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Chern classes of automorphic vector bundles*

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Dedicated to Yuri Manin on the occasion of his 80th birthday

Abstract: We prove that Chern classes in continuous ℓ-adic cohomology of automorphic bundles associated to representations of G on a projective Shimura variety with data (G, X) are trivial rationally. It is a consequence of Beilinson's conjectures which predict that the Chern classes in the Chow groups vanish rationally.

Introduction

Let X be a smooth projective variety defined over a number field k. Beilinson [Bei85, Conj. 2.4.2.1] conjectures that the rational Chow ring $CH(X)_{\mathbb{Q}}$ injects into Deligne cohomology of $X \otimes_k \mathbb{C}$. Concretely, if a class in $CH^n(X)_{\mathbb{Q}}$ vanishes in $H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$, it is expected to be 0. There is not a single example with dimension $X \geq 2$ with large Chow ring $CH(X \otimes_k \mathbb{C})$ for which this conjecture has been verified.

On the other hand, there are Chern classes reflecting the fact that X is defined over a number field. On proper models $\mathcal{X}_U$ over a non-trivial open U of $\mathrm{Spec}(\mathbb{Z})$, one has Chern classes in ℓ-adic cohomology $H^{2n}(\mathcal{X}_U, \mathbb{Q}_\ell(n))$. By taking the inductive limit over such $\mathcal{X}_U$, these yield the Chern classes in continuous ℓ-adic cohomology $H^{2n}_{\mathrm{cont}}(X, \mathbb{Q}(n))$ ([Jan87, Section 2]). Each space $H^{2n}_{\mathrm{cont}}(X, \mathbb{Q}(n))$ is filtered by the abutment of the Hochshild-Serre spectral sequence, which, by Deligne's argument [Del68, Thm. 1.5] using the strong Lefschetz theorem [Del80, Thm. 4.4.1], degenerates at E_2. Given that $H^1(k, H^{2n-1}(X_{\bar{k}}, \mathbb{Q}_\ell(n)))$, the first graded piece of the filtration, can be interpreted as the extension group of $\mathbb{Q}_\ell(0)$ by $H^{2n-1}(X_{\bar{k}}, \mathbb{Q}_\ell(n))$ in the category

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of Galois modules – just as $H^{2n-1}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{Q}(n))$ can be interpreted as the extension group of $\mathbb{Q}(0)$ by $H^{2n-1}(X_{\mathbb{C}}, \mathbb{Q}(n))$ in the category of Hodge structures over $\mathbb{Q}$ – Beilinson's conjecture predicts that

Conjecture 0.1. *With notation as above, if a class in $CH^n(X)_{\mathbb{Q}}$ vanishes in $H^i(k, H^{2n-i}(X_{\bar{k}}, \mathbb{Q}_\ell(n))$ for $i = 0, 1$, then it does for $i = 2$.*

This conjecture seems to be more modest than the general motivic one above. It is a fascinating problem in Galois cohomology, and it hasn't been studied at all. An analogous question for function fields over finite fields has been considered and proved to be true for 0-cycles in [Ras95, Thm. 0.1].

On the other hand, the Chern classes of flat bundles on a smooth projective variety X defined over $\mathbb{C}$ vanish in Deligne cohomology $H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$ for $n \geq 2$, due to Reznikov's theorem [Rez94, Thm. 1.1], giving a positive answer to Bloch's conjecture [Blo77, Intro.]. In particular, Beilinson's conjecture implies that the Chern classes of flat bundles on a smooth projective variety X defined over a number field vanish in the Chow groups $CH^n(X)_{\mathbb{Q}}$ for $n \geq 2$.

In addition, in [Esn96, 4.7] (in a vague form) and in [EV02, Intro.] (in a more precise form) the problem is posed whether Chern classes of Gauß-Manin bundles on a smooth variety X defined over a field of characteristic 0 vanish in the Chow groups $CH^n(X)_{\mathbb{Q}}$ for $n \geq 2$. It is proved to be the case for those of weight 1; that is, for the Gauß-Manin bundles of relative first de Rham cohomology of an abelian scheme over X ([vdG99, Thm. 1.1], [EV02, Thm. 1.1]). In fact, the weight 1 Gauß-Manin bundle is defined on $\mathcal{A}_g$, which is defined over $\mathbb{Q}$, and in [EV02, loc. cit.] it is proved that the Chern classes in the Chow groups of the Deligne extension on the toroidal compactification of $\mathcal{A}_g$ vanish. Thus this example confirms (in a weak sense) Beilinson's conjecture as well.

The moduli space $\mathcal{A}_g$ is a quasi-projective Shimura variety and the weight 1 Gauß-Manin bundle on it is an automorphic bundle associated to the tautological representation of $Sp(2g)$. A Shimura variety ${}_KS(G, X)$ (notation explained below) has a canonical model over its reflex field $E(G, X)$, which is a number field [Mil05, Section 14]. It carries a natural family of automorphic vector bundles that are defined on this model and themselves have models over explicit finite extensions of $E(G, X)$ [Har85, Thm. 4.8] (for unexplained notation see Section 1.1 below). An automorphic vector bundle $[\mathcal{E}]_K$ that comes from a representation of G is endowed canonically with a flat connection, and its Chern classes in Deligne cohomology $H^{2n}_{\mathcal{D}}({}_KS(G, X), \mathbb{Q}(n))$

vanish for $n \geq 1$ (Theorem 2.3, Remark 2.5 1)). Thus, at least when ${}_K S(G, X)$ is projective, Beilinson's conjecture implies that the Chern classes vanish even in $CH^n({}_K S(G, X))_{\mathbb{Q}}$ for $n \geq 1$. Unfortunately, we can not prove this. Instead we prove

Theorem 0.2. *If ${}_K S(G, X)$ is projective, the Chern classes of an automorphic bundle attached to a representation of G vanish in continuous ℓ-adic cohomology $H^{2n}_{\text{cont}}({}_K S(G, X), \mathbb{Q}_\ell(n))$ for $n \geq 1$.*

Stated differently, we prove Conjecture 0.1 in this particular case. The proof relies strongly on the purely algebraic definition of the automorphic bundles, as being associated to a representation of G. Indeed, all automorphic bundles, seen in the category of vector bundles on the Shimura variety, are eigenvectors for the so-called volume character of the Hecke algebra. The Hecke algebra acts semi-simply on (continuous) ℓ-adic cohomology, and the corresponding eigenspace $H^j({}_K S(G, X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)_v$ in ℓ-adic cohomology identifies with ℓ-adic cohomology $H^j(\hat{X}_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ of the compact dual $\hat{X}$ of X, which itself is generated by algebraic cycles. This allows us to compute the invariants in i-th Galois cohomology of $H^{2n-i}({}_K S(X, G)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n))$ (Theorem 3.3).

Finally, we remark that if every $\mathbb{Q}$-simple factor of G has real rank at least 2, then the super-rigidity theorem of Margulis, applied to the connected components of ${}_K S(G, X)$, implies that every flat vector bundle over ${}_K S(G, X)$ becomes isomorphic to an automorphic vector bundle after replacing K by an appropriate subgroup of finite index. Since the non-vanishing of Chern classes in the cohomology theories considered here is stable under finite coverings, this implies that for most Shimura varieties the vanishing holds for all flat vector bundles.

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Conventions

If G is a reductive algebraic group over $\mathbb{Q}$, by an *admissible irreducible representation* of $G(\mathbf{A})$ we will mean an irreducible admissible $(\mathfrak{g}, K) \times G(\mathbf{A}_f)$-module, where $\mathfrak{g}$ is the complexified Lie algebra of G, $K \subset G(\mathbb{R})$ is a connected subgroup generated by the center of $G(\mathbb{R})$ and a maximal compact connected subgroup, $G(\mathbf{A}_f)$ is the group of finite adèles of G. If π is such a representation then we will write

$$\pi \simeq \pi_\infty \otimes \pi_f$$

where π_∞ is an irreducible admissible $(\mathfrak{g}, K)$-module and π_f is an irreducible admissible representation of $G(\mathbf{A}_f)$.

1. Automorphic vector bundles and flag varieties

1.1. Review of automorphic vector bundles

Let (G, X) be a *Shimura datum*, in other words a datum defining a Shimura variety. We recall that this means that G is a connected reductive group over $\mathbb{Q}$ and that X is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ of real groups, where $\mathbb{S} = R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$ is $\mathbb{C}^\times$ viewed as an algebraic group over $\mathbb{R}$. The pair (G, X) must satisfy a list of familiar axioms that guarantee that X is a $G(\mathbb{R})$-equivariant finite union of hermitian symmetric spaces for the identity component of the derived subgroup of $G(\mathbb{R})$; see [Mil05]. In particular, we include the axiom that guarantees that the maximal $\mathbb{R}$-split torus in the center of G is also split over $\mathbb{Q}$; without this hypothesis the construction of automorphic vector bundles, as in (1.5), is not strictly true as stated, although there are ways to fix this. Then, for any open compact subgroup $K \subset G(\mathbf{A}_f)$, the double coset space

$${}_KS(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbf{A}_f)/K$$

is canonically the set of complex points of a quasiprojective algebraic variety that has a canonical model over a number field, usually denoted $E(G, X)$ and called the *reflex field* of (G, K). If $K' \subset K$ is a subgroup of finite index then the natural map

$$\pi_{K,K'} :_{K'} S(G, X) \ \to \ {}_KS(G, X)$$

is finite; if K' is a normal subgroup of K then $\pi_{K,K'}$ is the quotient map for the action of the group K/K' on the right. Moreover, if K is sufficiently small (*neat*, in the sense of [Pin90]), then the map $\pi_{K,K'}$ is finite étale.

The precise nature of the canonical model will not be considered in this paper; we will be concerned with ${}_KS(G,X)$ as a complex algebraic variety, and our aim is to study the Chern classes of a class of vector bundles on ${}_KS(G,X)$ that are defined canonically by reference to the origin of the variety in linear algebra. To this end, choose a base point $h \in X$ and let $K_h \subset G(\mathbb{R})$ be its stabilizer. Then K_h is the group of real points of a reductive subgroup of G, which we also denote K_h. We make the useful assumption that K_h is defined over a number field E_h; this is always possible, and we may even assume that E_h is a CM field and that every irreducible representation of K_h is defined over a CM field. In any case, K_h is reductive and there is a natural maximal parabolic subgroup $P_h \subset G_{\mathbb{C}}$ that contains $K_{h,\mathbb{C}}$ as a Levi factor. Let $\hat{X} = G/P_h$ be the corresponding flag variety. We view X as an analytic open subset of $\hat{X}$ by means of the Borel embedding β (this determines the choice of P_h among the two maximal parabolics containing K_h). In particular, the complex dimension of this analytic variety is the same as that of $\hat{X}$, which is the same as that of ${}_KS(G,X)$. Moreover, h may be viewed as a point of $\hat{X}$.

Let $E'_h \supset E_h$ denote a finite extension over which K_h becomes a split reductive group. Then every representation of K_h has a model over E'_h.

For any variety Z over $\mathbb{C}$, let $\mathrm{Vect}(Z)$ denote the exact category of complex vector bundles on Z. Let $\mathrm{Vect}_G(\hat{X})$ be the category of G-equivariant vector bundles on $\hat{X}$ with coefficients in $\mathbb{C}$; let

$$f : \mathrm{Vect}_G(\hat{X}) \ \to \ \mathrm{Vect}(\hat{X})$$

be the forgetful functor. Let $\mathrm{Vect}_G^{ss}(\hat{X}) \subset \mathrm{Vect}_G^{\hat{X}}$ denote the subcategory of semisimple G-equivariant bundles. If H is an algebraic group over a ring R and $k \supset R$ is another ring, let $\mathrm{Rep}_k(H)$ denote the category of algebraic representations (of finite type) of H on free modules over k. There is an equivalence of symmetric monoidal categories

$$r_P : \mathrm{Vect}_G(\hat{X}) \simeq \mathrm{Rep}_{\bar{\mathbb{Q}}}(P_h) \tag{1.1}$$

given by taking a vector bundle $B/\hat{X}$ to its fiber B_h at h, with the isotropy representation of the stabilizer P_h. Similarly, let $\mathrm{Vect}_G^{ss}(\hat{X}) \subset \mathrm{Vect}_G^{\hat{X}}$ denote the subcategory of semisimple G-equivariant vector bundles on $\hat{X}$. Then (1.1) restricts to an equivalence of symmetric monoidal categories

$$r : \mathrm{Vect}_G^{ss}(\hat{X}) \simeq \mathrm{Rep}_{\bar{\mathbb{Q}}}(K_h) \tag{1.2}$$

Evidently we have canonical isomorphisms

$$\begin{aligned} K_0(\mathrm{Vect}^{ss}_G(\hat{X})) &\xrightarrow{\sim} K_0(\mathrm{Vect}_G(\hat{X})); \\ K_0(\mathrm{Rep}_{\bar{\mathbb{Q}}}(K_h)) &\xrightarrow{\sim} K_0(\mathrm{Rep}_{\bar{\mathbb{Q}}}(P_h)) \end{aligned} \tag{1.3}$$

compatible with the isomorphisms (1.1) and (1.2).

Lemma 1.4. *Every simple object in* $\mathrm{Vect}_G(\hat{X})$ *has a model over* E'_h.

Proof. The fiber functor (1.2) at h is evidently rational over the number field E_h, so the claim comes down to the assertion that every irreducible representation of K_h has a model over E'_h, which we have already noted. □

On the other hand, for any $K \subset G(\mathbf{A}_f)$ as above, there is a functor

$$\mathcal{E} \mapsto [\mathcal{E}] : \mathrm{Vect}_G(\hat{X}) \to \mathrm{Vect}({}_K S(G,X)) \tag{1.5}$$

defined algebraically in [Har85, Thm. 4.8]. As a functor on complex vector bundles we have the explicit construction:

$$[\mathcal{E}] = [\mathcal{E}]_K = G(\mathbb{Q})\backslash\mathcal{E} \times G(\mathbf{A}_f)/K.$$

This is a monoidal functor and it satisfies the following property with respect to change of group: if $K' \subset K$ then there are canonical isomorphisms

$$\pi^*_{K,K'}([\mathcal{E}]_K) \xrightarrow{\sim} [\mathcal{E}]_{K'}; \quad \pi_{K,K',*}([\mathcal{E}]_{K'}) \xrightarrow{\sim} [\mathcal{E}]_K \otimes I^K_{K'}1, \tag{1.6}$$

where $I^K_{K'}1$ is the representation of K induced from the trivial representation of K'. On the left, this is by definition, and on the right, this is the projection formula. In particular,

$$\pi_{K,K',*}([\mathcal{E}]_{K'}) \xrightarrow{\sim} ([\mathcal{E}]_K)^{[K:K']} \tag{1.7}$$

as vector bundles.

In this paper we work systematically with Chow groups CH and the Grothendieck group K_0 of locally free sheaves with *rational* coefficients. For H and k as above, we let $K_0(\mathrm{Rep}_k(H))$ denote the Grothendieck group of $\mathrm{Rep}_k(H)$, tensored with $\mathbb{Q}$.

$$ch_{\hat{X}} : \mathrm{Vect}(\hat{X}) \to CH(\hat{X})_{\mathbb{Q}}, \quad ch_K : \mathrm{Vect}({}_K S(G,X)) \to CH({}_K S(G,X))_{\mathbb{Q}}$$

denote the respective Chern characters. We shall use the following proposition:

Proposition 1.8. *1) The map*

$$ch_{\hat{X}} \circ f \circ r^{-1} : \mathrm{Rep}_{\bar{\mathbb{Q}}}(K_h) \to \mathrm{Vect}(\hat{X})_{\mathbb{Q}} \to CH(\hat{X})_{\mathbb{Q}}$$

factors through the composite homomorphism

$$K_0(\mathrm{Rep}_{\overline{\mathbb{Q}}}(K_h))_{\mathbb{Q}} \to K_0(\hat{X})_{\mathbb{Q}} \to CH(\hat{X})_{\mathbb{Q}}.$$

2) The restriction of $ch_{\hat{X}}$ to $\mathrm{Vect}_G(\hat{X})$ generates $CH(\hat{X})_{\mathbb{Q}}$.
3) If we let $\mathrm{Rep}_{\overline{\mathbb{Q}}}(G) \to \mathrm{Rep}_{\overline{\mathbb{Q}}}(K_h)$ denote the restriction functor, then $ch_{\hat{X}} \circ r^{-1}$ induces an isomorphism

$$K_{\mathbb{Q}}(\mathrm{Rep}_{\overline{\mathbb{Q}}}(K_h)) \otimes_{K_{\mathbb{Q}}(\mathrm{Rep}_{\overline{\mathbb{Q}}}(G))} \mathbb{Q} \xrightarrow{\sim} CH(\hat{X})_{\mathbb{Q}}. \tag{1.9}$$

Here the map $K_{\mathbb{Q}}(\mathrm{Rep}_{\overline{\mathbb{Q}}}(G)) \to \mathbb{Q}$ is given by the augmentation, that is by the rank of a representation.

Proof. Point 1) is essentially a tautology: the Chern character obviously factors through $K_0(\hat{X})$ and r is an exact tensor functor. Point 2) is the main theorem of [Mar76]. Suppose V is a representation of G; then the corresponding homogeneous bundle on $\hat{X}$ is just $V \times \hat{X}$, with G acting diagonally. In particular, as a vector bundle it is a sum of $\dim V$ copies of the trivial bundle, hence the restriction to $K_{\mathbb{Q}}(\mathrm{Rep}_{\overline{\mathbb{Q}}}(G))$ of the Chern character factors through the augmentation map. Thus the surjection $ch_{\hat{X}} \circ r^{-1}$ factors through the left-hand side of (1.9). Now it follows from the main theorem of [Mar76] that this left hand side is of dimension $[W_G : W_{K_h}]$, where W_G (resp. W_{K_h}) is the absolute Weyl group of G (resp. K_h) relative to a common maximal torus. On the other hand, the Schubert cells form a basis for the right-hand side, and there are $[W_G : W_{K_h}]$ of them (cf. [Bri05, 3.4.2 (2)]). So the surjection is an isomorphism by comparing dimensions. □

The purpose of the present note is to provide some evidence for the following conjecture, which is an analogue of Proposition 1.8 for Shimura varieties.

Conjecture 1.10. *The map $c_K : \mathrm{Rep}_{\bar{\mathbb{Q}}}(K_h) \to CH({}_K S(G, X))_{\mathbb{Q}}$ defined by*

$$c_K(W) = ch_K([r^{-1}(W)])$$

induces an injective ring homomorphism

$$K_{\mathbb{Q}}(\mathrm{Rep}_{\overline{\mathbb{Q}}}(K_h)) \otimes_{K_{\mathbb{Q}}(\mathrm{Rep}_{\overline{\mathbb{Q}}}(G))} \mathbb{Q} \hookrightarrow CH({}_K S(G, X))_{\mathbb{Q}}.$$

We denote by $c_K^{>0}$ the composite of c_K with the projection

$$CH({}_KS(G,X))_{\mathbb{Q}} = \oplus_{n\geq 0} CH^n({}_KS(G,X))_{\mathbb{Q}} \to \oplus_{n>0} CH^n({}_KS(G,X))_{\mathbb{Q}}.$$

Claim 1.11. *Conjecture 1.10 is equivalent to*

$$c_K^{>0}|_{\mathrm{Rep}_{\bar{\mathbb{Q}}}(G)} = 0.$$

Proof. Indeed this condition is equivalent to saying that c_K induces a homomorphism

$$K_{\mathbb{Q}}(\mathrm{Rep}_{\overline{\mathbb{Q}}}(K_h)) \otimes_{K_{\mathbb{Q}}(\mathrm{Rep}_{\overline{\mathbb{Q}}}(G))} \mathbb{Q} \to CH({}_KS(G,X))_{\mathbb{Q}}.$$

On the other hand, given the Grothendieck-Riemann-Roch theorem, to say that it is injective is equivalent to saying that the ring homomorphism

$$K_0(\mathrm{Rep}_{\bar{\mathbb{Q}}}(K_h))_{\mathbb{Q}} \otimes_{K_{\mathbb{Q}}(\mathrm{Rep}_{\overline{\mathbb{Q}}}(G))} \mathbb{Q} \to K_0({}_KS(G,X))_{\mathbb{Q}}$$

induced by the functor $K_0(r^{-1})$ is injective. This is true, as follows from Proposition 1.20 and point 3) of Proposition 1.8. □

Remarks 1.12. 1) The only instance for which one knows that Conjecture 1.10 is true is when ${}_KS(G,X)$ is the Siegel domain $\mathcal{A}_g$ and the representation of $G = Sp(2g)$ is the tautological one (see [EV02, Thm. 1.1], [vdG99, Thm. 1.1]). Then the flat vector bundle $[\mathcal{E}]$ on $\mathcal{A}_g$ is the Gauß-Manin bundle of the relative de Rham cohomology H^1 of the universal abelian scheme. The family is defined only over a level structure, but the Gauß-Manin bundle, together with its Gauß-Manin connection, descends to $\mathcal{A}_g$. In this case the vanishing is even stronger: on the finite cover over which the local monodromies are unipotent, the Deligne extension of the Gauß-Manin bundle has vanishing Chern classes in the rational Chow groups.

2) To tie up with the conjecture on flat bundles alluded to in the introduction, we remark that the automorphic vector bundles $[\mathcal{E}]_K$ in the conjecture are those coming from $\mathrm{Rep}_{\bar{\mathbb{Q}}}(G)$ which have a $G(\mathbf{A}_f)$-equivariant integrable connection [Har85, Lemma 3.6].

3) Let $c_{\mathcal{D}} : CH^*({}_KS(G,X))_{\mathbb{Q}} \to H^{2*}_{\mathcal{D}}({}_KS(G,X),*)$ be the cycle homomorphism into Deligne cohomology [EV88, Section 7]. We prove in Theorem 2.3 that

$$c_{\mathcal{D}} \circ c_K^{>0}|_{\mathrm{Rep}_{\bar{\mathbb{Q}}}(G)} = 0.$$

Using Claim 1.11 in addition, one sees that Conjecture 1.10 when ${}_KS(G,X)$ is projective is a special case of Beilinson's motivic conjecture discussed in the introduction.

4) We are not able to prove Conjecture 1.10. In fact, apart from the example mentioned in 1), we can not prove vanishing in any other example.

Let us denote by ${}_KS(G,X)_{E(G,X)}$ the model of the Shimura variety over its reflex field. If

$$c_\ell : CH^*({}_KS(G,X)_{E(G,X)})_{\mathbb{Q}} \to H^{2*}_{\mathrm{cont}}({}_KS(G,X)_{E(G,X)}, \mathbb{Q}_\ell(*))$$

denotes the cycle homomorphism to continuous ℓ-adic cohomology [Jan87, Section 2], we prove

$$c_\ell \circ c_K^{>0}|_{\mathrm{Rep}_{\bar{\mathbb{Q}}}(G)} = 0.$$

in Theorem 3.3. (More precisely, we prove this after replacing $E(G,X)$ by the finite extension E'_h of Lemma 1.4.) In particular, we verify Conjecture 0.1 in this case. To our knowledge, the examples treated in this note are the first that confirm this prediction for cycle classes of flat bundles that do not depend on knowing in advance that the Chow class itself vanishes.

1.2. Hecke operators

Fix $K \subset G(\mathbf{A}_f)$. Let $g \in G(\mathbf{A}_f)$ and consider $K_g = K \cap gKg^{-1} \subset K$. Let

$$\pi_{1,g} = \pi_{K,K_g} : {}_{K_g}S(G,X) \ \to \ {}_KS(G,X),$$

defined as above. Right multiplication by g defines an isomorphism

$$r_g : {}_{gKg^{-1}}S(G,X) \ \xrightarrow{\sim} \ {}_KS(G,X).$$

Let

$$\pi_{2,g} = r_g \circ \pi_{gKg^{-1},K_g} : {}_{K_g}S(G,X) \ \to \ {}_KS(G,X).$$

We first observe that

$$T(g)[\mathcal{E}]_K \cong \oplus_1^{[K:K_g]}[\mathcal{E}]_K \tag{1.13}$$

in $K_0({}_KS(G,X))$, where $T(g) = \pi_{2,g*} \circ \pi_{1,g}^*$. Indeed, the definition implies $r_g^*[\mathcal{E}]_K \cong [\mathcal{E}]_{g^{-1}Kg}$, and formula (1.6) implies formula (1.13).

Both $\pi_{1,g}$ an $\pi_{2,g}$ are finite étale morphisms. So for any contravariant cohomology theory H which has push-downs for proper (or even only finite étale) morphisms, one can define the *Hecke operator*

$$T(g) : H({}_KS(G,X)) \xrightarrow{\pi_{2,g*}\circ\pi_{1,g}^*} H({}_KS(G,X)). \tag{1.14}$$

We shall use the Hecke operators on Chow groups, on continuous ℓ-adic cohomology, on Deligne cohomology, on syntomic cohomology. All of them are considered rationally. In particular, they are (possibly infinite) dimensional vector spaces over $\mathbb{Q}$, and the Hecke algebra splits those cohomologies as a sum of generalized eigenspaces.

A rational prime number q is *unramified* for K if there exists a $K_q \subset K$ with $K_q \subset G(\mathbb{Q}_q)$ a hyperspecial compact open subgroup, and *ramified* otherwise. There is a finite set $S(K)$ of ramified primes. We let $\mathcal{H}_K$ denote the $\mathbb{Q}$-subalgebra of the ring tensor $\mathbb{Q}$ of correspondences generated by the $T(g)$, where g runs through elements of $G(\mathbb{Q}_q)$ with $q \notin S(K)$; this is well-known to be a commutative algebra.

The following is obvious:

Lemma 1.15. *Let R be a ring, $a \in R$, and let $[a] : {}_KS(G,X) \to R$ be the constant function with value a. Then $[a]$ is an eigenfunction for every $T(g)$, with eigenvalue $v(K,g) = [K : K_g]$.*

We thus define the *volume character* $T(g) \mapsto v(K,g)$ of $\mathcal{H}_K$ by the formula

$$T(g)\cdot[1] = v(K,g)\cdot[1]$$

where $[1]$ is the constant function with value 1 on ${}_KS(G,X)$, as above. For any cohomology theory $H^*({}_KS(G,X))$, including the Chow groups, let $H^*({}_KS(G,X))_v \subset H^*({}_KS(G,X))$ be the eigenspace for the volume character. As $H^*({}_KS(G,X))$ is a possibly infinite dimensional $\mathbb{Q}$-vector space, so is $H^*({}_KS(G,X))_v$.

Lemma 1.16. *Let $\mathcal{E} \in \mathrm{Vect}_G(\hat{X})$. Then for any open compact subgroup $K \subset G(\mathbf{A}_f)$, $ch([\mathcal{E}]_K) \in CH({}_KS(G,X))_v$.*

Proof. As $\pi_{2,g}$ is finite étale, its Todd class is equal to 1, thus the Grothendieck-Riemann-Roch theorem implies

$$ch(\pi_{2,g*}\pi_{1,g}^*[\mathcal{E}]_K) = \pi_{2,g*}(ch(\pi_{1,g}^*[\mathcal{E}]_K)\cdot 1),$$

from which we conclude using formula (1.13)

$$ch(T(g)[\mathcal{E}]_K) = \pi_{2,g*}(ch(\pi_{1,g}^*[\mathcal{E}]_K)\cdot 1) = \pi_{2,g*}\pi_{1,g}^* ch([\mathcal{E}]_K) = T(g)ch([\mathcal{E}]_K).$$

Applying (1.6) and (1.7) one concludes

$$[K : K_g]ch([\mathcal{E}]_K) = T(g)ch([\mathcal{E}]_K). \tag{1.17}$$

□

Corollary 1.18. *Let H^* be a cohomology theory which, for any open compact subgroup $K \subset G(\mathbf{A}_f)$, admits a cycle map*

$$c_H^K : CH({}_K S(G, X)) \to H({}_K S(G, X))$$

which commutes with the action of $\mathcal{H}_K$. Then for any $\mathcal{E} \in \mathrm{Vect}_G(\hat{X})$,

$$c_H^K \circ ch([\mathcal{E}]_K) \in H^*({}_K S(G, X))_v.$$

The cohomology theories $H^i(-, j)$ with coefficients in a characteristic 0 field F considered in this note are all functorial. Thus to check that the cycle map commutes with the action of $\mathcal{H}_K$, it suffices to verify compatibility with the push-down via $\pi_{2,g*}$. In all those cohomology theories, the cycle map can be defined via purity: for $Z = \sum_i m_i Z_i$ a codimension n cycle on a smooth Y, where $m_i \in \mathbb{Z}$ and the codimension n cycles Z_i are prime, the Gysin morphism exists and is an isomorphism

$$\gamma : \oplus F \cdot [Z_i] \xrightarrow{\cong} H_Z^{2n}(Y, n).$$

See [EV88, Section 7] for Betti, de Rham and Deligne cohomology, [Jan88, Thm. 3.23] for continuous ℓ-adic cohomology. Thus compatibility reduces to showing

$$\gamma(\pi_*[Z_i]) = \gamma(\deg(k(Z_i)/k(\pi(Z_i))) \cdot [\pi(Z_i)]) = \pi_*\gamma([Z_i]) \in H_{p(Z_i)}^{2n}(Y', n)$$

for a finite surjective morphism $p : Y \to Y'$. In Deligne cohomology, it follows as the cohomology verifies the Bloch-Ogus axioms. In ℓ-adic cohomology, for lack of reference, we restrict ourselves to the case where Y is defined over a number field k. Let $\mathcal{Y}$ be a flat smooth model over a non-trivial open U in the spectrum of the ring of integers of k. Then the cycle class in $H^{2n}(\mathcal{Y}, n)$ is just the standard cycle class from [SGA4.5], for which the trace properties are known [SGA4.5, 2.3].

Corollary 1.19. *Corollary 1.18 holds true for Deligne cohomology and continuous ℓ-adic cohomology.*

In the next section we study $H^*({}_KS(G,X))_v$ for Betti and ℓ-adic cohomology.

1.3. Chern classes in cohomology

Henceforward we assume the Shimura variety ${}_KS(G,X)$ to be *projective*; equivalently, the derived subgroup G^{der} of G is anisotropic over $\mathbb{Q}$. In the non-compact case the automorphic theory naturally gives information about Chern classes of canonical extensions on toroidal compactifications on the one hand; on the other hand, the v-eigenspace most naturally appears in intersection cohomology of the minimal compactification. This has been worked out in detail by Goresky and Pardon in [GP02], and we expect to study the analogous questions for Chow groups in a second paper.

Although the point of the Goresky-Pardon paper is to study non-compact Shimura varieties, it still contains a convenient reference for our purposes. The following statement is well known.

Proposition 1.20. *Assume the derived subgroup G^{der} of G is anisotropic, so that ${}_KS(G,X)$ is projective. There is a canonical isomorphism of algebras*

$$H^*(\hat{X},\mathbb{Q}) \xrightarrow{\sim} H^*({}_KS(G,X),\mathbb{Q})_v.$$

Proof. We first show the corresponding statement over $\mathbb{C}$; thus we can compute $H^*({}_KS(G,X),\mathbb{C})$ using automorphic forms and Matsushima's formula. Say the space $\mathcal{A}(G)$ of automorphic forms on $G(\mathbb{Q})\backslash G(\mathbf{A})$ decomposes as the direct sum

$$\mathcal{A}(G) = \oplus_\pi m(\pi)\pi$$

where π runs over irreducible admissible representations of $G(\mathbf{A})$ and $m(\pi)$ is a non-negative integer, which is positive for a countable set of π. Then

$$H^i({}_KS(G,X),\mathbb{C}) \xrightarrow{\sim} \oplus_\pi\, m(\pi)H^i(\mathfrak{g},K_h;\pi_\infty)\otimes\pi_f^K.$$

Then

$$H^i({}_KS(G,X),\mathbb{C})_v \xrightarrow{\sim} \oplus_\pi\, m(\pi)H^i(\mathfrak{g},K_h;\pi_\infty)\otimes(\pi_f^K)_v,$$

where $(\pi_f^K)_v$ is the eigenspace in π_f^K for the volume character of $\mathcal{H}_K$. Write $\pi_f = \otimes_q'\pi_q$, where q runs over rational primes. Now if q is unramified for

K then $\pi_f^K = 0$ unless π_q is spherical; but the only spherical representation of $G(\mathbb{Q}_q)$ whose spherical subspace is an eigenspace for the (local) volume character is the *trivial* representation of $G(\mathbb{Q}_q)$. Thus $(\pi_f^K)_v = 0$ implies π_q is the trivial representation for all q that are unramified for K. It then follows from weak approximation that π is in fact the trivial representation. Thus for all i,

$$H^i({}_KS(G,X),\mathbb{C})_v \xrightarrow{\sim} H^i(\mathfrak{g},K_h;\mathbb{C}). \tag{1.21}$$

But this is equal to $H^i(\hat{X},\mathbb{C})$ by a standard calculation; see [GP02, Rmk. 16.6].

In particular, $H^i({}_KS(G,X),\mathbb{Q})_v = 0$ if i is odd. Now to prove that there is an isomorphism over $\mathbb{Q}$, it suffices to show that, for each m,

$$H^{2m}(\hat{X},\mathbb{Q})(m) \xrightarrow{\sim} H^{2m}({}_KS(G,X),\mathbb{Q})(m)_v, \tag{1.22}$$

where we write

$$H^{2m}(\hat{X},\mathbb{Q})(m) = H^{2m}(\hat{X},(2\pi i)^m\cdot\mathbb{Q}) \subset H^{2m}(\hat{X},\mathbb{C}).$$

But composing the isomorphism of Proposition 1.8 with the Chern class in cohomology, we obtain an isomorphism [Mar76], [GP02]

$$K_{\mathbb{Q}}(\mathrm{Rep}_{\overline{\mathbb{Q}}}(K_h)) \otimes_{K_{\mathbb{Q}}(\mathrm{Rep}_{\overline{\mathbb{Q}}}(G))} \mathbb{Q} \xrightarrow{\sim} H^{2m}(\hat{X},\mathbb{Q})(m). \tag{1.23}$$

Then (1.22) follows from the diagram in [GP02, Rmk. 16.6], where of course we are replacing intersection cohomology with ordinary cohomology. □

Remark 1.24. As the referee pointed out, the homomorphism

$$H^*(\hat{X},\mathbb{Q}) \to H^*({}_KS(G,X),\mathbb{Q})_v$$

can be constructed without reference to Lie algebra cohomology. More precisely, it can be described as the composition of quasiisomorphisms

$$R\Gamma_{dR}(\hat{X}) \to R\Gamma_{dR}(G_{\mathbb{C}}/K_{h,\mathbb{C}}) \leftarrow \Gamma(G_{\mathbb{C}}/K_{h,\mathbb{C}},\Omega^\bullet) \leftarrow \Gamma(G_{\mathbb{C}}/K_{h,\mathbb{C}},\Omega^\bullet)^{G_{\mathbb{C}}}$$

and the restriction map

$$\Gamma(G_{\mathbb{C}}/K_{h,\mathbb{C}},\Omega^\bullet)^{G_{\mathbb{C}}} \to \Gamma(X,\Omega^\bullet_{C^\infty})^{G(\mathbb{Q})\cap K_h(\mathbb{R})}$$

using the identification of X with $G(\mathbb{R})/K_h(\mathbb{R}) \subset G_{\mathbb{C}}/K_{h,\mathbb{C}}$. Here we have used standard notation. The fact that this defines an isomorphism, as stated in Proposition 1.20, seems to require reference to Matsushima's formula.

The proof of Proposition 1.20 gives additional information on the Galois action on ℓ-adic cohomology. The Shimura variety ${}_KS(G,X)$ and the flag variety $\hat{X}$ have canonical models over the *reflex field* $E(G,X)$. Theorem [Har85, Thm. 4.8] asserts, among other things, that the functor of (1.5) commutes with the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E(G,X))$. In addition, the Hecke correspondences $T(g)$ are all rational over $E(G,X)$. It follows that, for any prime ℓ, the ℓ-adic cohomology spaces $H^*(\hat{X}_{\bar{\mathbb{Q}}},\mathbb{Q}_\ell)$ and $H^*({}_KS(G,X)_{\bar{\mathbb{Q}}},\mathbb{Q}_\ell)_v$ carry an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E(G,X))$, where we denote by ${}_{\bar{\mathbb{Q}}}$ the base change of the models of the Shimura variety and of the flag variety over the reflex field $E(G,X)$.

Moreover, for any $\mathcal{E}\in \mathrm{Vect}_G(\hat{X})$, the i-th Chern class

$$c^i_\ell(\mathcal{E})\in H^{2i}(\hat{X}_{\bar{\mathbb{Q}}},\mathbb{Q}_\ell(i)),\ \text{resp. } c^i_\ell([\mathcal{E}]_K)\in H^{2i}({}_KS(G,X)_{\bar{\mathbb{Q}}},\mathbb{Q}_\ell(i))_v$$

generates a $\mathrm{Gal}(\overline{\mathbb{Q}}/E(G,X))$-subspace that is isotypic for the i-th power of the cyclotomic character.

Proposition 1.25. *Under the hypotheses of Proposition 1.20, the algebra isomorphism induces an algebra isomorphism*

$$\begin{gathered} H^{2*}(\hat{X}_{\bar{\mathbb{Q}}},\mathbb{Q}_\ell(*)) \xrightarrow{\sim} H^{2*}({}_KS(G,X)_{\bar{\mathbb{Q}}},\mathbb{Q}_\ell(*))_v \\ c^i_\ell(\mathcal{E})\mapsto c^i_\ell([\mathcal{E}]_K) \end{gathered}$$

in ℓ-adic cohomology over $\bar{\mathbb{Q}}$, which is equivariant for the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/E(G,X))$ *on both sides.*

Proof. Both sides are generated by Chern classes $c^i_\ell(\mathcal{E})$ and $c^i_\ell([\mathcal{E}]_K)$, and the isomorphism is defined on those Chern classes. $\square$

2. Chern classes in Deligne cohomology

We use the notations from Remarks 1.12 2). First we recall some of the basic properties of automorphic vector bundles in the image of $\mathrm{Rep}_{\overline{\mathbb{Q}}}(G)$.

Proposition 2.1. *Let $i:\mathrm{Rep}_{\overline{\mathbb{Q}}}(G)\to \mathrm{Vect}_G(\hat{X})$ be the composition of the natural inclusion $\mathrm{Rep}_{\overline{\mathbb{Q}}}(G)\to \mathrm{Rep}_{\overline{\mathbb{Q}}}(K_h)$ with the inverse of the equivalence (1.2). Let $\rho:G\to GL(V)$ be a finite-dimensional representation. Then*

- *(a) The vector bundle $[i(\rho)]$ on ${}_KS(G,X)$ has a canonical flat connection.*
- *(b) Let $[i(\rho)]^\nabla$ denote the local system on ${}_KS(G,X)$ that corresponds to the flat connection of (a). Let $Z\subset {}_KS(G,X)$ be a connected component, let $z\in Z$ be a base point, and let $r_z:\pi_1(Z,z)\to V$ be the monodromy representation attached to $[i(\rho)]^\nabla$. Then V has a model over the integers $\mathcal{O}_F$ of a number field F that is preserved by the image of r_z.*

(c) *If ρ is defined over $\mathbb{Q}$ then $[i(\rho)]$ is endowed with a canonical variation of Hodge structure, which is a direct sum of variations of pure Hodge structures.*

Proof. These points are all well-known. For (a), one can cite [Har85, Lem. 3.6]; for (c) see [Del79, Section 1.1], especially 1.1.13-1.1.17. For (b) we use Lemma 2.2, the proof of which was provided by Bruno Klingler. Indeed, the complex variety Z in (b) is the quotient of a connected component X^0 of X, which is contractible, by a congruence subgroup $\Gamma \subset G(\mathbb{Q})$, and the topological fundamental group $\pi_1(Z, z)$ can be identified with Γ. It suffices to assume (r_z, V) is irreducible. The reductive group G splits over a number field F, and by the theory of Chevalley groups, every irreducible representation of G has a model over F. Then $R_{F/\mathbb{Q}}(r_z, V)$ is a representation defined over $\mathbb{Q}$, to which Lemma 2.2 applies. The lattice L of Lemma 2.2 generates an $\mathcal{O}_F$ lattice in V which is invariant under Γ. □

Lemma 2.2. *Let G be a linear algebraic group over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{Q})$ an arithmetic subgroup. Let W be a vector space over $\mathbb{Q}$ and let $r : G \to GL(W)$ be a representation defined over $\mathbb{Q}$. Then there exists a lattice L in W such that $r(\Gamma) \subset GL(L)$.*

Proof. Let $H = r_z(G) \subset GL(W)$. By [PR94, Thm. 1.4], $r_z(\Gamma)$ is an arithmetic subgroup of H. The Lemma then follows from [PR94, Prop. 4.2], which is the special case where r is injective. □

Theorem 2.3. *Let ${}_KS(G, X)$ be a Shimura variety. Then*

$$c_{\mathcal{D}} \circ c_K^{>0}|_{\mathrm{Rep}_{\mathbb{C}}(G)} = 0.$$

Proof. Given Proposition 2.1, the theorem is a direct consequence of [CE05, Thm. 0.2], which in fact says more: the Chern-Simons invariants of the flat connection on $[\mathcal{E}]$ are torsion, for $\mathcal{E}$ coming from $\mathrm{Rep}_{\mathbb{C}}(G)$. □

However, for projective Shimura varieties, the theorem is a consequence of Lemma 1.19 and of the following proposition.

We denote by H_{dR} de Rham cohomology.

Proposition 2.4. *Let ${}_KS(G, X)$ be a projective Shimura variety. Then*

$$H^{2n}_{\mathcal{D}}({}_KS(G, X), \mathbb{Q}(n))_v \subset H^{2n}_{dR}({}_KS(G, X)).$$

Proof. Proposition 1.20 implies that $H^{2n-1}({}_KS(G,X),\mathbb{Q})_v = 0$. On the other hand, the action of $\mathcal{H}_K$ on $H^{2n}_{\mathcal{D}}({}_KS(G,X),\mathbb{Q}(n))$, via correspondences, is semi-simple. Thus it respects the exact sequence

$$0 \to H^{2n-1}({}_KS(G,X),\mathbb{C})/\big(H^{2n-1}({}_KS(G,X),\mathbb{Q}(n)) + F^n\big)$$
$$\to H^{2n}_{\mathcal{D}}({}_KS(G,X),\mathbb{Q}(n)) \to H^{2n}_{dR}({}_KS(G,X))$$

and on the left respects each of the three terms of the quotient. This finishes the proof. □

Remarks 2.5. 1) We have assumed ${}_KS(G,X)$ projective in order to apply Proposition 1.20 as stated above. For general Shimura varieties the map in Proposition 1.20 is in any case surjective – cf. [GP02] – and this suffices for the above proposition.

2) Theorem [CE05, Thm. 0.2] used in the proof of Theorem 2.3 is a variant of Reznikov's main theorem [Rez94] which also rests on the fact that some forms of odd weight do not exist.

3. Chern classes in continuous ℓ-adic cohomology

Recall the field E'_h introduced in §1. It follows from Lemma 1.4 that every ℓ-adic Chern class $c_K(W)$ belongs to $CH^*({}_KS(G,X)_{E'_h})$. Thus we can define the class $c_\ell \circ c_K(W) \in H^{2*}_{\text{cont}}({}_KS(G,X)_{E'_h}, \mathbb{Q}_\ell(*))$.

Remark 3.1. The discussion of the present section seems to depend on the choice of a CM point h. In fact, it is not difficult to see that, if we let $E(G)$ denote the extension of $\mathbb{Q}$ over which G splits, we can replace E'_h by $E(G)E(G,X)$, and Lemma 1.4 remains true. The point is that, up to twisting by a power of the canonical bundle of $\hat{X}$, every irreducible G-equivariant vector bundle on $\hat{X}$ can be obtained, by a construction that is defined over $E(G,X)$, as a canonical quotient of a G-equivariant vector bundle attached to an irreducible representation of G. This is a simple application of the theory of the highest weight, applied to K_h for variable $h \in \hat{X}$, and to G. Since for our purposes it suffices to prove the vanishing in rational continuous ℓ-adic cohomology after restriction to some finite extension, the choice of number field is immaterial.

We use the notation of Remarks 1.12 3), 4). The action of the Hecke algebra $\mathcal{H}_K$ commutes with the Galois action of $\mathrm{Gal}(\bar{\mathbb{Q}}/E(G,X))$. The Hecke algebra splits étale cohomology $H^i({}_KS(G,X)_{\mathbb{Q}}, \mathbb{Q}_\ell)$ into a sum of generalized eigenspaces which are Galois invariant. Thus $\mathcal{H}_K$ splits the filtration stemming from the Hochshild-Serre spectral sequence. In particular, one has a filtration on $H^{2n}_{\mathrm{cont}}({}_KS(G,X)_{E'_h}, \mathbb{Q}_\ell(n)))_v$ with 0-th graded quotient equal to $H^0(E'_h, H^{2n}({}_KS(G,X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n))_v)$, first graded quotient equal to $H^1(E'_h, H^{2n-1}({}_KS(G,X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n))_v)$ and last graded quotient being the subspace

$$H^2(E'_h, H^{2n-2}({}_KS(G,X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n))_v) \subset H^{2n}_{\mathrm{cont}}({}_KS(G,X)_{E'_h}, \mathbb{Q}_\ell(n)))_v.$$

The pendant on the ℓ-adic side of Proposition 2.4 is

Proposition 3.2. *Let ${}_KS(G,X)$ be a projective Shimura variety. Then*

$$\left(H^{2n}_{\mathrm{cont}}({}_KS(G,X)_{E'_h}, \mathbb{Q}_\ell(n)))/H^2(E'_h, H^{2n-2}({}_KS(G,X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n)))\right)_v \subset H^{2n}({}_KS(G,X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n)).$$

Proof. Proposition 1.25 implies that $H^{2n-1}({}_KS(G,X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n))_v = 0$. □

Theorem 3.3. *Let ${}_KS(G,X)$ be a projective Shimura variety. Then*

$$c_\ell \circ c_K^{>0}|_{\mathrm{Rep}_{\mathbb{Q}}(G)} = 0.$$

Proof. By Proposition 3.2, together with Corollary 1.19, $c_\ell \circ c_K^{>0}|_{\mathrm{Rep}_{\mathbb{Q}}(G)}$ has values in

$$H^2(E'_h, H^{2n-2}({}_KS(G,X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n))_v).$$

We now apply a variant of the proof of [Ras95, Prop. 2.3]. Let us denote by d the dimension of ${}_KS(G,X)$. Then Proposition 1.25 implies that the non-degenerate E'_h- equivariant cup-product

$$H^{2n-2}({}_KS(G,X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n)) \times H^{2d-2n+2}({}_KS(G,X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(d-n+1)) \to H^{2d}({}_KS(G,X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(d+1)) = \mathbb{Q}_\ell(1)$$

restricts to a non-degenerate E'_h-equivariant cup-product

$$H^{2n-2}({}_KS(G,X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n))_v \times H^{2d-2n+2}({}_KS(G,X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(d-n+1))_v \to H^{2d}({}_KS(G,X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(d+1))_v = H^{2d}({}_KS(G,X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(d+1))_v = \mathbb{Q}_\ell(1). \quad (3.4)$$

Write

$$h = \dim_{\mathbb{Q}_\ell} H^{2n-2}({}_K S(G,X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)_v = \dim_{\mathbb{Q}_\ell} H^{2d-2n+2}({}_K S(G,X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)_v.$$

Then (3.4) is written as

$$\oplus_1^h \mathbb{Q}(1) \times \oplus_1^h \mathbb{Q}_\ell(0) \to \mathbb{Q}_\ell(1)$$

as a non-degenerate E'_h-equivariant pairing. Indeed, by Proposition 1.25, for all $i \in \mathbb{N}$, $H^{2i}({}_K S(G,X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(i))_v$ is spanned as a $\mathbb{Q}_\ell$-vector space by the classes

$$c^i_\ell([\mathcal{E}]_{K,\bar{\mathbb{Q}}}) \in H^0(E'_h, H^{2i}({}_K S(G,X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(i))),$$

where $\mathcal{E}$ comes from a K_h-representation, so is algebraic. This implies that the pairing of $\mathbb{Q}_\ell$-vector spaces

$$H^2(E'_h, \oplus_1^h \mathbb{Q}_\ell(1)) \times H^0(E'_h, \oplus_1^h \mathbb{Q}_\ell(0)) \to H^2(E'_h, \mathbb{Q}_\ell(1)) \tag{3.5}$$

is non-degenerate.

On the other hand, for $W \in \mathrm{Rep}_{\bar{\mathbb{Q}}}(G)$, and $E \in \mathrm{Rep}(K_h)$ defining $[\mathcal{E}]_K$, one has

$$c^n_K(W) \cup ch^{d-n+1}([\mathcal{E}]_K) \in CH^{d+1}({}_K S(G,X))_{\mathbb{Q}} = 0.$$

(Recall here c_K is defined in Conjecture 1.10.) Thus

$$0 = c_\ell^{d+1}(c^n_K(W) \cup ch^{d-n+1}([\mathcal{E}]_K)) = \\ c^n_\ell(c^n_K(W)) \cup c_\ell^{d-n+1}([\mathcal{E}]_{K,\bar{\mathbb{Q}}}) \in H^2(E'_h, \mathbb{Q}_\ell(1)).$$

Applying (3.5) we conclude $c^n_K(W) = 0$. This finishes the proof. ☐

Remark 3.6 (Syntomic cohomology)**.** In addition to Deligne and continuous ℓ-adic cohomology, it is natural to consider syntomic cohomology as defined by Fontaine, Kato and Messing. Let us denote by $E'_{\mathfrak{p}}$ the p-adic completion of the number field E'_h at a place $\mathfrak{p}$. We assume that ${}_K S(G,X)_{E'_{\mathfrak{p}}}$ is proper and has a semi-stable model. Then with p-power torsion coefficients, the syntomic cohomology group $H^{2n}_{\mathrm{synt}}({}_K S(G,X)_{E'_{\mathfrak{p}}}, \mathbb{Z}/p^n\mathbb{Z}(n))$ is defined as the étale cohomology group of the $\tau_{\leq n}$-truncation of the vanishing cycle complex [KM92, Thm. 2.2]. This isomorphism lifts to continuous p-adic coefficients [NN16, Proof of Cor. 4.5] yielding a Hochshild-Serre spectral sequence

$$E_2^{st} = H^s_{st}(E'_{\mathfrak{p}}, H^t({}_K S(G,X)_{\bar{\mathbb{Q}}_{\mathfrak{p}}}, \mathbb{Q}_p(n)) \implies H^{s+t}_{\mathrm{synt}}({}_K S(G,X)_{E'_{\mathfrak{p}}}, \mathbb{Q}(n))$$

where *st* stands for ‘stable’, compatible with the one on continuous p-adic cohomology [NN16, Thm. 4.8]. Thus the same proof as in Theorem 3.3 yields the same result, with c_ℓ replaced by the syntomic Chern classes ([NN16, Section 5]).

Remark 3.7 (Construction of torsion classes in cohomology)**.** One motivation for studying the Chern classes of automorphic vector bundles is the hope that they might provide a way to construct interesting torsion classes in the Chow group, or that their ℓ-adic Abel-Jacobi classes in $H^1(E_h', H^{2n-1}({}_KS(G, X)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n))_v)$ might be torsion. It follows easily from (1.21) and (1.23) that any class in $K_0(\mathrm{Vect}_G(\hat{X})) \simeq K_0(\mathrm{Rep}_{\mathbb{Q}}(K_h))$ whose image in the Chow group $CH({}_KS(G,X))$ under all Chern classes of positive degree is torsion, must necessarily belong to the ideal generated by the kernel of the augmentation map $K_0(\mathrm{Rep}_{\mathbb{Q}}(G)) \to \mathbb{Z}$. Any torsion classes arising in this way would naturally be eigenvectors for the volume character of the Hecke algebra; in particular, the associated cohomology classes are *Eisenstein* classes, in the usual sense.

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