A Lefschetz theorem for overconvergent isocrystals with Frobenius structure

Tomoyuki Abe* and Hélène Esnault†

Abstract

We show a Lefschetz theorem for irreducible overconvergent F-isocrystals on smooth varieties defined over a finite field. We derive several consequences from it.

Introduction

Let X_0 be a normal geometrically connected scheme of finite type defined over a finite field $\mathbb{F}_q$, let $\mathcal{F}_0$ be an irreducible lisse Weil $\overline{\mathbb{Q}}_\ell$-sheaf with finite determinant (thus in fact $\mathcal{F}_0$ is an étale sheaf as well), where $\ell \neq p = \mathrm{char}(\mathbb{F}_q)$. In Weil II [Del80, Conj. 1.2.10], Deligne conjectured the following.

(i) The sheaf $\mathcal{F}_0$ is of weight 0.

(ii) There is a number field $E \subset \overline{\mathbb{Q}}_\ell$ such that for any $n > 0$ and $x \in X_0(\mathbb{F}_{q^n})$, the characteristic polynomial $f_x(\mathcal{F}_0, t) := \det(1 - tF_x \mid \mathcal{F}_{0,\overline{x}}) \in E[t]$, where F_x is the geometric Frobenius of x.

(iii) For any $\ell' \neq p$ and any embedding $\sigma\colon E \hookrightarrow \overline{\mathbb{Q}}_{\ell'}$, for any $n > 0$ and $x \in X_0(\mathbb{F}_{q^n})$, any root of $\sigma f_x(\mathcal{F}_0, t)$ is an ℓ'-adic unit.

(iv) For any σ as in (iii), there is an irreducible $\overline{\mathbb{Q}}_{\ell'}$-lisse sheaf $\mathcal{F}_{0,\sigma}$, called the σ-companion, such that $\sigma f_x(\mathcal{F}_0, t) = f_x(\mathcal{F}_{0,\sigma}, t)$.

(v) There is a crystalline version of (iv).

Deligne's conjectures (i)–(iv) have been proved by Lafforgue [Laf02, Thm. VII.6] when X_0 is a smooth curve, as a corollary of the Langlands correspondence, which is proven showing that automorphic forms are in some sense motivic.

When X_0 has dimension at least 2, the automorphic side on which one could rely to prove Deligne's conjectures is not available: there is no theory of automorphic forms in higher dimension. The problem then becomes how to reduce, by geometry, the statements to dimension 1. For (i) and (iii), one proves a Lefschetz theorem (see [Dri12, Thm. 2.15], [Del12, 1.5–1.9], [EK12, B1]):

*Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, Chiba, 277-8583, Japan; supported by Grant-in-Aid for Young Scientists (A) 16H05993.

†Freie Universität Berlin, Arnimallee 3, 14195, Berlin, Germany; supported by the Einstein program.
Mathematics Subject Classification (2010): 14F10, 14D20.

0.1 Theorem. — *On X_0 smooth, for any closed point x_0, there a smooth curve C_0 and a morphism $C_0 \to X_0$ such that $x_0 \to X_0$ lifts to $x_0 \to C_0$, and such that the restriction of $\mathcal{F}_0$ to C_0 remains irreducible.*

Using Theorem 0.1, Deligne proved (ii) ([Del12, Thm. 3.1]), and Drinfeld, using (ii), proved (iv) in ([Dri12, Thm. 1.1]), assuming in addition X_0 to be smooth. In particular Drinfeld proved in [Dri12, Thm. 2.5] the following key theorem.

0.2 Theorem. — *If X_0 is smooth, given a number field $E \subset \overline{\mathbb{Q}}_\ell$, and a place λ of E dividing ℓ, a collection of polynomials $f_x(t) \in E[t]$ indexed by any $n > 0$ and $x \in X_0(\mathbb{F}_{q^n})$, such that the following two conditions are satisfied:*

(i) for any smooth curve C_0 with a morphism $C_0 \to X_0$ and any $n > 0$ and $x \in C_0(\mathbb{F}_{q^n})$, there exists a lisse étale $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}_0^{C_0}$ on C_0 with monodromy in $\mathrm{GL}(r, E_\lambda)$ such that $f_x(t) = f_x(\mathcal{F}_0^{C_0}, t)$, where E_λ is the completion of E with respect to the place λ;

(ii) there exists a finite étale cover $X_0' \to X_0$ such that $\mathcal{F}_0^{C_0}$ is tame on all C_0 factoring through $X_0' \to X_0$.

Then there exists a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}_0$ on X_0 with monodromy in $\mathrm{GL}(r, E_\lambda)$, such that for any $n > 0$ and $x \in X_0(\mathbb{F}_{q^n})$, $f_x(t) = f_x(\mathcal{F}_0, t) \in E[t]$.

Here the notation E_λ means the completion of E at the place λ. Further, to realize the assumptions of Theorem 0.2 in order to show the existence of $\mathcal{F}_{0,\sigma}$, Drinfeld uses Theorem 0.1 in [Dri12, 4.1]. He constructs step by step the residual representations with monodromy in $\mathrm{GL}(r, \mathcal{O}_{E_\lambda}/\mathfrak{m}^n)$ for n growing, where $\mathcal{O}_{E_\lambda}$ is the ring of integers of E_λ and $\mathfrak{m}$ is its maximal ideal.

The formulation of (v) has been made explicit by Crew [Cre92, 4.13]. The conjecture is that the crystalline category analogous to the category of Weil $\overline{\mathbb{Q}}_\ell$-sheaves is the category of overconvergent F-isocrystals (see Section 1.1 for the definitions). In order to emphasize the analogy between ℓ and p, one slightly reformulates the definition of companions. One replaces σ in (iii) by an isomorphism $\sigma\colon \overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_{\ell'}$ (see [EK12, Thm. 4.4]), and keeps (iv) as it is. Here ℓ, ℓ' are any two prime numbers. For $\ell' = p$, $\ell \neq p$, and $\mathcal{F}$ an irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf, one requests the existence of an overconvergent F-isocrystal M_0 on X_0 with eigenpolynomial $f_x(M_0, t)$ such that $f_x(M_0, t) = \sigma f_x(\mathcal{F}, t) \in \sigma(E)[t]$ for any $n > 0$ and $x \in X_0(\mathbb{F}_{q^n})$, where $f_x(\mathcal{F}, t)$ is the characteristic polynomial of the geometric Frobenius at x on $\mathcal{F}$ (see Section 1.4 for the definitions). The isocrystal M_0 is called a *σ-companion to $\mathcal{F}$*. Given an irreducible overconvergent F-isocrystal M_0 with finite determinant on X_0, and σ as above, a lisse ℓ-adic Weil sheaf $\mathcal{F}$ on X_0 is a σ^{-1}-companion if $\sigma^{-1} f_x(M_0, t) = f_x(\mathcal{F}, t) \in E[t]$ at $x \in X_0(\mathbb{F}_{q^n})$ (see Definition 1.4). Similarly we can assume $p = \ell = \ell'$. This way we can talk on ℓ-adic or p-adic companions of either an M_0 or an $\mathcal{F}$. The companion correspondence should preserve the notions of irreducibility, finiteness of the determinant, the eigenpolynomials at closed points of X_0, and the ramification.

The conjecture in the strong form has been proven by the first author when X_0 is a smooth curve ([Abe13, Intro. Thm.]). The aim of this article is to prove the following analog of Theorem 0.1 on X smooth.

0.3 Theorem (Theorem 3.10)**.** — *Let X_0 be a smooth geometrically connected scheme over $\mathbb{F}_q$. Let M_0 be an irreducible overconvergent F-isocrystal with finite determinant.*

Then for every closed point $x_0 \to X_0$, there exists a smooth irreducible curve C_0 defined over k, together with a morphism $C_0 \to X_0$ and a factorization $x_0 \to C_0 \to X_0$, such that the pull-back of M_0 to C_0 is irreducible.

Theorem 0.3, together with [Del12, Rmk. 3.10], footnote 2, and [Abe13, Thm. 4.2.2] enable one to conclude that there is a number field $E \subset \overline{\mathbb{Q}}_p$ such that for any $n > 0$ and $x \in X_0(\mathbb{F}_{q^n})$, $f_x(M_0, t) \in E[t]$ (see Lemma 4.1). This yields the p-adic analog of (i) over a smooth variety X_0. (See Section 4.6 when X_0 is normal). Then Theorem 0.2 implies the existence of ℓ-adic companions to a given irreducible overconvergent F-isocrystal M_0 with finite determinant (see Theorem 4.2). We point out that the existence of ℓ-adic companions has already been proven by Kedlaya in [Ked16, Thm. 5.3] in a different way, using weights (see [Ked16, §4, Intro.]), however not their irreducibility. The Lefschetz theorem 3.10 implies that the companion correspondence preserves irreducibility.

Theorem 0.3 has other consequences (see Section 4), aside of the existence already mentioned of ℓ-adic companions. Deligne's finiteness theorem [EK12, Thm. 1.1] transposes to the crystalline side (see Corollary 4.3): on X_0 smooth, there are finitely many isomorphism classes of irreducible overconvergent F-isocrystals in bounded rank and bounded ramification, up to twist by a character of the finite field. One can also kill the ramification of an F-overconvergent isocrystal by a finite étale cover in Kedlaya's semistability reduction theorem (Remark 4.4).

We now explain the method of proof of Theorem 0.3. We replace M_0 by the full Tannakian subcategory $\langle M \rangle$ of the category of overconvergent F-isocrystals spanned by M over the algebraic closure $\overline{\mathbb{F}}_q$ (we drop the lower indices $_0$ to indicate this, see Section 1.1 for the definitions). We slightly improve the theorem ([DM82, Prop. 2.21 (a), Rmk. 2.29]) describing the surjectivity of an homomorphism of Tannaka groups in categorical terms in Lemma 1.6: the restriction functor $\langle M \rangle \to \langle M|_C \rangle$ to a curve $C \to X$ is an equivalence when it is fully faithful and any F-overconvergent isocrystal of rank 1 on C is torsion. Class field theory for F-overconvergent isocrystals ([Abe15, Lem. 6.1][(1)]) implies the torsion property. As for full faithfulness, the problem is of cohomological nature, one has to compute that the restriction homomorphism $H^0(X, N) \to H^0(C, N|_C)$ is an isomorphism for all objects N in $\langle M \rangle$. In the tame case, this is performed in Section 2 using the techniques developed in [AC13]. As a corollary, ℓ-adic companions exist in the tame case (see Proposition 2.8). In the wild case, Kedlaya's semistability reduction theorem asserts the existence of a good alteration $h\colon X_0'' \to X_0$ such that h^+M_0 becomes tame. One considers the ℓ-adic companion $\mathcal{F}_0$ of h^+M_0 and a finite étale cover $g\colon X_0' \to X_0''$ which is such that curves $C_0 \to X_0$ with non-disconnected pull-back $C_0 \times_{X_0} X_0'$ have the property that $C_0 \times_{X_0} X_0''$ preserves the irreducible constituents of $\mathcal{F}_0$ (see Lemma 3.9). It remains then to show that the dimensions of H^0 of M and of $\mathcal{F}$, which are the pull backs of M_0 and $\mathcal{F}_0$ over $\bar{\mathbb{F}}_q$, are the same (see Lemma 3.2).

Acknowledgments: The first author thanks Kiran S. Kedlaya for various discussions without which this article would not have existed. He thanks Atsushi Shiho for telling him the reference [Tsu12]. He is also grateful to Valentina Di Proietto for discussions, encouragements, and support while he was visiting Freie Universität. The second author thanks Moritz Kerz for the discussions while writing [EK12], which enabled her to better understand Deligne's ideas, Atsushi Shiho with whom she discussed various versions of

(1) See Remark 4.6 for a correction of a mistake in this lemma.

Lefschetz theorems for unit-root isocrystals, and Pierre Deligne who promptly answered a question concerning [Del12, Rmk. 3.10] (see Remark 1.5). We also thank the referee whose questions prompted us to write Section 4.6 and to make sure in Theorem 0.3 that one can assume the closed points to be anywhere on the variety.

Notations and conventions

Let $q = p^s$, and let k be a field with q elements. We fix once for all an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, the residue field $\overline{k}$ of $\overline{\mathbb{Q}}_p$ and an embedding $k \hookrightarrow \overline{k}$. For an integer $n \geq 0$, let k_n be the finite extension of k in $\overline{k}$ of degree $n+1$. By a curve we mean an irreducible scheme of finite type over k which is of dimension 1.

1. Generalities

1.1. Let us start with recalling basic concepts of p-adic coefficients used in [Abe13]. To be able to speak about the p-adic cohomology, we need to fix some data, called *base tuple* (cf. [Abe13, 1.4.10]).

A *geometric base tuple* is a set $\mathfrak{T}_\emptyset := (\kappa, R, K, L)$ where R is a complete discrete valuation ring, κ is the residue field, K is the field of fractions, and L is an algebraic extension of K. An *arithmetic base tuple* is a set $\mathfrak{T}_F := (\kappa, R, K, L, a, \sigma)$ where a is an integer and σ is an automorphism of L such that $\sigma|_K$, which is assumed to be an automorphism of K, is a lifting of the a-th absolute Frobenius automorphism on κ, and there exists a sequence of finite extensions K_n of K in L such that $\sigma(K_n) \subset K_n$. Let Z be a scheme of finite type over κ. Given these data, we defined the categories $\mathrm{Isoc}^\dagger(Z/\mathfrak{T}_\emptyset)$ and $\mathrm{Isoc}^\dagger(Z/\mathfrak{T}_F)$ in [Abe13, 1.4.11, 2.4.14].

Now, let k' be a finite extension of k in $\overline{k}$ of cardinality $p^{s'}$. In this article, we only consider particular base tuples $\mathfrak{T}_{k',\emptyset} := (k', R = W(k'), K = \mathrm{Frac}(R), \overline{\mathbb{Q}}_p)$ and $\mathfrak{T}_{k',F} := (k', R = W(k'), K = \mathrm{Frac}(R), \overline{\mathbb{Q}}_p, s', \mathrm{id})$. Let X_0 be a smooth scheme of finite type over k'. We recall now the definitions of $\mathrm{Isoc}^\dagger(X_0/\mathfrak{T}_{k',\emptyset})$ and $\mathrm{Isoc}^\dagger(X_0/\mathfrak{T}_{k',F})$.

In [Ber96, 2.3.6], Berthelot defined the category of overconvergent isocrystals, which we denote by $\mathrm{Isoc}^\dagger_{\mathrm{Ber}}(X_0/K)$. We extend the scalars from K to $\overline{\mathbb{Q}}_p$ in the following way analogous to [Abe13, 1.4] or [AM, 7.3]. Let L be a finite field extension of K in $\overline{\mathbb{Q}}_p$. Then an L-isocrystal is a pair (M, λ) where $M \in \mathrm{Isoc}^\dagger_{\mathrm{Ber}}(X_0/K)$, and $\lambda\colon L \to \mathrm{End}_{\mathrm{Isoc}^\dagger_{\mathrm{Ber}}(X_0/K)}(M)$ is a ring homomorphism which is called the *L-structure*. Homomorphisms between L-isocrystals are homomorphisms of isocrystals which are compatible with the L-structure. The category of L-isocrystals is denoted by $\mathrm{Isoc}^\dagger_{\mathrm{Ber}}(X_0/K) \otimes L$. Finally, taking the 2-inductive limit over all such L, we obtain $\mathrm{Isoc}^\dagger_{\mathrm{Ber}}(X_0/K) \otimes \overline{\mathbb{Q}}_p$. The "cohomology" of an object of the category, called the *rigid cohomology*, does not have suitable finiteness property in general. To be able to acquire this, we need, in addition, to define a "Frobenius structure".

Let $F\colon X_0 \to X_0$ be the s'-th Frobenius endomorphism of X_0, which is an endomorphism over k'. Recall[(2)] that we have the endofunctor F^+ on $\mathrm{Isoc}^\dagger_{\mathrm{Ber}}(X_0/K) \otimes \overline{\mathbb{Q}}_p$, which

[(2)] The functor F^+ is the same as the more familiar notation F^* in [Ber96] (cf. [Abe13, Rem 1.1.3]). Since our treatment of isocrystals is from the viewpoint of $\mathscr{D}$-modules, we borrow the notations from this theory.

is in fact an auto-equivalence by [Abe13, Lemma 1.1.3]. For an integer $n > 0$ and $M \in \mathrm{Isoc}^{\dagger}_{\mathrm{Ber}}(X_0/K)\otimes\overline{\mathbb{Q}}_p$, an *n-th Frobenius structure* is an isomorphism $\Phi\colon F^{n+}M \xrightarrow{\sim} M$. The category $\mathrm{Isoc}^{\dagger}(X_0/\mathfrak{T}_{k',F})$ is the category of pairs (M,Φ) where $M \in \mathrm{Isoc}^{\dagger}_{\mathrm{Ber}}(X_0/K)\otimes\overline{\mathbb{Q}}_p$, Φ is the 1-st Frobenius structure, and the homomorphisms are the ones compatible with Φ in an obvious manner.

We also consider the full subcategory $\mathrm{Isoc}^{\dagger}(X_0/\mathfrak{T}_{k',\emptyset})$ of $\mathrm{Isoc}^{\dagger}_{\mathrm{Ber}}(X_0/K)\otimes\overline{\mathbb{Q}}_p$ consisting of objects M such that for any constituent N of M, there exists $i > 0$ such that N can be endowed with i-th Frobenius structure. We recall the following result. Note that the objects of $\mathrm{Isoc}^{\dagger}(X_0/\mathfrak{T}_{k',F})$ are endowed with Frobenius structure as a part of data, whereas the objects of $\mathrm{Isoc}^{\dagger}(X_0/\mathfrak{T}_{k',\emptyset})$ are not.

Lemma ([Abe13, Lem. 1.4.11, Cor. 1.4.11])**.** — *Let k' be a finite extension of k, X_0 be a smooth scheme over k', and $X_{0,k'} := X_0 \otimes_k k'$, then we have canonical equivalences*

$$\mathrm{Isoc}^{\dagger}(X_0/\mathfrak{T}_{k,F}) \cong \mathrm{Isoc}^{\dagger}(X_0/\mathfrak{T}_{k',F}), \qquad \mathrm{Isoc}^{\dagger}(X_0/\mathfrak{T}_{k,\emptyset}) \cong \mathrm{Isoc}^{\dagger}(X_{0,k'}/\mathfrak{T}_{k'\emptyset}).$$

This lemma shows that the category $\mathrm{Isoc}^{\dagger}(X_0/\mathfrak{T}_{k',F})$ does not depend on the choice of the base field, and allows us to denote it simply by $\mathrm{Isoc}^{\dagger}(X_0)$. If X is a scheme of finite type over $\overline{k}$, there exists a scheme $X_{0,k'}$ of finite type over k' such that $X_{0,k'} \otimes_{k'} \overline{k} \cong X$. Then the lemma also tells us that $\mathrm{Isoc}^{\dagger}(X_{0,k'}/\mathfrak{T}_{k',\emptyset})$ does not depend on auxiliary choices, and may be denoted by $\mathrm{Isoc}^{\dagger}(X)$. We put $X_n := X_0 \otimes_k k_n$ and $X := X_0 \otimes_k \overline{k}$. As a convention, we put subscripts $\cdot_n$ for isocrystals on X_n. Let $M_n \in \mathrm{Isoc}^{\dagger}(X_n)$. Then the pull-back of M_n to $X_{n'}$ for $n' \geq n$ is denoted by $M_{n'}$, and the pull-back to $\mathrm{Isoc}^{\dagger}(X)$ is denoted by M.

For example, when $X_0 = \mathrm{Spec}(k)$, then $\mathrm{Isoc}^{\dagger}_{\mathrm{Ber}}(X_0/K)$ is the category of finite dimensional K-vector spaces. This implies that $\mathrm{Isoc}^{\dagger}(X_n)$ is the category of finite dimensional $\overline{\mathbb{Q}}_p$-vector spaces with an automorphism, and $\mathrm{Isoc}^{\dagger}(X)$ is the category of finite dimensional $\overline{\mathbb{Q}}_p$-vector spaces. The pull-back functor $\mathrm{Isoc}^{\dagger}(X_0) \to \mathrm{Isoc}^{\dagger}(X_n)$ sends an object (V,ϕ) to (V,ϕ^n), where V is a vector space and ϕ is its automorphism. This can be checked easily from the definition.

Remark. — (i) The category $\mathrm{Isoc}^{\dagger}(X_n)$ has another description by [Abe13, Lem. 1.4.11 (ii)]: it is equivalent to the category of isocrystals with n-th Frobenius structure in $\mathrm{Isoc}^{\dagger}_{\mathrm{Ber}}(X_0/K)\otimes\overline{\mathbb{Q}}_p$.

(ii) We caution that the notations $\mathrm{Isoc}^{\dagger}(X_n)$ and $\mathrm{Isoc}^{\dagger}(X)$ are not standard. Usually, the notations $\mathrm{Isoc}^{\dagger}_{\mathrm{Ber}}(X_n/K)$ and $\mathrm{Isoc}^{\dagger}_{\mathrm{Ber}}(X/\mathrm{Frac}(W(\overline{k})))$ are used. We also note that the categories $\mathrm{Isoc}^{\dagger}(X)$ and $\mathrm{Isoc}^{\dagger}_{\mathrm{Ber}}(X/\mathrm{Frac}(W(\overline{k})))$ differ; the former is actually the category of isocrystals on X_0 with some additional structures, but the latter is that on X.

(iii) Philosophically, the category $\mathrm{Isoc}^{\dagger}(X_n)$ is a p-adic analogue of the category of lisse (Weil) $\overline{\mathbb{Q}}_\ell$-sheaves on X_n. The category $\mathrm{Isoc}^{\dagger}(X)$ is an analogue of the category of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on X which can be written as an extension of lisse sheaves which descends to X_n for some n.

1.2. For later use, we fix the terminology of tameness.

Definition. — Let X_0 be a smooth curve, and $\overline{X}_0$ be the smooth compactification of X_0. An isocrystal $M \in \mathrm{Isoc}^{\dagger}(X)$ is said to be *tame* if it is log-extendable along the boundary $\overline{X}_0 \setminus X_0$, cf. [Shi10, §1]. For a general scheme X_0 of finite type over k, $M \in \mathrm{Isoc}^{\dagger}(X)$ is said to be *tame* if for any smooth curve C_0 and any morphism $\varphi\colon C_0 \to X_0$, the pull-back $\varphi^+(M)$ is tame. We say $M_0 \in \mathrm{Isoc}^\dagger(X_0)$ is *tame* if M is tame.

Remark. — Let X_0 be a smooth scheme which admits a smooth compactification whose divisor at infinity has strict normal crossings. Then one has the curve criterion: $M \in \mathrm{Isoc}^\dagger(X)$ is tame if and only if M is log-extendable along the divisor at infinity. The "if" part is easy to check, and the "only if" part is a consequence of [Shi11, Thm. 0.1]. We need to be careful as Shiho is assuming the base field to be uncountable. However, in our situation, because of the presence of Frobenius structure, this assumption is not needed as explained in [Abe13, footnote (4) of 2.4.13].

1.3. In this note, we freely use the formalism of arithmetic $\mathscr{D}$-modules developed in [Abe13]. For a separated scheme of finite type X_0 over k, in *ibid.*, the triangulated category $D^{\mathrm{b}}_{\mathrm{hol}}(X_0/\mathfrak{T}_{k,\emptyset})$ (resp. $D^{\mathrm{b}}_{\mathrm{hol}}(X_0/\mathfrak{T}_{k,F})$) with t-structure is defined. Define $X := X_0 \otimes_k \overline{k}$ as usual. To harmonize with the notation of the category of isocrystals in this article, this category is denoted by $D^{\mathrm{b}}_{\mathrm{hol}}(X)$ (resp. $D^{\mathrm{b}}_{\mathrm{hol}}(X_0)$). A justification for dropping the base from the notation goes exactly in the same manner as for $\mathrm{Isoc}^\dagger$. Its heart is denoted by $\mathrm{Hol}(X)$ (resp. $\mathrm{Hol}(X_0)$), see [Abe13, 1.1]. The cohomology functor for this t-structure is denoted by $\mathscr{H}^*$. When X (resp. X_0) is smooth, $\mathrm{Isoc}^\dagger(X)$ (resp. $\mathrm{Isoc}^\dagger(X_0)$) is fully faithfully embedded into $\mathrm{Hol}(X)[-d]$ (resp. $\mathrm{Hol}(X_0)[-d]$) where d is the dimension of X (resp. X_0), and we identify $\mathrm{Isoc}^\dagger(X)$ (resp. $\mathrm{Isoc}^\dagger(X_0)$) with its essential image in $\mathrm{Hol}(X)[-d]$ (resp. $\mathrm{Hol}(X_0)[-d]$). Let $\epsilon\colon X_0 \to \mathrm{Spec}(k)$ be the structural morphism. Recall the functor $\epsilon_+\colon D^{\mathrm{b}}_{\mathrm{hol}}(X_0/\mathfrak{T}_{k,\emptyset}) \to D^{\mathrm{b}}_{\mathrm{hol}}(\mathrm{Spec}(k)/\mathfrak{T}_{k,\emptyset})$ defined in [Abe13, 2.3.10]. For $M \in \mathrm{Isoc}^\dagger(X_0/\mathfrak{T}_{k,\emptyset})$ (or more generally in $D^{\mathrm{b}}_{\mathrm{hol}}(X_0/\mathfrak{T}_{k,\emptyset})$), we set

$$H^i(X_0/\mathfrak{T}_{k,\emptyset}, M) := H^i(\epsilon_+ M)$$

which is a finite dimensional $\overline{\mathbb{Q}}_p$-vector space. Let X be a scheme of finite type over $\overline{k}$ and $M \in \mathrm{Isoc}^\dagger(X)$. We may take a scheme of finite type X_n over k_n such that $X_n \otimes_{k_n} \overline{k} = X$. Then $H^i(X_n/\mathfrak{T}_{k_n,\emptyset}, M)$ only depends on X. Indeed, by [Abe13, Cor. 1.4.11], we have an equivalence $D^{\mathrm{b}}_{\mathrm{hol}}(X_0/\mathfrak{T}_{k,\emptyset}) \cong D^{\mathrm{b}}_{\mathrm{hol}}(X_0/\mathfrak{T}_{k',\emptyset})$, which is compatible with push-forwards, and the claim for $X = \mathrm{Spec}(k)$ is easy to check. Thus, we may denote $H^i(X_n/\mathfrak{T}_{k_n,\emptyset}, M)$ simply by $H^i(X, M)$.

1.4. Let X_0 be a smooth scheme over k. Let $x \in X_0(k')$ for some finite extension k' of k in $\overline{k}$, and $i_x\colon \mathrm{Spec}(k') \to X_0$ be the corresponding morphism. We have the pull-back functor $i_x^+\colon \mathrm{Isoc}^\dagger(X_0) \to \mathrm{Isoc}^\dagger(\mathrm{Spec}(k'))$. The category $\mathrm{Isoc}^\dagger(\mathrm{Spec}(k'))$ is equivalent to the category of finite dimensional $\overline{\mathbb{Q}}_p$-vector spaces V endowed with an automorphism Φ of V. Given $M_0 \in \mathrm{Isoc}^\dagger(X_0)$, we denote by $f_x(M_0, t) \in \overline{\mathbb{Q}}_p[t]$ the eigenpolynomial

$$f_x(M_0, t) = \det\bigl(1 - t\Phi_x \mid i_x^+(M_0)\bigr) \in \overline{\mathbb{Q}}_p[t]$$

of the automorphism Φ_x of $i_x^+(M_0)$. Similarly, for a lisse Weil $\overline{\mathbb{Q}}_\ell$-sheaf ${}_\ell M_0$ on X_0, we denote the characteristic polynomial of the geometric Frobenius F_x at x by $f_x({}_\ell M_0, t)$:

$$f_x({}_\ell M_0, t) = \det\bigl(1 - tF_x \mid {}_\ell M_{0,\overline{x}}\bigr) \in \overline{\mathbb{Q}}_\ell[t],$$

where $\overline{x}$ is a $\overline{k}$-point above x.

Definition. — Let $M_0 \in \mathrm{Isoc}^\dagger(X_0)$.

1. Let $\iota\colon \overline{\mathbb{Q}}_p \to \mathbb{C}$ be a field isomorphism. The isocrystal M_0 is said to be *ι-pure of weight $w \in \mathbb{C}$* if the absolute value of any root of $f_x(M_0, t) \in \overline{\mathbb{Q}}_p[t] \xrightarrow[\iota]{\sim} \mathbb{C}[t]$ equals to $q_x^{w/2}$ for any $x \in X_0(k')$ with residue field of cardinality q_x. The isocrystal M_0 is said to be *ι-pure* if it is ι-pure of weight w for some w.

2. The isocrystal M_0 is said to be *algebraic* if $f_x(M_0, t) \in \overline{\mathbb{Q}}[t] \subset \overline{\mathbb{Q}}_p[t]$ for any $x \in X_0(k')$.

3. Given an isomorphism $\sigma\colon \overline{\mathbb{Q}}_p \to \overline{\mathbb{Q}}_\ell$ for a prime $\ell \neq p$, the isocrystal M_0 is said to be *σ-unit-root* if any root of $\sigma f_x(M_0, t)$ is an ℓ-adic unit for any $x \in X_0(k')$.

4. Given an automorphism σ of $\overline{\mathbb{Q}}_p$, a *σ-companion to M_0* is an $M_0^\sigma \in \mathrm{Isoc}^\dagger(X_0)$ such that $\sigma f_x(M_0, t) = f_x(M_0^\sigma, t)$ for any $x \in X_0(k')$. One says that M_0^σ is a *p-companion of M_0*.

5. Given an isomorphism $\sigma\colon \overline{\mathbb{Q}}_p \to \overline{\mathbb{Q}}_\ell$ for a prime $\ell \neq p$, a *σ-companion of M_0* is a lisse Weil $\overline{\mathbb{Q}}_\ell$-sheaf ${}_\ell M_0^\sigma$ on X_0 such that $\sigma f_x(M_0, t) = f_x({}_\ell M_0^\sigma, t)$ for any $x \in X_0(k')$. One says that ${}_\ell M_0^\sigma$ is an *ℓ-companion of M_0*. Abusing notations, we also write ${}_\ell M_0$ for an ℓ-companion.

1.5. We recall the following theorem by Deligne.

Theorem ([Del12, Prop. 1.9 + Rmk. 3.10]). — *Let X_0 be a connected scheme of finite type over k. Assume given a function $t_n\colon X_0(k_n) \to \overline{\mathbb{Q}}_\ell[t]$ for all $n \geq 0$ such that*

> (*) *for any morphism $\varphi\colon C_0 \to X_0$ from a smooth curve C_0, there exists a lisse Weil $\overline{\mathbb{Q}}_\ell$-sheaf ${}_\ell M[\varphi]$ on C_0 such that for any n and $x \in C_0(k_n)$, we have*
>
> $$f_x({}_\ell M[\varphi], t) = t_n(\varphi(x)).$$

(i) Assume there exists $x \in X_0(k_n)$ such that $t_n(x) \in \overline{\mathbb{Q}}[t] \subset \overline{\mathbb{Q}}_\ell[t]$ (resp. any root of $t_n(x) = 0$ is ℓ-adic unit). Then $t_n(x) \in \overline{\mathbb{Q}}[t] \subset \overline{\mathbb{Q}}_\ell[t]$ for any $x \in X_0(k_n)$ (resp. any root of $t_n(x) = 0$ is ℓ-adic unit for any $x \in X_0(k_n)$).

(ii) Assume that there exists a finite étale cover $X' \to X$ such that for any φ as in (*) above, the pull-back of ${}_\ell M[\varphi]$ to $C' = X' \times_X C_0$ is tamely ramified. If in addition, there exists $x \in X_0(k_n)$ such that $t_n(x) \in \overline{\mathbb{Q}}[t] \subset \overline{\mathbb{Q}}_\ell[t]$, namely the assumption of (i) holds, then there exists a number field E in $\overline{\mathbb{Q}}_\ell$ such that t_n takes value in $E[t]$ for any n.

Remark. — 1) Deligne assumes in (*) the $\overline{\mathbb{Q}}_\ell$-sheaf to be lisse. However, in an email to the authors, he pointed out that it is enough to assume the $\overline{\mathbb{Q}}_\ell$-sheaf to be a Weil sheaf, without changing a single word of his proof.

2) In [Del12, Prop. 1.9], the assertion is formulated slightly differently: if there exists one closed point $x \in X_0(k_n)$ such that any root of $t_n(x)$ is a Weil number of weight 0, then the same property holds for any point of X_0. Deligne's argument shows that if $t_n(x)$ is algebraic (resp. any root of $t_n(x) = 0$ is ℓ-adic unit) at one closed point x, then it is algebraic (resp. any root is ℓ-adic unit) at all closed points.

1.6. Let $(\mathcal{T}, \omega)$ be a neutral Tannakian category where $\mathcal{T}$ is a Tannakian category, and ω is a fiber functor. We denote by $\pi_1(\mathcal{T}, \omega)$ the associated fundamental group. We shall use the following lemma on Tannakian categories to show our Lefschetz theorem 3.10.

Lemma. — *Let $\Phi\colon (\mathcal{T}, \omega = \omega' \circ \Phi) \to (\mathcal{T}', \omega')$ be a tensor functor between neutral Tannakian categories over $\overline{\mathbb{Q}}_p$ (or any field of characteristic 0). If*

> $(\star)$ *for any rank 1 object $L \in \mathcal{T}'$, there exists an integer $m > 0$ such that $L^{\otimes m}$ is in the essential image of Φ,*

then the induced functor $\Phi^*\colon \pi_1(\mathcal{T}', \omega') \to \pi_1(\mathcal{T}, \omega)$ is faithfully flat if and only if Φ is fully faithful.

Proof. By [DM82, Prop. 2.21 (a)], we just have to show the "if" part, which itself is a slight refinement of [DM82, Rmk. 2.29]. The functor Φ is fully faithful if and only its restriction to $\langle M \rangle$, for every object $M \in \mathcal{T}$, induces an equivalence with $\langle \Phi(M) \rangle$, where $\langle M \rangle$ is the full Tannakian subcategory spanned by M (*i.e.* the full subcategory consisting of subquotient objects of $M^{\otimes m} \otimes M^{\vee \otimes m'}$ and their direct sums). By Tannaka duality, this is equivalent to $\Phi^*\colon \pi_1(\langle \Phi(M) \rangle, \omega') \to \pi_1(\langle M \rangle, \omega)$ being an isomorphism, which by [DM82, Prop. 2.21 (b)] is a closed embedding of group schemes of finite type over $\overline{\mathbb{Q}}_p$. Chevalley's theorem ([Bri09, Thm. 1.15], [Del82, Prop. 3.1.(b)]) asserts that $\pi_1(\langle \Phi(M) \rangle, \omega')$ is the stabilizer of a line l in a finite dimensional representation V of $\pi_1(\langle M \rangle, \omega)$. In particular, l is a one dimensional representation of $\pi_1(\langle \Phi(M) \rangle, \omega')$. Let N_V (resp. L) be the Tannakian dual of V in $\mathcal{T}$ (resp. l in $\mathcal{T}'$). Then the $\pi_1(\langle \Phi(M) \rangle, \omega')$-equivariant inclusion $l \subset V$ induces the inclusion $i\colon L \subset \Phi(N_V)$ in $\mathcal{T}'$. By $(\star)$, there is an integer $m > 0$, and an object $\widetilde{L} \in \mathcal{T}$, such that $L^{\otimes m} = \Phi(\widetilde{L})$. By full faithfulness, there is a uniquely defined inclusion $j\colon \widetilde{L} \subset N_V^{\otimes m}$ such that $i^{\otimes m} = \Phi(j)\colon L^{\otimes m} \subset \Phi(N_V^{\otimes m}) = \Phi(N_V)^{\otimes m}$. Thus the stabilizer of $l^{\otimes m} \subset V^{\otimes m}$ is $\pi_1(\langle M \rangle, \omega)$. On the other hand, if $g \in \pi_1(\langle M \rangle, \omega)(\overline{\mathbb{Q}}_p)$ acts on $l^{\otimes m} \subset V^{\otimes m}$ with eigenvalue $\lambda \in \overline{\mathbb{Q}}_p$, it acts on $l \subset V$ with eigenvalue $\frac{\lambda}{m} \in \overline{\mathbb{Q}}_p$. This implies that $\pi_1(\langle \Phi(M) \rangle, \omega')$ is the stabilizer of $l^{\otimes m}$ in $V^{\otimes m}$, thus $\Phi^*\bigl(\pi_1(\langle \Phi(M) \rangle, \omega')\bigr) = \pi_1(\langle M \rangle, \omega)$. This finishes the proof. ∎

1.7. Let X be a smooth *connected* scheme over $\overline{k}$. Let $x\colon \mathrm{Spec}(\overline{k}) \to X$ be a geometric point. The pull-back i_x^+ functor induces a fiber functor $\mathrm{Isoc}^\dagger(X) \to \mathrm{Vec}_{\overline{\mathbb{Q}}_p}$, which endows the Tannakian category $\mathrm{Isoc}^\dagger(X)$ with a neutralization. The Tannakian fundamental group is denoted by $\pi_1^{\mathrm{isoc}}(X, x)$. This is also independent of the base field. The detailed construction is written in [Abe13, 2.4.17]. For $M \in \mathrm{Isoc}^\dagger(X)$, we denote by $\langle M \rangle$ the Tannakian subcategory of $\mathrm{Isoc}^\dagger(X)$ generated by M. Its fundamental group is denoted by $\mathrm{DGal}(M, x)$, but as the base point chosen is irrelevant for the further discussion, we just write $\mathrm{DGal}(M)$.

Remark. — The fundamental group $\pi_1^{\mathrm{isoc}}(X, x)$ is very close to the one defined by Crew in [Cre92], but he used $\mathrm{Isoc}^\dagger_{\mathrm{Ber}}(X_0/K)$ to define the group contrary to $\mathrm{Isoc}^\dagger(X)$ here.

1.8 Proposition. — *Let $f\colon Y_0 \to X_0$ be a morphism between smooth schemes, and put $X := X_0 \otimes_k \overline{k}$, $Y := Y_0 \otimes_k \overline{k}$. Assume that X_0 is geometrically connected. Let $M \in \mathrm{Isoc}^\dagger(X)$, and assume that* *for any $N \in \langle M \rangle$, the induced homomorphism*

$$H^0(X, N) \to H^0(Y, f^+ N) \tag{1.8.1}$$

is an isomorphism.

Then Y_0 is geometrically connected, and the homomorphism $\mathrm{DGal}(f^+ M) \to \mathrm{DGal}(M)$ *is an isomorphism.*

Proof. First of all, Y_0 is geometrically connected since

$$\dim_{\overline{\mathbb{Q}}_p} H^0(Y, \overline{\mathbb{Q}}_{p,Y}) = \dim_{\overline{\mathbb{Q}}_p} H^0(X, \overline{\mathbb{Q}}_{p,X}) = 1$$

by (1.8.1). By [DM82, Prop. 2.21 (b)], the homomorphism in question is a closed immersion. Let $N, N' \in \langle M \rangle$. We have

$$\mathrm{Hom}(N, N') \cong H^0\big(X, \mathcal{H}om(N, N')\big) \xrightarrow{\sim} H^0\big(X, f^+ \mathcal{H}om(N, N')\big) \cong \mathrm{Hom}\big(f^+ N, f^+ N'\big),$$

where the second isomorphism holds by assumption since $\mathcal{H}om(N, N') \in \langle M \rangle$. This implies that the functor $f^+ \colon \langle M \rangle \to \langle f^+ M \rangle$ is fully faithful. By Lemma 1.6, it suffices to show that for any rank 1 object N in $\mathrm{Isoc}^\dagger(X)$, there exists an integer $m > 0$ such that $N^{\otimes m}$ is trivial. By definition, there exists an integer $n \geq 0$ such that N is the pull-back of $N_n \in \mathrm{Isoc}^\dagger(X_n)$. Then by [Abe15, Lem. 6.1], there exists a rank 1 isocrystal $L_0 \in \mathrm{Isoc}^\dagger(\mathrm{Spec}(k))$ and an integer $m > 0$ such that $(N_n \otimes L_n)^{\otimes m}$ is trivial. As L is trivial in $\mathrm{Isoc}^\dagger(X)$, $N^{\otimes m}$ is trivial as well. ■

Remark. — Since we use class field theory in the proof, our argument works only when the base field is finite.

1.9 Corollary. — *Let $f \colon Y_0 \to X_0$ be a morphism between smooth schemes over k, and put $X := X_0 \otimes_k \overline{k}$, $Y := Y_0 \otimes_k \overline{k}$. Assume that X_0 is geometrically connected over k. Let $M_0 \in \mathrm{Isoc}^\dagger(X_0)$ such that for any $N \in \langle M \rangle$, the homomorphism* (1.8.1) *is an isomorphism. Then the functor f^+ induces an equivalence of categories $\langle M_0 \rangle \xrightarrow{\sim} \langle f^+ M_0 \rangle$.*

Proof. First, the proposition tells us that $f_* \colon \mathrm{DGal}(f^+ M) \xrightarrow{\sim} \mathrm{DGal}(M)$. Now, for any isocrystal N on a smooth geometrically connected scheme Z over k, the pull-back functor $F_Z^+ \colon D^{\mathrm{b}}_{\mathrm{hol}}(Z) \to D^{\mathrm{b}}_{\mathrm{hol}}(Z)$ is an auto-equivalence by [Abe13, Lem. 1.1.3], and induces an auto-equivalence on $\mathrm{Isoc}^\dagger(Z)$. Thus, we have an isomorphism $\mathrm{DGal}(F_Z^+ N) \xrightarrow{\sim} \mathrm{DGal}(N)$, where F_Z is the s-th Frobenius endomorphism on Z, defined by the functor F_Z^+. If we have a Frobenius structure N_0 on N, it induces an automorphism φ of $\mathrm{DGal}(N)$. Using this automorphism, let φ- $\langle N \rangle$ be the category of couples (ρ, α) such that ρ is a representation of $\mathrm{DGal}(N)$ and $\alpha \colon \rho \circ \varphi \cong \rho$. Then $\langle N_0 \rangle$ is equivalent to the Tannakian subcategory of φ- $\langle N \rangle$ generated by N_0.

Let us come back to the situation of the corollary. The Frobenius structure on M_0 (resp. $f^+ M_0$) induces an automorphism φ_X (resp. φ_Y) of the group $\mathrm{DGal}(M)$ (resp. $\mathrm{DGal}(f^+ M)$), and these automorphisms coincide via the isomorphism f_*. Thus, we have the equivalence φ_X- $\langle M \rangle \xrightarrow{\sim} \varphi_Y$- $\langle f^+ M \rangle$. The categories $\langle M_0 \rangle$ and $\langle f^+ M_0 \rangle$ are the Tannakian subcategories generated by M_0 and $f^+ M_0$ respectively, thus the claim follows. ■

2. Cohomological Lefschetz theorem

This section is devoted to showing the existence of ℓ-adic companions for tame isocrystals.

2.1. Let X be a smooth variety defined over an algebraic closure of finite field $\overline{k}$. Let $Z \subset X$ be a closed subscheme, we denote by $j\colon X \setminus Z \hookrightarrow X$ the open immersion, and $i\colon Z \hookrightarrow X$ the closed immersion. We introduce the following four functors from $D^{\mathrm{b}}_{\mathrm{hol}}(X)$ to itself:

$$(+Z) := j_+ \circ j^+, \quad (!Z) := j_! \circ j^+, \quad \mathbb{R}\underline{\Gamma}^+_Z := i_+ \circ i^+, \quad \mathbb{R}\underline{\Gamma}^!_Z := i_+ \circ i^!.$$

Note that the pairs (j^+, j_+), $(j_!, j^+)$, (i^+, i_+), $(i_+, i^!)$ are adjoint pairs. For $M \in D^{\mathrm{b}}_{\mathrm{hol}}(X)$, we sometimes denote $j^+(M)$ by $M|_{X\setminus Z}$. The properties of functors i_+, $i^!$, j_+, $j_!$ are summarized in [Abe13, 1.1.3].

2.2 Proposition. — *Let X be a smooth variety of dimension ≥ 2, and Z be a smooth divisor. Let a closed subscheme $C \subset X$ be a smooth curve, intersecting with Z transversally. Let M be in $D^{\mathrm{b}}_{\mathrm{hol}}(X)$ such that $M|_{X\setminus Z}$ is in $\mathrm{Isoc}^\dagger(X \setminus Z)$ and is tame along Z with nilpotent residues. Then we have a canonical isomorphism in $D^{\mathrm{b}}_{\mathrm{hol}}(X)$:*

$$(!C)(+Z)(M) \cong (+Z)(!C)(M).$$

Proof. First, let us construct the homomorphism $(!C)(+Z)(M) \to (+Z)(!C)(M)$. Let $j_C\colon X \setminus C \to X$, $j_Z\colon X \setminus Z \to X$ and $j'_Z\colon X \setminus (Z \cup C) \to X \setminus C$ be the open immersion. We have

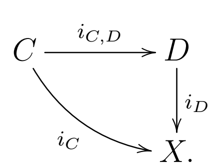
$$j_C^+ j_{Z+} j_Z^+ \cong j'_{Z+} j'^+_Z j_C^+ \cong j'_{Z+} j'^+_Z j_C^+ j_{!C} j_C^+ \cong j_C^+ j_{Z+} j_Z^+ j_{!C} j_C^+,$$

where the first and the last isomorphisms hold since j_C is an open immersion. Since $(j_{C!}, j_C^+)$ is an adjoint pair, we get the desired homomorphism.

It suffices to show the dual statement

$$\rho_{(X,Z,C)}(M)\colon (!Z)(+C)(M) \xrightarrow{\sim} (+C)(!Z)(M).$$

By arguing componentwise, we may assume that X is connected. We use the induction on the dimension of X. The base of the induction is the case where $\dim(X) = 1$. This case is excluded in the proposition, but we take Z to be any divisor in X, and $C = X$. Then the proposition is obvious in this case since $(+C) = 0$ as functors. Assume that the statement is known for $\dim(X) \leq d$. We show the lemma for $\dim(X) = d + 1$. Since to check that a homomorphism in $D^{\mathrm{b}}_{\mathrm{hol}}(X)$ is an isomorphism is local (which can be seen easily from the definition or one can refer to [AC13, 1.3.11]), the claim is local and we may assume that there exists a system of local coordinates $\{t_0, t_1, \ldots, t_d\}$ such that $Z = V(t_0)$, $C = V(t_1, \ldots, t_d)$. We put $D := V(t_1)$. To simplify the notation, we denote the boundaries by $Z_C := Z \cap C$, $Z_D := Z \cap D$. Moreover, we introduce notations of morphisms as follows:



First, let us show that

$$\alpha\colon (!Z_D) \circ i_D^!(M) \to i_D^! \circ (!Z)(M)$$

is an isomorphism. The exact triangle $(!D) \to \mathrm{id} \to \mathbb{R}\underline{\Gamma}_D^+ \to$ induces the following commutative diagram of exact triangles in $D^{\mathrm{b}}_{\mathrm{hol}}(D)$:

$$\begin{array}{ccccccc}
(!Z_D)\circ i_D^! \circ (!D)(M) & \longrightarrow & (!Z_D)\circ i_D^!(M) & \longrightarrow & (!Z_D)\circ i_D^! \circ \mathbb{R}\underline{\Gamma}_D^+(M) & \xrightarrow{+} & \\
\downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_1} & & \\
i_D^! \circ (!Z)\circ (!D)(M) & \longrightarrow & i_D^! \circ (!Z)(M) & \longrightarrow & i_D^! \circ (!Z)\circ \mathbb{R}\underline{\Gamma}_D^+(M) & \xrightarrow{+} &
\end{array}$$

We claim that α_1 is an isomorphism. Indeed, we have

$$\begin{aligned}
(!Z_D)\circ i_D^! \circ \mathbb{R}\underline{\Gamma}_D^+ &= (!Z_D)\circ i_D^! \circ i_{D+}\circ i_D^+ \cong (!Z_D)\circ i_D^+,\\
i_D^! \circ (!Z)\circ \mathbb{R}\underline{\Gamma}_D^+ &= i_D^! \circ (!Z)\circ i_{D+}\circ i_D^+ \cong i_D^! \circ i_{D+}\circ (!Z_D)\circ i_D^+ \cong (!Z_D)\circ i_D^+.
\end{aligned}$$

Here we use the isomorphism $i_D^! \circ i_{D+} \cong \mathrm{id}$ twice, and $i_{D!} \cong i_{D+}$ in the second line. Since α_1 is the identity on $D \setminus Z_D$, the claim is proven.

Thus, it remains to show that α_2 is an isomorphism. It is obvious that $\alpha_2|_{D\setminus Z_D}$ is an isomorphism. This implies that it is enough to check

$$\mathbb{R}\underline{\Gamma}_{Z_D}^+ \circ i_D^! \circ (!Z)\circ (!D)(M) \cong \mathbb{R}\underline{\Gamma}_{Z_D}^+ \circ i_D^! \circ (!D\cup Z)(M) = 0.$$

Since $M|_{X\setminus Z}$ is assumed to be tame with nilpotent residues, we use [AC13, (3.4.12.1)] to conclude.

We now complete the proof. The exact triangle $\mathbb{R}\underline{\Gamma}_C^! \to \mathrm{id} \to (+C) \xrightarrow{+1}$ induces the following commutative diagram of exact triangles:

$$\begin{array}{ccccccc}
(!Z)\circ \mathbb{R}\underline{\Gamma}_C^!(M) & \longrightarrow & (!Z)(M) & \longrightarrow & (!Z)\circ (+C)(M) & \xrightarrow{+} & \\
\downarrow{\scriptstyle \beta} & & \| & & \downarrow{\scriptstyle \rho_{(X,Z,C)}(M)} & & \\
\mathbb{R}\underline{\Gamma}_C^! \circ (!Z)(M) & \longrightarrow & (!Z)(M) & \longrightarrow & (+C)\circ (!Z)(M) & \xrightarrow{+} & .
\end{array}$$

This implies that $\rho_{(X,Z,C)}$ is an isomorphism if and only if β is an isomorphism. We have the following commutative diagram:

$$\begin{array}{ccc}
i_{C+}\circ (!Z_C)\circ i_C^!(M) & \longrightarrow & i_{C+}\circ i_C^! \circ (!Z)(M) \\
\wr & & \| \\
(!Z)\circ \mathbb{R}\underline{\Gamma}_C^!(M) & \xrightarrow[\beta]{} & \mathbb{R}\underline{\Gamma}_C^! \circ (!Z)(M).
\end{array}$$

Since i_C is a closed immersion, β is an isomorphism if and only if

$$\rho'_{(X,Z,C)}(M)\colon (!Z_C)\circ i_C^!(M) \to i_C^! \circ (!Z)(M)$$

is an isomorphism. Namely,

$(\star)$ $\qquad \rho_{(X,Z,C)}(M)$ is an isomorphism $\Leftrightarrow \rho'_{(X,Z,C)}(M)$ is an isomorphism.

The homomorphism $\rho'_{(X,Z,C)}(M)$ can be computed as follows:

$$\begin{aligned}
(!Z_C)\circ i_C^!(M) \cong (!Z_C)\circ i_{C,D}^! \circ i_D^!(M) &\underset{\rho'}{\xrightarrow{\sim}} i_{C,D}^! \circ (!Z_D)\circ i_D^!(M)\\
&\underset{\alpha}{\xrightarrow{\sim}} i_{C,D}^! \circ i_D^! \circ (!Z)(M) \cong i_C^! \circ (!Z)(M),
\end{aligned}$$

where $\rho' := \rho'_{(D,Z_D,C)}\big(i_D^!(M)\big)$, which is an isomorphism by the induction hypothesis $(\star)$ applied to (D, Z_D, C). ■

2.3 Lemma. — *Let X be a smooth projective variety of dimension $d \geq 2$, let C be a curve which is a smooth complete intersection of ample divisors. Then, for any $M \in D^{\mathrm{b}}_{\mathrm{hol}}(X)$ such that $\mathscr{H}^i(M) = 0$ (cf. 1.3 for $\mathscr{H}^*$) for $i < d$, one has $H^n\big(X, (!C)(M)\big) = 0$ for $n = 0, 1$.*

Proof. We use induction on the dimension of X. When $d = 2$, the structural morphism $\epsilon\colon X \setminus C \to \mathrm{Spec}(k)$ is affine. By [AC13, Prop. 1.3.13], this implies that ϵ_+ is left t-exact (with respect to the t-structure introduced in 1.3). By the vanishing condition on the cohomologies of M, the lemma follows in this case.

Assume $d \geq 3$. Let $H \subset X$ be an ample divisor containing C. The localization triangle for H induces the exact triangle exact triangle

$$(!C) \circ (!H)(M) \to (!C)(M) \to (!C) \circ \mathbb{R}\underline{\Gamma}^{+}_{H}(M) \xrightarrow{+1}$$

Using [AC13, Prop. 1.3.13] again and by the assumption on M, $\mathscr{H}^i((!H)M) = \mathscr{H}^i(M) = 0$ for $i < d$. Using the localization sequence, $\mathscr{H}^i\mathbb{R}\underline{\Gamma}^{+}_{H}(M) = 0$ for $i < d-1$, thus $\mathscr{H}^i(i^{+}_{H}M) = 0$ for $i < d-1$. This implies that

$$H^i\big(X, (!C) \circ \mathbb{R}\underline{\Gamma}^{+}_{H}(M)\big) \cong H^i\big(H, (!C)(i^{+}_{H}M)\big) = 0$$

for $i = 0, 1$ by induction hypothesis. Moreover, we have

$$H^i\big(X, (!C) \circ (!H)(M)\big) \xrightarrow{\sim} H^i\big(X, (!H)(M)\big) = 0$$

for $i = 0, 1$, where the first isomorphism holds since $C \subset H$, and the second since $X \setminus H$ is affine (cf. *ibid.*). This finishes the proof. ■

2.4 Corollary. — *Let X be a smooth projective variety of dimension ≥ 2, Z be a simple normal crossings divisor. Let C be a smooth curve which is a complete intersection of ample divisors in good position with respect to Z, and $i_C : C \setminus Z_C \to X \setminus Z$ (where $Z_C := C \cap Z$ as before) be the closed embedding. Then, for any tame isocrystal M with nilpotent residues of* $\mathrm{Isoc}^{\dagger}(X \setminus Z)$*, the homomorphism $H^0(X \setminus Z, M) \to H^0(C \setminus Z_C, i^{+}_{C}M)$ is an isomorphism.*

Proof. Consider the long exact sequence

$$H^0\big(X\backslash Z, (!C\backslash Z_C)(M)\big) \to H^0(X\backslash Z, M) \xrightarrow{i^*} H^0(C\backslash Z_C, i^{+}_{C}M) \to H^1\big(X\backslash Z, (!C\backslash Z_C)(M)\big).$$

In order to prove that i^* is an isomorphism, it is sufficient to prove that the left and the right terms are both zero. Now, let $j\colon X \setminus Z \hookrightarrow X$ be the open immersion. We have

$$H^i\big(X \setminus Z, (!C \setminus Z_C)(M)\big) \cong H^i\big(X, (+Z) \circ (!C)(j_+M)\big) \cong H^i\big(X, (!C) \circ (+Z)(j_+M)\big),$$

where the last isomorphism holds by Proposition 2.2. To finish the proof, we note that j is an affine immersion, which implies that $\mathscr{H}^i(j_+M) = 0$ for $i \neq d$ by [AC13, Prop. 1.3.13]. This enables us to apply Lemma 2.3. ■

2.5 Remark. — In Proposition 2.2, Lemma 2.3, and Corollary 2.4, we do not need k to be a finite field. The argument holds for any base tuple.

2.6 Theorem. — *Let X_0 be a smooth projective variety over k, Z_0 be a simple normal crossings divisor, and x_0 be a closed point of $X_0 \setminus Z_0$. Let C_0 be a smooth curve in X_0 passing through x_0, which is a complete intersection of ample divisors in good position with respect to Z_0, and let $i_C : C_0 \setminus Z_{C,0} \to X_0 \setminus Z_0$ (where $Z_{C,0} := C_0 \cap Z_0$) be the closed immersion. Then, for any irreducible $M_0 \in \mathrm{Isoc}^\dagger(X_0 \setminus Z_0)$, which is tame with nilpotent residues along the boundary, $i_C^+ M_0$ is irreducible.*

Proof. We may assume X_0 is connected by arguing componentwise. Let k' be the field of constants of X_0, namely the algebraic closure of k in $\Gamma(X_0, \mathcal{O}_{X_0})$. Since the category $\mathrm{Isoc}^\dagger(X_0 \setminus Z_0)$ does not depend on the base, we may replace k by k', and thus assume that X_0 is geometrically connected. Then, the theorem follows from Corollary 2.4, combined with Corollary 1.9. ■

Remark. — The existence of the curve C_0 follows from [Poo04, Thm. 1.3].

2.7 Theorem. — *Let X_0 be a scheme of finite type over k, then any object in $D^{\mathrm{b}}_{\mathrm{hol}}(X_0)$ (cf. 1.3 for the notation) is ι-mixed (cf. [AC13, 2.2.2]).*

Proof. By a dévissage argument, it suffices to show the theorem when X_0 is smooth and for objects M_0 in $\mathrm{Isoc}^\dagger(X_0)$.

We first assume that X_0 admits a smooth compactification with a simple normal crossings boundary divisor, and that M_0 is tame. To check that M_0 is ι-mixed, it suffices to check that any constituent is ι-pure, thus we may assume that M_0 is irreducible. Twisting by a character, we may further assume that the determinant of M_0 is of finite order ([Abe15, Thm. 6.1]). Let $x_0 \in X_0$ be a closed point, $i \colon C_0 \hookrightarrow X_0$ be a smooth curve passing through x_0 as in Remark 2.6. Since $i^+ M_0$ is irreducible by Theorem 2.6, it is ι-pure of weight 0 by [Abe13, Thm. 4.2.2]. Thus, M_0 is ι-pure of weight 0.

We now treat the general case. Let $h \colon X_0' \to X_0$ be a semistable reduction with respect to M_0 ([Ked11, Thm. 2.4.4]). Then $h^+ M_0$ is ι-mixed. As M_0 is a direct factor of $h_+ h^+ M_0$ by the trace formalism [Abe13, Thm. 1.5.1], we conclude that M_0 is ι-mixed by [AC13, Thm. 4.2.3]. ■

2.8 Proposition. — *Let X_0 be a smooth projective variety over k, Z_0 be a simple normal crossings divisor, and M_0 be an isocrystal in $\mathrm{Isoc}^\dagger(X_0 \setminus Z_0)$, which is tame along Z_0 with nilpotent residues. Then ℓ-adic companions of M_0 exist, and they are tame along Z_0.*

Proof. We may assume that M_0 is irreducible. Let us fix $\sigma \colon \overline{\mathbb{Q}}_p \to \overline{\mathbb{Q}}_\ell$. Arguing componentwise, we may assume that X_0 is connected. Twisting by a character, we may assume that the determinant of M_0 is of finite order. Let us show that M_0 is algebraic and σ-unit root. Let $x_0 \in X_0 \setminus Z_0$ be a closed point, and C_0 be as in Theorem 2.6 (see Remark 2.6). Using the notation of *ibid.*, $i_C^+ M_0$ is irreducible with determinant of finite order. This implies that $i_C^+ M_0$ is algebraic and σ-unit-root by [Abe13, Thm. 4.2.2], thus the claim follows.

Now, we wish to apply Drinfeld's theorem 0.2 to construct the companions. Deligne's theorem 1.5 (ii) shows that there exists a number field E in $\overline{\mathbb{Q}}_p$ such that $f_x(M_0, t) \in E[t]$ for any finite extension k' of k and $x \in X_0(k')$. Let λ be the place of $\sigma(E)$ over ℓ corresponding the embedding $\sigma(E) \subset \overline{\mathbb{Q}}_\ell$. Put $f_x(t) := f_x(M_0, t)$, and let us show that this collection of functions satisfies the assumptions of Theorem 0.2. Since M_0 is σ-unit-root, for any smooth curve C_0 and a morphism $\varphi \colon C_0 \to X_0$, we have a σ-companion

$_\ell(\varphi^+M_0)$ of φ^+M_0 which is a *lisse étale* $\overline{\mathbb{Q}}_\ell$*-sheaf* by [Abe13, Thm. 4.2.2]. By using [Dri12, Lem. 2.7] and [Dri12, §2.3], there is a finite extension F of $\sigma(E)_\lambda$ such that the monodromy of $_\ell(\varphi^+M_0)$ is in $\mathrm{GL}(r,F)$ for any C_0 and φ, thus the assumption of Theorem 0.2 (i) is satisfied.

Let us check (ii). We put $X'_0 := X_0$. Take a smooth curve $\varphi\colon C_0 \to X_0$, then since M_0 is assumed to be tame, the pull-back φ^+M_0 is tame. This implies that a companion $_\ell(\varphi^+M_0)$ is tame as well by the same argument as [Del12, Lem. 2.3]. (Alternatively, we may also argue that since the local epsilon factors coincide by the Langlands correspondence, and since local epsilon factors detect the irregularity, the irregularity and Swan conductor coincide at each point.) In conclusion, the assumption of Theorem 0.2 (ii) is satisfied as well, and we may apply Drinfeld's theorem to construct the desired companion. ■

3. Wildly ramified case

In this section, we show the Lefschetz type theorem for isocrystals by reduction to the tame case. We keep the same notations as in the previous section, notably X_0, $k \subset k_n \subset \overline{k}$, X_n, X. If $M_0 \in \mathrm{Isoc}^\dagger(X_0)$, we denote by M_0^{ss} the semisimplification in $\mathrm{Isoc}^\dagger(X_0)$, and likewise for $M \in \mathrm{Isoc}^\dagger(X)$ and M^{ss}.

3.1. First, we recall the following well-known consequences of the Weil conjectures (see [Laf02, Cor. VI.3] and [Del80, Thm. 3.4.1 (iii)] for an ℓ-adic counterpart of the theorem).

Theorem ([Abe13, Prop. 4.3.3], [AC13, Thm. 4.3.1])**.** — *(i) Let X_0 be a geometrically connected smooth scheme over k. Let $M_0, M'_0 \in \mathrm{Isoc}^\dagger(X_0)$ be ι-pure F-isocrystals on X_0. Assume that M'_0 is irreducible. Then, the multiplicity of M'_0 in M_0, in other words $\dim \mathrm{Hom}(M'_0, M_0^{\mathrm{ss}})$, is equal to the order of pole of $L(X, M_0 \otimes M_0'^{\vee}, t)$ at $t = q^{-\dim(X_0)}$.*

(ii) Let $M_0 \in \mathrm{Isoc}^\dagger(X_0)$ be ι-pure. Then M, the pull-back of M_0 to $\mathrm{Isoc}^\dagger(X)$, is semisimple.

3.2 Lemma. — *Let X_0 be a geometrically connected smooth scheme over k. Let M_0 be an ι-pure isocrystal, and $_\ell M_0$ be an ℓ-adic companion. Then*

$$\dim H^0(X, M) = \dim H^0(X, {}_\ell M).$$

Proof. First of all let us recall notations. We fix isomorphisms $\overline{\mathbb{Q}}_p \cong \mathbb{C} \cong \overline{\mathbb{Q}}_\ell$ which gives the companion $_\ell M_0$. We regard numbers in $\mathbb{C}$ as numbers in $\overline{\mathbb{Q}}_p$ or $\overline{\mathbb{Q}}_\ell$ via this fixed isomorphisms if there is nothing to confuse. For $t \in \mathbb{C}$, we denote by $\overline{\mathbb{Q}}_{p,X_n}(t)$ the isocrystal in $\mathrm{Isoc}^\dagger(X_n)$ whose underlying object is $\overline{\mathbb{Q}}_p$ and the Frobenius structure is defined by

$$F^{n+}\overline{\mathbb{Q}}_p \cong \overline{\mathbb{Q}}_p \xrightarrow[\sim]{q^{-tn}} \overline{\mathbb{Q}}_p$$

where the first isomorphism is the canonical isomorphism, and q^{-tn} denotes the homomorphism sending 1 to $q^{-tn} \in \mathbb{C} \cong \overline{\mathbb{Q}}_p$. For $n' \geq n$, the functor $\mathrm{Isoc}^\dagger(X_n) \to \mathrm{Isoc}^\dagger(X_{n'})$ sends $\overline{\mathbb{Q}}_{p,X_n}(t)$ to $\overline{\mathbb{Q}}_{p,X_{n'}}(t)$. We denote $\overline{\mathbb{Q}}_{p,X_n}(t)$ by $\overline{\mathbb{Q}}_p(t)$ for simplicity. We also denote $\overline{\mathbb{Q}}_p(0)$ by $\overline{\mathbb{Q}}_p$.

There exists an integer n such that the number of constituents of M_n and M coincide. Let $\alpha_n \subset \mathbb{C}$ be the kernel of the homomorphism of groups $\mathbb{C} \to \mathbb{C}^\times$ sending t to q^{-nt}, thus

$\mathbb{C}/\alpha_n \cong \mathbb{C}^\times$. One has

$$\dim H^0(X,M) = \dim H^0(X,M^{\mathrm{ss}}) = \dim \mathrm{Hom}(\overline{\mathbb{Q}}_p, M^{\mathrm{ss}}) \\ = \sum_{s\in\mathbb{C}/\alpha_n} \dim\bigl(\mathrm{Hom}(\overline{\mathbb{Q}}_p(s), (M_n)^{\mathrm{ss}})\bigr),$$

where the first equality holds since M is semisimple by Theorem 3.1 (ii), the middle one is by definition, and the last one since for any isocrystal N_n in $\mathrm{Isoc}^\dagger(X_n)$ such that $N \cong \overline{\mathbb{Q}}_p$, there exists $s \in \mathbb{C}$ such that $N_n \cong \overline{\mathbb{Q}}_p(s)$. Indeed, fix an isomorphism $N \cong \overline{\mathbb{Q}}_p$. With this identification, N_n yields an isomorphism $\Phi\colon F^{n+}\overline{\mathbb{Q}}_p \xrightarrow{\sim} \overline{\mathbb{Q}}_p$. Let $\mathrm{can}\colon F^{n+}\overline{\mathbb{Q}}_p \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ be canonical isomorphism, or in other words the Frobenius structure of $\overline{\mathbb{Q}}_p \in \mathrm{Isoc}^\dagger(X_n)$. Giving Φ is equivalent to giving $\Phi\circ\mathrm{can}^{-1}(1)$, which uniquely determines s up to multiplication by elements of α_n such that $N_n \cong \overline{\mathbb{Q}}_p(s)$.

By Theorem 3.1 (i), the dimension of $\mathrm{Hom}(\overline{\mathbb{Q}}_p(s), M_n^{\mathrm{ss}})$ is equal to the order of pole of $L(X_n, M_n(-s), t)$ at $t = q^{-dn}$, where d denotes the dimension of X_0. The similar result holds for ${}_\ell M_n$, by increasing n if needed, so the lemma holds since $M_n(s)$ and ${}_\ell M_n(s)$ have the same L-function. ■

3.3. Let X_0 be a smooth scheme over k. Let M (resp. ${}_\ell M$) be in $\mathrm{Isoc}^\dagger(X)$ (resp. lisse Weil ℓ-adic sheaf on X). We say M (resp. ${}_\ell M$) *satisfies* (C) *with respect to an alteration* $X' \to X$ if it satisfies the following condition:

> (*) Let $U \subset X$ be the biggest open dense subscheme over which X' is finite étale, and put $U' := U \times_X X'$. For any connected smooth curve $i\colon C_0 \to X_0$ such that
> $$\#\pi_0(C_0 \times_{X_0} U') = \#\pi_0(U'),$$
> the pull-back homomorphism $H^0(X,M) \to H^0(C, i^+M)$ (resp. $H^0(X, {}_\ell M) \to H^0(C, i^*{}_\ell M)$) is an isomorphism, where $C = C_0 \times_{X_0} X$.

3.4 Lemma. — *Let X_0 be a geometrically connected smooth scheme, and M_n be an ι-pure isocrystal. Let ${}_\ell M_n$ be an ℓ-adic companion. Then if ${}_\ell M_n$ satisfies* (C) *with respect to $X' \to X$, so does M_n.*

Proof. Take C_0 as in (*). Since X_0 is assumed to be geometrically connected, C_0 is geometrically connected as well. By definition, $i^*{}_\ell M$ is an ℓ-adic companion of i^+M, and these are ι-pure. Thus, we have

$$\dim H^0(X,M) = \dim H^0(X, {}_\ell M) = \dim H^0(C, i^*{}_\ell M) = \dim H^0(C, i^+M),$$

where the first and the last equality hold by Lemma 3.2, and the middle one by assumption. ■

3.5 Lemma. — *Assume X_0 is smooth and geometrically connected over k. Let ${}_\ell M_0$ be a lisse Weil $\overline{\mathbb{Q}}_\ell$-sheaf on X_0. Then there exists a connected finite étale cover $g\colon X' \to X$ such that any ${}_\ell N \in \langle {}_\ell M\rangle$ satisfies* (C) *with respect to g.*

Proof. Let $\rho\colon \pi_1(X) \to \mathrm{GL}(r, \overline{\mathbb{Q}}_\ell)$ be the representation corresponding to ${}_\ell M$, and set $G := \mathrm{Im}(\rho)$. The argument of [EK12, B.2] holds also for schemes over $\overline{k}$, since only the finiteness of $H^1_{\text{ét}}(-, \mathbb{Z}/\ell)$ is used. Thus, there exists a connected finite Galois cover

$g\colon X'\to X$ such that for any profinite group K mapping continuously to $\pi_1(X)$, such that the composite $K\to\pi_1(X)\to\mathrm{Gal}(X'/X)$ is surjective, the composite $K\to\pi_1(X)\to G$ is surjective as well. Now, ${}_\ell N$ is a representation of G, and the geometric condition on C asserts that the continuous composite homomorphism $\pi_1(C)\to\pi_1(X)\to\mathrm{Gal}(X'/X)$ is surjective, so so is the continuous composite homomorphism $\pi_1(C)\to\pi_1(X)\to G$. Thus for any ${}_\ell N\in\langle{}_\ell M\rangle$, $H^0(X,{}_\ell N)={}_\ell N^{\pi_1(X)}={}_\ell N^{\pi_1(C)}=H^0(C,{}_\ell N)$, that is, any ${}_\ell N$ in $\langle{}_\ell M\rangle$ satisfies (C) with respect to g. ■

3.6 Lemma. — *Assume X_0 is smooth and geometrically connected over k. Let N_n be an ι-pure isocrystal on X_n, and ${}_\ell N_n$ be an ℓ-adic companion. Then if N_n is geometrically irreducible (i.e. N is irreducible), so is ${}_\ell N_n$.*

Proof. We may assume $N_n\neq 0$. An ι-pure lisse Weil $\overline{\mathbb{Q}}_\ell$-sheaf L_n on X_n is geometrically irreducible if and only if $\mathrm{End}(L)$ is of dimension 1. Indeed, it suffices to show the if part. As L_n is assumed to be ι-pure, L is semisimple ([Del80, Thm. 3.4.1 (iii)]), thus $\dim\mathrm{End}(L)=\dim\mathrm{End}(L^{\mathrm{ss}})$. If L_n were not geometrically irreducible, we would have $\dim\mathrm{End}(L^{\mathrm{ss}})>1$, contradicting with the assumption that $\mathrm{End}(L)=1$. To conclude the proof, we have $\dim\mathrm{End}({}_\ell N)=\dim\mathrm{End}(N)$ by Lemma 3.2, and the latter is equal to 1 since N is assumed to be irreducible. ■

3.7 Corollary. — *Let X_0 be a smooth and geometrically connected scheme over k. Let M_0 be an ι-pure isocrystal on X_0. Assume that for any $n\in\mathbb{N}_{>0}$, any $N_n\in\langle M_n\rangle$ has an ℓ-adic companion. Then there exists a connected finite étale cover $g\colon X'\to X$ such that any $N\in\langle M\rangle$ satisfies* (C) *with respect to g.*

Proof. Let $X'\to X$ be a finite étale cover as in Lemma 3.5 for the ℓ-adic sheaf ${}_\ell M$. Our goal is to show that this cover satisfies the required condition.

Take $N\in\langle M\rangle$. Since the category $\langle M\rangle$ is semisimple, it is enough to check (C) for any irreducible N. Then we can find $N_n\in\langle M_n\rangle$ which induces N for some $n\geq 0$. Take integers m, m' such that N_n is a subquotient of $M_n^{m,m'}:=M_n^{\otimes m}\otimes M_n^{\vee\otimes m'}$. By Lemma 3.4, it remains to show that its ℓ-adic companion ${}_\ell N_n$ is in $\langle{}_\ell M_n\rangle$. By Lemma 3.6, we know that ${}_\ell N_n$ is irreducible as well. Since N_n is assumed to be a subquotient of $M_n^{m,m'}$, $L(X_n,M_n^{m,m'}\otimes N_n^\vee,t)$ has a pole at $t=q^{-nd}$ by Theorem 3.1 (i). This implies that $L(X_n,{}_\ell M_n^{m,m'}\otimes{}_\ell N_n^\vee,t)$ has a pole at $t=q^{-nd}$ as well. It follows again by Theorem 3.1 (i) that ${}_\ell N_n$ is a subquotient of ${}_\ell M_n^{m,m'}$, as ${}_\ell M_n^{m,m'}$ is ι-pure, and finishes the proof. ■

3.8 Theorem. — *Let X_0 be a smooth geometrically connected scheme over k. Let $M_0\in\mathrm{Isoc}^\dagger(X_0)$ be ι-pure. There exists a generically étale alteration $g\colon X'\to X$ such that any $N\in\langle M\rangle$ satisfies* (C) *with respect to g.*

Proof. By [Ked11, Thm. 2.4.4], there exists a generically étale alteration $h\colon X_0''\to X_0$ such that X_0'' is smooth geometrically connected and admits a smooth compactification such that the divisor at infinity has strict normal crossings, and such that $h^+(M_0)$ is log-extendable with nilpotent residues. Then $h^+(M_0)$ possesses an ℓ-adic companion by Proposition 2.8. Thus, we may take a connected finite étale cover $g\colon X'\to X'':=X_0''\otimes_k\overline{k}$ which satisfies (C) for any object in $\langle h^+M_0\rangle$ by Corollary 3.7. Now, take $U_0\subset X_0$ over which h is a finite étale cover. Then $h^+(M_0|_{U_0})$ satisfies (C) with respect to $g|_{h^{-1}(U)}$. Namely, we are in the following situation: we have a generically étale alteration $X_0''\xrightarrow{h}X_0$, and a connected étale cover $X' \xrightarrow{g} X''$ such that any object in $\langle h^+M_0|_{U_0}\rangle$ satisfies (C) with respect to $g|_{h^{-1}(U)}$.

We check now that any object in $\langle M\rangle$ satisfies (C) with respect to $h\circ g$. Let $U\subset X$ be the biggest dense open subscheme over which X'' is finite étale, and let U' and U'' are pull-backs to X' and X''. Let C_0 be a curve such that $\#\pi_0(C_0\times_{X_0}U')=\#\pi_0(U')$. Take $N\in\langle M\rangle$. Then $h^+N\in\langle h^+M\rangle$. We have the following diagram:

$$\begin{array}{ccccccc} & & H^0(U'',h^+N) & \xrightarrow{\alpha} & H^0(C\times_X U'',h^+N) & & \\ & & \uparrow\downarrow \mathrm{tr} & & \uparrow\downarrow \mathrm{tr} & & \\ H^0(X,N) & \xrightarrow[\star]{} & H^0(U,N) & \xrightarrow[\alpha']{} & H^0(C\times_X U,N) & \xleftarrow[\star]{} & H^0(C,N), \end{array}$$

where tr denotes the trace map [Abe13, Thm. 1.5.1]. The homomorphism α is an isomorphism by construction. Since the trace is functorial, it makes α' a direct summand of α. Thus α' is an isomorphism as well. The homomorphisms marked $\star$ are also isomorphisms. Indeed, if Y_0 is a smooth scheme over k and N is an isocrystal in $\mathrm{Isoc}^\dagger(Y)$, then for any open dense subscheme $U_0\subset Y_0$, the restriction $H^0(Y,N)\to H^0(U,N)$ is an isomorphism (in fact, the restriction homomorphism $\pi_1^{\mathrm{isoc}}(U)\to\pi_1^{\mathrm{isoc}}(Y)$ is surjective: see the proof of [Abe13, 2.4.20]). This implies that the homomorphism $H^0(X,N)\to H^0(C,N)$ is an isomorphism, and finishes the proof. ■

3.9. Finally, the existence of the curve C_0 is guaranteed by the following lemma.

Lemma. — *Let X_0 be a smooth geometrically connected scheme over k. Let $U_0\subset X_0$ be a non-empty open subscheme, and let $h\colon U'\to U$ be a connected finite étale morphism. For any finite set of closed points $x^{(j)}\to X_0$, there is a smooth connected curve $C_0\to X_0$ with a factorization $x^{(j)}\to C_0\to X_0$ such that $U'\times_{X_0}C_0$ is non-empty and irreducible.*

Proof. By the Hasse-Weil bounds, we know that U_0 has rational points for any large enough degree extension of k. So by possibly adding more points $x^{(j)}$ to our collection, we may assume that the g.c.d. of the degrees of the points $x^{(j)}$ in U_0 is 1. The morphism h is defined over a finite extension k_n/k in $\bar{k}$. Let $h_n\colon U'_n\to U_n$ be the descended morphism, so that $h=h_n\otimes_{k_n}\bar{k}$. Let $\lambda_n\colon U'_n\xrightarrow{h_n}U_n\to U_0$ be the composition with the base change $U_n=U_0\otimes_k k_n\to U_0$. Let $\mu_n\colon V_n\to U_0$ be the Galois closure of λ_n. It defines an open finite index subgroup H of $\pi_1(U_0)$. The field of constants of V_n is still k_n. By [Dri12, Thm. 2.15 (i)] applied to $(H,U_0\subset X_0,\{x_0^{(j)}\})$, there is a smooth curve $C_0\hookrightarrow X_0$, such that $x_0^{(j)}\in C_0$ and such that $V_n\times_{X_0}C_0$ is irreducible. By the degree assumption on the points $x^{(j)}$, C_0 is geometrically irreducible, thus the field of constants of $V_n\times_{X_0}C_0$ is k_n. Thus a fortiori, $U'_n\times_{X_0}C_0$ is irreducible with field of constants equal to k_n. This finally implies that $(U'_n\times_{X_0}C_0)\otimes_{k_n}\bar{k}=U'\times_{X_0}C_0$ is irreducible. ■

3.10 Theorem (Lefschetz theorem for isocrystals)**.** — *Let X_0 be a smooth scheme over k, and $M_0\in\mathrm{Isoc}^\dagger(X_0)$ be irreducible. Then for any finite set of closed points $x^{(j)}\to X_0$, there is a smooth curve $C_0\to X_0$ with a factorization $x^{(j)}\to C_0\to X_0$ such that the pull-back of M_0 to C_0 is irreducible.*

Proof. Arguing componentwise, we may assume X_0 to be connected, and moreover, geometrically connected by changing k if needed. By Theorem 2.7, M_0 is ι-pure. By Theorem 3.8, there is an alteration $g\colon X'\to X$ such that $N\in\langle M\rangle$ satisfies (C) with respect to

g. Let $U_0 \subset X_0$ be the maximum open dense subscheme such that $U' := X' \times_{X_0} U_0 \to U$ is finite étale. One takes C_0 as in Lemma 3.9, so with factorization $x^{(j)} \to C_0 \to X_0$ for all j, and such that the dominant component of $C_0 \times_{X_0} U'$ is irreducible. Since M satisfies (C) with respect to g, by Corollary 1.9, this implies that the pull-back of M_0 to C_0 is irreducible as required. ■

4. Remarks and applications

4.1 Lemma. — *Let X_0 be a smooth scheme over k, and $M_0 \in \mathrm{Isoc}^\dagger(X_0)$. Assume that M_0 is algebraic. Then there is a number field $E \subset \overline{\mathbb{Q}}_p$ such that $f_x(M_0, t) \in E[t]$ for any finite extension k' and $x \in X_0(k')$.*

Proof. We argue by induction on the dimension of X_0. The curve case has already been treated. We assume the lemma is known for smooth schemes of dimension less than $\dim(X_0)$. By [Ked11, Thm. 2.4.4], there exists an alteration $h\colon X_0' \to X_0$ such that X_0' is smooth and admits a smooth compactification such that the divisor at infinity has strict normal crossings, and such that $h^+(M_0)$ is log-extendable. Let $U_0 \subset X_0$ be a dense open subscheme such that $h|_{h^{-1}(U_0)}$ is finite étale. Using Theorem 1.5 (ii), there exists a number field E_U such that $f_x(M_0, t) \in E_U[t]$ for any $x \in U_0(k')$. Now, there exists a finite stratification $\{X_{i,0}\}_{i \in I}$ of $X_0 \setminus U_0$ by smooth schemes. Since algebraicity is an absolute notion, the restriction $M_0|_{X_{i,0}}$ is algebraic as well. Thus, by induction hypothesis, there exists a number field E_i such that $f_x(M_0, t) \in E_i[t]$ for any $x \in X_{i,0}(k')$. Take E to be a number field which contain E_U and E_i for $i \in I$. Then E is a desired number field. ■

We now formulate the existence of ℓ-adic companions in general. This has been proven by Kedlaya in [Ked16, Thm. 5.3]. However, two additional properties follow from our method: ℓ-adic companions of irreducible overconvergent isocrystals with finite determinant are irreducible, and they are ℓ-adic étale sheaves, not only Weil sheaves.

4.2 Theorem. — *Let X_0 be a smooth geometrically connected scheme over k, and $M_0 \in \mathrm{Isoc}^\dagger(X_0)$ be irreducible with finite determinant. Then ℓ-adic companions exist and they are irreducible lisse étale $\overline{\mathbb{Q}}_\ell$-sheaves.*

Proof. Using Theorem 3.10, there is a smooth curve $\varphi\colon C_0 \to X_0$ such that $\varphi^+(M_0)$ is irreducible. This implies that $\varphi^+(M_0)$ is irreducible with finite determinant, so it is algebraic and σ-unit-root by [Abe13, Thm. 4.2.2]. Thus there exists a closed point x such that $f_x(M_0, t)$ is algebraic and any root is ℓ-adic unit. By Theorem 1.5 (i), M_0 is algebraic and σ-unit-root at any point of X_0 and we can apply Lemma 4.1 to conclude the existence of E. Further, the existence of the companions follows from Theorem 0.2 and from the semistable reduction theorem as the proof of Proposition 2.8. As for irreducibility, since $\varphi^+(M_0)$ is irreducible, the pull-back of an ℓ-adic companion to C_0 is irreducible as well, else a strict subobject would produce a strict subobject of $\varphi^+(M_0)$. This finishes the proof. ■

4.3 Corollary. — *Let X_0 be a smooth scheme over k. Let $X_0 \hookrightarrow \overline{X}_0$ be a normal compactification, D be an effective Cartier divisor with support $\overline{X}_0 \setminus X_0$, $\sigma\colon \overline{\mathbb{Q}}_p \to \overline{\mathbb{Q}}_\ell$ is an isomorphism for a prime $\ell \neq p$. Then there are finitely many isomorphism classes of* *irreducible* $M_0 \in \mathrm{Isoc}^\dagger(X_0)$ *of rank* r*, such that* ${}_\ell M_0^\sigma$ *has ramification bounded by* D*, up to twist by rank* 1 *objects in* $\mathrm{Isoc}^\dagger(\mathrm{Spec}(k))$*.*

Proof. This is a direct application of Theorem 4.2 and Deligne's finiteness theorem [EK12, Thm. 1.1], once one knows that the correspondence $M_0 \to {}_\ell M_0^\sigma$ is injective, which follows from Lemma 3.2. ■

We end with two remarks.

4.4 Remark. — Kedlaya's semistable reduction can be made finite étale, at least in the case where the base field is finite, as asked in [Ked09, Rmk. A.1.2]. Let X_0 be a smooth scheme over k, and $M_0 \in \mathrm{Isoc}^\dagger(X_0)$. Then there exists a finite étale cover $g\colon X_0' \to X_0$ such that $g^+(M_0)$ is tamely ramified. Indeed, let ${}_\ell M_0$ be an ℓ-adic companion. We take a finite étale cover $g\colon X_0' \to X_0$ such that $g^* {}_\ell M_0$ is tamely ramified on X'. Then we claim that $g^+ M_0$ is tame. Indeed, it suffices to check that for any smooth curve C_0 and a morphism $i\colon C_0 \to X_0$, the restriction $i^+(g^+M_0)$ is tame by Definition 1.2. Now, $i^+(g^+M_0)$ and $i^*(g^* {}_\ell M_0)$ are companion, and the local epsilon factors coincide, thus $i^+(g^+M_0)$ is tame since $i^*(g^* {}_\ell M_0)$ is tame by construction.

4.5 Remark. — If $M \in \mathrm{Isoc}^\dagger(X)$ is irreducible, it is coming from an irreducible $M_n \in \mathrm{Isoc}^\dagger(X_n)$. An ℓ-adic companion ${}_\ell M_n$ has to be irreducible as well by Lemma 3.6. If in addition $\pi_1(X) = \{1\}$, then ${}_\ell M_n$ comes from k_n. This implies that M_n comes from k_n as well. As $\mathrm{Isoc}^\dagger(X)$ is semisimple, this shows a (very) weak version of de Jong's conjecture ([ES15, Conj. 2.1]): $\pi_1(X) = \{1\}$ implies that objects in $\mathrm{Isoc}^\dagger(X)$ come from $\mathrm{Isoc}^\dagger(\mathrm{Spec}(k))$. Here "very weak" refers to the fact that we restrict de Jong's conjecture to the case where the ground field is finite and the isocrystals considered have a Frobenius structure.

4.6. So far, we have treated overconvergent isocrystals on smooth varieties, even though they are defined in a more general context. This is partly because, for the moment, we do not have an embedding $\mathrm{Isoc}^\dagger(X) \to D^{\mathrm{b}}_{\mathrm{hol}}(X)$ at disposal when X is not smooth. This sometimes causes technical difficulties. However, we check purity for normal varieties in Corollary 4.6 using Tsuzuki's work.

The definitions in 1.1 can be carried out without any changes for separated schemes X_0 of finite type over k, and in particular, the category $\mathrm{Isoc}^\dagger(X_0)$ makes sense. Given a morphism $f\colon Y_0 \to X_0$ between separated schemes of finite type, Berthelot constructed the pull-back functor $f^*\colon \mathrm{Isoc}^\dagger_{\mathrm{Ber}}(X_0/K) \to \mathrm{Isoc}^\dagger_{\mathrm{Ber}}(Y_0/K)$ in [Ber96, 2.3.6]. This induces a functor $f^*\colon \mathrm{Isoc}^\dagger(X_0) \to \mathrm{Isoc}^\dagger(Y_0)$. When X_0 and Y_0 are smooth, this functor coincides with f^+ (cf. [Abe13, 2.4.15]). With this pull-back, the notion of purity can be defined exactly in the same way as in 1.4 except that we replace i_x^+ by i_x^*. This purity is the $\overline{\mathbb{Q}}_p$-coefficient variant of Crew's purity notion in [Cre92, just before 5.6].

Corollary. — *Let* X_0 *be normal, and* M_0 *be an irreducible object in* $\mathrm{Isoc}^\dagger(X_0)$ *such that the determinant is finite. Then* M_0 *is pure of weight* 0*.*

Remark. — There is a mistake in [Abe15, Lem. 6.1]. Let us use the notation of *ibid.*. In fact, we believe that it is false for arbitrary scheme of finite type X, but we need to assume that X is geometrically unibranch (*e.g.* normal). Let $U \subset X$ be a smooth open dense subscheme. Without assuming X to be geometrically unibranch, it can happen that an ℓ-adic smooth sheaf L is non-trivial even though if $L|_U$ is. In fact, the homomorphism $\pi_1^{\text{ét}}(U) \to \pi_1^{\text{ét}}(X)$ is not in general surjective if U is the smooth open subscheme of X, as easily seen for example on a rational nodal curve, where the map is zero. We think a similar phenomenon occurs also in the p-adic situation. However [Abe15, Lem. 6.1] holds under the assumption by [Tsu12, Cor. 1.2] since the restriction functor is fully faithful.

Proof. Let $U_0 \subset X_0$ be a non-trivial smooth dense subscheme. First, we show that $\pi_1^{\text{isoc}}(U_0) \to \pi_1^{\text{isoc}}(X_0)$ is faithfully flat. The functor $\text{Isoc}^\dagger(X) \to \text{Isoc}^\dagger(U)$ is fully faithful by the result of Tsuzuki cited in Remark 4.6. Arguing as in Proposition 1.8, the homomorphism $\pi_1^{\text{isoc}}(U) \to \pi_1^{\text{isoc}}(X)$ is faithfully flat. Again, similarly to Corollary 1.9, this implies that $M_0|_{U_0}$ is irreducible. Thus, Theorem 4.2 implies that $M_0|_{U_0}$ is pure of weight 0.

Now, let $i_x \colon x \to X_0$ be a closed point. It suffices to show that $i_x^*(M_0)$ is of weight 0. By Theorem 0.3, there exists a morphism $\varphi \colon C_0 \to X_0$ from a smooth curve such that $x \in \varphi(C_0)$, $\varphi(C_0) \cap U_0 \neq \emptyset$ and $\varphi^*(M_0)$ is irreducible. This implies that $\varphi^*(M_0)$ is pure of weight 0. Thus, $i_x^* M_0$ is of weight 0 as required. ■

References

[Abe13] Abe, T.: *Langlands correspondence for isocrystals and existence of crystalline companion for curves*, available at `arXiv:1310.0528`, to appear in JAMS.

[Abe15] Abe, T.: *Langlands program for p-adic coefficients and the petits camarades conjecture*, to appear in Crelle.

[AC13] Abe, T., Caro, D.: *Theory of weights in p-adic cohomology*, preprint available at `arXiv:1303.0662`, to appear in Amer. J. Math..

[AM] Abe, T., Marmora, A.: *On p-adic product formula for epsilon factors*, J. Inst. Math. Jussieu **14**, p.275–377 (2015).

[Ber96] Berthelot, P.: *Cohomologie rigide et cohomologie rigide à supports propres. Première partie*, Prépublication IRMAR 96-03 (1996), 83 pages, `https://perso.univ-rennes1.fr/pierre.berthelot/`.

[Bri09] Brion, M.: *Introduction to actions of algebraic groups*, CIRM, Luminy, (2009), `https://www-fourier.ujf-grenoble.fr/~mbrion/notes.html`.

[Cre92] Crew, R.: *F-isocrystals and their monodromy group*, Ann. Sci. Éc. Norm. Sup. 4ième série **25** (1992), 717–763.

[Del80] Deligne, P.: *La conjecture de Weil II*, Publ. Math. I.H.É.S. **52** (1980), 137–252.

[Del82] Deligne, P.: *Hodge cycles on abelian varieties*, in "Hodge cycles, motives, and Shimura varieties", Lecture Notes in Mathematics **900** (1982), 9–100.

[Del12] Deligne, P.: *Finitude de l'extension de $\mathbb{Q}$ engendrée par des traces de Frobenius, en caractéristique finie*, Mosc. Math. J. **12** (2012), no.3, 497–514.

[DM82] Deligne, P., Milne, J.S.: *Tannakian categories*, in "Hodge cycles, motives, and Shimura varieties", Lecture Notes in Mathematics **900** (1982), 101–228.

[Dri12] Drinfeld, V.: *On a conjecture of Deligne*, Mosc. Math. J. **12** (2012), no.3, 515–542.

[EK12] Esnault, H., Kerz, M.: *A finiteness theorem for Galois representations of function fields over finite fields (after Deligne)*, Acta Mathematica Vietnamica **37** 4 (2012), 531–562.

[ES15] Esnault, H., Shiho, A.: *Convergent isocrystals on simply connected varieties*, preprint 2015, 36 pages, `http://www.mi.fu-berlin.de/users/esnault/preprints/helene/116-EsnShi.pdf`

[Ked09] Kedlaya, K.S.: *Semistable reduction for overconvergent F-isocrystals, III: Local semistable reduction at nonmonomial valuations*, Compos. Math. **145** (2009), 143–172.

[Ked11] Kedlaya, K.S.: *Semistable reduction for overconvergent F-isocrystals, IV: local semistable reduction at monomial valuations*, Compos. Math. **147** (2011), 467–523.

[Ked16] Kedlaya, K.S.: *Étale and crystalline companions*, preprint available at `http://kskedlaya.org/papers/companions.pdf`.

[KS10] Kerz, M., Schmidt, A.: *On different notions of tameness in arithmetic geometry*, Math. Ann. **346** 3 (2010), 641–668.

[Laf02] Lafforgue, L.: *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. math. **147** (2002), no 1, 1–241.

[Poo04] Poonen, B.: *Bertini theorems over finite fields*, Ann. Math. **160** (2004), 1099–1127.

[Shi10] Shiho, A.: *On logarithmic extension of overconvergent isocrystals*, Math. Ann. **348** (2010), no.2, 467–512.

[Shi11] Shiho, A.: *Cut-by-curves criterion for the log extendability of overconvergent isocrystals*, Math. Z. **269** (2011), no.1-2, 59–82.

[Tsu12] Tsuzuki, N.: *A note on the first rigid cohomology group for geometrically unibranch varieties*, Rend. Sem. Mat. Univ. Padova **128** (2012), 17–53.