

A RELATIVE VERSION OF GIESEKER'S PROBLEM ON STRATIFICATIONS IN CHARACTERISTIC $p > 0$

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ABSTRACT. We prove that the vanishing of the functoriality morphism for the étale fundamental group between smooth projective varieties over an algebraically closed field of characteristic $p > 0$ forces the same property for the fundamental groups of stratifications.

1. INTRODUCTION

In analogy with complex geometry, where according to Malčev-Grothendieck's theorem ([Mal40], [Gro70]), a simply connected algebraic complex manifold has no non-trivial regular singular connections, Gieseker conjectured in [Gie75, p. 8] that simply connected smooth projective varieties defined over an algebraically closed characteristic $p > 0$ field have no non-trivial stratified bundles. This has been solved in the positive in [EM10, Thm. 1.1]. Gieseker in *loc.cit.* also conjectured that if the commutator of the étale fundamental group is a pro- p -group, irreducible stratified bundles have rank 1, and if the étale fundamental group is abelian with no non-trivial p -power quotient, then the category of stratified bundles is semi-simple with irreducible objects of rank 1. This is proven in [ES14, Thm. 3.9]. In [ES16, Thm. 3.2] we extended [EM10, Thm. 1.1] to the regular locus of a normal variety, assuming in addition that the ground field is \mathbb{F}_p .

In this article we prove a relative version.

Theorem 1.1. *Let $\pi : Y \rightarrow X$ be a morphism between smooth projective varieties over an algebraically closed field k of characteristic $p > 0$, such that for any finite étale cover $Z \rightarrow X$, the pull-back cover $Y \times_X Z \rightarrow Y$ splits completely. Then for any stratified vector bundle \mathbb{M} on X , the pull-back stratified bundle $\pi^*\mathbb{M}$ on Y is trivial.*

We can of course translate the theorem using the fundamental groups: if the homomorphism $f_* : \pi_1^{\text{ét}}(Y) \rightarrow \pi_1^{\text{ét}}(X)$ on the étale fundamental groups is the zero map, then so is the homomorphism $f_* : \pi_1^{\text{strat}}(Y) \rightarrow \pi_1^{\text{strat}}(X)$ on the k -Tannaka group-schemes of stratified bundles (we omit all the base points as this is irrelevant here).

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As in the other proofs in *loc.cit.*, the key tool is the theorem of Hrushovski [Hru04, Cor. 1.2] on the density of preperiodic points by a correspondence over \mathbb{F}_p . A complete proof of it in the framework of arithmetic geometry is due to Varshavsky [Var04, Thm. 0.1]. However, with this tool in the back ground, one needs a slightly different strategy as compared to the other proofs in *loc. cit.*. Indeed, one cannot argue directly on the moduli of objects the triviality of which one wants to prove. One important point here is that the triviality of a stratified bundle is recognized by the dimension of its space of stratified sections. This enables one to argue by semi-continuity arguments (see Lemma 2.1). Another point is that in the non-irreducible case, one has to deal with extensions which are not recognized on the moduli space. This also requires a new argument. See Section 5.

Acknowledgements: The theorem for irreducible stratified bundles was proved by Xiaotao Sun on 2013 in a unpublished note. He told us on January of 2017 that he and his student knew how to prove the general case based on an application of Simpson's theorem on moduli of framed bundles, which so far exists in characteristic 0 only. We hope very much that he and his student shall soon write a characteristic $p > 0$ version of Simpson's theorem and that this way one shall have a proof different from the one presented here. We thank him warmly for communicating his idea to us.

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2. SOME PROPERTIES OF VECTOR BUNDLES AND THEIR MODULI

We use the standard notations: if $f : Y \rightarrow X$ and $X' \rightarrow X$ are any morphisms, one denotes by $Y_{X'} \rightarrow X'$ the base change $Y \times_X X' \rightarrow X'$. If Y has characteristic $p > 0$, one denotes by $F_X : X \rightarrow X$ the absolute Frobenius, by $f' : Y' =$

$Y \times_{X, F_X} X \rightarrow X$ the pull-back of f by F_X , by $Y \xrightarrow{F_{Y/X}} Y'$ the relative Frobenius

$$\begin{array}{ccc} Y & \xrightarrow{F_{Y/X}} & Y' \\ & \searrow f & \downarrow f' \\ & & X \end{array}$$

above X .

If E is a vector bundle over Y , one denotes by $E_{X'}$ the pull-back vector bundle over $Y_{X'}$ via $Y_{X'} \rightarrow Y$.

Recall that, from the theory of the Harder-Narasimhan filtration, any vector bundle E on a smooth projective curve over an algebraically closed field has a unique maximal subbundle $F \subset E$ of maximal slope, and the slope of this subbundle F is denoted by $\mu_{\max}(E)$. In particular, if E has degree 0, then $\mu_{\max} \mathcal{E} \geq 0$, with equality precisely when E is μ -semistable.

Lemma 2.1. *Let $C \rightarrow T$ be a smooth projective morphism, where T is a Noetherian scheme and where the fibers are geometrically connected smooth projective curves. Let E be a vector bundle of rank r on C . Then the following holds.*

- (i) *There is a constructible subset $T_0 \subset T$ so that for a point t of T , it holds that*

$$H^0(C_t, E_t) \otimes_{k(t)} \mathcal{O}_{C_t} \xrightarrow{\cong} E_t \text{ is a trivial bundle } \Leftrightarrow t \text{ is a point of } T_0.$$

This subset T_0 is stable under any base change $T' \rightarrow T$.

- (ii) *If moreover for every geometric point $t \in T$, E_t is a semistable vector bundle of degree 0, then T_0 is closed. In particular, if T_0 contains a dense set of closed points of T , then $T_0 = T$.*
- (iii) *As t ranges over the points of T , the set of numbers $\mu_{\max}(E_t)$ is bounded.*

Proof. The semicontinuity theorem [EGAIII, Thm. 7.7.5] gives (i), and also gives that for any geometric point t in the closure of T_0 , $\dim_{k(t)} H^0(C_t, E_t) \geq r$. For (ii), we note that $\dim_{k(t)} H^0(C_t, V) \leq r$ for any semistable vector bundle V of rank r and degree 0 on C_t , and equality holds if and only if $H^0(C_t, V) \otimes_{k(t)} \mathcal{O}_{C_t} \xrightarrow{\cong} V$.

For (iii), note that if T is integral, η is a generic point, then for an open dense subset $U \subset T$, and any point $t \in U$, one has $\mu_{\max}(E_\eta) = \mu_{\max}(E_t)$. Now (iii) follows in general by Noetherian induction. \square

Definition 2.2. Let X be a smooth projective variety over a field k , and let E be a vector bundle of rank r on X .

- (i) We say E is *unipotent* if E has a filtration by subbundles, the associated graded bundle of which is a trivial vector bundle.
- (ii) We say E is *F_X -nilpotent of index N* for some natural number $N > 0$, if the N -th iterated Frobenius pullback $F_X^{N*} E$ is a trivial vector bundle.

Lemma 2.3. *Let X be a smooth projective variety over a field k of characteristic $p > 0$. Let $g = \dim H^1(X, \mathcal{O}_X)$, and let r be a fixed positive integer. Then there is an $N = N(g, r)$, depending only on g, r , such that if E is a unipotent vector bundle of rank r , which is also F_X -nilpotent of some index, then in fact E is F_X -nilpotent of index N .*

Proof. We do induction on r , where the case $r = 1$ is trivial. If E is unipotent of rank $r > 1$, there is an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow E' \rightarrow 0$$

where E' is unipotent of rank $r - 1$. Since E is F -nilpotent of some index, it is easy to see that E' is F_X -nilpotent of the same index, by taking that particular iterated Frobenius pullback of the above exact sequence, and noting that the quotient of a trivial bundle on X by a trivial subbundle is also a trivial bundle.

Hence by induction, there exists $N' = N'(g, r - 1)$ so that $F_X^{N'} E'$ is trivial. Then $F_X^{N'} E$ determines an element of

$$\mathrm{Ext}_X^1(H^0(X, F_X^{N'} E') \otimes_k \mathcal{O}_X, \mathcal{O}_X) = H^0(X, F_X^{N'} E')^\vee \otimes_k H^1(X, \mathcal{O}_X).$$

The Frobenius F_X sends a vector $v \otimes_k \gamma$ in this tensor product to $F_X^* v \otimes_{F_k^* k} F_X^* \gamma$. It therefore suffices to note that if a vector of a g -dimensional vector space over k is killed by the N -th power of a p -linear map, it is killed by its g -th power. \square

Corollary 2.4. *Let $C \rightarrow T$ be a smooth projective morphism, where T is a Noetherian scheme and where the fibers are geometrically connected smooth projective curves. Let E be a filtered vector bundle of rank r on C . Then the set of points $t \in T$ such that the filtered vector bundle E_t is unipotent, and is also F -nilpotent, is constructible. This constructible subset is stable under any base-change $T' \rightarrow T$.*

Proof. This follows immediately on combining Lemmas 2.1 and 2.3. \square

It is convenient for us to record the following, which may be of independent interest.

Lemma 2.5. *Let X be a smooth projective variety over a finite field \mathbb{F}_q , and let E be a vector bundle with a given filtration by subbundles, and associated graded bundle $gr(E)$. Suppose $gr(E)$ is trivialised on a finite étale covering of X . Then there is a positive integer c such that $F_X^{c*}(E)$ is also trivialised on some finite étale cover of X .*

Proof. Since $gr(E)$ is trivialised on a finite étale cover of X , we know that $F_X^{a*}(gr(E)) \cong gr(E)$, for some positive integer a (see [LS77, Satz 1.4]). Consider now the sequence of filtered vector bundles $\{F_X^{b a*}(E)\}_{b \geq 1}$. All the bundles in this sequence then have the same associated graded bundle, up to isomorphism.

However, over \mathbb{F}_q , there are only finitely many isomorphism classes of filtered vector bundles with a given associated graded bundle. Hence the above sequence of vector bundles has only finitely many distinct isomorphism types, and so there must exist positive integers $b_1 > b_2$ so that $F_X^{b_1 a*}(E) \cong F_X^{b_2 a*}(E)$. Taking $c = b_2 a$, we get that $F_X^{m*}(F_X^{c*}(E)) \cong (F_X^{c*}(E))$, where $m = (b_1 - b_2)a$. Thus the isomorphism class of the vector bundle $F_X^{c*}(E)$ is invariant under some iterated Frobenius pullback, and so $F_X^{c*}(E)$ is trivialised on a finite étale covering of X , again by *loc.cit.* \square

We also note for reference the following two properties. The first one is an easy consequence of the fact that the slope of a vector bundle gets multiplied by p under the Frobenius pull-back and the second one is ultimately a consequence of this as well.

Lemma 2.6. *Let X be a smooth projective variety over an algebraically closed field k of characteristic $p > 0$, with a chosen polarization, and let E be a vector*

bundle with numerically trivial Chern classes. Let $E^{(m)}$ be a vector bundle such that $F_X^{m*} E^{(m)} \cong E$, where $p^m > \mu_{\max}(E) \cdot \text{rank}(E)$. Then $E^{(m)}$ is μ -semistable.

Recall that on a smooth projective scheme X_S over a scheme S , a stratified bundle \mathbb{M} is defined by an infinite Frobenius descent sequence (E^n, τ^n) , where E^n is a vector bundle on the n -th Frobenius twist $X_S^{(n)}$ of X_S over S with isomorphisms $F_{X_S^{(n)}/S}^* E^n \xrightarrow{\tau^{n-1}} E^{n-1}$. If $S = \text{Spec}(k)$ where k is a perfect characteristic $p > 0$ field, as an object we can also write $\mathbb{M} = (E_m, \sigma_m)_{m \geq 0}$ where E_m is a vector bundle on $X_k = X$ and $F_X^* E_n \xrightarrow{\sigma^{n-1}} E_{n-1}$. If \mathbb{M} is given, and n is a natural number, we denote by $\mathbb{M}(n)$ the stratified bundle $\mathbb{M}(n) = (E_{n+m}, \sigma_{n+m})_{m \geq 0}$.

Lemma 2.7 ([EM10], Proposition 3.2). *Let X be a smooth projective variety. Any stratified bundle is filtered by stratified subbundles $\mathbb{M}^i \subset \mathbb{M}^{i-1} \dots \subset \mathbb{M}^0 = \mathbb{M}$ such that the associated graded stratified bundle $\bigoplus_i \mathbb{M}^i / \mathbb{M}^{i+1}$ has the property that $\mathbb{M}^i / \mathbb{M}^{i+1}$ is irreducible and for some n , all the underlying vector bundles of $(\mathbb{M}^i / \mathbb{M}^{i+1})(n)$ are μ -stable with trivial numerical Chern classes.*

In view of Lemma 2.6 and Lemma 2.7, we shall consider the moduli scheme of μ -stable vector bundles. Given a natural number $r > 0$, $X_S \rightarrow S$ a smooth projective morphism, there is a coarse moduli quasi-projective scheme $M(r, X_S) \rightarrow S$ of μ -stable bundles of rank r with trivial numerical Chern classes, which universally corepresents the functor of families of geometrically stable bundles ([Lan04, Thm. 4.1]). In particular, if $T \rightarrow S$ is any morphism of schemes, the base change is an isomorphism

$$(2.1) \quad M(r, X_T) \xrightarrow{\cong} M(r, X_S) \times_S T.$$

While applied to $F_S : T = S \rightarrow S$, this yields the isomorphism

$$(2.2) \quad M(r, X'_S) \xrightarrow{\cong} M(r, X_S) \times_{S, F_S} S.$$

We finally note the following, which is immediate from Lemma 2.3.

Corollary 2.8. *If X is a smooth projective variety over an algebraically closed field k of characteristic $p > 0$, and \mathbb{M} is a stratified bundle, such that for the underlying sequence (E_m, σ_m) , each E_m is an F -nilpotent vector bundle, then \mathbb{M} is trivial as a stratified vector bundle.*

3. REDUCTION TO THE CURVE CASE

Lemma 3.1. *It suffices to prove Theorem 1.1 in the special case when $Y = C$ is a smooth projective curve, and X is a smooth projective surface.*

Proof. Let $C \subset Y$ be a nonsingular complete intersection of very ample divisors in Y . By [ES16, Thm. 3.5], the homomorphism $\pi_1^{\text{strat}}(C) \rightarrow \pi_1^{\text{strat}}(Y)$ is surjective. On the other hand, the homomorphism $\pi_1^{\text{ét}}(C) \rightarrow \pi_1^{\text{ét}}(X)$ factors through

$\pi_1^{\text{ét}}(Y) \rightarrow \pi_1^{\text{ét}}(X)$, and so $C \rightarrow X$ satisfies the hypotheses of Theorem 1.1. This proves the first part of the lemma. We assume now $Y = C$.

On the other hand, we can make an embedding resolution of the closed embedding $\pi(C) \subset X$ by a sequence of blow ups at closed points. So the morphism $\pi : C \rightarrow X$ factors as a composition

$$C \xrightarrow{\tilde{\pi}} \tilde{X} \xrightarrow{f} X,$$

where $\tilde{\pi} : C \rightarrow \tilde{X}$ has smooth image, and $f : \tilde{X} \rightarrow X$ is a composition of blow ups at closed points. Thus $f_* : \pi_1^{\text{ét}}(\tilde{X}) \rightarrow \pi_1^{\text{ét}}(X)$ is an isomorphism. Hence, it suffices to prove Theorem 1.1 for morphisms $\pi : C \rightarrow X$, where C is a smooth curve, and $\pi(C) = D \subset X$ is a smooth curve as well. By Bertini theorem [Joa83, Thm. 6.3], there is a smooth projective surface $X' \subset X$ which is a complete intersection of very ample divisors, and with $D \subset X'$. Then $\pi_1^{\text{ét}}(X') \rightarrow \pi_1^{\text{ét}}(X)$ is an isomorphism [SGA2, X, Thm. 3.10]. This finishes the proof. \square

4. PROOF OF THEOREM 1.1 IN THE IRREDUCIBLE CASE

The aim of this section is to prove Theorem 1.1 for stratified bundles which are irreducible. So $\pi : C \rightarrow X$ is a projective morphism between a smooth projective curve C and a smooth projective surface over an algebraically closed field k of characteristic $p > 0$. We have the property for any connected étale covering $Z \rightarrow X$, the pullback covering $Z \times_X C \rightarrow C$ splits completely.

As the triviality of $\pi^*\mathbb{M}$ is equivalent to the triviality of $\pi^*\mathbb{M}(n)$ for any natural number n , Lemma 2.7 enables us to assume that $\mathbb{M} = (E_m, \sigma_m)_{m \geq 0}$ has underlying E_m being μ -stable with trivial numerical Chern classes.

Let $\mathcal{S}_X(r)$ be the set of isomorphism classes $[\mathbb{M}]$ of irreducible stratified vector bundles $\mathbb{M} = (E_m, \sigma_m)_{m \geq 0}$ of rank r , all of whose underlying vector bundles E_m are μ -stable.

Let $S_r(n) \subset M(r, X)(k)$ be the set of all moduli points $[E_m] \in M(r, X)(k)$ such that E_m is one of the underlying bundles of $\mathbb{M}(n)$, for some $[M]$ of $\mathcal{S}_X(r)$. Equivalently, if E is a stable vector bundle of rank r on X with trivial numerical Chern classes, so that we have a point $[E] \in M(r, X)(k)$, then $[E] \in S_r(n)$ if and only if there exists a stratified bundle $\mathbb{M} = (E_m, \sigma_m)_{m \geq 0}$, with $[\mathbb{M}] \in \mathcal{S}_X(r)$, and such that $E \cong \overline{E_m}$ for some $m \geq n$.

We denote by $\overline{S_r(n)} \subset M(r, X)$ the Zariski closure, which is a reduced closed subscheme. This defines a decreasing sequence of closed subschemes

$$M(r, X) \supset \overline{S_r(0)} \supset \overline{S_r(1)} \supset \overline{S_r(2)} \supset \dots$$

which, by Noetherianity, becomes stationary at some finite stage. Set

$$\mathcal{N}_r = \mathcal{N} = \bigcap_{n \geq 0} \overline{S_r(n)} = \overline{S_r(N)} \text{ for some } N \geq 0.$$

The assignment

$$(4.1) \quad \Phi : \mathcal{N} \rightarrow \mathcal{N}, \quad E \mapsto F_X^* E$$

induces a dominant rational morphism which does not commute with the structure morphism $\mathcal{N} \rightarrow \text{Spec}(k)$. One can reinterpret Φ as follows. Let $\mathcal{N}' \subset M(r, X')$ be defined in the same way as $\mathcal{N} \subset M(r, X)$. Via the diagram (2.2), one obtains a morphism

$$(4.2) \quad F_k : \mathcal{N} \times_{k, F_k} k \rightarrow \mathcal{N}',$$

which is an isomorphism over k (as we assumed k to be algebraically closed). One has a factorization

$$(4.3) \quad \begin{array}{ccccc} & & \Phi & & \\ & \nearrow & & \searrow & \\ \mathcal{N} & \xrightarrow{F_k^{-1}} & \mathcal{N}' & \xrightarrow{\Phi'} & \mathcal{N} \\ \downarrow & & \downarrow & \swarrow & \\ \text{Spec}(k) & \xrightarrow{F_k^{-1}} & \text{Spec}(k) & & \end{array}$$

Now Φ' is a dominant morphism of k -schemes

$$(4.4) \quad \Phi' : \mathcal{N}' \rightarrow \mathcal{N}, [E] \mapsto [F_{X/k}^* E].$$

Let W be a scheme of finite type over \mathbb{F}_p over which our data $(\pi : C \rightarrow X, \mathcal{N}' \subset M(r, X'), \mathcal{N} \subset M(r, X), \Phi')$ have a flat model. We use the notation $(\pi_W : C_W \rightarrow X_W, \mathcal{N}'_W \subset M(r, X_W), \mathcal{N}_W \subset M(r, X_W), \Phi'_W : \mathcal{N}'_W \rightarrow \mathcal{N}_W)$, where we use (2.2) for $M(r, X_W)$ and $M(r, X'_W)$. We assume in addition that $C_W \rightarrow W$ and $X_W \rightarrow W$ are smooth and projective. By Grothendieck's specialisation theorem [SGA1, Thm. 3.8], any finite étale connected cover $Y_s \rightarrow X_s$, for $s \rightarrow W$ a geometric point, splits completely over C_s , that is $Y_s \times_{W_s} C_s \rightarrow C_s$ is a disjoint union of isomorphisms.

The rational map of W -schemes

$$(4.5) \quad \Phi'_W : \mathcal{N}'_W \rightarrow \mathcal{N}_W, [E] \mapsto [F_{X_W/W}^* E]$$

is dominant. So shrinking W if necessary, for all closed points $t \in W$, the restriction $\Phi_t : \mathcal{N}'_t \rightarrow \mathcal{N}_t$ is a dominant rational map of $k(t)$ -schemes which is defined by $[E] \mapsto [F_{X_t/t}^* E]$. Let p^a be the cardinality of $k(t)$. Then

$$(4.6) \quad \Phi_t^a : \mathcal{N}_t \rightarrow \mathcal{N}_t, [E] \mapsto [F_{X_t/t}^{a*} E]$$

is rational dominant.

We claim that the set of closed moduli points $[E] \in \mathcal{N}_t$ which are stable under a power of the Frobenius of the residue field $k(t)$, and in particular are trivialized by a finite étale cover of X_t , is dense. (We note that such bundles E have trivial pullback to C_t , by our hypothesis.)

To prove this claim, we make use of Hrushovski's essential theorem [Hru04, Cor. 1.2]. We argue as in [EM10, Thm. 3.14]: we first take a finite extension of $k(t)$ so as to separate the components of \mathcal{N}_t , and then take the corresponding

power of Φ_t which stabilizes the components; we then apply Hrushovski's theorem to the graph of the resulting iterates of Φ_t .

Let $T_W \rightarrow \mathcal{N}_W$ be a surjective morphism induced by a bundle E on $X_W \times_W T_W$ such that the induced moduli map $T_W \rightarrow M(r, X_W)$ factors through $\mathcal{N}_W \hookrightarrow M(r, X_W)$. Let $V = (\pi_W \times_W \text{id}_T)^* E$ be the pull-back of E on $C_W \times_W T_W$. Lemma 2.1 (i) applied to V and the projection $C_W \times_W T_W \rightarrow T_W$ implies that there is a constructible subset $T_W^0 \subset T_W$ such that for all points $t \in T_W^0$, V_t is the trivial bundle on C_t . Then Lemma 2.6, together with the above density claim and Lemma 2.1 (ii), imply that T_W^0 contains the pull-back in T_W of a non-empty open dense subscheme $\mathcal{N}^0 \hookrightarrow \mathcal{N}$.

By Lemma 2.1 (iii) applied to V and the morphism $C_W \times_W T_W \rightarrow T_W$, and Lemma 2.6, we may assume T_W^0 contains $S_r(N_r)$ for some sufficiently large N_r . Thus the restriction to C of any irreducible stratified bundle on X of rank r is trivial. This finishes the proof.

Remark 4.1. We single out the last property for later use: there is a non-empty open dense subscheme $\mathcal{N}^0 \hookrightarrow \mathcal{N}$ such that $S_r(N_r) \subset \mathcal{N}^0$, and for any $[E] \in \mathcal{N}^0 =: \mathcal{N}_r^0$ (thus on k), $\pi^* E = H^0(C, \pi^* E) \otimes_k \mathcal{O}_C$.

5. PROOF OF THEOREM 1.1 IN THE GENERAL CASE

The aim of this section is to prove Theorem 1.1 in the general case. The notations are as in Section 4. Applying Lemma 2.7, we are reduced to considering all stratified bundles $\mathbb{M} = (E_m, \sigma_m)_{m \geq 0}$ on X of some rank r , which are filtered by stratified subbundles $\mathbb{M}^i \subset \mathbb{M}^{i-1} \dots \subset \mathbb{M}^0 = \mathbb{M}$ such that the associated graded stratified bundle $\oplus_{i=0}^{s-1} \mathbb{M}^i / \mathbb{M}^{i+1}$ has the property that $\mathbb{M}^i / \mathbb{M}^{i+1} = (E_{im}, \sigma_{im})_{m \geq 0}$ is irreducible of rank say r_i , and all the underlying vector bundles E_{im} are μ -stable with trivial numerical Chern classes. By Theorem 1.1 in the irreducible case, we then have the property that $\pi^* \mathbb{M}^i / \mathbb{M}^{i+1}$ is trivial.

We now start the proof. We fix a partition $r = r_0 + \dots + r_{s-1}$. We set

(5.1)

$$\Phi' := \Phi'_0 \times_k \dots \times_k \Phi'_{s-1} : \mathcal{N}' = \mathcal{N}'_{r_0} \times_k \dots \times_k \mathcal{N}'_{r_{s-1}} \rightarrow \mathcal{N} := \mathcal{N}_{r_0} \times_k \dots \times_k \mathcal{N}_{r_{s-1}}$$

and denote by $U' = V'_0 \times_k \dots \times_k V'_{s-1} \subset \mathcal{N}'$ the dense product open on which the dominant rational map Φ' is defined, and is given by $([E_0], \dots, [E_{s-1}]) \mapsto ([F_k^* E_0], \dots, [F_k^* E_{s-1}])$, $V_i = F_k(V'_i)$ via (4.3) and $U = V_0 \times_k \dots \times_k V_{s-1}$. We note that $\mathcal{N}_{r_i}^0 \subset V_i$ for each $0 \leq i < s$.

Definition 5.1. In $\mathcal{N}(k)$, define the set

$$A = \{([E_0], \dots, [E_{s-1}]) \in \mathcal{N}'(k)\}$$

such that

- 1) there is a bundle E with a filtration such that its associated graded is isomorphic to $\oplus_{i=1}^{s-1} E_i$;

2) π^*E is not F -nilpotent.

We define $A' = F_k^{-1}(A) \subset \mathcal{N}(k)$ via (4.3).

Lemma 5.2. *The following properties hold.*

- (i) *The set A is a constructible subset of \mathcal{N} .*
- (ii) *$\Phi'(U' \cap A') \subset A$, $\Phi(U \cap A) \subset A$.*

Proof. The property (ii) is by definition. For (i), note that for $s = 1$, there is nothing to prove ($A = \emptyset$). So we may assume $s \geq 2$. Repeatedly using standard properties of Ext groups, we can find a surjective morphism $h : T \rightarrow \mathcal{N}$ (where T may not be connected), together with a vector bundle E_T on $X \times_k T$, with the property that E_T has an s -step filtration by subbundles, with quotients $E_{0,T}, \dots, E_{s-1,T}$, satisfying the following property:

- 1) for each geometric point t of T , the vector bundles $E_{0,t}, \dots, E_{s-1,t}$ obtained by restriction to X_t are stable, of ranks (r_0, \dots, r_{s-1}) respectively, and the induced morphism $T \rightarrow M(r_0, X) \times_k \dots \times_k M(r_s, X)$ (determined by the sequence $(E_{0,T}, \dots, E_{s-1,T})$, regarded as a sequence of flat families of vector bundles on X) is the given morphism h composed with the inclusion $\mathcal{N} \subset M(r_0, X) \times_k \dots \times_k M(r_s, X)$;
- 2) if E is any vector bundle on X , with an s -step filtration, and quotients E_0, \dots, E_{s-1} with $[E_i] \in \mathcal{N}_{r_i} \subset M(r_i, X)$, then there is a point t of T such that $E \cong E_t$.

By Corollary 2.4, the pull-back vector bundle $(\pi \times_k \text{Id}_T)^*E_T$ on $C \times_k T$ has the property that the set of points $t \in T$, such that $\bigoplus_i \pi^*E_{i,t}$ is trivial and π^*E_t is F -nilpotent, is a constructible subset of T . This implies that its complement Σ in T is constructible as well. By definition, $A = h(\Sigma) \subset \mathcal{N}$. Thus A is constructible. \square

We note for future use the existence of the constructible set Σ considered above, and the corresponding filtered vector bundle E_Σ on $X_\Sigma = X \times_k \Sigma$, with the property that for any closed point $\tau \in \Sigma(k)$, the filtered vector bundle $E_\tau = E_\Sigma \times_\Sigma \tau$ on X has the property that π^*E_τ is not F -nilpotent on C . By construction, $h : \Sigma \rightarrow A$ is surjective (we continue to use h to denote the restriction of h to Σ).

In order to finish the proof, it will be shown that A is not dense in \mathcal{N} (a more precise assertion is given in Lemma 5.3). To this aim, we use Lemma 5.2 (ii) to yet again define in it a locus stable by Φ . By construction, U is a dense open subset of \mathcal{N} , and $\Phi : U \rightarrow \mathcal{N}$ is a morphism with dense image. Set $U_1 = U$, and for $m \geq 2$, inductively define dense open subsets $U_m \subset \mathcal{N}$ by

$$U_m = \Phi^{-1}(U_{m-1}).$$

Then

$$V_0 \times_k \dots \times_k V_{s-1} \supset U = U_1 \supset U_2 \supset U_3 \supset \dots$$

is a decreasing sequence of open sets in \mathcal{N} .

Lemma 5.3. *There is a natural number $m_1 \neq 0$ such that $U_m \cap A = \emptyset$, for all $m \geq m_1$.*

Proof. Suppose $U_m \cap A \neq \emptyset$ for all m . Define

$$A_m = \overline{U_m \cap A} \subset A,$$

where the Zariski closure is taken relative to A . Then

$$A = A_0 \supset A_1 \supset A_2 \supset \dots$$

is a decreasing sequence of closed, nonempty subsets. Hence by the Noetherian property, there exists m_0 so that

$$A_{m_0} = A_{m_0+1} = \dots =: A_\infty$$

is a nonempty closed subset of A . By construction, $U_r \cap A_m$ is dense for any $m \geq r$, and so the rational map Φ^r is defined as a rational map on A_m , for all $m \geq r$, and satisfies $\Phi^r(A_m) \subset A_{m-r}$. Hence $\Phi^r : A_\infty \rightarrow A_\infty$ is a well-defined rational map, for each $r \geq 0$. Define

$$B_r = \overline{\Phi^r(A_\infty)} \neq \emptyset,$$

where the closure is again relative to A . Then

$$A_\infty = B_0 \supset B_1 \supset B_2 \supset \dots$$

is a decreasing sequence of nonempty closed subsets. Hence there exists m_1 such that

$$B_{m_1} = B_{m_1+1} = \dots =: B_\infty \neq \emptyset$$

is a closed subset of A . By construction, Φ is a rational map defined on B_r , for each $r \geq 0$, and further satisfies

$$\overline{\Phi(B_r)} = \overline{\Phi(\overline{\Phi^r(A_\infty)})} = \overline{\Phi(\Phi^r(A_\infty))} = \overline{\Phi^{r+1}(A_\infty)} = B_{r+1}.$$

The first equality is by definition, the second equality is true for all continuous maps of topological spaces, the third one is again by definition. Hence $\Phi : B_\infty \rightarrow B_\infty$ is a dominant rational self-map.

We now define $B'_\infty = F_k(B_\infty) \subset \mathcal{N}'$ via (4.3). Then

$$(5.2) \quad \Phi' : B'_\infty \rightarrow B_\infty, \quad \oplus_{i=0}^{s-1} [E_i] \rightarrow \oplus_{i=0}^{s-1} [F_{X/k}^* E_i]$$

is a dominant rational map. One defines Σ' as the Frobenius twist of Σ , and the bundle $E_{\Sigma'}$ on $X' \times_k \Sigma'$ as the pull back of E_Σ on $X \times_k \Sigma$ via the Frobenius twist $F_k : X' \times_k \Sigma' \rightarrow X \times_k \Sigma$. Then (5.2) implies that the filtered bundle $(F_{X' \times_k \Sigma' / \Sigma'_{B'_\infty}})^* E_{\Sigma'_{B'_\infty}}$ is such that its graded pieces, regarded as families of bundles on X , determine a morphism $\Sigma_{B_\infty} \setminus \Delta \rightarrow B_\infty$ with $\Delta \subset \Sigma_{B_\infty}$ mapping to a codimension ≥ 1 constructible subset of B_∞ .

Applying again Hrushovski's theorem [Hru04, Cor. 1.2] as in Section 4, with (5.2) replacing (4.4), we obtain a model $B_{\infty, W}$ of B_∞ , and for any closed point

of $t \in W$, a dense set of closed points $e = [\oplus_{i=0}^{s-1} E_i]$ in $B_{\infty,t}$ which are (each) stabilized by a power of the Frobenius of $k(t)$ (the power depends on e).

Thus, for any graded vector bundle $gr(E)$ on X_t which corresponds to the point e in the product of moduli spaces, the iterated Frobenius pullback graded bundle $F^{a(e)*}(gr(E))$ corresponds to the same moduli point; hence from [EK16, Lem. 2.4], the bundles $gr(E)$ and $F^{a(e)*}(gr(E))$ are isomorphic. It follows from [LS77, Satz 1.4], which is applicable as the ground field is finite, that such a graded bundle $gr(E)$ is trivialised on a finite étale covering of X_t , and hence also has trivial pullback to C_t .

We choose now W such that $h : \Sigma_{B_{\infty}} \rightarrow B_{\infty}$ has a model $h_W : \Sigma_{B_{\infty},W} \rightarrow B_{\infty,W}$, along with a filtered vector bundle $E_{\Sigma_{B_{\infty},W}}$ on $X_W \times_W \Sigma_{B_{\infty},W}$, which base changes to the original filtered bundle $E_{\Sigma_{B_{\infty}}}$ on $X \times \Sigma_{B_{\infty}}$.

By Lemma 2.5, it follows that for any closed point $t \in W$, and any $e \in B_{\infty,t}$ (obtained as above using Hrushovsky's theorem), and any closed point $\tau \in \Sigma_e := h_W^{-1}(e)$, the corresponding vector bundle $E_{\Sigma_{B_{\infty},W}} \times_e \tau =: E_{\tau}$ has the following property: some Frobenius pullback $F^{a*}(E_{\tau})$ is trivialised on a finite étale cover of X_{τ} . In particular, the pullback filtered vector bundle $\pi_{\tau}^* E_{\tau}$ on $C_{\tau} = C_t \times_t \tau$ is unipotent, and also F -nilpotent.

Let $V = (\pi_W \times_W \text{id}_{\Sigma_{B_{\infty},W}})^* E_{\Sigma_{B_{\infty},W}}$ be the pull-back of $E_{\Sigma_{B_{\infty},W}}$ on $C_W \times_W \Sigma_{B_{\infty},W}$. Then Corollary 2.4 applied to V and the projection $C_W \times_W \Sigma_{B_{\infty},W} \rightarrow \Sigma_{B_{\infty},W}$ implies that there is a constructible subset $T^0 \subset \Sigma_{B_{\infty},W}$, the image of which in $B_{\infty,W}$ contains a dense set of closed points $e \in B_{\infty,W}$ (as above), such that for all points $\tau \in T^0$, V_{τ} is an F -nilpotent bundle on C_{τ} . Thus the (dense) image in $B_{\infty,W}$ of this constructible set T^0 contains a dense open set, and hence also points lying in the image of $B_{\infty} \rightarrow B_{\infty,W}$, which contradicts the definition of $h : \Sigma \rightarrow B_{\infty}$.

This finishes the proof. \square

Now if \mathbb{M} is any stratified bundle on X of rank r with irreducible filtered constituents $\mathbb{M}^i/\mathbb{M}^{i+1}$ of ranks r_i , for $0 \leq i < s$, we may, after replacing \mathbb{M} by some $\mathbb{M}(n)$ if necessary, assume that the associated sequences of vector bundles E_{im} (for $m \geq 0$) satisfy that $(\dots, [E_{im}], \dots) \in \mathcal{N}^0 \cap U_{m_1}$, where m_1 is as in Lemma 5.3, and hence lies in the complement of A . Thus \mathbb{M} has corresponding sequence (E_m, σ_m) such that $\pi^* E_m$ is F -nilpotent on C . Hence by Corollary 2.8, $\pi^* \mathbb{M}$ is trivial as a stratified bundle.

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