrow KS(G,X) \). One defines

\[
T(g)[\mathcal{E}]_K = r_g^* \circ \pi_{gKg^{-1},K_g} \circ \pi_{K,K_g}^* [\mathcal{E}]_K.
\]

(3.4)

The projection formula implies

\[
T(g)[\mathcal{E}]_K = [\mathcal{E}]_K \otimes \pi_{K,K_g}^* \mathcal{O}_{K_gS(G,X)}.
\]

As one has pull-backs and push-downs for \( K_0,\mathbb{Q} \) and for \( \ell \)-adic cohomology, one has on them an action of the Hecke algebra \( \mathcal{H}_K \), spanned by the \( T(g) \) with values in a ring \( R \), is the character sending \( T(g) \) to \([K : K_g] \) viewed as an element in \( R \) ([EH17], Lemma 1.15). Thus (3.5) is saying that classes of automorphic bundles in \( K_0(K_S(G,X))_\mathbb{Q} \) are eigenvectors for the volume character under the action of \( \mathcal{H}_K \).

Let \( \ell \neq p \) be a prime number.

Lemma 3.6. The isomorphism \( r_g \) induces an isomorphism

\[
r_g^{\min} : KS(G,X)^{\min} \sim \rightarrow gKg^{-1}S(G,X)^{\min}.
\]

The finite morphism \( \pi_{K,K_g} : K_gS(G,X) \rightarrow KS(G,X) \) extends to a finite morphism

\[
\pi_{K,K_g}^{\min} : K_gS(G,X)^{\min} \rightarrow KS(G,X)^{\min}
\]

and the action of \( \mathcal{H}_K \) on \( H^{2n}(KS(G,X),\mathbb{Z}_\ell(n)) \) extends to an action on \( H^{2n}(KS(G,X)^{\min},\mathbb{Z}_\ell(n)) \).
Proof. When \( G = GL(2)_{\mathbb{Q}} \) the Shimura varieties are finite unions of modular curves, and the result in that case is standard. We may thus assume \( G^{\text{ad}} \) contains no factor isomorphic to \( PGL(2)_{\mathbb{Q}} \). Then we know by Theorem 10.14 of [BB66] that \( H^0(K_S(G, X), \omega^n) \) is finite dimensional for any \( n \geq 0 \), where \( \omega \) is the dualizing sheaf. The isomorphism \( r_g \) induces an algebra isomorphism

\[
\oplus_{n \in \mathbb{N}} H^0(K_S(G, X), \omega^n) \xrightarrow{\sim} \oplus_{n \in \mathbb{N}} H^0(g_{K_g^{-1}}S(G, X), \omega^n).
\]

Similarly the finite cover \( \pi_{K, K_g} \) induces an injective algebra morphism

\[
\oplus_{n \in \mathbb{N}} H^0(K_S(G, X), \omega^n) \hookrightarrow \oplus_{n \in \mathbb{N}} H^0(K_S(G, X), \omega^n)
\]

Since \( K_S(G, X)_{\min} \) (resp. \( K_S(G, X) \)) is \( \text{Proj} \) of the left-hand side (resp. right-hand side) of the last diagram ([BB66], Theorem 10.11), the induced map on the \( \text{Proj} \) defines the extensions \( \pi_{K, K_g}^{\min} \) and \( \pi_{K, K_g}^{\min} \).

On the other hand, pull-back on cohomology is defined, while the trace map \( \text{Tr} : \pi_{K, K_g} \mathbb{Z}_\ell \rightarrow \mathbb{Z}_\ell \) extends to \( j_K \ast \pi_{K, K_g} \mathbb{Z}_\ell = \pi_{K, K_g}^{\min} \ast j_K \ast \mathbb{Z}_\ell = \pi_{K, K_g}^{\min} \mathbb{Q}_\ell \rightarrow j_K \ast \mathbb{Z}_\ell = \mathbb{Z}_\ell \). This proves the second part.

Let \( E(G, X) := E \) be the reflex field. Fix an embedding \( E \hookrightarrow \overline{\mathbb{Q}} \). We choose a place \( v \) of \( E \) dividing the prime \( p \) which we assume to be different from \( \ell \). We extend \( E \hookrightarrow \overline{\mathbb{Q}} \) to the completion \( E_v \hookrightarrow \mathbb{Q}_v \). We denote by \( G_{E_v}, G_E \) the Galois groups of \( E_v, E \). We add a subscript on the lower right to indicate the field over which we regard the varieties considered. We denote by \( H^i(K_S(G, X)^{\min}_{\overline{\mathbb{Q}}}, \mathbb{Q}_v) = H^i(K_S(G, X)^{\min}_{\mathbb{Q}_p}, \mathbb{Q}_\ell) \) the eigenspace under the volume character of the action of \( \mathcal{H}_K \).

Through the end of this section, we set \( K_0 = K \) for the neat subgroup \( K \subset G(\mathbb{A}_f) \) and use the notation \( H^i(\Lambda(j)) \) for étale cohomology with coefficients \( \Lambda \), which are either \( \mathbb{Z}_\ell \) or \( \mathbb{Q}_\ell \).

**Theorem 3.7.** Let \( (G, X) \) be a Shimura datum of Hodge type. For any prime \( \ell \neq p \), any \( \mathcal{E} \in \text{Vect}^{\text{ss}}_G(\hat{X}_E) \), we define \( \ell \)-adic Chern classes

\[
c_{n, \ell}(\mathcal{E})_{K_r} \in H^{2n}(K_r, S(G, X)^{\min}_{\overline{\mathbb{Q}}}, \Lambda(n))^{G_{E_v}}, \ n \in \mathbb{N}
\]

where \( \Lambda = \mathbb{Q}_\ell \) for \( r = 0 \) and \( \mathbb{Z}_\ell \) for \( r \geq 1 \), with the following properties:

1. \( j_{K_r}^*(c_{n, \ell}(\mathcal{E})_{K_r}) = c_{n, \ell}(\mathcal{E})_{K_r} \in H^{2n}(K_r, S(G, X)^{\min}_{\overline{\mathbb{Q}}}, \Lambda(n)) \), where the right hand side is the \( \ell \)-adic Chern class of \( \mathcal{E} \) on \( K_r, S(G, X) \).

2. If \( K' \subset K \) then \( \pi_{K_r, K'_r}^{\min}(c_{n, \ell}(\mathcal{E})_{K_r}) = c_{n, \ell}(\mathcal{E})_{K'_r} \).

3. For \( g \in G(\mathbb{A}_f) \) unramified, \( r_g^{\min}(c_{n, \ell}(\mathcal{E})_{K_r}) = c_{n, \ell}(\mathcal{E})_{gK_r, g^{-1}} \).

4. Whitney product formula: If \( 0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{E}_2 \rightarrow 0 \) is an exact sequence in \( \text{Vect}^{\text{ss}}_G(\hat{X}_E) \), then

\[
c_{n, \ell}(\mathcal{E})_{K_r} = \oplus_{a+b = n} c_{a, \ell}(\mathcal{E}_1)_{K_r} \ast c_{b, \ell}(\mathcal{E}_2)_{K_r}.
\]
(5) If \((G', X') \to (G, X)\) is a morphism of Shimura data of Hodge type, and \(K' \to K\) is a compatible level, then
\[
f^*(c_{n,\ell}([\mathcal{E}]_p)|_{K'}) = (c_{n,\ell}([f^*\mathcal{E}]_p)|_{K'}) \in H^{2n}(K'_r S(G', X')_{\eta_{p}}, \Lambda(n))^{G_{E_v}}.
\]

Proof. We first construct \(c_{n,\ell}([\mathcal{E}]_p)|_{K'}\), without the \(v\) invariance. The vector bundle \([\mathcal{E}]_p\) from (3.1) has Chern classes
\[
c_{n,\ell}([\mathcal{E}]_p) \in \lim_{\longrightarrow} H^{2n}(K_r S(G, X)^{\min}, \mathbb{Z}/\ell^m(n)), \ n \in \mathbb{N}
\]
by Proposition A.1. On the other hand, \([\mathcal{E}]_p\) is defined as \(\pi_H^*\mathcal{E}\), where \(\mathcal{E}\) is viewed as a vector bundle on the \(C\)-adic space \(\hat{X}\). But \(\mathcal{E}\) is already defined on \(\hat{X}_E\). In particular, the classes \(c_{n,\ell}([\mathcal{E}]_p)\) are invariant under the action of \(K_{p,r} \subset G(\mathbb{Q}_p)\) for all \(r\) and by \(G_{E_v}\). By Proposition A.3 it follows that the classes \(c_{n,\ell}([\mathcal{E}]_p)\) uniquely define classes in \(H^{2n}(K_r S(G, X)^{\min}, \mathbb{Z}/\ell^m(n))^{G_{E_v}}\) for \(r \geq 1\). Since \(K/K_1\) is finite, we can descend to \(\mathbb{Q}_\ell\)-cohomology at level \(K\). The image \(c_{n,\ell}([\mathcal{E}]_p)|_{K'} \in H^{2n}(K_r S(G, X)^{\min}, \mathbb{Z}/\ell^m(n))^{G_{E_v}}\) defines the desired classes.

Property (1) is then a direct consequence of Proposition 2.19. Property (2) follows directly from the construction using descent via Proposition A.3. As for Property (3), using descent again, it is enough to see that \(r_g^{\text{min}}\) extends to an isomorphism
\[
r_g^{\text{perf}, \text{min}} : K_r S(G, X)^{\min} \to g_{K, g^{-1}} S(G, X)^{\min}
\]
of perfectoid spaces, which respects \([\mathcal{E}]_p\). (See [Sch15], Theorem 4.1.1 (iv) for the Siegel modular case; the general case is identical.) Our notation \([\mathcal{E}]_p\) does not refer to \(K_p\). One should replace the notation \([\mathcal{E}]_p\) with \([\mathcal{E}]_p^{K_p}\). Then the compatibility means \(r_g^{\text{perf}, \text{min}}([\mathcal{E}]_p^{K_p g^{-1}}) = [\mathcal{E}]_p^{K_p}\), which follows from Theorem 2.13 together with the addendum Theorem 2.15. Property (4) follows directly from the Whitney product formula for the Chern classes of \([\mathcal{E}]_p\) and Property (5) of the functoriality of the construction of \(K_r S(G, X)^{\min}\) ([Sch15], loc. cit, comes from the construction, even if this is not explicitly mentioned). It remains to see that \(c_{n,\ell}([\mathcal{E}]_p)|_{K'} \in H^{2n}(K_r S(G, X)^{\min}, \Lambda(n))_{\mathcal{V}}\). By construction, this follows from
\[
c_{n,\ell}([\mathcal{E}]_p) \in \lim_{\longrightarrow} H^{2n}(K_r S(G, X)^{\min}, \mathbb{Z}/\ell^m(n))_{\mathcal{V}}, \ n \in \mathbb{N}
\]
which again follows by the projection formula from the trivial relation
\[
T(g)[\mathcal{E}]_p = [\mathcal{E}]_p \otimes_{\mathcal{O}_{K_p S(G, X)^{\min}}} \mathcal{O}_{K_g^{V} S(G, X)^{\min}}.
\]
\(\square\)
Notation 3.8. The functor
\[ \text{Rep}_Q(K_h) \otimes_{\text{Rep}_Q(G)} Q \to \{ \text{Vector Bundles on } K^p S(G, X)_{\min} \}, \]
\[ \mathcal{E} \mapsto [\mathcal{E}]_{p} \]
is a tensor functor, inducing the Chern character
\[ K_0(\text{Rep}_Q(K_h) \otimes_{\text{Rep}_Q(G)} Q) \to H^{2*}(K^p S(G, X)_{\min}, Q_{\ell}(\ast))^G_{\mathcal{E}} \]
\[ \subset \left( \lim_{\leftarrow m} H^{2*}(K^p S(G, X)_{\min}, \mathbb{Z}/\ell^m(\ast)) \right) \otimes Q. \]

3.2. The image of \( H^*(K^p S(G, X)_{\min}) \) in \( H^*(K^p S(G, X)_{\Sigma}) \). We now return to the topological setting. In this section, \( K^p S(G, X)_{\min} \) and \( K^p S(G, X)_{\Sigma} \) are identified with the analytic spaces underlying their \( \mathbb{C} \)-valued points. We fix \( \Sigma \), use the notation (2.1) and let \( \varphi = \varphi_\Sigma : K^p S(G, X)_{\Sigma} \to K^p S(G, X)_{\min} \) for the desingularization map. It is a projective morphism. Recall from the general theory ([dCM05], Theorem 2.8.1) that the choice of a polarization \( \mathcal{L} \) for \( \varphi \) induces a factorization
\[ (3.9) \quad H^i(K^p S(G, X)_{\min}, Q) \xrightarrow{\text{natural map}} IH^i(K^p S(G, X)_{\min}, Q) \]
\[ \xrightarrow{\varphi_*} H^i(K^p S(G, X)_{\Sigma}, Q) \]
in which all morphisms are compatible with the polarized Hodge structure, and \( \varphi_* \) is injective. The image of \( \varphi_* \) might depend on the choice of \( \mathcal{L} \), see [dCM05], Example 2.9. Furthermore, \( IH^i(K^p S(G, X)_{\min}, Q) \), thus \( \varphi_*^{\prime}(IH^i(K^p S(G, X)_{\min}, Q)) \subset H^i(K^p S(G, X)_{\Sigma}, Q) \) is pure of weight \( i \), and \( \varphi^{\ast}(H^i(K^p S(G, X)_{\min}, Q)) \subset H^i(K^p S(G, X)_{\Sigma}, Q) \) is the maximal pure weight \( i \) quotient of \( H^i(K^p S(G, X)_{\min}, Q) \) by [Del74], Prop. 8.2.5.

For a prime \( \ell \neq p \) and any \( r \geq 0 \), the diagram (3.9) has an \( \ell \)-adic version
\[ (3.10) \quad H^i(K^p S(G, X)_{\min}, \mathbb{Z}_\ell(j)) \xrightarrow{\text{natural map}} IH^i(K^p S(G, X)_{\min}, \mathbb{Z}_\ell(j)) \]
\[ \xrightarrow{\varphi_*} H^i(K^p S(G, X)_{\Sigma}, \mathbb{Z}_\ell(j)) \]

**Theorem 3.11.** For any \( \mathcal{E} \in \text{Vect}_G^c(\hat{X}) \), any neat level subgroup \( K \subset G(\mathbb{A}_f) \) and all \( r \geq 0 \), we have
\[ \varphi^\ast((c_{n,\ell}([\mathcal{E}]_{p})_{K}) = c_{n,\ell}([\mathcal{E}]_{\text{can}}(K),) \in H^{2n}(K^p S(G, X)_{\Sigma}, \mathbb{Z}_\ell(j)) \otimes_{\mathbb{Z}_\ell(j)} \Lambda(n))^{G_E}. \]
Here $\Lambda = \mathbb{Q}_\ell$ for $r = 0$ and $\mathbb{Z}_\ell$ for $r \geq 1$, as in Theorem 3.7.

Proof. By diagram 2.18 and [Sch12, Theorem 7.17], the topological space $K^{\nu} S(G,X)^{\min}$ is the inverse limit of the topological spaces $K, S(G, X)^{\Sigma}$, thus

$$K^{\nu} S(G, X)^{\text{tor}, r} := \varprojlim K^{\nu} S(G, X)^{\min} \times_{K, S(G, X)^{\Sigma}} K, S(G, X)^{\Sigma}$$

(see (2.18)) is the inverse limit of the topological spaces

$$\varprojlim K^{\nu} S(G, X)^{\min} \times_{K, S(G, X)^{\Sigma}} K, S(G, X)^{\Sigma}$$

for $r' \geq r$. On the other hand, the map

$$H^i(K, S(G, X)^{\Sigma}, \Lambda(j)) \to H^i(K^{\nu} S(G, X)^{\min} \times_{K, S(G, X)^{\Sigma}} K, S(G, X)^{\Sigma}, \Lambda(j))$$

factors

$$\iota(r, r') : H^i(K, S(G, X)^{\Sigma}, \Lambda(j)) \to H^i(K^{\nu} S(G, X)^{\min} \times_{K, S(G, X)^{\Sigma}} K, S(G, X)^{\Sigma}, \Lambda(j))$$

$$H^i(K^{\nu} S(G, X)^{\Sigma}, \Lambda(j)) \to H^i(K^{\nu} S(G, X)^{\text{tor}, r}, \mathbb{Z}/\ell^m(j))$$

which is injective as $K, S(G, X)^{\Sigma} \to K, S(G, X)^{\Sigma}$ is proper and $K, S(G, X)^{\Sigma}$ is smooth. We conclude that the map

$$\iota(r) : H^i(K, S(G, X)^{\Sigma}, \mathbb{Z}_\ell(j)) \to \varprojlim H^i(K^{\nu} S(G, X)^{\text{tor}, r}, \mathbb{Z}/\ell^m(j))$$

for $r \geq 1$ is injective. Thus the map

$$H^i(K, S(G, X)^{\Sigma}, \mathbb{Q}_\ell(j)) \to \left( \varprojlim H^i(K^{\nu} S(G, X)^{\text{tor}, 0}, \mathbb{Z}/\ell^m(j)) \right) \otimes_{\mathbb{Z}} \mathbb{Q},$$

is injective as well, as post-composed with

$$\left( \varprojlim H^i(K^{\nu} S(G, X)^{\text{tor}, 0}, \mathbb{Z}/\ell^m(j)) \right) \otimes_{\mathbb{Z}} \mathbb{Q} \to \left( \varprojlim H^i(K^{\nu} S(G, X)^{\text{tor}, 1}, \mathbb{Z}/\ell^m(j)) \right) \otimes_{\mathbb{Z}} \mathbb{Q}$$

it is the composition $\iota(1) \circ \iota(0, 1)$. The theorem follows applying Corollary 3.3 and Theorem 2.20 and observing that the classes $c_{n,t}([\mathcal{E}]^{\text{can}}_{K, r})$ lie in the invariant subgroup

$$H^{2n}(K, S(G, X)^{\Sigma, \bar{Q}}, \Lambda(n))^G \subset H^{2n}(K, S(G, X)^{\Sigma, \bar{Q}}, \Lambda(n))$$

as $[\mathcal{E}]^{\text{can}}$ is defined over $E$.

Let

$$H^{2s}_{\text{Chern}}(K, S(G, X)^{\min}_{\bar{Q}_p}, \mathbb{Q}_\ell(*)) \subset H^{2s}(K, S(G, X)^{\min}_{\bar{Q}_p}, \mathbb{Q}_\ell(*))$$

denote the $\mathbb{Q}_\ell$-subalgebra generated by the images of $c_{s,t}([\mathcal{E}]_{p,K}$, and

$$H^{2s}_{\text{Chern}}(K, S(G, X)^{\Sigma, \bar{Q}_p}, \mathbb{Q}_\ell(*)) \subset H^{2s}(K, S(G, X)^{\Sigma, \bar{Q}_p}, \mathbb{Q}_\ell(*))$$


denote the $\mathbb{Q}_\ell$-subalgebra generated by the images of $c_*(\mathcal{E}^{can})_K$. Here, when $G$ satisfies (i) of Definition 2.9, we take $\mathcal{E}$ to belong to $\text{Vect}_G(\hat{X})^+$. 

**Proposition 3.12.** The commutative diagram (3.10) restricts to the commutative diagram of $\mathbb{Q}_\ell$-algebras

\[
\begin{array}{ccc}
K_0(\text{Rep}_{\overline{\mathbb{Q}}}(K_h)) & \otimes & K_0(\text{Rep}_{\overline{\mathbb{Q}}}(G)) \mathbb{Q}_\ell^+ \\
\sim (\text{ch} \circ (1.1)) & & \\
H^2(\hat{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(\ast))^+ & \sim & I H^2(\hat{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(\ast))^+_v \\
\sim \text{(Prop. 2.3)} & & \\
H^2_{\text{Chern}}(K S(G, X)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(\ast)) & \sim & H^2_{\text{Chern}}(K S(G, X)_{\Sigma, \overline{\mathbb{Q}}}, \mathbb{Q}_\ell(\ast)) \\
\phi^* & & \\
\end{array}
\]

in which $\phi^*_\Sigma$, $\phi^*$, the natural map and $\phi^*$ are isomorphisms of $\mathbb{Q}_\ell$-algebras. In particular

\[\phi^*_\Sigma(I H^2(K S(G, X)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(\ast))^+_v) \subset H^2(K S(G, X)_{\Sigma, \overline{\mathbb{Q}}}, \mathbb{Q}_\ell(\ast))\]

is a $\mathbb{Q}_\ell$-sub-vectorspace which does not depend on the choice of the relative polarization $\mathcal{L}$.

**Proof.** By Theorem 2.12, the image of the Goresky-Pardon Chern classes in $I H^2(K S(G, X)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(\ast))^+_v$ lies in $I H^2(K S(G, X)_{\Sigma, \overline{\mathbb{Q}}}, \mathbb{Q}_\ell(\ast))^+_v$ and spans this subalgebra over $\mathbb{Q}$. In particular, the image of

\[H^2(K S(G, X)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(\ast))^+_v \rightarrow I H^2(K S(G, X)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(\ast))^+_v\]

contains $I H^2(K S(G, X), \mathbb{Q}_\ell(\ast))^+_v$, and the map

\[\phi^*_\Sigma : I H^2(K S(G, X), \mathbb{Q}_\ell(\ast))^+_v \rightarrow H^2_{\text{Chern}}(K S(G, X)_{\Sigma}, \mathbb{Q}_\ell(\ast))^+_v\]

is surjective. By comparison with $\ell$-adic intersection cohomology, the image of

\[H^2(K S(G, X)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(\ast))^+_v \rightarrow I H^2(K S(G, X)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(\ast))^+_v\]

contains $I H^2(K S(G, X)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(\ast))^+_v$, and

\[\phi^*_\Sigma : I H^2(K S(G, X)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(\ast))^+_v \rightarrow H^2_{\text{Chern}}(K S(G, X)_{\Sigma, \overline{\mathbb{Q}}}, \mathbb{Q}_\ell(\ast))\]

is surjective as well.
We have no direct way to compare Goresky-Pardon’s classes and our classes from Theorem 3.7 on $K S(G,X)_{\bar{Q}_p}$. However, applying Proposition 3.11 for $r = 0$, i.e. for $K_0 = K$, one has

$$\varphi_L^*(H^*_\text{Chern}(K S(G,X)_{\bar{Q}_p}^{\min}, Q_\ell(\ast))) = H^*_\text{Chern}(K S(G,X)_{\Sigma,\bar{Q}_p}, Q_\ell(\ast)).$$

As $\varphi_L^*$ is injective, we deduce that the natural map sends the $Q_\ell$-subalgebra $H^*_\text{Chern}(K S(G,X)_{\bar{Q}_p}^{\min}, Q_\ell(\ast))$ onto $IH^*_\text{Chern}(K S(G,X)_{\bar{Q}_p}^{\min}, Q_\ell(\ast))^+_v$. This proves that the natural map in the proposition and $\varphi^*$ are surjective. It remains to prove that they are injective as well. The Chern character has by definition values in $H^*_\text{Chern}(K S(G,X)_{\bar{Q}_p}^{\min}, Q_\ell(\ast))$ and the whole diagram commutes. This proves that $\varphi^*$ is injective as well. It follows that the homomorphisms $ch_\ell$, $\varphi^*$ and the natural map are isomorphisms. Finally, the image of $\varphi^*$ being equal to the image of $\varphi^*_L$, the latter does not depend on $L$. This finishes the proof.

□

**Remarks 3.14.** (i) If we ignore the action of the Galois group, the Chern classes constructed above depend only on the connected components of $K S(G,X)^{\min}$. Thus the results above extend to general Shimura varieties of abelian type. We leave the precise formulation to the reader.

(ii) We naturally expect that Proposition 3.12 remains true without the superscripts $^+$, which in any case change nothing when $G$ satisfies condition (iv) of Theorem 2.7. In the excluded case, it is easy to see that the classes denoted $e_r$ in §2.1.1 have non-trivial images in $IH^n(K S(G,X)_{\bar{Q}_p}^{\min}, Q_\ell(\ast))$; indeed, this follows from the results of [GP02] and the equality $e_r^2 = c_{1,r}^n$ for each $r$. However, we also have

$$(c_{1,r}^{k-1})^2 = c_{1,r}^n,$$

so it is possible that $c_{1,r}$ and $\pm e_r$ have the same image in $IH^n$. We have not attempted to prove that this is not the case.

4. **Comparison of torsors**

We begin with the following observation, which is a simple consequence of the definitions:

**Lemma 4.1.** The tensor functor

$$\text{Vect}_G^s(\hat{X}_{\bar{Q}_p})[\simeq \text{Rep}_{\bar{Q}_p}(K h)] \to \{\text{Vector bundles over } K_p S(G,X)^{\text{tor}}\},$$

$$\mathcal{E} \mapsto q^* [\mathcal{E}]_p,$$
where $E \mapsto [E]_p$ is defined by the $K_h$-torsor $M_p$, coincides with the tensor functor

$$E \mapsto q^* \pi^*_H [E].$$

Let $(G, X) = (GSp(2g), X_{2g})$, write $X_{2g}$ for the compact dual of $X_{2g}$, fix a base point $h \in X_{2g} \subset \hat{X}_{2g}$, and let $\Omega_h$ denote the fiber at $h$ of the cotangent bundle to $\hat{X}_{2g}$. Let $V_g$ denote the vector group $G_m$. Let $K_{h,g} \subset G$ denote the stabilizer of $h$. Then $K_{h,g}$ can be identified with $GL(g) = \text{Aut}(V_g)$ in such a way that the isotropy action of $K_{h,g}$ on $\Omega_h$ is equivalent to $\text{Sym}^2(V_g)$. We take $St$ to be the standard representation $K_{h,g} \mapsto \text{Aut}(V_g)$, and let $E_{St}$ be the corresponding equivariant vector bundle on $\hat{X}_{2g}$. Let $K_g = K_{g,p} \cdot K^p_g$ a neat open compact subgroup of $GSp(2g)(A_f)$, and fix a toroidal datum $\Sigma$ for $K$. We write $q_g : K_p S(GSp(2g), X_{2g})^{\text{tor}} \to K_g S(GSp(2g), X_{2g})^{\text{min}}$ for the morphism denoted $q$ above.

**Proposition 4.2 ([PS16]).** There is a canonical isomorphism

$$\theta_g : q_g^* [E_{St}]_p \sim \sim [E_{St}]_{dR, \Sigma}$$

of vector bundles over $K_g S(GSp(2g), X_{2g})^{\text{tor}}$.

**Proof.** This is essentially equivalent to Corollaire 1.16 to Proposition 1.15 of [PS16]. More precisely, the generic fiber of the bundle denoted $\omega^\text{mod}_{A}$ in [PS16] is exactly $[E_{St}]_{dR, \Sigma}$. Indeed, with our notation, the restriction of $[E_{St}]^{\text{can}}$ to the open Siegel modular variety in (neat) finite level $K$ is the sheaf of relative sections of the relative 1-forms of the universal polarized abelian scheme, and its canonical extension to the toroidal compactification is isomorphic to the sheaf of relative sections of the relative invariant 1-forms on the corresponding semi-abelian scheme; see [Lan12], Proposition 6.9, or [Lan17], (1.3.3.15). \qed

We restrict now the information to Shimura varieties of Hodge type. Let $(G, X)$ be a Shimura datum of Hodge type, with compact dual $\hat{X}$. We fix a symplectic embedding

$$\tau : (G, X) \to (GSp(2g), X_{2g}).$$

Let $h \in X$ be a base point, let $K_h \subset G$ be its stabilizer, let $K_{h,g} \subset GSp(2g)$ be the stabilizer of $\tau(h)$, and denote by $\tau$ the inclusion of $K_h$ in $K_{h,2g}$. Let $St_h = St \circ \tau : K_h \to \text{Aut}(V_g)$, with $St$ the standard faithful representation of $K_{h,g}$. Finally, let $E_{St_h}$ be the equivariant vector bundle on $\hat{X}$ with isotropy representation $St_h$. Let $K = K_p \cdot K^p$ be a neat open compact subgroup of $G(A_f)$.
Corollary 4.3. The morphism θₙ induces by restriction for any r a canonical isomorphism

\[ \theta(r) : \nu(r)^*q^*[E_{Sth}]_p \sim \to [E_{Sth}]_{dR,\Sigma,r} \]

of vector bundles over \( K^\mathbb{P}_S(G,X)^{tor,r} \).

Proof of Theorem 2.20. As Stₕ is faithful, any irreducible representation V of Kₕ defined over \( \overline{\mathbb{Q}} \) is a direct factor of the representation \( Stₕ^{\odot m} \otimes Stₕ^{\odot n} \) for some pair of natural numbers \( (m,n) \) ([DMOS82], I. Proposition 3.1 (a), II, Proposition 2.23). Write

\[ V \to Stₕ^{\odot m} \otimes Stₕ^{\odot n} \to V \]

for the splitting. The isomorphism of vector bundles \( \theta(r) \) induces an isomorphism of vector bundles

\[ \theta(r)^{\odot m} \otimes \theta(r)^{\odot n} : \nu(r)^*q^*([E_{Sth}]^{\odot m}_p \otimes [E_{Sth}]^{\odot n}) \sim \to [E_{Sth}]^{\odot m}_{dR,\Sigma,r} \otimes [E_{Sth}]^{\odot n}_{dR,\Sigma,r} \]

while s and t induce the splittings of vector bundles

\[ [\mathcal{V}]_p \xrightarrow{s_p} [E_{Sth}]^{\odot m}_p \otimes [E_{Sth}]^{\odot n}_p \xrightarrow{t_p} [\mathcal{V}]_p \]

\[ [\mathcal{V}]_{dR,\Sigma,r} \xrightarrow{s_{dR,\Sigma,r}} [E_{Sth}]^{\odot m}_{dR,\Sigma,r} \otimes [E_{Sth}]^{\odot n}_{dR,\Sigma,r} \xrightarrow{t_{dR,\Sigma,r}} [\mathcal{V}]_{dR,\Sigma,r} \]

This yields morphisms of coherent sheaves

\[ \alpha := t_{dR,\Sigma,r} \circ \theta(r)^{\odot m} \otimes \theta(r)^{\odot n} \circ \nu(r)^*q^*s_p : \nu(r)^*q^*[\mathcal{V}]_p \to [\mathcal{V}]_{dR,\Sigma,r} \]

\[ \beta := \nu(r)^*q^*t_p \circ (\theta(r)^{\odot m} \otimes \theta(r)^{\odot n})^{-1} \circ s_{dR,\Sigma,r} : [\mathcal{V}]_{dR,\Sigma,r} \to \nu(r)^*q^*[\mathcal{V}]_p \]

which are inverse isomorphisms in restriction to \( K^\mathbb{P}_S(G,X) \). Thus

\[ \beta \circ \alpha - Id_{\nu(r)^*q^*[\mathcal{V}]_p} : \nu(r)^*q^*[\mathcal{V}]_p \to \nu(r)^*q^*[\mathcal{V}]_p, \]

\[ \alpha \circ \beta - Id_{[\mathcal{V}]_{dR,\Sigma,r}} : [\mathcal{V}]_{dR,\Sigma,r} \to [\mathcal{V}]_{dR,\Sigma,r} \]

are homomorphisms of coherent sheaves which are zero on \( K^\mathbb{P}_S(G,X) \). As \( \nu(r)^*q^*[\mathcal{V}]_p \) and \( [\mathcal{V}]_{dR,\Sigma,r} \) are vector bundles, thus are both torsion free, we conclude that both maps are 0. This finishes the proof.

□

A. Appendix

We collect here the statements of some standard results in the theory of schemes, in versions adapted to perfectoid spaces. The proof are due to Peter Scholze.
Proposition A.1 (Scholze). Let $\mathcal{W}$ be a vector bundle of rank $r$ over a perfectoid space $\mathcal{X}$ over $C$. Let $\mathbb{P}(\mathcal{W})$ denote the corresponding projective bundle over $\mathcal{X}$, viewed as an adic space over $C$. Then $\mathcal{O}_{\mathbb{P}(\mathcal{W})}(1)$ has a first Chern class $(z_m) \in \varprojlim_m H^2(\mathbb{P}(\mathcal{W}), \mathbb{Z}/\ell^m(1))$ such that the homomorphism

$$\varprojlim_m H^b(\mathbb{P}(\mathcal{W}), \mathbb{Z}/\ell^m(1)) \leftarrow \bigoplus_{i=0}^{r-1} \varprojlim_m H^{b-2i}(\mathcal{X}, \mathbb{Z}/\ell^m(a-i)) \cdot (z_m)^i$$

is an isomorphism.

We define the Chern class $c_i(\mathcal{W})$ by Grothendieck’s standard equation

$$\varprojlim_m H^{2r}(\mathbb{P}(\mathcal{W}), \mathbb{Z}/\ell^m(r)) \ni \sum_{i=0}^{r} (-1)^i c_i(\mathcal{W}) \cdot (z_m)^{r-i} = 0,$$

$$c_i(\mathcal{W}) \in \varprojlim_m H^{2i}(\mathbb{P}(\mathcal{W}), \mathbb{Z}/\ell^m(i)).$$

Proof. One defines $z$ as usual, as the projective limit over $m$ of the images $z_m \in H^2(\mathbb{P}(\mathcal{W}), \mathbb{Z}/\ell^m(1))$ of the class of $\mathcal{O}_{\mathbb{P}(\mathcal{W})}(1)$ in $H^1(\mathbb{P}(\mathcal{W}), \mathbb{G}_m)$ via the connecting homomorphism of the étale Kummer exact sequence $1 \to \mu_{\ell^m} \to \mathbb{G}_m \to \mathbb{G}_m \to 1$. To prove the statement, it is enough to prove that the map

$$H^b(\mathbb{P}(\mathcal{W}), \mathbb{Z}/\ell^m(a)) \leftarrow \bigoplus_{i=0}^{n-1} H^{b-2i}(\mathcal{X}, \mathbb{Z}/\ell^m(a-i)) \cdot z_i^m$$

is an isomorphism. This is a local property on $\mathcal{X}$, reduced by [CS17], Lemma 4.4.1 to the computation of the étale cohomology of $\mathbb{P}(\mathcal{W})$ over a geometric point $\bar{x} = \text{Spa}(C(\bar{x}), C(\bar{x})^+)$, which then is the standard computation.

\[\square\]

Remark A.2. We avoid here the delicate question whether the surjection

$$H^3(\mathcal{X}, \mathbb{Z}_\ell(i)) \to \varprojlim_m H^3(\mathcal{X}, \mathbb{Z}/\ell^m(i))$$

is an isomorphism as it is irrelevant for our purpose.

Recall that we have level subgroups $K_{p,r} \subset G(\mathbb{Q}_p)$. We denote by $\Sigma^H$ the invariants under a group $H$ acting on the set $\Sigma$.

Proposition A.3 (Scholze). The morphisms of ringed spaces

$$\xymatrix{ K_p \cdot K_{p,r} \mathcal{S}(G, X)_{\text{min}} \ar[d] \ar[r] & K_{p,r} \mathcal{S}(G, X)_{\text{min}} \ar[r] & K_p \cdot K_{p,r} \mathcal{S}(G, X)_{\text{min}} }$$
induce for all pairs of integers \((i, j)\) and all \(r \geq 1\) homomorphisms
\[
H^j(K_pS(G, X)^{\min}, \mathbb{Z}/\ell^m(i))^{K_p, r} \rightarrow H^j(K_p, K_{p, r}, S(G, X)^{\min}, \mathbb{Z}/\ell^m(i))
\]
and
\[
\left[ \lim_{m} H^j(K_pS(G, X)^{\min}, \mathbb{Z}/\ell^m(i)) \right]^{K_p, r} = \lim_{m} \left[ H^j(K_pS(G, X)^{\min}, \mathbb{Z}/\ell^m(i)) \right]^{K_p, r}
\]
\[
\rightarrow H^j(K_p, K_{p, r}, S(G, X)^{\min}, \mathbb{Z}/\ell^m(i)).
\]

**Proof.** By [Sch12, Cor. 7.18], one has
\[
H^j(K_pS(G, X)^{\min}, \mathbb{Z}/\ell^m(i)) = \lim_{r} H^j(K_rS(G, X)^{\min}, \mathbb{Z}/\ell^m(i))
\]
\[
= \lim_{r} H^j(K_rS(G, X)^{\min}, \mathbb{Z}/\ell^m(i)).
\]

On the other hand, for \(r' > r \geq 1\) one has \(K_r/K_{r'} = K_{p, r}/K_{p, r'}\), which is a \(p\)-group. It follows that the transition maps in the inductive system are injective as \(K_{r+1}S(G, X)^{\min} \rightarrow K_rS(G, X)^{\min}\) is finite surjective of degree prime to \(\ell\). This implies
\[
H^j(K_rS(G, X)^{\min}, \mathbb{Z}/\ell^m(i))^{K_{p, r}/K_{p, r'}} = \lim_{r'} H^j(K_{r'}S(G, X)^{\min}, \mathbb{Z}/\ell^m(i))^{K_{p, r}/K_{p, r'}}.
\]

Passing to the normalization \(K_rS(G, X)^{\min} \rightarrow K_rS(G, X)^{\min}\), one obtains a map
\[
\lim_{r'} H^j(K_rS(G, X)^{\min}, \mathbb{Z}/\ell^m(i))^{K_{p, r}/K_{p, r'}} \rightarrow \lim_{r'} H^j(K_{r'}S(G, X)^{\min}, \mathbb{Z}/\ell^m(i))^{K_{p, r}/K_{p, r'}}
\]
where in the inductive system on the right, the transition maps are again injective. Finally one has
\[
H^j(K_rS(G, X)^{\min}, \mathbb{Z}/\ell^m(i)) = \lim_{r'} H^j(K_{r'}S(G, X)^{\min}, \mathbb{Z}/\ell^m(i))^{K_{p, r}/K_{p, r'}}.
\]

So composing, one obtains the homomorphism
\[
\begin{align*}
H^j(K_pS(G, X)^{\min}, \mathbb{Z}/\ell^m(i))^{K_{p, r}} & \rightarrow \\
H^j(K_rS(G, X)^{\min}, \mathbb{Z}/\ell^m(i)) & = H^j(K_rS(G, X)^{\min}, \mathbb{Z}/\ell^m(i))
\end{align*}
\]
which is an isomorphism in case \(K_rS(G, X)^{\min}\) is normal. Finally, by definition, one has
\[
\lim_{m} H^j(K_pS(G, X)^{\min}, \mathbb{Z}/\ell^m(i))^{K_{p, r}} = \lim_{m} H^j(K_pS(G, X)^{\min}, \mathbb{Z}/\ell^m(i))^{K_{p, r}}.
\]
which thus maps to
\[
\lim_{m} H^j(K, S(G, X)^{\text{min}}_{\mathbb{Z}_p}, \mathbb{Z}/\ell^m(i)) = H^j(K, S(G, X)^{\text{min}}_{\mathbb{Z}_p}, \mathbb{Z}(i)).
\]

\[\square\]

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