Questions in Arithmetic Algebraic Geometry

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Individual contributions below

1 (Hélène Esnault and Atsushi Shiho): Existence of locally free lattices of isocrystals.

1.1. Let X be a smooth variety defined over a perfect field k. Let $W := W(k)$ be the ring of Witt vectors of k, $K = \mathrm{Frac}(W)$ be its field of fractions. One defines the category $\mathrm{Crys}(X/W)$ of crystals, and the functor

$$\mathrm{Crys}(X/W) \xrightarrow{\mathbb{Q}\otimes(-)} \mathrm{Crys}(X/W)_{\mathbb{Q}}$$

to its $\mathbb{Q}$-linearization, the category of isocrystals.

It is easily seen that any isocrystal $\mathcal{E} \in \mathrm{Crys}(X/W)_{\mathbb{Q}}$ admits a lattice $E \in \mathrm{Crys}(X/W)$, that is a *p-torsion free* crystal, such that $\mathcal{E} = \mathbb{Q} \otimes E$ ([1, Rmk.2.4]).

Question 1.2. *We ask whether any isocrystal admits a locally free lattice.*
In fact, this question has been asked to us by other mathematicians as well, notably by Tomoyuki Abe, Bhargav Bhatt, Peter Scholze, Nobuo Tsuzuki.

1.3. Using that coherent reflexive sheaves on regular schemes of dimension ≤ 2 are locally free, as well as coherent reflexive sheaves of rank 1 on regular schemes in any dimension, one shows that the question has a positive answer if X is affine and has dimension 1 or in general if $\mathcal{E}$ is a successive extension of rank 1 isocrystals ([1, Prop.2.10] for the latter case, for the former one it goes similarly). In the latter case, it is easy to glue [1, Lem.2.11] so there is a positive answer in the non-affine case as well. In the former case, if X is not affine, it is projective and one uses [1, Prop.4.1], as on X, local freeness is the same as torsionfreeness. To summarize: in all those cases, the procedure is the same. One starts with a random lattice, then one shows that making it 'reflexive' in the sense explained in the proof of [1, Prop.2.10] does it.

However, this procedure can not be general, as is demonstrated by the following example. Here $X = \mathbb{A}^2$, with coordinates (x, y), and one uses the equivalence of categories between the quasi-nilpotent connections on the formal scheme $\mathbb{A}^2$ over W and $\mathrm{Crys}(X/W)$. One defines E by the exact sequence

$$0 \to \mathcal{O} \cdot e \xrightarrow{e \mapsto (x^p e_1, y^p e_2, p e_3)} \oplus \mathcal{O} \cdot e_i \to E \to 0.$$

and the connection by the matrix

$$\Omega = \begin{pmatrix} 0 & 0 & -x^{p-1}dx \\ 0 & 0 & -y^{p-1}dy \\ 0 & 0 & 0 \end{pmatrix}$$

E is reflexive but not locally free. However, since $\mathbb{Q} \otimes E$ is the trivial isocrystal, we can find another lattice which is locally free.

If X is in addition projective, under some extra assumptions it is proved in [1] that the question has a positive answer.

References

[1] H. Esnault and A. Shiho, – *Convergent isocrystals on simply connected varieties.* `http://www.mi.fu-berlin.de/users/esnault/preprints/helene/116-EsnShi.pdf`