

Convergent isocrystals on simply connected varieties

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April 12, 2016

Abstract

It is conjectured by de Jong that, if X is a connected smooth projective variety over an algebraically closed field k of characteristic $p > 0$ with trivial étale fundamental group, any isocrystal on X is constant. We prove this conjecture under certain additional assumptions.

Contents

1 Preliminaries	5
2 Statement of the main results	16
3 Proof of Theorem 2.6	21
4 Proof of Theorem 2.8	29

Introduction

The fundamental group is an important invariant in topology, algebraic geometry and arithmetic geometry. For a complex connected smooth projective variety X , the topological fundamental group $\pi_1(X)$ (based at some point), which classifies all the coverings of X , is defined in a topological, non-algebraic way. But there are (at least) two approaches to define the fundamental group of X in an algebraic way. One

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Mathematics Subject Classification (2010): 14F10, 14D20.

is the étale fundamental group $\pi_1^{\text{ét}}(X)$ [SGA1, V] (based at some geometric point), which classifies all the finite étale coverings of X . It is isomorphic to the profinite completion of $\pi_1(X)$. Another one is the category of \mathcal{O}_X -coherent \mathcal{D}_X -modules, which is equivalent to the category of finite dimensional complex linear representations of $\pi_1(X)$ via the Riemann-Hilbert correspondence. As for the relation between these two algebraic approaches, Mal'cev [Mal40] and Grothendieck [Gro70] proved that, if $\pi_1^{\text{ét}}(X) = \{1\}$, then there are no non-constant \mathcal{O}_X -coherent \mathcal{D}_X -modules.

This leads to the question of an analog for a connected smooth projective variety X over an algebraically closed field k of characteristic $p > 0$, namely, the question to compare the étale fundamental group $\pi_1^{\text{ét}}(X)$ of X and the category of \mathcal{O}_X -coherent \mathcal{D} -modules on X . Due to the absence of the topological fundamental group of X , the relation between them is more mysterious.

One issue to precisely formulate the question above is that there are many versions of \mathcal{D} -modules which are defined on X . One can consider the full ring of differential operators, or the ring of PD-differential operators. One can consider with or without thickenings to the Witt ring W of k , and one can impose various nilpotence or convergence conditions.¹

When we consider the full ring of differential operators \mathcal{D}_X on X in usual sense (without any thickenings to W), the category of \mathcal{O}_X -coherent \mathcal{D}_X -modules is equivalent to the category $\text{Inf}(X/k)$ of crystals of finite presentation on the infinitesimal site $(X/k)_{\text{inf}}$ of X over k . In this case, Gieseker [Gie75] conjectured in 1975 that, on a connected smooth projective variety X over an algebraically closed field k of characteristic $p > 0$ with $\pi_1^{\text{ét}}(X) = \{1\}$, there are no non-constant \mathcal{O}_X -coherent \mathcal{D}_X -modules. This conjecture was answered affirmatively in [EM10, Thm. 1.1].

When we consider the full ring of differential operators on X with thickenings to W , we obtain the category $\text{Inf}(X/W)$ of crystals of finite presentation on the infinitesimal site $(X/W)_{\text{inf}}$ of X over W (see Section 1 for the definition). This is a W -linear category which lifts $\text{Inf}(X/k)$. Because this category contains p -power torsion objects which cannot be constant even when $\pi_1^{\text{ét}}(X) = \{1\}$, it is natural to consider the \mathbb{Q} -linearization $\text{Inf}(X/W)_{\mathbb{Q}}$, which we call the category of infinitesimal isocrystals. This category is known to be too small to capture all the geometric objects, but it is still an interesting category because it contains the geometric objects coming from finite étale coverings of X .

The category of (certain) modules on the ring of PD-differential operators on X with thickenings to W and quasi-nilpotence condition is studied most extensively, which is defined as the category $\text{Crys}(X/W)$ of crystals of finite presentation on

¹Also, we can consider \mathcal{D} -modules on X with Frobenius structure. In this case, a p -adic analogue of the Riemann-Hilbert correspondence, which gives an equivalence between the category of p -adic representations of $\pi_1^{\text{ét}}(X)$ and the category of unit-root convergent F -isocrystals on X , is known by Crew [Cre87]. Although it is also interesting to consider the case with Frobenius structure, we concentrate to the case without Frobenius structure in this article.

the crystalline site $(X/W)_{\text{crys}}$ of X over W (see Section 1 for the definition). As before, it is natural to consider the \mathbb{Q} -linearization $\text{Crys}(X/W)_{\mathbb{Q}}$, which we call the category of isocrystals.

After [EM10], de Jong conjectured in 2010 that, on a connected smooth projective variety X over an algebraically closed field k of characteristic $p > 0$ with $\pi_1^{\text{ét}}(X) = \{1\}$, there are no non-constant isocrystals on X .

In this article, we consider the conjecture of de Jong for a closely related and slightly smaller category $\text{Conv}(X/K)$ (where K is the fraction field of W), the category of convergent isocrystals on X over K , which is introduced by Ogus [Ogu84, Defn. 2.7]. This corresponds to the category of (certain) modules on the ring of differential operators on X with thickenings to W , tensorization with \mathbb{Q} and the convergence condition, which is slightly stronger than the quasi-nilpotence condition. Although the category $\text{Conv}(X/K)$ is slightly smaller than $\text{Crys}(X/W)_{\mathbb{Q}}$, it is large enough to contain the objects coming from geometry (e.g., the Gauß-Manin convergent isocrystals defined by Ogus [Ogu84, Thm. 3.7]) and enjoys nice topological properties such as proper descent. Over an algebraically closed field, it shares many properties with the category of lisse $\bar{\mathbb{Q}}_{\ell}$ -sheaves.

The conjecture of de Jong (for convergent isocrystals) is not trivial even when X is liftable to a smooth projective scheme X_W over $\text{Spec } W$, because the étale fundamental group of the geometric generic fiber of X_W need not be trivial. On the other hand, for a proper smooth morphism $f : Y \rightarrow X$ which is liftable to a proper smooth morphism $f_W : Y_W \rightarrow X_W$, we can prove the constancy of the Gauß-Manin convergent isocrystal $R^i f_{\text{conv}*} \mathcal{O}_{Y/K}$ [Ogu84, Thm. 3.7] rather easily, in the following way. If we denote by $f_L : Y_L \rightarrow X_L$ the base change of f_W to a field L containing W , it suffices to prove the constancy of $R^i f_{K, \text{dR}*} \mathcal{O}_{Y_K}$ as a module with an integrable connection, by [Ogu84, Thm. 3.10]. Then we may assume that $K \subseteq \mathbb{C}$ and it suffices to prove the constancy of $R^i f_{\mathbb{C},*}^{\text{an}} \mathbb{C}_{Y_{\mathbb{C}}^{\text{an}}}$, where $f_{\mathbb{C}}^{\text{an}} : Y_{\mathbb{C}}^{\text{an}} \rightarrow X_{\mathbb{C}}^{\text{an}}$ is the analytification of $f_{\mathbb{C}}$. This is reduced to the constancy of $R^i f_{\mathbb{C},*} \mathbb{Q}_{\ell}$ for a prime $\ell \neq p$ by Artin's comparison theorem [SGA4, XVI,4], and reduces to the constancy of $R^i f_{W,*} \mathbb{Q}_{\ell}$, which is true by Grothendieck's base change theorem on the étale fundamental group $\{1\} = \pi_1^{\text{ét}}(X) \xrightarrow{\cong} \pi_1^{\text{ét}}(X_W)$ [EGAIV4, Thm. 18.1.2]. Thus we find this conjecture interesting enough.

Our main result is a partial solution to the conjecture of de Jong for convergent isocrystals, which is stated as follows.

Theorem 0.1. *Let X be a connected smooth projective variety over k with trivial étale fundamental group. Then*

- (1) *any convergent isocrystal which is filtered so that the associated graded is a sum of rank 1 convergent isocrystals, is constant;*
- (2) *if the maximal slope of the sheaf of 1-forms on X is non-positive, then any convergent isocrystal is constant.*

We refer to Theorem 2.8 (1) and Corollary 2.9 for this formulation. Theorem 2.8 is more precisely formulated: In (2), positive slopes of the sheaf of 1-forms on X are allowed, according to the maximal rank of the irreducible constituents of the given convergent isocrystal.

We now explain the main ideas of the proof. Convergent isocrystals are known to be Frobenius divisible, although p -torsion free crystals in one isocrystal class (called *lattices* of an isocrystal) is not. Using this, one proves in Proposition 3.1 that the Chern classes of the value E_X on X of a crystal E over W vanish when E is a lattice of a convergent isocrystal.

If one assumes in addition that E_X is strongly μ -semistable, one sees that the subquotients associated to some filtration of E_X yield points in the moduli of χ -stable sheaves with vanishing Chern classes. Then, assuming now that X has trivial fundamental group, it is proved in Propositions 3.2, 3.3, 3.4 by a noetherianity argument, that not only infinitely Frobenius divisible sheaves are constant (Gieseker's conjecture proved in [EM10]), but also strongly μ -semistable ones with vanishing Chern classes which admit a large enough Frobenius divisibility.

This, together with the crystalline deformation theory in Propositions 3.6, 3.7, Corollary 3.8 which allows to prove the constancy modulo p^n from that on X , leads to the theorem (see Theorem 2.6).

Theorem 0.2. *Let X be a connected smooth projective variety over k with trivial étale fundamental group and let \mathcal{E} be a convergent isocrystal. If, for any $n \in \mathbb{N}$, the F^n -division $\mathcal{E}^{(n)}$ of \mathcal{E} admits a lattice such that its value on X as a coherent \mathcal{O}_X -module is strongly μ -semistable, then \mathcal{E} is trivial.*

Also, one proves a Langton type theorem in Proposition 4.1, claiming the existence of a lattice whose restriction to a crystal on X over k is μ -semistable. One proves Theorem 0.1, together with its refinements not discussed in the introduction, using Theorem 0.2 and the slope condition on the sheaf of 1-forms which forces the requested stability conditions (see Proposition 4.2).

As another consequence of Theorem 0.2, we have the following corollary, which confirms the conjecture of de Jong for infinitesimal isocrystals (see Corollary 2.7).

Corollary 0.3. *Let X be a connected smooth projective variety over k with trivial étale fundamental group. Then any infinitesimal isocrystal on X is constant.*

Acknowledgement: The first named author thanks Johan de Jong for inviting her in the fall of 2010 to lecture on [EM10] at Columbia University, after which he formulated the conjecture discussed here. She thanks Arthur Ogus for subsequent fruitful discussions which solved the abelian case (see Proposition 2.10 for a precise statement), Yves André for discussions on the moduli points of isocrystals in Simpson's moduli of flat connections on a characteristic 0 lift of X , Adrian Langer for enlightening discussions on [Lan14, Thm. 5.1]. She thanks Takeshi Saito for offering

hospitality in summer 2014 at the Graduate School of Mathematical Sciences, the University of Tokyo, where this work was started (see [Shi14]). The second named author thanks Kirti Joshi for his suggestion to consider the condition $\mu_{\max}(\Omega_X^1) \leq 0$. Both authors thank one of the anonymous referees for gracious, constructive and helpful comments.

1 Preliminaries

In this section, we review some facts on (iso)crystals, infinitesimal (iso)crystals, convergent isocrystals, and Cartier transform of Ogus-Vologodsky.

Throughout the article, we fix an algebraically closed field k of characteristic $p > 0$. Let W be the Witt ring of k and K be the fraction field of W . For $n \in \mathbb{N}$, put $W_n := W/p^n W$. Let $\sigma : k \rightarrow k$ be the Frobenius map $a \mapsto a^p$ on k . Let $\sigma_W : W \rightarrow W$ be the automorphism which lifts σ and let $\sigma_K : K \rightarrow K$ be the automorphism induced by σ_W .

First we summarize a few facts on (iso)crystals from [BO78, Sections 5/6/7], [Ber74, III/IV]. For a scheme X of finite type over k , let $(X/W)_{\text{crys}}$ (resp. $(X/W_n)_{\text{crys}}$) be the *crystalline site* on X/W (resp. X/W_n). An object is a pair $(U \hookrightarrow T, \delta)$, where $U \hookrightarrow T$ is a closed immersion over W_n for some n (resp. over W_n) from an open subscheme U of X to a scheme T and δ is a PD-structure on $\text{Ker}(\mathcal{O}_T \rightarrow \mathcal{O}_U)$, compatible with the canonical PD-structure on pW_n . Morphisms are the obvious ones. For the definition of coverings, see [BO78, page 5.2]. The structure sheaf $\mathcal{O}_{X/W}$ on $(X/W)_{\text{crys}}$ (resp. \mathcal{O}_{X/W_n} on $(X/W_n)_{\text{crys}}$) is defined by the rule $(U \hookrightarrow T, \delta) \mapsto \Gamma(T, \mathcal{O}_T)$.

A sheaf E of $\mathcal{O}_{X/W}$ -modules (resp. \mathcal{O}_{X/W_n} -modules) on $(X/W)_{\text{crys}}$ (resp. on $(X/W_n)_{\text{crys}}$) is equivalent to the datum of a sheaf of \mathcal{O}_T -modules E_T in the Zariski topology of T for each object $T := (U \hookrightarrow T, \delta)$, and of an $\mathcal{O}_{T'}$ -linear morphism $\varphi^* E_T \rightarrow E_{T'}$ for each morphism $\varphi : (U' \hookrightarrow T', \delta') \rightarrow (U \hookrightarrow T, \delta)$, which is an isomorphism when $\varphi : T' \rightarrow T$ is an open immersion and U' is equal to $U \times_T T'$. The sheaf E_T is called the *value* of E at T . Via the module structure of \mathcal{O}_T over itself, $\mathcal{O}_{X/W}$ (resp. \mathcal{O}_{X/W_n}) is a sheaf of $\mathcal{O}_{X/W}$ -modules (resp. \mathcal{O}_{X/W_n} -modules).

A sheaf E of $\mathcal{O}_{X/W}$ -modules (resp. of \mathcal{O}_{X/W_n} -modules) on $(X/W)_{\text{crys}}$ (resp. on $(X/W_n)_{\text{crys}}$) is a *crystal* if the morphisms $\varphi^* E_T \rightarrow E_{T'}$ are all isomorphisms. A crystal is of *finite presentation* if its value E_T is an \mathcal{O}_T -module of finite presentation for any $(U \hookrightarrow T, \delta)$. The *category of crystals of finite presentation* on $(X/W)_{\text{crys}}$ (resp. on $(X/W_n)_{\text{crys}}$) is denoted by $\text{Crys}(X/W)$ (resp. $\text{Crys}(X/W_n)$), as a full subcategory of the category of sheaves of $\mathcal{O}_{X/W}$ -modules (resp. of \mathcal{O}_{X/W_n} -modules). The structure sheaf $\mathcal{O}_{X/W}$ (resp. \mathcal{O}_{X/W_n}) is a crystal. It is known [Ber74, IV Prop. 1.7.6] that $\text{Crys}(X/W)$, $\text{Crys}(X/W_n)$ are abelian categories. Furthermore, the categories $\text{Crys}(X/W)$, $\text{Crys}(X/W_n)$ satisfy the descent property for Zariski coverings of X , that is, crystals ‘glue’ in the Zariski topology.

If we denote the topos associated to $(X/W)_{\text{crys}}$ by $(X/W)_{\text{crys}}^{\sim}$, it is functorial with respect to X/W , namely, if we have a commutative diagram

$$(1.1) \quad \begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ \text{Spf}(W) & \xrightarrow{f'} & \text{Spf}(W) \end{array}$$

with X' of finite type over k , we have the canonical morphism of topoi $(X'/W)_{\text{crys}}^{\sim} \rightarrow (X/W)_{\text{crys}}^{\sim}$ [BO78, §5]. It induces the morphism of ringed topoi $((X'/W)_{\text{crys}}^{\sim}, \mathcal{O}_{X'/W}) \rightarrow ((X/W)_{\text{crys}}^{\sim}, \mathcal{O}_{X/W})$ and the pullback functor $f^* : \text{Crys}(X/W) \rightarrow \text{Crys}(X'/W)$. Similar functoriality holds also for the the ringed topos $((X/W_n)_{\text{crys}}^{\sim}, \mathcal{O}_{X/W_n})$ associated to $(X/W_n)_{\text{crys}}$ and the category $\text{Crys}(X/W_n)$.

The natural inclusion of sites $(X/W_n)_{\text{crys}} \hookrightarrow (X/W)_{\text{crys}}$ induces the restriction functor

$$(1.2) \quad \text{Crys}(X/W) \longrightarrow \text{Crys}(X/W_n); \quad E \mapsto E_n.$$

Since $(X/W)_{\text{crys}}$ is the 2-inductive limit of the sites $(X/W_n)_{\text{crys}}$ [BO78, page 7-22], we have the equivalence

$$(1.3) \quad \text{Crys}(X/W) \xrightarrow{\simeq} \varprojlim_n \text{Crys}(X/W_n); \quad E \mapsto (E_n)_n.$$

The functors (1.2), (1.3) are also functorial with respect to X/W .

For any object $(U \hookrightarrow T, \delta)$ in $(X/W)_{\text{crys}}$ (resp. $(X/W_n)_{\text{crys}}$), the *functor of evaluation at T*

$$(1.4) \quad \text{Crys}(X/W) \longrightarrow \text{Coh}(\mathcal{O}_T) \quad (\text{resp. } \text{Crys}(X/W_n) \longrightarrow \text{Coh}(\mathcal{O}_T)); \quad E \mapsto E_T$$

is defined, where $\text{Coh}(\mathcal{O}_T)$ denotes the category of sheaves of \mathcal{O}_T -modules of finite type. It is known to be right exact. This follows from [Ber74, IV Rem. 1.7.8] and [Ber74, III Prop. 1.1.5].

If $U \hookrightarrow Y$ is a closed immersion from an open subscheme U of X into a smooth scheme Y over W_n and $T := (U \hookrightarrow T, \delta)$ is its PD-envelope [BO78, page 3.20], the functor (1.4) is exact [Ber74, IV Prop. 1.7.5]. Moreover, we have an equivalence of categories lifting (1.4), which we explain now. The derivation $d : \mathcal{O}_Y \rightarrow \Omega_Y^1$ is extended canonically to a PD-derivation $d : \mathcal{O}_T \rightarrow \Omega_T^1 := \mathcal{O}_T \otimes_{\mathcal{O}_Y} \Omega_Y^1$, and we have the notion of *\mathcal{O}_T -modules of finite presentation with integrable connection on T with respect to this PD-derivation* [BO78, Section 4]. With the obvious morphisms, we denote the category of such objects by $\text{MIC}(T)$, and denote the full subcategory consisting of quasi-nilpotent ones by $\text{MIC}(T)^{\text{qn}}$. (For the definition of the quasi-nilpotence, see [BO78, Def. 4.10].) For $T = (U \hookrightarrow T, \delta)$ as above and $E \in \text{Crys}(U/W_n)$, E_T is naturally endowed with a quasi-nilpotent integrable

connection $\nabla_{E_T} : E_T \rightarrow E_T \otimes \Omega_T^1$, and we have a natural equivalence of abelian categories

$$(1.5) \quad \text{Crys}(U/W_n) \xrightarrow{\simeq} \text{MIC}(T)^{\text{qn}}; \quad E \mapsto (E_T, \nabla_{E_T}).$$

We use the functors (1.4), (1.5) in the following cases. First, for a smooth variety X over k , we have the right exact functors

$$(1.6) \quad \text{Crys}(X/W) \longrightarrow \text{Coh}(\mathcal{O}_X),$$

$$(1.7) \quad \text{Crys}(X/W_n) \longrightarrow \text{Coh}(\mathcal{O}_X)$$

of evaluation at $X := (X \xrightarrow{\text{id}} X, 0)$. When $n = 1$, the functor (1.7) is exact. The functor (1.6) (resp. (1.7)) is functorial with respect to X/W (resp. X/W_n).

Next, let X be a smooth variety over k and assume that we have a lifting of X to a p -adic smooth formal scheme X_W over W . As X/k is smooth, there always exists such a lifting on affine open subschemes of X . If we put $X_n := X_W \otimes_W W_n$, the evaluation at $X_n := (X \hookrightarrow X_n, \text{canonical PD-structure on } p\mathcal{O}_{X_n})$ induces the equivalence

$$(1.8) \quad \text{Crys}(X/W_n) \xrightarrow{\simeq} \text{MIC}(X_n)^{\text{qn}}.$$

So we have an equivalence of categories

$$(1.9) \quad \begin{aligned} \text{Crys}(X/W) &\xrightarrow{\simeq} \varprojlim_n \text{MIC}(X_n)^{\text{qn}} =: \text{MIC}(X_W)^{\text{qn}}; \\ E &\mapsto \varprojlim_n (E_{X_n}, \nabla_{E_{X_n}}) =: (E_{X_W}, \nabla_{E_{X_W}}). \end{aligned}$$

In addition, there exists a full embedding

$$(1.10) \quad \text{MIC}(X_W)^{\text{qn}} \hookrightarrow (\mathcal{O}_{X_W}\text{-coherent left } \widehat{\mathcal{D}}_{X_W/W}^{(0)}\text{-modules)},$$

where $\widehat{\mathcal{D}}_{X_W/W}^{(0)} := \varprojlim \mathcal{D}_{X_n/W_n}^{(0)}$ and $\mathcal{D}_{X_n/W_n}^{(0)}$ is the ring of PD-differential operators, by [BO78, Thm. 4.8]. When there exists a local basis x_1, \dots, x_d of X_W over W , $\widehat{\mathcal{D}}_{X_W/W}^{(0)}$ is topologically generated by

$$\left(\frac{\partial}{\partial x} \right)^n := \left(\frac{\partial}{\partial x_1} \right)^{n_1} \left(\frac{\partial}{\partial x_2} \right)^{n_2} \cdots \left(\frac{\partial}{\partial x_d} \right)^{n_d} \quad (n := (n_1, \dots, n_d) \in \mathbb{N}^d)$$

over \mathcal{O}_{X_W} [BO78, Section 4].

The functors (1.9), (1.10) are functorial with respect to X_W/W , namely, if there exists a diagram

$$(1.11) \quad \begin{array}{ccc} X'_W & \xrightarrow{f_W} & X_W \\ \downarrow & & \downarrow \\ \text{Spf}(W) & \xrightarrow{f'} & \text{Spf}(W) \end{array}$$

lifting (1.1), the functor (1.9) is compatible with the pullback f^* on the left hand side and the pullback f_W^* on the right hand side. Also, the functor (1.10) is compatible with the pullback f_W^* .

We say that an object $(E_{X_W}, \nabla_{E_{X_W}})$ (resp. $(E_{X_n}, \nabla_{E_{X_n}})$) in $\text{MIC}(X_W)^{\text{qn}}$ (resp. in $\text{MIC}(X_n)^{\text{qn}}$) is p -torsion if so is E_{X_W} (resp. E_{X_n}). Since the restriction functors

$$\begin{aligned} (p\text{-torsion objects in } \text{MIC}(X_W)^{\text{qn}}) &\longrightarrow \text{MIC}(X)^{\text{qn}}, \\ (p\text{-torsion objects in } \text{MIC}(X_n)^{\text{qn}}) &\longrightarrow \text{MIC}(X)^{\text{qn}} \end{aligned}$$

are equivalences, we see by (1.8), (1.9) and Zariski descent that the restriction functors

$$\begin{aligned} (p\text{-torsion objects in } \text{Crys}(X/W)) &\longrightarrow \text{Crys}(X/k), \\ (p\text{-torsion objects in } \text{Crys}(X/W_n)) &\longrightarrow \text{Crys}(X/k) \end{aligned}$$

are equivalences.

When X is projective over k and we are given a fixed closed k -immersion $\iota : X \hookrightarrow \mathbb{P}_k^N$, we denote the PD-envelope of $X \xrightarrow{\iota} \mathbb{P}_k^N \hookrightarrow \mathbb{P}_{W_n}^N$ by $D_n := (X \hookrightarrow D_n, \delta_n)$. Then the equivalence of categories (1.5) becomes globally defined on X [BO78, Thm. 6.6]

$$(1.12) \quad \text{Crys}(X/W_n) \xrightarrow{\simeq} \text{MIC}(D_n)^{\text{qn}}; \quad E \mapsto (E_{D_n}, \nabla_{E_{D_n}}).$$

A crystal $E \in \text{Crys}(X/W)$ (resp. $\text{Crys}(X/W_n)$) is called *locally free* if, for any object $(U \hookrightarrow T, \delta)$ in $(X/W)_{\text{crys}}$ (resp. $\text{Crys}(X/W_n)$), E_T is locally free of finite rank.

A crystal $E \in \text{Crys}(X/W)$ is said to be *p -torsion free* if the multiplication by p on E is injective.

A crystal $E \in \text{Crys}(X/W_n)$ is called *flat* over W_n if, for any $1 \leq i \leq n-1$, the morphism $E/p^{n-i}E \rightarrow E$ induced by the multiplication by p^i is an isomorphism onto its image $p^i E \subset E$.

When we have a lifting of X to a p -adic smooth formal scheme X_W over W , $E \in \text{Crys}(X/W)$ is locally free (resp. p -torsion free) if and only if E_{X_W} is locally free (resp. p -torsion free), and $E \in \text{Crys}(X/W_n)$ is flat over W_n if and only if E_{X_n} is flat over W_n , where $X_n := X_W \otimes_W W_n$. Therefore, $E \in \text{Crys}(X/W)$ is locally free if and only if its value $E_X \in \text{Coh}(X)$ is locally free, if and only if its restriction to $\text{Crys}(X/k)$ is locally free. Also, $E \in \text{Crys}(X/W)$ is p -torsion free if and only if E_n is flat over W_n for each $n \in \mathbb{N}$, where $(E_n)_n \in \varprojlim_n \text{Crys}(X/W_n)$ is the object corresponding to E via (1.3).

For a smooth variety X over k , let $\text{Crys}(X/W)_{\mathbb{Q}}$ be the \mathbb{Q} -linearization of the category $\text{Crys}(X/W)$, which is called the category of *isocrystals* on X . This means that the objects of $\text{Crys}(X/W)_{\mathbb{Q}}$ are those of $\text{Crys}(X/W)$ and that the morphisms

of $\text{Crys}(X/W)_{\mathbb{Q}}$ are those of $\text{Crys}(X/W)$ tensored with \mathbb{Q} . So one has a natural functor $\text{Crys}(X/W) \xrightarrow{\mathbb{Q} \otimes} \text{Crys}(X/W)_{\mathbb{Q}}$ which is the identity on objects. The image of \mathcal{E} by this functor is denoted by $\mathbb{Q} \otimes \mathcal{E}$. When X is liftable to a p -adic smooth formal scheme X_W over W , the functors (1.9) and (1.10) induce the full embedding

$$(1.13) \quad \text{Crys}(X/W)_{\mathbb{Q}} \hookrightarrow ((\mathbb{Q} \otimes \mathcal{O}_{X_W})\text{-coherent left } (\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W}^{(0)})\text{-modules}),$$

which is functorial with respect to X_W/W .

Next, we recall the basic facts on infinitesimal (iso)crystals. Basic references are [Gro68], [Ogu75], [BO78, §2]. For a scheme X of finite type over k , let $(X/W)_{\text{inf}}$ (resp. $(X/W_n)_{\text{inf}}$) be the *infinitesimal site* on X/W (resp. X/W_n). An object is a nilpotent closed immersion $U \hookrightarrow T$ over W_n for some n (resp. over W_n) from an open subscheme U of X to a scheme T . Morphisms are the obvious ones and the covering is defined in the same way as in the case of crystalline site. Thus $(X/W)_{\text{inf}}$ contains $(X/W_n)_{\text{crys}}$ as a full subcategory.

We can define the structure sheaf $\mathcal{O}_{X/W}$ (resp. \mathcal{O}_{X/W_n}) and the notion of crystals of finite presentation on $(X/W)_{\text{inf}}$ (resp. $(X/W_n)_{\text{inf}}$), which we call *infinitesimal crystals* on X/W (resp. X/W_n), in the same way as in the case of crystalline site. We denote the category of infinitesimal crystals on X/W (resp. X/W_n) by $\text{Inf}(X/W)$ (resp. $\text{Inf}(X/W_n)$). The categories $\text{Inf}(X/W)$, $\text{Inf}(X/W_n)$ also satisfy the descent property for Zariski coverings of X .

The topos $(X/W)_{\text{inf}}^{\sim}$ associated to $(X/W)_{\text{inf}}$ is also functorial with respect to X/W . To prove it, we need to prove the analogue of [BO78, Lem. 5.11, 5.12, 5.13] for infinitesimal site. The proof of [BO78, Lem. 5.11, 5.13] works as it is (and we don't need the argument on PD-structure). The proof of [BO78, Lem. 5.12] works if we define T there to be the N -th infinitesimal neighborhood of U in $T_1 \times_Y T_2$ for $N \gg 0$. As a consequence, if we are given a diagram (1.1), we have the morphism of topoi $(X'/W)_{\text{inf}}^{\sim} \rightarrow (X/W)_{\text{inf}}^{\sim}$, the morphism of ringed topoi $((X'/W)_{\text{inf}}^{\sim}, \mathcal{O}_{X'/W}) \rightarrow ((X/W)_{\text{inf}}^{\sim}, \mathcal{O}_{X/W})$ and the pullback functor $f^* : \text{Inf}(X/W) \rightarrow \text{Inf}(X'/W)$. Similar functoriality holds also for the the ringed topos $((X/W_n)_{\text{inf}}^{\sim}, \mathcal{O}_{X/W_n})$ associated to $(X/W_n)_{\text{inf}}$ and the category $\text{Inf}(X/W_n)$.

As in the case of crystalline site, the natural inclusion of sites $(X/W_n)_{\text{inf}} \hookrightarrow (X/W)_{\text{inf}}$ induces the restriction functor

$$(1.14) \quad \text{Inf}(X/W) \longrightarrow \text{Inf}(X/W_n); \quad E \mapsto E_n,$$

which induces the equivalence

$$(1.15) \quad \text{Inf}(X/W) \xrightarrow{\cong} \varprojlim_n \text{Inf}(X/W_n); \quad E \mapsto (E_n)_n.$$

The functors (1.14), (1.15) are also functorial with respect to X/W . Also, for any object $U \hookrightarrow T$ in $(X/W)_{\text{inf}}$ (resp. $(X/W_n)_{\text{inf}}$), the *functor of evaluation* at T

$$(1.16) \quad \text{Inf}(X/W) \longrightarrow \text{Coh}(\mathcal{O}_T) \quad (\text{resp. } \text{Inf}(X/W_n) \longrightarrow \text{Coh}(\mathcal{O}_T)); \quad E \mapsto E_T$$

is defined.

When X is a smooth variety over k and there exists a lifting of X to a p -adic smooth formal scheme X_W over W , we have an equivalence of categories

$$(1.17) \quad \text{Inf}(X/W_n) \xrightarrow{\simeq} (\mathcal{O}_{X_n}\text{-coherent left } \mathcal{D}_{X_n/W_n}\text{-modules}),$$

where $X_n := X_W \otimes_W W_n$ and \mathcal{D}_{X_n/W_n} is the full ring of differential operators of X_n over W_n . Thus we have an equivalence of categories

$$(1.18) \quad \text{Inf}(X/W) \xrightarrow{\simeq} (\mathcal{O}_{X_W}\text{-coherent left } \widehat{\mathcal{D}}_{X_W/W}\text{-modules}),$$

where $\widehat{\mathcal{D}}_{X_W/W} := \varprojlim_n \mathcal{D}_{X_n/W_n}$, and the action of $\widehat{\mathcal{D}}_{X_W/W}$ on objects on the right hand side is assumed to be continuous. When there exists a local basis x_1, \dots, x_d of X_W over W , $\widehat{\mathcal{D}}_{X_W/W}$ is topologically generated by

$$\frac{1}{n!} \left(\frac{\partial}{\partial x} \right)^n := \frac{1}{n_1!} \left(\frac{\partial}{\partial x_1} \right)^{n_1} \frac{1}{n_2!} \left(\frac{\partial}{\partial x_2} \right)^{n_2} \cdots \frac{1}{n_d!} \left(\frac{\partial}{\partial x_d} \right)^{n_d} \quad (n := (n_1, \dots, n_d) \in \mathbb{N}^d)$$

over \mathcal{O}_{X_W} [BO78, Section 2].

We give a proof of the equivalence (1.17), which seems to be missing in the literature. For $n, m, r \in \mathbb{N}$, let $X_n(r)_m$ (resp. $X_n(r)'_m$) be the m -th infinitesimal neighborhood of X (resp. X_n) in $X_n(r) := \underbrace{X_n \times_{W_n} \cdots \times_{W_n} X_n}_{r+1}$. Also, for $i = 1, 2$, let

$p_{i,m} : X_n(1)_m \rightarrow X_n$ (resp. $p'_{i,m} : X_n(1)'_m \rightarrow X_n$) be the morphism induced by the i -th projection $X_n(1) \rightarrow X_n$, and for $1 \leq i < j \leq 3$, let $p_{i,j,m} : X_n(2)_m \rightarrow X_n(1)_m$ (resp. $p'_{i,j,m} : X_n(2)'_m \rightarrow X_n(1)'_m$) be the morphism induced by the projection $X_n(2) \rightarrow X_n(1)$ into the i -th and j -th factors. We denote by $\text{Str}(X/W_n)$ (resp. $\text{Str}(X_n/W_n)$) be the category of pairs $(E, \{\epsilon_m\}_m)$, where E is a coherent \mathcal{O}_{X_n} -module and $\{\epsilon_m : p_{2,m}^* E \xrightarrow{\simeq} p_{1,m}^* E\}_m$ (resp. $\{\epsilon_m : p'_{2,m}^* E \xrightarrow{\simeq} p'_{1,m}^* E\}_m$) is a compatible family of linear isomorphisms such that $\epsilon_0 = \text{id}_E$ and $p_{1,2,m}^* \epsilon \circ p_{2,3,m}^* \epsilon = p_{1,3,m}^* \epsilon$ (resp. $p'_{1,2,m}^* \epsilon \circ p'_{2,3,m}^* \epsilon = p'_{1,3,m}^* \epsilon$). Such a datum is usually called a *stratification* on E .

Then, one has the functor

$$(1.19) \quad \begin{aligned} \text{Inf}(X/W_n) &\longrightarrow \text{Str}(X/W_n), \\ E &\mapsto (E_{X_n}, \{p_{2,m}^* E_{X_n} \xrightarrow{\simeq} E_{X_n(1)_m} \xleftarrow{\simeq} p_{1,m}^* E_{X_n}\}_m). \end{aligned}$$

We can also define the functor

$$(1.20) \quad \text{Str}(X/W_n) \longrightarrow \text{Inf}(X/W_n)$$

of converse direction as follows. If we are given $(E, \{\epsilon_m\}_m) \in \text{Str}(X/W_n)$ and an object $U \hookrightarrow T$ in $(X/W_n)_{\text{inf}}$, we define the coherent \mathcal{O}_T -module E_T in the following way. Since there exists a morphism $\varphi : T \rightarrow X_n$ over W_n lifting the closed immersion $X \hookrightarrow X_n$ locally on T , E_T is defined by $E_T := \varphi^* E$ locally. If

we have two morphisms $\varphi, \varphi' : T \rightarrow X_n$ as above, $\varphi \times \varphi'$ induces a morphism $\psi : T \rightarrow X_n(1)_m$ for some m , and $\psi^* \epsilon_m$ defines a gluing data for the sheaf E_T defined locally as above. Thus E_T is defined globally on T by descent. Then the family $\{E_T\}_{U \hookrightarrow T}$ gives an object of $\text{Inf}(X/W_n)$. Thus the functor (1.20) is defined. One can check that it defines a quasi-inverse of (1.19), and so (1.19) is an equivalence.

Next, because the canonical closed immersions $X_n(r)_m \hookrightarrow X_n(r)'_m$ induce the isomorphism $\{X_n(r)_m\}_m \xrightarrow{\cong} \{X_n(r)'_m\}_m$ as ind-schemes, we have the canonical equivalence of categories

$$(1.21) \quad \text{Str}(X_n/W_n) \xrightarrow{\cong} \text{Str}(X/W_n).$$

Finally, we have an equivalence

$$(1.22) \quad \text{Str}(X_n/W_n) \xrightarrow{\cong} (\mathcal{O}_{X_n}\text{-coherent left } \mathcal{D}_{X_n/W_n}\text{-modules})$$

by [BO78, Prop. 2.11, Rmk. 2.13]. By combining (1.19), the quasi-inverse of (1.21) and (1.22), we obtain the equivalence (1.17).

By construction, the functors (1.19), (1.21) and (1.22) are functorial with respect to X_W/W , namely, if we are given a diagram as (1.11), we have the pullback by f_W modulo p^n on $\text{Str}(X/W_n), \text{Str}(X_n/W_n)$, and the category of \mathcal{O}_{X_n} -coherent left \mathcal{D}_{X_n/W_n} -modules, and the functors are compatible with respect to f^* on $\text{Inf}(X/W_n)$ and the above pullback functors. Thus the functors (1.17), (1.18) are also functorial with respect to X_W/W .

For any infinitesimal crystal E on X/W or X/W_n , the value E_T of E at any $U \hookrightarrow T$ is locally free. To prove this, it suffices to consider the case of infinitesimal crystals on X over k , and in this case, the claim follows from the equivalence (1.17), Katz' theorem [Gie75, Thm. 1.3] and [DS07, Lem. 6].

For a smooth variety X over k , let $\text{Inf}(X/W)_{\mathbb{Q}}$ be the \mathbb{Q} -linearization of the category $\text{Inf}(X/W)$, which is called the category of *infinitesimal isocrystals* on X . As in the case of crystalline site, one has a natural functor $\text{Inf}(X/W) \xrightarrow{\mathbb{Q} \otimes} \text{Inf}(X/W)_{\mathbb{Q}}$. When X is liftable to a p -adic smooth formal scheme X_W over W , the functor (1.18) induces the full embedding

$$(1.23) \quad \text{Inf}(X/W)_{\mathbb{Q}} \hookrightarrow ((\mathbb{Q} \otimes \mathcal{O}_{X_W})\text{-coherent left } (\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W})\text{-modules}),$$

which is functorial with respect to X_W/W . For objects in the right hand side, the action of $\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W}$ is assumed to be continuous.

Next we recall the basic facts on convergent isocrystals [Ogu84], [Ogu90]. On a scheme X of finite type over k , the category $\text{Enl}(X/W)$ of *enlargements* is defined in [Ogu84, Defn. 2.1]. Objects are pairs (T, z_T) where T is a p -adic formal flat scheme of finite type over $\text{Spf}(W)$ together with a morphism $(T \otimes_W k)_{\text{red}} \xrightarrow{z_T} X$. Morphisms in the category are the obvious ones. A *convergent isocrystal* [Ogu84, Defn. 2.7]

on X/K is a crystal on $\text{Enl}(X/W)$ with value on (T, z_T) in the \mathbb{Q} -linearization $\text{Coh}(\mathcal{O}_T)_{\mathbb{Q}}$ of the category $\text{Coh}(\mathcal{O}_T)$. This defines a category $\text{Conv}(X/K)$ with the obvious morphisms, which is abelian [Ogu84, Cor. 2.10]. We denote the structure convergent isocrystal on X/K by $\mathcal{O}_{X/K}$. The category $\text{Conv}(X/K)$ is functorial with respect to X/W , and it has descent property for Zariski coverings.

When X is liftable to a p -adic smooth formal scheme X_W over W , we have the equivalence

$$(1.24) \quad \text{Conv}(X/K) \xrightarrow{\cong} ((\mathbb{Q} \otimes \mathcal{O}_{X_W})\text{-coherent left } \mathcal{D}_{X_W, \mathbb{Q}}^{\dagger}\text{-modules}),$$

where $\mathcal{D}_{X_W, \mathbb{Q}}^{\dagger} = \varinjlim_m \mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W}^{(m)}$, $\widehat{\mathcal{D}}_{X_W/W}^{(m)} := \varprojlim_n \mathcal{D}_{X_n/W_n}^{(m)}$ ($X_n := X_W \otimes W_n$) and $\mathcal{D}_{X_n/W_n}^{(m)}$ is the ring of PD-differential operators of level m [Ber96, Prop. 4.1.4]. It is functorial with respect to X_W/W .

For a smooth variety X over k , Ogus defines in [Ogu90, Thm. 0.7.2] a fully faithful functor

$$(1.25) \quad \Phi : \text{Conv}(X/K) \longrightarrow \text{Crys}(X/W)_{\mathbb{Q}}$$

using a nice system of objects in $\text{Enl}(X/W)$ and the local nature of isocrystals [Ogu90, Lem. 0.7.5], such that, for any $\mathcal{E} \in \text{Conv}(X/K)$, the convergent cohomology $H_{\text{conv}}^i(X/K, \mathcal{E})$ (defined in [Ogu90, Section 4]) and the crystalline cohomology $H_{\text{crys}}^i(X/W, \Phi(\mathcal{E}))$ coincide [Ogu90, Thm. 0.7.7]. The functor Φ is functorial with respect to X/W . Also, when X is liftable to a p -adic smooth formal scheme X_W over W , Φ is compatible with the canonical functor

$$(1.26) \quad \begin{aligned} & ((\mathbb{Q} \otimes \mathcal{O}_{X_W})\text{-coherent left } \mathcal{D}_{X_W, \mathbb{Q}}^{\dagger}\text{-modules}) \\ & \longrightarrow ((\mathbb{Q} \otimes \mathcal{O}_{X_W})\text{-coherent left } (\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W}^{(0)})\text{-modules}), \end{aligned}$$

via (1.13) and (1.24). The functor (1.26) is obviously functorial with respect to X_W/W . In the following, we omit to write the functor Φ and regard a convergent isocrystal \mathcal{E} on X as an isocrystal on $(X/W)_{\text{crys}}$ via the functor Φ . For the whole theory of convergent isocrystals, we also refer to [Ber96b], [LeS07].

For a smooth variety X over k , we have the full embedding of sites $(X/W_n)_{\text{crys}} \hookrightarrow (X/W)_{\text{inf}}$, which induces the functors

$$\Phi' : \text{Inf}(X/W) \longrightarrow \text{Crys}(X/W), \quad \Phi'_{\mathbb{Q}} : \text{Inf}(X/W)_{\mathbb{Q}} \longrightarrow \text{Crys}(X/W)_{\mathbb{Q}}.$$

They are functorial with respect to X/W . When X is liftable to a p -adic smooth formal scheme X_W over W , $\Phi' \cdot \Phi'_{\mathbb{Q}}$ are compatible with the canonical functors

$$(1.27) \quad \begin{aligned} & (\mathcal{O}_{X_W}\text{-coherent left } \widehat{\mathcal{D}}_{X_W/W}\text{-modules}) \\ & \longrightarrow (\mathcal{O}_{X_W}\text{-coherent left } \widehat{\mathcal{D}}_{X_W/W}^{(0)}\text{-modules}), \end{aligned}$$

$$(1.28) \quad \begin{aligned} & ((\mathbb{Q} \otimes \mathcal{O}_{X_W})\text{-coherent left } (\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W})\text{-modules}) \\ & \longrightarrow ((\mathbb{Q} \otimes \mathcal{O}_{X_W})\text{-coherent left } (\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W}^{(0)})\text{-modules}) \end{aligned}$$

via (1.9), (1.10), (1.13), (1.18) (1.23), because the constructions involved are done in a parallel way for (iso)crystals and infinitesimal (iso)crystals. The functors (1.27) and (1.28) are functorial with respect to X_W/W .

We prove that Φ' is fully faithful. To see this, we may work locally by Zariski descent for morphisms in $\text{Inf}(X/W)$ and $\text{Crys}(X/W)$. So we may assume that X lifts to a p -adic smooth formal scheme X_W over W . Thus we are reduced to proving the full faithfulness of (1.27). Noting the local freeness of the values of any object in $\text{Inf}(X/W)$, we are reduced to proving the equality

$$M^{\widehat{\mathcal{D}}_{X_W/W}=0} \xrightarrow{\simeq} M^{\widehat{\mathcal{D}}_{X_W/W}^{(0)}=0}$$

of horizontal elements for any \mathcal{O}_{X_W} -locally free $\widehat{\mathcal{D}}_{X_W/W}$ -module M . This is clear because any such M is flat over W and the image of $\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathcal{D}}_{X_W/W}^{(0)}$ is dense in $\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathcal{D}}_{X_W/W}$ because, in terms of local coordinates $x := (x_1, \dots, x_d)$, the former contains the sections $\frac{1}{n!} \left(\frac{\partial}{\partial x} \right)^n$ ($n \in \mathbb{N}^d$) which topologically generates the latter. As a consequence, we see that the functor $\Phi'_{\mathbb{Q}}$ is also fully faithful.

Also, we prove that the functor $\Phi'_{\mathbb{Q}}$ factors through Φ and thus induces the functor

$$\text{Inf}(X/W)_{\mathbb{Q}} \longrightarrow \text{Conv}(X/K)$$

which we denote also by $\Phi'_{\mathbb{Q}}$. To prove it, we may work locally by Zariski descent for $\text{Conv}(X/K)$ and full faithfulness of Φ . So we may assume that X lifts to a p -adic smooth formal scheme X_W over W , and in this case, the claim follows from the fact that the functor (1.28) factors through (1.26). In the following, we omit to write also the functor $\Phi'_{\mathbb{Q}}$ and regard an infinitesimal isocrystal on X as a convergent isocrystal on X/K (hence an isocrystal on X) via the functor $\Phi'_{\mathbb{Q}}$.

We recall the functoriality of the categories discussed above with respect to the absolute Frobenius morphism $F : X \rightarrow X$. By applying the functoriality with respect to the diagram (1.1) in the case $f = F$ and $f' = \sigma_W$, we obtain the pullback functors

- $\text{Crys}(X/W) \rightarrow \text{Crys}(X/W)$, • $\text{Crys}(X/W_n) \rightarrow \text{Crys}(X/W_n)$,
- $\text{MIC}(X)^{\text{an}} \rightarrow \text{MIC}(X)^{\text{an}}$, • $\text{MIC}(X) \rightarrow \text{MIC}(X)$,
- $\text{Coh}(X) \rightarrow \text{Coh}(X)$, • $\text{Crys}(X/W)_{\mathbb{Q}} \rightarrow \text{Crys}(X/W)_{\mathbb{Q}}$,
- $\text{Inf}(X/W) \rightarrow \text{Inf}(X/W)$, • $\text{Inf}(X/W_n) \rightarrow \text{Inf}(X/W_n)$,
- $\text{Inf}(X/W)_{\mathbb{Q}} \rightarrow \text{Inf}(X/W)_{\mathbb{Q}}$, • $\text{Conv}(X/K) \rightarrow \text{Conv}(X/K)$,

which we all denote by F^* . By the functoriality discussed above, the functors (1.2), (1.3), (1.6), (1.7), (1.8) for $n = 1$, (1.14), (1.15), Φ , Φ' , $\Phi'_\mathbb{Q}$ are compatible with the various Frobenius pullbacks F^* . In particular, the pullback F^* by Frobenius on $\text{Crys}(X/W)_\mathbb{Q}$ respects the full subcategories $\text{Conv}(X/K)$, $\text{Inf}(X/W)_\mathbb{Q}$.

When X is liftable to a p -adic smooth formal scheme X_W over W and F is liftable to a morphism $F_W : X_W \rightarrow X_W$ over σ_W , F_W and σ_W induce the pullback functors

- $\text{MIC}(X_n)^{\text{qn}} \rightarrow \text{MIC}(X_n)^{\text{qn}}$, • $\text{MIC}(X_W)^{\text{qn}} \rightarrow \text{MIC}(X_W)^{\text{qn}}$,
- $\text{Str}(X/W_n) \rightarrow \text{Str}(X/W_n)$, • $\text{Str}(X_n/W_n) \rightarrow \text{Str}(X_n/W_n)$,
- $(\mathcal{O}_{X_W}\text{-coherent left } \widehat{\mathcal{D}}_{X_W/W}^{(0)}\text{-modules}) \rightarrow (\mathcal{O}_{X_W}\text{-coherent left } \widehat{\mathcal{D}}_{X_W/W}^{(0)}\text{-modules})$,
- $((\mathbb{Q} \otimes \mathcal{O}_{X_W})\text{-coherent left } (\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W}^{(0)})\text{-modules})$
 $\rightarrow ((\mathbb{Q} \otimes \mathcal{O}_{X_W})\text{-coherent left } (\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W}^{(0)})\text{-modules})$,
- $(\mathcal{O}_{X_n}\text{-coherent left } \mathcal{D}_{X_n/W_n}\text{-modules}) \rightarrow (\mathcal{O}_{X_n}\text{-coherent left } \mathcal{D}_{X_n/W_n}\text{-modules})$,
- $(\mathcal{O}_{X_W}\text{-coherent left } \widehat{\mathcal{D}}_{X_W/W}\text{-modules}) \rightarrow (\mathcal{O}_{X_W}\text{-coherent left } \widehat{\mathcal{D}}_{X_W/W}\text{-modules})$,
- $((\mathbb{Q} \otimes \mathcal{O}_{X_W})\text{-coherent left } (\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W})\text{-modules})$
 $\rightarrow ((\mathbb{Q} \otimes \mathcal{O}_{X_W})\text{-coherent left } (\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W})\text{-modules})$,
- $((\mathbb{Q} \otimes \mathcal{O}_{X_W})\text{-coherent left } \mathcal{D}_{X_W, \mathbb{Q}}^\dagger\text{-modules})$
 $\rightarrow ((\mathbb{Q} \otimes \mathcal{O}_{X_W})\text{-coherent left } \mathcal{D}_{X_W, \mathbb{Q}}^\dagger\text{-modules})$,

which we all denote by F_W^* . The functors (1.8), (1.9), (1.10), (1.13), (1.17), (1.18), (1.19), (1.21), (1.22), (1.23), (1.24), (1.26), (1.27), (1.28) are compatible with the various pullbacks by Frobenius or its lifting.

We give a short review on *Cartier descent* and the *inverse Cartier transform* after Ogus-Vologodsky [OV07].

For $(E, \nabla) \in \text{MIC}(X)$, one defines the p -curvature $\beta : S^\bullet T_X \rightarrow \mathcal{E}nd_{\mathcal{O}_X}(E)$, which is an F^* -linear algebra homomorphism. We say that (E, ∇) has *zero p -curvature* (resp. has *nilpotent p -curvature of length $p - 1$*) if $\beta(S^n T_X) = 0$ for $n \geq 1$ (resp. $n \geq p$). We denote the full subcategory of $\text{MIC}(X)$ of modules with integrable connection with zero p -curvature (resp. nilpotent p -curvature of length $p - 1$) by $\text{MIC}_0(X)$ (resp. $\text{MIC}_{p-1}(X)$). The forgetful functor $\text{MIC}_s(X) \rightarrow \text{Coh}(X)$, $s = 0, p - 1$, yields the abelian structure on $\text{MIC}_s(X)$ with respect to which the functor is exact.

A *Higgs module* is a pair (H, θ) consisting of a coherent \mathcal{O}_X -module H and an \mathcal{O}_X -linear morphism $\theta : H \rightarrow H \otimes \Omega_X^1$ satisfying the integrability condition $\theta \wedge \theta = 0$. The map θ , called the Higgs field, induces an \mathcal{O}_X -algebra homomorphism $S^\bullet T_X \rightarrow \mathcal{E}nd_{\mathcal{O}_X}(H)$, denoted by the same symbol θ . We say that (H, θ) has *nilpotent Higgs field of length $\leq p - 1$* if $\theta(S^n T_X) = 0$ for $n \geq p$. With the obvious

morphisms, we denote the category of Higgs modules with Higgs field zero (resp. nilpotent Higgs field of length $\leq p - 1$) by $\text{HIG}_0(X)$ (resp. $\text{HIG}_{p-1}(X)$). The forgetful functor $\text{HIG}_s(X) \rightarrow \text{Coh}(X)$, $s = 0, p - 1$, yields the abelian structure on $\text{HIG}_{p-1}(X)$ with respect to which the functor is exact, and is an equivalence of categories for $s = 0$.

For a coherent \mathcal{O}_X -module E , F^*E is uniquely endowed with an integrable connection ∇_{can} with zero p -curvature which is characterized by the condition that $F^{-1}(E) \subset F^*E$ is the subsheaf of abelian groups of flat sections. The functor

$$(1.29) \quad \text{Coh}(\mathcal{O}_X) \longrightarrow \text{MIC}_0(X), \quad E \mapsto (F^*E, \nabla_{\text{can}})$$

is an equivalence of categories. This fact is called *Cartier descent* ([OV07, Section 2]). It is easy to see by direct calculation that the functor $F^* : \text{MIC}(X) \rightarrow \text{MIC}(X)$ factors through (1.29) and hence it has the form

$$(1.30) \quad F^* : \text{MIC}(X) \xrightarrow{(E, \nabla) \mapsto E} \text{Coh}(X) \xrightarrow{(1.29)} \text{MIC}_0(X) \subset \text{MIC}(X), \\ (E, \nabla) \mapsto (F^*E, \nabla_{\text{can}}).$$

Ogus-Vologodsky generalized the equivalence (1.29) when X admits a smooth lifting \tilde{X} over W_2 . (In [OV07], they assume the existence of a smooth lifting \tilde{X}' of the Frobenius twist $X' := X \otimes_{\sigma} k$ over W_2 , but this is equivalent to the condition above because k is perfect.) Assuming the existence of \tilde{X} , they generalized the equivalence (1.29) to the equivalence

$$(1.31) \quad C^{-1} : \text{HIG}_{p-1}(X) \longrightarrow \text{MIC}_{p-1}(X),$$

which is called *the inverse Cartier transform*. (Precisely speaking, the functor C^{-1} here is the functor $C_{\mathcal{X}/S}^{-1} \circ \pi_{\mathcal{X}/S}^*$ used in [OV07, (4.16.1)] for $\mathcal{X}/S = (X, \tilde{X}')/\text{Spec } W_2$ and $S = \text{Spec } k$.) Note that any object (E, ∇) in $\text{MIC}(X)^{\text{qn}}$ with E torsion free of rank $\leq p$ is contained in $\text{MIC}_{p-1}(X)$.

Finally, we recall some terminologies on (semi)stability. When X is projective, we fix a k -embedding $\iota : X \hookrightarrow \mathbb{P}_k^N$ of X into a projective space and denote the pullback of $\mathcal{O}_{\mathbb{P}_k^N}(1)$ to X by $\mathcal{O}_X(1)$. For a coherent, torsion free \mathcal{O}_X -module E , the slope $\mu(E) = \text{deg}(E)/\text{rank}(E)$ and the reduced Hilbert polynomial $p_E(n) = \chi(X, E \otimes_{\mathcal{O}_X} \mathcal{O}_X(n))/\text{rank}(E)$ are defined with respect to $\mathcal{O}_X(1)$, as well as μ - (Mumford-Takemoto) or χ - (Gieseker-Maruyama) (semi)stability. As usual, $E \in \text{Coh}(X)$ is said to be μ -(*semi*)*stable* if it is torsion free and $\mu(E') < \mu(E)$ ($\mu(E') < \mu(E)$) for any strict subobject $0 \neq E' \subset E$, and it is said to be *strongly μ -semistable* if $(F^n)^*(E)$ is μ -semistable for all natural numbers n . Similarly for χ -(semi)stability.

We say that an object E in $\text{Crys}(X/k) = \text{MIC}(X)^{\text{qn}}$ is μ -*semistable* as a crystal if E is torsion free as an \mathcal{O}_X -module and $\mu(E') \leq \mu(E)$ for any non-zero subobject

E' of E in $\text{Crys}(X/k) = \text{MIC}(X)^{\text{an}}$. Similarly, we say that an object (E, ∇) in $\text{MIC}_{p-1}(X)$ (resp. in $\text{HIG}_{p-1}(X)$) is μ -(*semi*)*stable* if $E \in \text{Coh}(X)$ is torsion free and $\mu(E') < \mu(E)$ ($\mu(E') \leq \mu(E)$) for any strict subobject $0 \neq (E', \nabla')$ of (E, ∇) in $\text{MIC}_{p-1}(X)$ (resp. in $\text{HIG}_{p-1}(X)$).

2 Statement of the main results

We say that $\mathcal{E} \in \text{Crys}(X/W)_{\mathbb{Q}}$ (resp. $\text{Conv}(X/K)$, $\text{Inf}(X/W)_{\mathbb{Q}}$, $\text{Inf}(X/k)$) is *constant* when it is isomorphic to a finite sum of the structure isocrystal $\mathbb{Q} \otimes \mathcal{O}_{X/W}$ (resp. the structure convergent isocrystal $\mathcal{O}_{X/K}$, the structure isocrystal $\mathbb{Q} \otimes \mathcal{O}_{X/W}$, the structure crystal $\mathcal{O}_{X/k}$). As a p -adic version of Gieseker's conjecture, according to which if a smooth projective variety X over an algebraically closed field k has a trivial étale fundamental group, then infinitesimal crystals on X/k , that is \mathcal{O}_X -coherent \mathcal{D}_X -modules, are constant (see [EM10] for a positive answer), Johan de Jong posed the following conjecture in October 2010:

Conjecture 2.1 (de Jong). *Let X be a connected smooth projective variety over an algebraically closed field k of characteristic $p > 0$ and assume that the étale fundamental group of X is trivial. Then any isocrystal $\mathcal{E} \in \text{Crys}(X/W)_{\mathbb{Q}}$ is constant.*

By the fully faithful functor $\Phi : \text{Conv}(X/K) \rightarrow \text{Crys}(X/W)_{\mathbb{Q}}$ defined in (1.25), Conjecture 2.1 contains the sub-conjecture

Conjecture 2.2. *Let X be a connected smooth projective variety over an algebraically closed field k of characteristic $p > 0$ and assume that the étale fundamental group of X is trivial. Then any convergent isocrystal $\mathcal{E} \in \text{Conv}(X/K)$ is constant.*

The aim of this article is to discuss Conjecture 2.2. We restrict our attention to Conjecture 2.2 rather than Conjecture 2.1 mainly because we shall strongly use the following proposition, which is a special case of [Ogu84, Cor.4.10]:

Proposition 2.3. *Let X be a smooth variety over k . Then the pullback functor $F^* : \text{Conv}(X/K) \rightarrow \text{Conv}(X/K)$ is an equivalence of categories.*

Hence, for any convergent isocrystal \mathcal{E} on X/K and for any $n \in \mathbb{N}$, there is the unique convergent isocrystal $\mathcal{E}^{(n)}$ on X/K with $F^{*n}\mathcal{E}^{(n)} = \mathcal{E}$ up to isomorphism. We call this $\mathcal{E}^{(n)}$ the F^n -*division* of \mathcal{E} .

Remark 2.4. The category $\text{Conv}(X/K)$ is characterized as the intersection

$$\bigcap_n \text{Im}(F^*)^n$$

of the essential images of $(F^*)^n$ ($n \in \mathbb{N}$) in $\text{Crys}(X/W)_{\mathbb{Q}}$. The inclusion $\text{Conv}(X/K) \subseteq \bigcap_n \text{Im}(F^*)^n$ follows from Proposition 2.3. To prove the inclusion in the other direction, we take $\mathcal{E} \in \bigcap_n \text{Im}(F^*)^n$. To prove that \mathcal{E} belongs to $\text{Conv}(X/K)$, we may

work locally, and so we may assume that X is liftable to a p -adic smooth formal scheme X_W over W endowed with a lift $F_W : X_W \rightarrow X_W$ of Frobenius morphism F on X . Then F_W induces an equivalence of categories

$$\begin{aligned} (F_W^*)^n : & ((\mathbb{Q} \otimes \mathcal{O}_{X_W})\text{-coherent left } (\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W}^{(0)})\text{-modules}) \\ & \xrightarrow{\simeq} ((\mathbb{Q} \otimes \mathcal{O}_{X_W})\text{-coherent left } (\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W}^{(n)})\text{-modules}) \end{aligned}$$

such that the composition of it with the canonical functor

$$\begin{aligned} & ((\mathbb{Q} \otimes \mathcal{O}_{X_W})\text{-coherent left } (\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W}^{(n)})\text{-modules}) \\ & \longrightarrow ((\mathbb{Q} \otimes \mathcal{O}_{X_W})\text{-coherent left } (\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W}^{(0)})\text{-modules}) \end{aligned}$$

is equal to the pullback functor $(F_W^*)^n$ on the category of $(\mathbb{Q} \otimes \mathcal{O}_{X_W})$ -coherent left $(\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W}^{(0)})$ -modules, by [Ber00, Thm. 4.1.3, Rmq. 4.1.4(v)]. From this and the compatibility of the functor (1.13) with F^* and F_W^* , we see that the $(\mathbb{Q} \otimes \mathcal{O}_{X_W})$ -coherent left $(\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W}^{(0)})$ -module corresponding to \mathcal{E} admits an action of $\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W}^{(n)}$ for any n . Such actions for $n \in \mathbb{N}$ are consistent because $\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W}^{(0)}$ is dense in $\mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W}^{(n)}$. Thus the actions induce the action of $\mathcal{D}_{X_W, \mathbb{Q}}^\dagger := \varinjlim_n \mathbb{Q} \otimes \widehat{\mathcal{D}}_{X_W/W}^{(n)}$. Hence \mathcal{E} belongs to $\text{Conv}(X/K)$ by the equivalence (1.24).

Remark 2.5. Proposition 2.3 does not necessarily extend to $\text{Crys}(X/W)_\mathbb{Q}$ via (1.25) even when X is projective and smooth, as is shown in Proposition 2.13 below.

Given \mathcal{E} in $\text{Conv}(X/K)$ or $\text{Crys}(X/W)_\mathbb{Q}$, a crystal $E \in \text{Crys}(X/W)$ is called a *lattice of \mathcal{E}* if it is p -torsion free and $\mathcal{E} = \mathbb{Q} \otimes E$ in $\text{Crys}(X/W)_\mathbb{Q}$. Out of any choice $E \in \text{Crys}(X/W)$ with $\mathcal{E} = \mathbb{Q} \otimes E$ in $\text{Crys}(X/W)_\mathbb{Q}$, one constructs a lattice as follows. The surjective morphisms $E/\text{Ker}(p^{n+1}) \rightarrow E/\text{Ker}(p^n)$ in $\text{Crys}(X/W)$ become isomorphisms for n large. Indeed, as X is of finite type, it is enough to show it on an affine X , for which one applies (1.9). Then $E/\text{Ker}(p^n)$ for n large is a lattice of \mathcal{E} . Clearly, there are then many lattices of the same \mathcal{E} .

We now formulate the first main result, proved in Section 3 (compare with [Shi14, Thm. 1.7, Cor. 1.10]).

Theorem 2.6. *Let X be a connected smooth projective variety over k with trivial étale fundamental group. If $\mathcal{E} \in \text{Conv}(X/K)$ is such that for any $n \in \mathbb{N}$, the F^n -division $\mathcal{E}^{(n)}$ of \mathcal{E} admits a lattice $E^{(n)}$ with $E_X^{(n)} \in \text{Coh}(X)$ strongly μ -semistable, then \mathcal{E} is constant.*

We have the following corollary confirming the conjecture of de Jong for infinitesimal isocrystals, which will be also proved in Section 3.

Corollary 2.7. *Let X be a connected smooth projective variety over k with trivial étale fundamental group. Then any infinitesimal isocrystal on X is constant.*

A non-zero $\mathcal{E} \in \text{Conv}(X/K)$ is called *irreducible* if it is in its category (recall it is abelian), i.e. if it does not admit any non-zero strict subobject. In general, every object admits a Jordan-Hölder filtration. Its irreducible subquotients are called *irreducible constituents*. Using Theorem 2.6, we shall prove the following theorem in Section 4.

Theorem 2.8. *Let X be a connected smooth projective variety over k with trivial étale fundamental group. If $\mathcal{E} \in \text{Conv}(X/K)$ satisfies either of the following conditions:*

- (1) *any irreducible constituent of \mathcal{E} is of rank 1;*
- (2) *$\mu_{\max}(\Omega_X^1) < 2$ and any irreducible constituent of \mathcal{E} is of rank ≤ 2 ;*
- (3) *$\mu_{\max}(\Omega_X^1) < 1$ and any irreducible constituent of \mathcal{E} is of rank ≤ 3 ;*
- (4) *$r \geq 4$, $\mu_{\max}(\Omega_X^1) < \frac{1}{N(r)}$ and any irreducible constituent of \mathcal{E} is of rank $\leq r$,
where $N(r) := \max_{a,b \geq 1, a+b \leq r} \text{lcm}(a, b)$.*

then \mathcal{E} is constant.

Corollary 2.9. *Let X be a connected smooth projective variety over k with trivial étale fundamental group. Then*

- (1) *any $\mathcal{E} \in \text{Conv}(X/K)$ of rank 1 is constant;*
- (2) *if $\mu_{\max}(\Omega_X^1) \leq 0$, then any $\mathcal{E} \in \text{Conv}(X/K)$ is constant.*

By [Lan15b, Thm.0.1], simply connected non-uniruled Calabi-Yau varieties of dimension d in characteristic $\geq (d-1)(d-2)$ fulfill condition (2). However, even if many examples of Calabi-Yau varieties which are not liftable to characteristic 0 are known, it is not clear whether some of them are not uniruled. If X is simply connected and uniruled, it is likely that one can show that $\mathcal{E} \in \text{Conv}(X/K)$ is constant directly by geometric method.

When $p \geq 3$, there is a purely cohomological proof of Corollary 2.9 (1). The following proposition is due to the first named author and Ogus.

Proposition 2.10. *Let X be a connected smooth projective variety over k with trivial étale fundamental group. Then*

- (1) *for $p \geq 3$, any locally free $E \in \text{Crys}(X/W)$ of rank 1 is constant;*
- (2) *extensions in $\text{Crys}(X/W)_{\mathbb{Q}}$ of $\mathbb{Q} \otimes \mathcal{O}_{X/W}$ by itself are constant.*

Proof. Since the étale fundamental group is trivial, $H^1(X, \mathcal{O}_X^\times)$ has no torsion of order prime to p . So $\text{Pic}_{\text{red}}^0(X) = 0$ and so $H_{\text{crys}}^1(X/W, \mathcal{O}_{X/W}) = 0$. This proves (2).

To prove (1), it suffices to prove the similar assertion for crystals on the nilpotent crystalline site $(X/W)_{\text{Ncrys}}$, by [Ber74, IV Rmq. 1.6.6].

One has the exact sequences on $(X/W)_{\text{Ncrys}}$,

$$(2.1) \quad 1 \longrightarrow K \longrightarrow \mathcal{O}_{X/W}^\times \longrightarrow \mathcal{O}_X^\times \longrightarrow 1,$$

$$(2.2) \quad 0 \longrightarrow J \longrightarrow \mathcal{O}_{X/W} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

defining J and K . The p -adic logarithm $\log : K \rightarrow J$ and exponential $\exp : J \rightarrow K$ functions are well defined and are isomorphisms on $(X/W)_{\text{Ncrys}}$. Hence, from the exact sequences (2.1), (2.2), we obtain on $(X/W)_{\text{Ncrys}}$ the exact sequences

$$(2.3) \quad 0 \rightarrow H_{\text{Ncrys}}^1(X/W, J) \rightarrow H_{\text{Ncrys}}^1(X/W, \mathcal{O}_{X/W}^\times) \xrightarrow{\alpha} H^1(X, \mathcal{O}_X^\times) \xrightarrow{\beta} H_{\text{Ncrys}}^2(X/W, J),$$

and

$$(2.4) \quad 0 \rightarrow H_{\text{Ncrys}}^1(X/W, J) \xrightarrow{\gamma} H_{\text{Ncrys}}^1(X/W, \mathcal{O}_{X/W}).$$

From (2.4) and the vanishing $H_{\text{Ncrys}}^1(X/W, \mathcal{O}_{X/W}) = H_{\text{crys}}^1(X/W, \mathcal{O}_{X/W}) = 0$ (where the first equality follows from [Ber74, V 2.4]), one deduces $H_{\text{Ncrys}}^1(X/W, J) = 0$. Hence α is injective. Moreover, by (2.3), we obtain the commutative diagram

$$(2.5) \quad \begin{array}{ccccc} H_{\text{Ncrys}}^1(X, \mathcal{O}_{X/W}^\times) & \xrightarrow{\alpha} & H^1(X, \mathcal{O}_X^\times) & \xrightarrow{\gamma \circ \log \circ \beta} & H_{\text{Ncrys}}^2(X/W, \mathcal{O}_{X/W}) \\ & & \delta \downarrow & & \parallel \\ & & H^1(X, \mathcal{O}_X^\times) \otimes \mathbb{Z}_p & \xrightarrow{\epsilon} & H_{\text{Ncrys}}^2(X/W, \mathcal{O}_{X/W}), \end{array}$$

where δ is induced by the inclusion $\mathbb{Z} \rightarrow \mathbb{Z}_p$ and ϵ is the \mathbb{Z}_p -linearization of $\gamma \circ \log \circ \beta$. By construction, the upper horizontal line is a complex. On the other hand, δ is injective because $H^1(X, \mathcal{O}_X^\times)$ has no torsion of order prime to p . Also, by the commutativity of the diagram [Gro85, I (5.1.7)], ϵ is identified with the map $\text{NS}(X) \otimes \mathbb{Z}_p \rightarrow H_{\text{crys}}^2(X/W, \mathcal{O}_{X/W})$ given in [Ill79, II Prop. 6.8] and so it is injective. Then an easy diagram chase shows that $H_{\text{Ncrys}}^1(X, \mathcal{O}_{X/W}^\times) = 0$. This proves (1). \square

Proposition 2.10 together with the following proposition implies Corollary 2.9 (1) when $p \geq 3$.

Proposition 2.11. *Let X be a smooth variety over k . Then any isocrystal on X which is filtered such that the associated graded isocrystal is a sum of rank 1 isocrystals admits a locally free lattice.*

Proof. We start with the rank 1 case. The idea of the proof is simple. Locally it uses the equivalence (1.9) and the fact that the reflexive hull of a rank 1 coherent sheaf on a regular scheme is locally free.

So assume first X lifts to an affine p -adic smooth formal scheme $X_W = \text{Spf}(A)$ (hence $X = \text{Spec}(A/p)$). Let E be a lattice of a rank 1 isocrystal \mathcal{E} . Via (1.9), writing $M = \Gamma(X_W, E)$, $E \in \text{Crys}(X/W)$ is given by an integrable quasi-nilpotent connection

$$(2.6) \quad \nabla_M : M \rightarrow M \otimes_A \hat{\Omega}_A^1$$

where $\hat{\Omega}_A^1 := \varprojlim_n \Gamma(X_n, \Omega_{X_n}^1)$. Then A , $\hat{\Omega}_A^1$, M are sheafified to \mathcal{O} , Ω^1 , \mathcal{M} on $\text{Spec}(A)$ by the standard procedure.

As M is p -torsion free and $A[p^{-1}] \otimes_A M$ is a projective module over $A[p^{-1}]$ of rank 1, there is a closed codimension ≥ 2 subscheme $C \subset \text{Spec}(A)$ such that \mathcal{M} restricted to $\text{Spec}(A) \setminus C$ is locally free of rank 1. Define $N = \Gamma(\text{Spec}(A) \setminus C, \mathcal{M})$. Then, as A is regular, N is a projective A -module of rank 1 and, as $\hat{\Omega}_A^1$ is a projective A -module, projection formula implies that (2.6) induces an integrable connection

$$(2.7) \quad \nabla_N : N \rightarrow N \otimes_A \hat{\Omega}_A^1$$

which is quasi-nilpotent, as this condition is local on sections of \mathcal{M} . The connection (2.7) can be seen as a connection on $X_W = \text{Spf}(A)$, and so it defines a locally free lattice E in $\text{Crys}(X/W)$ of \mathcal{E} .

Then we can glue the locally defined lattices of \mathcal{E} by Lemma 2.12 below, by replacing the lattices E by $p^n E$ for suitable n 's. This finishes the proof of the rank 1 case.

Let $0 \rightarrow \mathcal{E}' \xrightarrow{a} \mathcal{E} \xrightarrow{b} \mathcal{E}'' \rightarrow 0$ be an exact sequence in $\text{Crys}(X/W)_{\mathbb{Q}}$, with $\mathcal{E} \neq 0, \mathcal{E}'' \neq 0$, both $\mathcal{E}', \mathcal{E}''$ satisfying the assumptions of the proposition. Let E be a lattice of \mathcal{E} . Then $0 \rightarrow a^{-1}(\text{Ker}(b)) \xrightarrow{a} E \xrightarrow{b} b(E) \rightarrow 0$ is an exact sequence ϵ in $\text{Crys}(X/W)$, while $a^{-1}(\text{Ker}(b))$ is a lattice of \mathcal{E}' , and $b(E)$ is a lattice of \mathcal{E}'' . By induction on the rank of \mathcal{E} , there are locally free lattices E' of \mathcal{E}' and E'' of \mathcal{E}'' , which we can rescale by multiplication with p -powers such that they are injective maps $E'' \xrightarrow{i} b(E)$ and $a^{-1}(\text{Ker}(b)) \xrightarrow{j} E'$. If we pull back ϵ by i and push the resulting extension by j , we obtain the extension $0 \rightarrow E' \rightarrow E''' \rightarrow E'' \rightarrow 0$ of E'' by E' in $\text{Crys}(X/W)$ such that $\mathbb{Q} \otimes E''' = \mathcal{E}$. Moreover, E''' is locally free. This finishes the proof. \square

Lemma 2.12. *Let X be a connected smooth variety over k and let E, E' be rank 1 locally free crystals on $(X/W)_{\text{crys}}$ endowed with an isomorphism $\varphi : \mathbb{Q} \otimes E \xrightarrow{\cong} \mathbb{Q} \otimes E'$ in $\text{Crys}(X/W)_{\mathbb{Q}}$. Then, for some $n \in \mathbb{Z}$, φ induces an isomorphism $p^n E \xrightarrow{\cong} E'$ in $\text{Crys}(X/W)$.*

Proof. By assumption, φ induces the isomorphism $\mathbb{Q} \otimes (E^\vee \otimes E') \xrightarrow{\cong} \mathbb{Q} \otimes \mathcal{O}_{X/W}$ in $\text{Crys}(X/W)_{\mathbb{Q}}$. Thus $H_{\text{crys}}^0(X/W, E^\vee \otimes E') = \text{Hom}_{\text{Crys}(X/W)}(E, E') = p^n W$ for some $n \in \mathbb{Z}$. Hence φ induces an invertible morphism $p^n E \rightarrow E'$, thus an isomorphism. \square

We shall reprove in Theorem 4.4 by a different method a weaker version of Proposition 2.11, together with statements in the higher rank case.

Finally in this section, we provide an example for which F^* is not surjective on $\text{Crys}(X/W)_{\mathbb{Q}}$, which we promised in Remark 2.5.

Proposition 2.13. *Assume $p \geq 3$ and let X be a supersingular elliptic curve. Then the pullback functor*

$$F^* : \text{Crys}(X/W)_{\mathbb{Q}} \longrightarrow \text{Crys}(X/W)_{\mathbb{Q}}$$

is not essentially surjective.

Proof. Let X_W be a formal lift of X over $\text{Spf } W$ and let ω be a generator of $H^1(X_W, \Omega_{X_W/W}^1)$. Then $(\mathcal{O}_{X_W}, d + p\omega)$ defines a module with integrable connection on X_W over W and one can check that it is quasi-nilpotent. Hence it defines a non-constant locally free crystal of rank 1 on $(X/W)_{\text{crys}}$, which we denote by E . We prove

Claim 2.14. $(\mathcal{O}_{X_W}, d + p\omega) \in \text{Crys}(X/W)_{\mathbb{Q}}$ is not infinitely F^* -divisible.

Proof. By Lemma 2.12, when E' is a locally free crystal of rank 1 on $(X/W)_{\text{crys}}$ such that $\mathbb{Q} \otimes E \cong \mathbb{Q} \otimes E'$, then E and E' are isomorphic crystals. So, if E is infinitely F^* -divisible in $\text{Crys}(X/W)_{\mathbb{Q}}$, then so in $\text{Crys}(X/W)$. Hence it suffices to prove that E is not infinitely F^* -divisible in $\text{Crys}(X/W)$.

By the restriction functor from $\text{Crys}(X/W)$ to the category of crystals on $(X/W)_{\text{Ncrys}}$, it is enough to show that E on $(X/W)_{\text{Ncrys}}$ is not infinitely F^* -divisible.

Assume that $E = (F^*)^n E'_n$ with $E'_n \in H_{\text{Ncrys}}^1(X/W, \mathcal{O}_{X/W}^{\times})$ for $n \in \mathbb{N}$. Then $(E'_n)_X \in H^1(X, \mathcal{O}_X^{\times})$ is a p^n -torsion line bundle, thus is constant as X is supersingular. Thus via (2.3), $E, E'_n \in H_{\text{Ncrys}}^1(X/W, J)$, and via (2.4), $0 \neq \gamma(E) = (F^*)^n \gamma(E'_n) \in H_{\text{Ncrys}}^1(X/W, \mathcal{O}_{X/W}) = H_{\text{crys}}^1(X/W, \mathcal{O}_{X/W})$. As the slopes of F^* on $H_{\text{crys}}^1(X/W, \mathcal{O}_{X/W})$ are strictly positive, this is impossible. □

□

3 Proof of Theorem 2.6

In this section, we prove Theorem 2.6, so throughout, X is a smooth projective variety of dimension d over an algebraically closed field k of characteristic $p > 0$.

A coherent \mathcal{O}_X -module E has crystalline Chern classes $c_i^{\text{crys}}(E)$ in crystalline cohomology $H_{\text{crys}}^{2i}(X/W)$, a module of finite type over W . In [Lan11, §1.1], numerical Chern classes $c_i(E)$ are defined in the group $Z_{d-i}(X)/\sim$, where \sim is the numerical equivalence relation on the free group on dimension $(d-i)$ -points. Denoting by $H_{\text{alg}}^i \subset H_{\text{crys}}^{2i}(X/W)$ the sub- W -module spanned by the $c_i^{\text{crys}}(E)$'s, one has a group homomorphism $H_{\text{alg}}^i \rightarrow (Z_{d-i}(X)/\sim) \otimes_{\mathbb{Z}} W$. Since $Z_{d-i}(X)/\sim$ is a free \mathbb{Z} -module of finite rank, $c_i^{\text{crys}}(E) = 0$ implies $c_i(E) = 0$. As the (reduced) Hilbert polynomial depends only on $c_i(E)$, if $c_i(E) = 0$ for all $i \geq 1$, then $p_E = p_{\mathcal{O}_X}$.

Proposition 3.1. *Let X be a smooth projective variety over k , let \mathcal{E} be a convergent isocrystal on X/K and let $E \in \text{Crys}(X/W)$ be a lattice of \mathcal{E} . Then $c_i^{\text{crys}}(E_X) = 0$ for any $i > 0$, and, if E_X is torsion free, $p_{E_X} = p_{\mathcal{O}_X}$, $\mu(E_X) = 0$.*

Proof. For $n \in \mathbb{N}$, let $\mathcal{E}^{(n)}$ be the F^n -division of \mathcal{E} and let $E^{(n)} \in \text{Crys}(X/W)$ be a lattice of $\mathcal{E}^{(n)}$. As X is smooth, there exists locally a lift X_W of X to a smooth p -adic formal scheme over W and a local lift of F on X_W , which is faithfully flat. Thus the equivalence (1.9) implies that the p -torsion freeness of a crystal is preserved by F^* and so $(F^*)^n E^{(n)}$ is a lattice of \mathcal{E} . In addition $((F^*)^n E^{(n)})_X = (F^*)^n E_X^{(n)}$. Hence, if we prove $c_i^{\text{crys}}(E_X) = c_i^{\text{crys}}((F^*)^n E_X^{(n)})$ in this situation, we have $c_i^{\text{crys}}(E_X) = c_i^{\text{crys}}((F^*)^n E_X^{(n)}) = p^{ni} c_i^{\text{crys}}(E_X^{(n)})$ for all $n \geq 1$, thus $c_i^{\text{crys}}(E_X) = 0$ in the finite type W -module $H_{\text{crys}}^{2i}(X/W)$, as claimed. Therefore, it suffices to prove that $c_i^{\text{crys}}(E_X)$ does not depend on the choice of the lattice E .

So let E' be another lattice of \mathcal{E} . Then, replacing E' by $p^a E'$ for some $a \in \mathbb{N}$, one has $p^n E \subseteq E' \subseteq E$ for some $n \in \mathbb{N}$, where $p^n E$ is the image of $p^n : E \rightarrow E$, and it suffices to treat this case. For $0 \leq i \leq n$, let E'^i be the image of the map $E' \oplus p^i E \rightarrow E$ defined as the sum of inclusions. Then we have $E'^0 = E$, $E'^n = E'$ and $pE'^{i-1} \subseteq E'^i \subseteq E^{i-1}$ ($1 \leq i \leq n$). So to prove the equality $c_i^{\text{crys}}(E) = c_i^{\text{crys}}(E')$, we may assume that $pE \subseteq E' \subseteq E$. If we denote $Q := \text{Coker}(E' \rightarrow E)$, we have the following commutative diagram with exact horizontal lines in $\text{Crys}(X/W)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 0 \\ & & p \downarrow & & p \downarrow & & 0 \downarrow & & \\ 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 0. \end{array}$$

By the snake lemma, we obtain the exact sequence

$$0 \longrightarrow Q \longrightarrow E'/pE' \longrightarrow E/pE \longrightarrow Q \longrightarrow 0$$

in $\text{Crys}(X/W)$. Since all the objects in the above sequence are p -torsion, we can regard it as an exact sequence in $\text{Crys}(X/k)$. By evaluating this sequence at X and noting the equalities $(E/pE)_X = E_X/pE_X = E_X$, $(E'/pE')_X = E'_X/pE'_X = E'_X$, we obtain the exact sequence

$$0 \longrightarrow Q_X \longrightarrow E'_X \longrightarrow E_X \longrightarrow Q_X \longrightarrow 0$$

of coherent \mathcal{O}_X -modules. Hence $[E_X] = [E'_X]$ in $K_0(X)$ and so $c_i^{\text{crys}}(E_X) = c_i^{\text{crys}}(E'_X)$ for all $i \in \mathbb{N}$. \square

The following proposition, which uses Gieseker's conjecture, proven in [EM10], is the key step for the proof.

Proposition 3.2. *Let X be a connected smooth projective variety over k with trivial étale fundamental group. Let r be a positive integer. Then there exists a positive integer $a = a(X, r)$ satisfying the following condition: For any sequence of χ -stable sheaves $\{E_i\}_{i=0}^a$ on X with $\text{rank } E_0 \leq r$, $p_{E_i} = p_{\mathcal{O}_X}$ ($0 \leq i \leq a$) and $F^* E_i = E_{i+1}$ ($0 \leq i \leq a-1$), E_a is necessarily of rank 1 and isomorphic to \mathcal{O}_X .*

Proof. By standard base change argument, we may assume that k is uncountable. For $1 \leq s \leq r$, let M_s be the moduli of χ -stable sheaves on X with rank s and reduced Hilbert polynomial $p_{\mathcal{O}_X}$, which is constructed by Langer ([Lan04], [Lan04b]). It is a quasi-projective scheme over k . Also, let $M_{s,n}$ be the locus of closed points consisting of χ -stable sheaves G such that $(F^*)^n G$ remains χ -stable. It is known to be an open subvariety of M_s endowed with the reduced structure. (See discussion in the beginning of [EM10, §3].) The pullback by F induces the morphism V (over σ) called Verschiebung

$$\cdots \longrightarrow M_{s,2} \xrightarrow{V} M_{s,1} \xrightarrow{V} M_s.$$

Let $\text{Im}V^n$ be the image of $V^n : M_{s,n} \longrightarrow M_s$, which is a constructible set of the topological space M_s . Then, $\dim \text{Im}V^n$ is stable for $n \gg 0$, which we denote by f . Assume $f > 0$. Then the generic point of some irreducible closed subscheme of dimension f remains contained in $\text{Im}V^n$ ($n \in \mathbb{N}$). Pick such an irreducible closed subscheme and denote it by C . Then $C \cap \text{Im}V^n$ is non-empty for any $n \in \mathbb{N}$ and it contains an open subscheme of C . So there exists a closed subscheme $D_n \subsetneq C$ of smaller dimension such that $C \cap \text{Im}V^n \supseteq C \setminus D_n$. Then $C \cap (\bigcap_n \text{Im}V^n) \supseteq C \setminus (\bigcup_n D_n)$, and $C \setminus (\bigcup_n D_n)$ contains at least two k -rational points P, P' , because k is uncountable. On the other hand, the k -rational points of $\bigcap_n \text{Im}V^n$ are moduli points of infinitely F -divisible torsion free sheaves, thus they are locally free infinitely F -divisible sheaves. By the affirmation [EM10, Thm. 1.1] of Gieseker's conjecture, $\bigcap_n \text{Im}V^n$ is either empty or consists only of the moduli point of \mathcal{O}_X . Since P, P' are different k -rational points of $\bigcap_n \text{Im}V^n$, this is a contradiction. So $\text{Im}V^n$ consists of a finite set of points (possibly empty) for some n . Then, since $\bigcap_n \text{Im}V^n$ is empty (if $s \geq 2$) or consists of one point corresponding to \mathcal{O}_X (if $s = 1$), it is equal to $\text{Im}V^{a(s)}$ for some $a(s) \in \mathbb{N}$. Let us define a to be the maximum of natural numbers $a(s)$ ($s \leq r$). Then, if we are given a sequence $\{E_i\}_{i=0}^a$ as in the statement of the proposition with $s := \text{rank}E_0 \leq r$, E_a defines a k -rational point of $\text{Im}V^{a(s)} \subseteq M_s$. Then s should be equal to 1 and E_a should be isomorphic to \mathcal{O}_X . \square

We also use the following proposition.

Proposition 3.3. *Let X be a connected projective smooth variety over k with trivial étale fundamental group. Then there exists a positive integer $b = b(X)$ satisfying the following condition: For any sequence of locally free sheaves $\{E_i\}_{i=0}^{b(r-1)}$ on X with $\text{rank}E_0 = r$, $F^*E_i = E_{i+1}$ ($0 \leq i \leq b(r-1) - 1$) such that E_0 is an iterated extension of \mathcal{O}_X , $E_{b(r-1)}$ is isomorphic to \mathcal{O}_X^r .*

Proof. The proof is similar to that in [EM10, Prop. 2.4]. By [Mum70, Cor. on p.143], one has the decomposition

$$H^1(X, \mathcal{O}_X) = H^1(X, \mathcal{O}_X)_{\text{nilp}} \oplus H^1(X, \mathcal{O}_X)_{\text{ss}}$$

of $H^1(X, \mathcal{O}_X)$ as k -vector spaces such that the absolute Frobenius F^* acts nilpotently on $H^1(X, \mathcal{O}_X)_{\text{nilp}}$ and as a bijection on $H^1(X, \mathcal{O}_X)_{\text{ss}}$. Moreover, one has

$$(3.1) \quad \begin{aligned} H^1(X, \mathcal{O}_X)_{\text{ss}} &= H^1(X, \mathcal{O}_X)^{F=1} \otimes_{\mathbb{F}_p} k \\ &= H^1_{\text{ét}}(X, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k = \text{Hom}(\pi_1^{\text{ét}}(X), \mathbb{F}_p) \otimes_{\mathbb{F}_p} k = 0, \end{aligned}$$

and there exists some $b \in \mathbb{N}$ such that $(F^*)^b$ acts by 0 on $H^1(X, \mathcal{O}_X)_{\text{nilp}}$, since $H^1(X, \mathcal{O}_X)_{\text{nilp}}$ is finite-dimensional. So $(F^*)^b$ acts by 0 on $H^1(X, \mathcal{O}_X)$. We prove the proposition for this choice of b .

By assumption on E_0 , there exists a filtration

$$0 = E_{0,0} \subset E_{0,1} \subset \cdots \subset E_{0,r} = E_0$$

the graded quotients of which are isomorphic to \mathcal{O}_X . By pulling back to E_i via $(F^*)^i$, we obtain the filtration

$$0 = E_{i,0} \subset E_{i,1} \subset \cdots \subset E_{i,r} = E_i$$

of E_i still with graded quotients isomorphic to \mathcal{O}_X . We prove that $E_{b(\ell-1),\ell}$ is isomorphic to \mathcal{O}_X^ℓ by induction on ℓ . Assume that $E_{b(\ell-1),\ell} \cong \mathcal{O}_X^\ell$. Then, for $b(\ell-1) \leq n \leq b\ell$, consider the extension class e_n of the exact sequence

$$0 \longrightarrow E_{n,\ell} \longrightarrow E_{n,\ell+1} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

in $H^1(X, E_{n,\ell}) = H^1(X, \mathcal{O}_X)^\ell$. The family of classes $\{e_n\}_{n=b(\ell-1)}^{b\ell}$ defines an element of the inverse limit of the diagram

$$H^1(X, \mathcal{O}_X)^\ell \xrightarrow{F^*} \cdots \xrightarrow{F^*} H^1(X, \mathcal{O}_X)^\ell$$

of length b with last component $e_{b\ell}$. Then, by definition of b , $e_{b\ell} = 0$. So $E_{b\ell,\ell+1}$ is isomorphic to $\mathcal{O}_X^{\ell+1}$. This finishes the proof. \square

We use Propositions 3.2 and 3.3 to proceed towards the proof of Theorem 2.6.

Proposition 3.4. *Let X be a connected smooth projective variety over k with trivial étale fundamental group. Let r be a positive integer. Let $\mathcal{E} \in \text{Conv}(X/K)$ be of rank r and let E be a lattice of \mathcal{E} such that $E_X \in \text{Coh}(X)$ is strongly μ -semistable. Then, there exists a positive integer $c = c(X, r)$ such that $((F^*)^c E)_X \in \text{Crys}(X/k)$ is constant.*

The following result of Langer [Lan11, Theorem 4.1] plays a crucial rôle in the proof of Proposition 3.4.

Theorem 3.5 (Langer). *Let X be a smooth projective variety over k , let E be a strongly μ -semistable sheaf with vanishing Chern classes. Then there exists a filtration of E whose graded quotients are μ -stable, strongly μ -semistable locally free sheaves with vanishing Chern classes.*

Proof of Proposition 3.4. Take $a = a(X, r), b = b(X) \in \mathbb{N}$ so that the statement of Propositions 3.2, 3.3 are satisfied, and set $c := br(r-1) + ar + 1$. We prove that the proposition is true for this choice of c .

First, note that $(F^*)^n E_X = ((F^*)^n E)_X$ for any $n \in \mathbb{N}$. Hence $(F^*)^n E_X$ has vanishing Chern classes by Proposition 3.1, and is strongly μ -semistable by assumption, for all $n \geq 0$.

For $0 \leq n \leq c-1 = br(r-1) + ar$, define a filtration $\{((F^*)^n E_X)_q\}_{q=0}^{q_n}$ of $(F^*)^n E_X$ whose graded quotients are μ -stable, strongly μ -semistable with vanishing Chern classes, in the following way. First, when $n=0$, take a filtration $\{(E_X)_q\}_{q=0}^{q_0}$ of E_X whose graded quotients are μ -stable, strongly μ -semistable with vanishing Chern classes. (Such a filtration exists by Theorem 3.5 because E_X has vanishing Chern classes by Proposition 3.1 and strongly μ -semistable by assumption.) When we defined $\{((F^*)^{n-1} E_X)_q\}_{q=0}^{q_{n-1}}$, the pull-back $\{F^*((F^*)^{n-1} E_X)_q\}_{q=0}^{q_{n-1}}$ of it by F^* defines a filtration of $(F^*)^n E_X$ whose graded quotients are strongly μ -semistable with vanishing Chern classes. Then, using Theorem 3.5 for the graded quotients, we can refine this filtration to a filtration $\{((F^*)^n E_X)_q\}_{q=0}^{q_n}$ of $(F^*)^n E_X$ whose graded pieces are μ -stable, strongly μ -semistable with vanishing Chern classes.

By definition, we have

$$1 \leq q_0 \leq q_1 \leq \cdots \leq q_{c-1} = q_{br(r-1)+ar} \leq r.$$

So, there exists some j with $0 \leq j \leq b(r-1)^2 + a(r-1)$ such that $q_j = \cdots = q_{j+b(r-1)+a} (= Q)$.

Put $G_{n,q} := ((F^*)^n E_X)_q / ((F^*)^n E_X)_{q-1}$. Then, for each $1 \leq q \leq Q$, any subsequence of length a of the sequence $\{G_{n,q}\}_{n=j}^{j+b(r-1)+a}$ satisfies the assumption of Proposition 3.2. Hence $\{G_{n,q}\}_{n=j+a}^{j+b(r-1)+a}$'s ($1 \leq q \leq Q$) are isomorphic to the constant sequence $\{\mathcal{O}_X\}_{n=j+a}^{j+b(r-1)+a}$. Then, we can apply Proposition 3.3 to the sequence $\{(F^*)^n E_X\}_{n=j+a}^{j+b(r-1)+a}$. So $(F^*)^{j+b(r-1)+a} E_X = \mathcal{O}_X^r$ (hence $(F^*)^{c-1} E_X = \mathcal{O}_X^r$) in $\text{Coh}(X)$.

Therefore, $((F^*)^{c-1} E)_X$ has the form $(\mathcal{O}_X^r, \nabla)$ when regarded as an object in $\text{MIC}(X)^{\text{an}}$ via the equivalence (1.8). Then, by (1.30), one has

$$((F^*)^c E)_X = F^*(\mathcal{O}_X^r, \nabla) = (\mathcal{O}_X^r, d)$$

in $\text{MIC}(X)^{\text{an}}$. So $((F^*)^c E)_X \in \text{Crys}(X/k)$ is constant. \square

Proposition 3.4 deals with the value of a lattice of an isocrystal in $\text{Crys}(X/k)$. To go up to $\text{Crys}(X/W_n)$, we consider the deformation theory of crystals.

Let X be a smooth projective variety over k and fix $n, r \in \mathbb{N}$. Let us denote the restriction functor $\text{Crys}(X/W_{n+1}) \rightarrow \text{Crys}(X/W_n)$ by $G \mapsto \overline{G}$. Let \mathcal{D} be the set of pairs (G, φ) consisting of $G \in \text{Crys}(X/W_{n+1})$ and an isomorphism $\varphi : \mathcal{O}_{X/W_n}^r \xrightarrow{\cong} \overline{G}$ in $\text{Crys}(X/W_n)$. Then \mathcal{D} is a pointed set, whose distinguished element is $(\mathcal{O}_{X/W_{n+1}}, \text{id})$. The pullback $(G, \varphi) \mapsto (F^*G, F^*\varphi)$ by F^* defines a morphism of

pointed sets $F^* : \mathcal{D} \rightarrow \mathcal{D}$. We denote by $H_{\text{crys}}^n(X/k)$ the crystalline cohomology of X over k , which is the same as the de Rham cohomology $H^n(X, \Omega_{X/k}^\bullet)$.

Proposition 3.6. *Let the notations be as above. Then there is an isomorphism of pointed sets*

$$e : \mathcal{D} \xrightarrow{\cong} H_{\text{crys}}^1(X/k)^{r^2}.$$

Moreover, the following diagram is commutative:

$$(3.2) \quad \begin{array}{ccc} \mathcal{D} & \xrightarrow{e} & H_{\text{crys}}^1(X/k)^{r^2} \\ F^* \downarrow & & \downarrow F^* \\ \mathcal{D} & \xrightarrow{e} & H_{\text{crys}}^1(X/k)^{r^2}. \end{array}$$

Proof. For $\ell = n, n+1$, let D_ℓ be the PD-envelope of the closed immersion $X \xrightarrow{\iota} \mathbb{P}_k^N \hookrightarrow \mathbb{P}_{W_\ell}^N$. Using the equivalence $\text{Crys}(X/W_\ell) \cong \text{MIC}(D_\ell)^{\text{qm}}$ of (1.12), we consider the pointed set \mathcal{D} in terms of objects in $\text{MIC}(D_\ell)^{\text{qm}}$. Namely, we denote the restriction $\text{MIC}(D_{n+1}) \rightarrow \text{MIC}(D_n)^{\text{qm}}$ by $(G, \nabla) \mapsto (\overline{G}, \overline{\nabla})$ and we regard \mathcal{D} as the set of pairs $((G, \nabla), \varphi)$ consisting of $(G, \nabla) \in \text{MIC}(D_{n+1})^{\text{qm}}$ and an isomorphism $\varphi : (\mathcal{O}_{D_n}, d) \xrightarrow{\cong} (\overline{G}, \overline{\nabla})$ in $\text{MIC}(D_n)^{\text{qm}}$.

Assume given an object $G := ((G, \nabla), \varphi)$ in \mathcal{D} . Take an affine open covering $\mathcal{U} = \{U_\alpha\}_\alpha$ of D_{n+1} and for each α , take an isomorphism $\psi_\alpha : \mathcal{O}_{U_\alpha}^r \xrightarrow{\cong} G|_{U_\alpha}$ which lifts $\varphi|_{D_n \times_{D_{n+1}} U_\alpha}$. On each U_α , the connection $\psi_\alpha^*(\nabla)$ is written as $d + p^n A_\alpha$, where $A_\alpha \in M_r(\Gamma(U_\alpha, \Omega_{D_1}^1))$. On each $U_{\alpha\beta} := U_\alpha \cap U_\beta$, the gluing $(\psi_\alpha|_{U_{\alpha\beta}})^{-1} \circ (\psi_\beta|_{U_{\alpha\beta}})$ is given by $1 + p^n B_{\alpha\beta}$, where $B_{\alpha\beta} \in M_r(\Gamma(U_{\alpha\beta}, \mathcal{O}_{D_1}))$. Then, $dA_\alpha = 0$ by the integrability of the connection $\psi_\alpha^*(\nabla)$ and $B_{\beta\gamma} - B_{\alpha\gamma} + B_{\alpha\beta} = 0$ by the cocycle condition for the maps $(\psi_\alpha|_{U_{\alpha\beta}})^{-1} \circ (\psi_\beta|_{U_{\alpha\beta}})$. Also, by the compatibility of the connection with the gluing, we have the equality

$$(1 + p^n B_{\alpha\beta})^{-1} d(1 + p^n B_{\alpha\beta}) + (1 + p^n B_{\alpha\beta})^{-1} p^n A_\alpha (1 + p^n B_{\alpha\beta}) = p^n A_\beta.$$

We see from this the equality $A_\beta - A_\alpha = dB_{\alpha\beta}$. So $(\{A_\alpha\}, \{B_{\alpha\beta}\})$ defines a 1-cocycle of $\text{Tot } \Gamma(\mathcal{U}, \Omega_{D_1}^\bullet)^{r^2}$. We define $e(G)$ to be the class of this 1-cocycle in the cohomology $H^1(\text{Tot } \Gamma(\mathcal{U}, \Omega_{D_1}^\bullet)^{r^2}) = H_{\text{crys}}^1(X/k)^{r^2}$.

In order to show that this is well-defined, we need to check that $e(G)$ is independent of the choice of the affine open covering $\mathcal{U} = \{U_\alpha\}_\alpha$ and the isomorphisms $\{\psi_\alpha\}_\alpha$. If we choose another set of isomorphisms $\{\psi'_\alpha\}_\alpha$, we have another set of matrices $(\{A'_\alpha\}, \{B'_{\alpha\beta}\})$. Then, on each U_α , the map $\psi_\alpha^{-1} \circ \psi'_\alpha$ is given by $1 + p^n C_\alpha$, where $C_\alpha \in M_r(\Gamma(U_\alpha, \mathcal{O}_{D_1}))$, and we see by direct calculation the equalities $dC_\alpha = A'_\alpha - A_\alpha$, $C_\beta - C_\alpha = B'_{\alpha\beta} - B_{\alpha\beta}$. So the class $e(G)$ does not depend on the choice of the isomorphisms $\{\psi_\alpha\}_\alpha$. One can prove the independence of the choice of affine open covering $\mathcal{U} = \{U_\alpha\}_\alpha$ by taking a refinement. So we obtain the map $e : \mathcal{D} \rightarrow H_{\text{crys}}^1(X/k)^{r^2}$, and it is easily seen that this is a map of pointed sets. One can prove the bijectivity of e by considering the above argument in reverse direction.

Finally, we prove the commutativity of the diagram (3.2). Let $F_{\mathbb{P}} : \mathbb{P}_{W_{n+1}}^N \longrightarrow \mathbb{P}_{W_{n+1}}^N$ be the $\sigma_{W_{n+1}}^*$ -linear map which sends the coordinates to their p -th powers. Then, there exists a unique PD-morphism $F_{D_{n+1}} : D_{n+1} \longrightarrow D_{n+1}$ which makes the following diagram commutative:

$$\begin{array}{ccccc} X & \longrightarrow & D_{n+1} & \longrightarrow & \mathbb{P}_{W_{n+1}}^N \\ F \downarrow & & F_{D_{n+1}} \downarrow & & F_{\mathbb{P}} \downarrow \\ X & \longrightarrow & D_{n+1} & \longrightarrow & \mathbb{P}_{W_{n+1}}^N. \end{array}$$

Because $F_{D_{n+1}} \bmod p$ is equal to the Frobenius map F_{D_1} for D_1 , we see (from the above expression of cocycle) that the class $e(G) = [(\{A_\alpha\}, \{B_{\alpha\beta}\})]$ is sent by F^* (on cohomology) to $[(\{F_{D_1}^* A_\alpha\}, \{F_{D_1}^* B_{\alpha\beta}\})] = e(F^*(G))$. From this, we see the desired commutativity. \square

Proposition 3.7. *Let X be a connected smooth projective variety over k with trivial étale fundamental group. Let $\mathcal{E} \in \text{Conv}(X/K)$. Let E be a lattice of \mathcal{E} such that the restriction $E_X \in \text{Crys}(X/k)$ is constant. Then there exists a positive integer $d = d(X)$ such that, for any $n \in \mathbb{N}$, the restriction $((F^*)^{d(n-1)} E)_n$ of $(F^*)^{d(n-1)} E \in \text{Crys}(X/W)$ to $\text{Crys}(X/W_n)$ is constant.*

Proof. We have the decomposition $H_{\text{crys}}^1(X/k) = H_{\text{crys}}^1(X/k)_{\text{nilp}} \oplus H_{\text{crys}}^1(X/k)_{\text{ss}}$ of $H_{\text{crys}}^1(X/k)$ as in the proof of Proposition 3.3, where $H^1(X, \mathcal{O}_X)$ is replaced by $H_{\text{crys}}^1(X/k)$. As F^* is 0 on the image of $H^0(X, \Omega_{X/k}^1)$ in $H_{\text{crys}}^1(X/k)$, one has $H_{\text{crys}}^1(X/k)_{\text{ss}} \subset H^1(X, \mathcal{O}_X)_{\text{ss}} = 0$ by (3.1), and there exists some $d \in \mathbb{N}$ such that $(F^*)^d$ acts by 0 on $H_{\text{crys}}^1(X/k)_{\text{nilp}}$, since $H_{\text{crys}}^1(X/k)_{\text{nilp}}$ is finite-dimensional. So $(F^*)^d$ acts by 0 on $H_{\text{crys}}^1(X/k)$. We prove the proposition for this choice of d , by induction on n .

Assume that $((F^*)^{c+d(n-1)} E)_n$ is constant. Then $((F^*)^{c+d(n-1)} E)_{n+1}$ defines the class $e(((F^*)^{c+d(n-1)} E)_{n+1})$ in $H_{\text{crys}}^1(X/k)^{r^2}$ by Proposition 3.6. Then, by definition of d , we have $0 = (F^*)^d e(((F^*)^{c+d(n-1)} E)_{n+1}) = e(((F^*)^{c+dn} E)_{n+1})$, and so $((F^*)^{c+dn} E)_{n+1}$ is constant again by Proposition 3.6. This finishes the proof. \square

Combining Propositions 3.4 and 3.7, we obtain the following:

Corollary 3.8. *Let X be a connected smooth projective variety over k with trivial étale fundamental group. Let r be a positive integer. Let $\mathcal{E} \in \text{Conv}(X/K)$ of rank r and let E be a lattice of \mathcal{E} such that $E_X \in \text{Coh}(X)$ is strongly μ -semistable. Let $c = c(X, r)$ and $d = d(X)$ be as in Proposition 3.4 and Proposition 3.7. Then, for any $n \in \mathbb{N}$, the restriction $((F^*)^{c+d(n-1)} E)_n$ of $(F^*)^{c+d(n-1)} E \in \text{Crys}(X/W)$ to $\text{Crys}(X/W_n)$ is constant.*

Now we can finish the proof of Theorem 2.6:

Proof of Theorem 2.6. Let r be the rank of \mathcal{E} and let $c = c(X, r)$, $d = d(X)$ be as in Proposition 3.4, Proposition 3.7 for X and r . Apply Corollary 3.8 to $\mathcal{E}^{(c+d(n-1))}$

and its lattice $E^{(c+d(n-1))}$ for each $n \geq 1$. Then we see that the restriction of $G^{(n)} := (F^*)^{c+d(n-1)} E^{(c+d(n-1))}$ to $\text{Crys}(X/W_n)$, which we denote by $G_n^{(n)}$, is constant. Note that, for any $n \geq 1$, this is a lattice of $(F^*)^{c+d(n-1)} \mathcal{E}^{(c+d(n-1))} = \mathcal{E}$.

We put $E := G^{(1)}$ so that it is a lattice of \mathcal{E} with $E_1 \in \text{Crys}(X/k)$ constant. We may assume by replacing $G^{(n)}$ by $p^{m_n} G^{(n)}$ for suitable $m_n \in \mathbb{Z}$ that $G^{(n)} \subseteq E$, $G^{(n)} \not\subseteq pE$. We further fix a natural number $m \geq 1$. Then, for any $n \geq m$, the image of the composite map

$$(3.3) \quad \begin{aligned} H_{\text{crys}}^0(X/W_n, G_n^{(n)}) &\longrightarrow H_{\text{crys}}^0(X/W_n, E_n) \\ &\longrightarrow H_{\text{crys}}^0(X/W_n, E_n/p^m E_n) = H_{\text{crys}}^0(X/W_m, E_m) \end{aligned}$$

is not zero: Otherwise, as $G_n^{(n)}$ is constant, it would be contained in $p^m E_n \subset pE_n$. Hence $G^{(n)}$ is contained in pE , which is a contradiction. So, for $n \geq m$, the map

$$H_{\text{crys}}^0(X/W_n, E_n) \longrightarrow H_{\text{crys}}^0(X/W_n, E_n/p^m E_n) = H_{\text{crys}}^0(X/W_m, E_m)$$

is non-zero. Hence

$$\{\text{Im}(H_{\text{crys}}^0(X/W_n, E_n) \longrightarrow H_{\text{crys}}^0(X/W_m, E_m))\}_{n \geq 1}$$

is a decreasing family of non-zero W_m -submodules of the finite type W_m -module $H_{\text{crys}}^0(X/W_m, E_m)$. So, W_m being an Artinian ring, the family is stationary, thus non-zero, and

$$0 \neq \bigcap_{m \leq n \in \mathbb{N}} \text{Im}(H_{\text{crys}}^0(X/W_n, E_n) \longrightarrow H_{\text{crys}}^0(X/W_m, E_m)).$$

Thus the system $\{H_{\text{crys}}^0(X/W_n, E_n)\}_n$ satisfies the Mittag-Leffler condition and

$$0 \neq H_{\text{crys}}^0(X/W, E) = \varprojlim_n H_{\text{crys}}^0(X/W_n, E_n).$$

As E is p -torsion free, $H_{\text{crys}}^0(X/W, E)$ is a free module of rank s over W , for some $1 \leq s \leq r$. If $s < r$, then the quotient $Q := E/(H_{\text{crys}}^0(X/W, E) \otimes_W \mathcal{O}_{X/W}) \in \text{Crys}(X/W)$ is nonzero. The p -torsion of Q is identified with the kernel of the homomorphism

$$(H_{\text{crys}}^0(X/W, E)/p) \otimes_k \mathcal{O}_{X/k} = \mathcal{O}_{X/k}^s \rightarrow E_1 = H_{\text{crys}}^0(X/k, E_1) \otimes_k \mathcal{O}_{X/k} = \mathcal{O}_{X/k}^r$$

in $\text{Crys}(X/k)$, which is zero. Thus $Q \in \text{Crys}(X/W)$ is p -torsion free. By multiplying the composite map $G^{(n)} \hookrightarrow E \rightarrow Q$ with a suitable p -power, we obtain a map $G^{(n)} \rightarrow Q$ whose image is not contained in pQ . Then the diagram (3.3) with E_n replaced by Q_n shows that $H_{\text{crys}}^0(X/W, Q) \neq 0$. On the other hand, one has the exact sequence $0 \rightarrow H_{\text{crys}}^0(X/W, E) \otimes_W \mathcal{O}_{X/W} \xrightarrow{\iota} E \xrightarrow{q} Q \rightarrow 0$ in $\text{Crys}(X/W)$. By definition, $H_{\text{crys}}^0(\iota)$ is an isomorphism and by Proposition 2.10 (2), $H_{\text{crys}}^0(q)$ is surjective. Thus $H_{\text{crys}}^0(X/W, Q) = 0$, a contradiction. Thus $s = r$ and E is constant in $\text{Crys}(X/W)$, thus \mathcal{E} is constant in $\text{Crys}(X/W)_{\mathbb{Q}}$. This finishes the proof. \square

We give a proof of Corollary 2.7.

Proof of Corollary 2.7. We check that any infinitesimal isocrystal $\mathcal{E} = \mathbb{Q} \otimes E \in \text{Inf}(X/W)_{\mathbb{Q}}$ satisfies the assumption of Theorem 2.6. By Proposition 3.9 below, the functor $F^* : \text{Inf}(X/W) \rightarrow \text{Inf}(X/W)$ is an equivalence. Thus the F^n -division $\mathcal{E}^{(n)}$ of \mathcal{E} has the form $\mathbb{Q} \otimes E^{(n)}$ for some $E^{(n)} \in \text{Inf}(X/W)$. Then the value $E_X^{(n)}$ of $E^{(n)}$ at X has the structure of an object in $\text{Inf}(X/k)$, which is constant by the affirmation [EM10] of Gieseker's conjecture. So $E_X^{(n)}$ is isomorphic to \mathcal{O}_X^r for some r and hence strongly μ -semistable. \square

Proposition 3.9. *For a smooth variety X over k , the functor*

$$F^* : \text{Inf}(X/W) \rightarrow \text{Inf}(X/W)$$

is an equivalence.

Proof. Because the category $\text{Inf}(X/W)$ satisfies the Zariski descent property, we may work locally. So we may assume that X lifts to a p -adic smooth formal scheme X_W over W on which there exists a lift $F_W : X_W \rightarrow X_W$ of Frobenius morphism on X . Then we have the equivalence (1.18) in which the functor F^* on the left hand side is compatible with the pull-back F_W^* by F_W on the right hand side. Thus it suffices to see that F_W^* is an equivalence, which is proven in [Ber12, Thm. 2.1]. \square

4 Proof of Theorem 2.8

In this section, we prove Theorem 2.8. The following proposition, which is a crystalline version of Langton's theorem [Lan75], is a key step for the proof:

Proposition 4.1. *Let X be a smooth projective variety over k and let $\mathcal{E} \in \text{Crys}(X/W)_{\mathbb{Q}}$ be irreducible. Then there exists $E \in \text{Crys}(X/W)$ with $\mathcal{E} = \mathbb{Q} \otimes E$ such that $E_X \in \text{Crys}(X/k) = \text{MIC}(X)^{\text{qn}}$ is μ -semistable.*

Proof. We follow the proof of Langer [Lan14, Thm. 5.1] and Huybrechts-Lehn's book [HL97, 2.B]. Let us consider the following two claims:

- (A) There exists $E \in \text{Crys}(X/W)$ with $\mathcal{E} = \mathbb{Q} \otimes E$ such that $E_X \in \text{Coh}(X)$ is torsion free.
- (B) There exists $E \in \text{Crys}(X/W)$ with $\mathcal{E} = \mathbb{Q} \otimes E$ such that $E_X \in \text{Crys}(X/k) = \text{MIC}(X)^{\text{qn}}$ is μ -semistable.

To prove the proposition, we first prove the claim (A) and then prove the claim (B). However, since the proof of (A) and that of (B) are parallel, we will describe them simultaneously in the following.

First take a p -torsion free crystal $E \in \text{Crys}(X/W)$ with $\mathcal{E} = \mathbb{Q} \otimes E$ in the case (A), and take a p -torsion free crystal $E \in \text{Crys}(X/W)$ with $\mathcal{E} = \mathbb{Q} \otimes E$ and E_X

torsion free in the case (B). (This is possible because, when we prove (B), we can assume the claim (A).) Put $E^0 := E$. If E^0 does not satisfy the conclusion of the claim, let B^0 be the maximal torsion \mathcal{O}_X -submodule of E_X^0 in the case (A) and the maximal destabilizing subobject of E_X^0 in the category $\text{Crys}(X/k)$ in the case (B). In the case (A), one can check (by looking at E_X^0 as an object (E_X^0, ∇) in $\text{MIC}(X)$ and noting the fact that $fe = 0$ ($e \in E_X^0, f \in \mathcal{O}_X$) implies $f^2 \nabla(e) = 0$) that B^0 is actually an object in $\text{Crys}(X/k)$. Let E^1 be the kernel of $E^0 \rightarrow E_X^0 \rightarrow E_X^0/B^0$. If E^1 satisfies the conclusion of the claim, we are done. Otherwise, let B^1 be the maximal torsion \mathcal{O}_X -submodule (actually an object in $\text{Crys}(X/k)$) of E_X^1 in the case (A) and the maximal destabilizing subobject of E_X^1 in the category $\text{Crys}(X/k)$ in the case (B). If the claim is not true, we obtain a sequence

$$E = E^0 \supset E^1 \supset E^2 \supset \dots$$

Let $G^n := E_X^n/B^n = E^n/E^{n+1}$. Note that in the case (A), the rank of G^n is the same as the rank of E_X^n , which is the same as the rank of \mathcal{E} . In addition, as $B^n \subset E_X^n$ is the maximal torsion submodule, G^n is torsion free in $\text{Coh}(X)$. In the case (B), G^n is nonzero by definition of B^n , and torsion free by the maximality of B^n . By definition, one has exact sequences $0 \rightarrow E^{n+1} \rightarrow E^n \rightarrow G^n \rightarrow 0$ and $0 \rightarrow pE^n/pE^{n+1} \rightarrow E_1^{n+1} \rightarrow E^{n+1}/pE^n \rightarrow 0$, both in $\text{Crys}(X/W)$. As $pE^n/pE^{n+1} \cong G^n$ and $E^{n+1}/pE^n = B^n$ in $\text{Crys}(X/W)$, this yields the exact sequences

$$(4.1) \quad 0 \rightarrow B^n \rightarrow E_X^n \rightarrow G^n \rightarrow 0, \quad 0 \rightarrow G^n \rightarrow E_X^{n+1} \rightarrow B^n \rightarrow 0$$

in $\text{Crys}(X/k)$. From these, we see that the slope $\mu(E_X^n)$ of E_X^n is constant and so equal to $\mu(E_X)$ in the case (B).

Let C^n be the kernel of the composite $B^{n+1} \rightarrow E_X^{n+1} \rightarrow B^n$. It is nothing but $B^{n+1} \cap G^n$, and this is zero in the case (A) because B^{n+1} is torsion while G^n is torsion free. In the case (B), if $C^n = 0$, $\mu(B^{n+1}) \leq \mu(B^n)$ due to the maximality of B^n . If $C^n \neq 0$, $\mu(C^n) \leq \mu_{\max}(G^n) < \mu(B^n)$ because C^n is a subobject of G^n and B^n is the maximal destabilizing subobject. So, if $\mu(B^{n+1}) \leq \mu(C^n)$, we obtain the inequality $\mu(B^{n+1}) < \mu(B^n)$. On the other hand, if $\mu(B^{n+1}) > \mu(C^n)$, we have $\mu(B^{n+1}) < \mu(B^{n+1}/C^n) \leq \mu(B^n)$ because B^n is μ -semistable as a crystal. Hence $\mu(B^{n+1}) < \mu(B^n)$ when $C^n \neq 0$. In conclusion, $\mu(B^n)$ ($n \in \mathbb{N}$) is non-increasing, and strictly decreasing when $C^n \neq 0$. But the latter case can happen only finitely many times, because $\mu(B^n)$ should be contained in $\frac{1}{r_1}\mathbb{Z}$ (where r is the rank of E) and $> \mu(E_1)$. Therefore, $C^n = 0$ for $n \gg 0$ in the case (B).

So we may assume that $C^n = 0$, namely, $B^{n+1} \cap G^n = 0$. This implies that we have the inclusions

$$(4.2) \quad \dots \supseteq B^n \supseteq B^{n+1} \supseteq \dots, \quad \dots \subseteq G^n \subseteq G^{n+1} \subseteq \dots$$

We may assume also that the rank of G^n is constant and that $\mu(B^n)$ ($n \in \mathbb{N}$), $\mu(G^n)$ ($n \in \mathbb{N}$) are constant in the case (B). Note also that $G^n = G^{n+1}$ if and only if $B^n = B^{n+1}$.

Next we prove that G^n is constant for $n \gg 0$. In the case (A), the support of B^n is non-increasing and so it is constant for $n \gg 0$. So, for $n \gg 0$, $B^n = B^{n+1}$ outside some codimension 2 closed subscheme of X . Indeed, if the support of the B_n for n large is in codimension ≥ 2 , there is nothing to prove, else it is a divisor, and B_n on each generic point of the divisor is eventually constant. So $G^n = G^{n+1}$ outside a codimension 2 closed subscheme. Hence the double dual of G^n is constant and, as G^n is torsion free, contains all the G^n . So the right tower in (4.2) is stationary and then G^n is constant for $n \gg 0$. In the case (B), the constancy of the rank and the slope and the torsion freeness of G^n imply the equality $G^n = G^{n+1}$.

So we may assume that B^n, G^n are constant. So we write it by B, G , respectively. Then the exact sequences (4.1) split, and so $E_X^n = B \oplus G$. Now define $Q_n := E/E^n$. Then Q has a natural filtration whose graded quotients are $E^i/E^{i+1} \cong G$. This implies that Q_n is nonzero and when regarded as an object in $\text{Crys}(X/W_n)$, it is flat over W_n . So $Q = (Q_n)_n \in \text{Crys}(X/W)$ is a nonzero p -torsion free crystal. Also, we have the canonical surjection $E \rightarrow Q$, hence the surjection $\mathcal{E} \rightarrow \mathbb{Q} \otimes Q$. In the case (A), if it is not an isomorphism, this contradicts the irreducibility of \mathcal{E} . If it is an isomorphism, Q gives the lattice such that $Q_X = Q_1 = G$ is torsion free. In the case (B), since B is non-zero and torsion free, $\mathcal{E} \rightarrow \mathbb{Q} \otimes Q$ is not an isomorphism, and this contradicts the irreducibility of \mathcal{E} . This finishes the proof. \square

Proposition 4.2. *Let X be a smooth projective variety over k and let $G \in \text{Crys}(X/k)$ be of rank r and μ -semistable. Assume moreover one of the following conditions:*

- (1) $r = 1$.
- (2) $\mu_{\max}(\Omega_X^1) < 2$, $r = 2$ and $\mu(G) = 0$.
- (3) $\mu_{\max}(\Omega_X^1) < 1$, $r = 3$ and $\mu(G) = 0$.
- (4) $\mu_{\max}(\Omega_X^1) < \frac{1}{N(r)}$, where $N(r) := \max_{a,b \geq 1, a+b \leq r} \text{lcm}(a, b)$.

Then G is strongly μ -semistable in $\text{Coh}(X)$.

Proof. In the case (1), the only point of the assertion is that the Frobenius pull backs of G remain torsion free, which is trivial because torsion freeness of \mathcal{O}_X -modules is preserved by F^* as X is smooth so F^* is faithfully flat. So we will prove the proposition in the cases (2), (3) or (4). The proof is a variant of that in [MR83, Thm. 2.1].

First we prove the claim that any μ -semistable sheaf H of rank r is strongly μ -semistable in $\text{Coh}(X)$ under one of the following conditions:

- (a) $\mu_{\max}(\Omega_X^1) < 2$, $r = 2$ and $\mu(H) = 0$.
- (b) $\mu_{\max}(\Omega_X^1) < 1$, $r = 3$ and $\mu(H) = 0$.
- (c) $\mu_{\max}(\Omega_X^1) < \frac{1}{N(r)}$. In this case, we prove it by induction on r .

For this, it suffices to prove that F^*H is μ -semistable in $\text{Coh}(X)$. Assume the contrary and let $H' \subset F^*H$ be the maximal destabilizing subsheaf of H . Let $H'' := H/H'$. Then the connection $\nabla_{\text{can}} : F^*H \rightarrow F^*H \otimes \Omega_X^1$ in (1.29) induces a linear map $\bar{\nabla}_{\text{can}} : H' \rightarrow H'' \otimes \Omega_X^1$. If we prove $\bar{\nabla}_{\text{can}} = 0$, $(H', \bar{\nabla}_{\text{can}}|_{H'})$ defines a submodule with integrable connection of $(F^*H, \nabla_{\text{can}})$ and so there exists a \mathcal{O}_X -submodule H'_0 of H with $H' = F^*H'_0$. Then we have $p\mu(H'_0) = \mu(H') > \mu(F^*H) = p\mu(H)$ and this contradicts the μ -semistability of H . So it suffices to prove the equality $\bar{\nabla}_{\text{can}} = 0$. To prove it, we may replace H'' by its graded quotients with respect to Harder-Narasimhan filtration. So we may assume that H', H'' are μ -semistable and $\mu(H') > \mu(H'')$. Also, it suffices to prove that the map

$$f : T_X \rightarrow \mathcal{H}om(H', H'')$$

(where T_X denotes the tangent sheaf on X) induced by $\bar{\nabla}_{\text{can}}$ is equal to zero. Since T_X is locally free and $\mathcal{H}om(H', H'')$ is torsion free as H'' is, it suffices to prove $f = 0$ outside some codimension 2 closed subscheme of X . Until the end of the proof of the claim, we will consider sheaves and morphisms of sheaves up to some codimension 2 subscheme in X . Then $\mathcal{H}om(H', H'') \cong H'^{\vee} \otimes H''$. When (a) or (b) is satisfied, at least one of H', H'' is of rank 1. So $H'^{\vee} \otimes H''$ is μ -semistable of slope $-\mu(H') + \mu(H'')$, which is ≤ -2 in the case (a) and ≤ -1 in the case (b) (we use the assumption $\mu(H) = 0$ here). Hence $-\mu(H') + \mu(H'') < -\mu_{\max}(\Omega_X^1) = \mu_{\min}(T_X)$. So we see that $f = 0$.

When (c) is satisfied, H', H'' are strongly μ -semistable by induction hypothesis. Then, by [RR84, Thm. 3.23] (see also [Lan04b, Cor. A.3.1]), $H'^{\vee} \otimes H''$ is μ -semistable of slope $-\mu(H') + \mu(H'') \leq \frac{-1}{\text{lcm}(\text{rank}H', \text{rank}H'')} \leq -\frac{1}{N(r)} < -\mu_{\max}(\Omega_X^1) = \mu_{\min}(T_X)$. So we see that $f = 0$ also in this case.

Now we prove the proposition. In the proof, we regard G as an object in $\text{MIC}(X)^{\text{qn}}$ and so we denote it by (G, ∇) . By the argument above, it suffices to prove that G is μ -semistable as sheaf. Assume the contrary and let $H' \subset G$ be the maximal destabilizing subsheaf of G . Let $H'' := G/H'$. Then the connection $\nabla : G \rightarrow G \otimes \Omega_X^1$ induces a linear map $\bar{\nabla} : H' \rightarrow H' \otimes \Omega_X^1$. It suffices to prove that $\bar{\nabla} = 0$: Indeed, if this is the case, $(H, \bar{\nabla}|_H)$ defines a destabilizing subobject of (G, ∇) , which is a contradiction.

We prove that $\bar{\nabla} = 0$ in a similar way to the proof of $\bar{\nabla}_{\text{can}} = 0$ above. We replace H'' by its graded quotients with respect to Harder-Narasimhan filtration so that H', H'' are μ -semistable and $\mu(H') > \mu(H'')$, and we prove that the map

$$f : T_X \rightarrow \mathcal{H}om(H', H'')$$

induced by $\bar{\nabla}$ is equal to zero outside some codimension 2 closed subscheme of X . Working again up to some codimension 2 subscheme in X , we have $\mathcal{H}om(H', H'') \cong H'^{\vee} \otimes H''$. When (1) or (2) is satisfied, at least one of H', H'' is of rank 1. So $H'^{\vee} \otimes H''$ is μ -semistable of slope $-\mu(H') + \mu(H'')$, which is ≤ -2 in the case (1)

and ≤ -1 in the case (2) (we use the assumption $\mu(G) = 0$ here). In the case (3), H', H'' are strongly μ -semistable by the claim we proved above. Then, $H'^{\vee} \otimes H''$ is μ -semistable of slope $-\mu(H') + \mu(H'') \leq \frac{-1}{\text{lcm}(\text{rank}H', \text{rank}H'')} \leq -\frac{1}{N(r)} < -\mu_{\max}(\Omega_X^1) = \mu_{\min}(T_X)$. So we see that $f = 0$ also in this case. \square

Now we give a proof of Theorem 2.8:

Proof of Theorem 2.8. First assume that \mathcal{E} is irreducible. In this case, any F^n -division $\mathcal{E}^{(n)}$ of \mathcal{E} is also irreducible. Then, by Propositions 3.1, 4.1 and 4.2, each $\mathcal{E}^{(n)}$ admits a lattice $E^{(n)}$ such that $E_X^{(n)}$ is strongly μ -semistable as \mathcal{O}_X -module. So, by Theorem 2.6, we see that \mathcal{E} is constant.

In the general case, \mathcal{E} has a filtration whose graded quotients are irreducible. So, by the previous case, \mathcal{E} can be written as an iterated extension of constant convergent isocrystals. Since we have $H_{\text{conv}}^1(X/K, \mathcal{O}_{X/K}) = \mathbb{Q} \otimes H_{\text{crys}}^1(X/W, \mathcal{O}_{X/W}) = 0$, where the second equality is proven in Proposition 2.10 (2), this finishes the proof. \square

We give another application of Proposition 4.1. It seems that the following question is frequently asked among experts:

Question 4.3. *Let X be a smooth variety of finite type over k and let $\mathcal{E} \in \text{Conv}(X/K)$. Does there exist a locally free $E \in \text{Crys}(X/W)$ with $\mathcal{E} = \mathbb{Q} \otimes E$?*

We give the following partial answer to this question, using Proposition 4.1:

Theorem 4.4. *Let X be a smooth projective variety over k , let $\mathcal{E} \in \text{Conv}(X/K)$. Assume one of the following:*

- (1) *The rank of irreducible constituents of \mathcal{E} are 1.*
- (2) *$\mu_{\max}(\Omega_X^1) < 2$ and the rank of irreducible constituents of \mathcal{E} are ≤ 2 .*
- (3) *$\mu_{\max}(\Omega_X^1) < 1$ and the rank of irreducible constituents of \mathcal{E} are ≤ 3 .*
- (4) *$r \geq 4$, $\mu_{\max}(\Omega_X^1) < \frac{1}{N(r)}$ and the rank of irreducible constituents of \mathcal{E} are $\leq r$, where $N(r) := \max_{a,b \geq 1, a+b \leq r} \text{lcm}(a, b)$.*
- (5) *X lifts to a smooth scheme \tilde{X} over W_2 and the rank of irreducible constituents of \mathcal{E} are $\leq p$.*

Then there exists $E \in \text{Crys}(X/W)$ locally free with $\mathcal{E} = \mathbb{Q} \otimes E$.

The case (1) reproves a weaker version of Proposition 2.11. Also, when $\mu_{\max}(\Omega_X^1) \leq 0$, any convergent isocrystal \mathcal{E} on X admits a locally free crystal E on $(X/W)_{\text{crys}}$ with $\mathcal{E} = \mathbb{Q} \otimes E$.

Proof. First we prove the theorem in the cases (1), (2), (3) or (4) by induction on the rank of \mathcal{E} . When \mathcal{E} is irreducible, there exists a lattice E of \mathcal{E} such that $E_X \in \text{Coh}(X)$ is strongly μ -semistable, by Propositions 3.1, 4.1 and 4.2. This, together with Proposition 3.1, Theorem 3.5 implies that E_X is locally free. Hence E is also locally free. When \mathcal{E} is not irreducible, we have an irreducible convergent subisocrystal $\mathcal{E}' \subsetneq \mathcal{E}$. Put $\mathcal{E}'' := \mathcal{E}/\mathcal{E}'$. Then, by induction hypothesis, there exist locally free lattices E', E'' of $\mathcal{E}', \mathcal{E}''$, respectively. Then $H_{\text{conv}}^1(X/K, \mathcal{E}''^\vee \otimes \mathcal{E}') = \mathbb{Q} \otimes H_{\text{crys}}^1(X/W, E''^\vee \otimes E')$, and from this we see that there exists an extension E of E' by E'' in $\text{Crys}(X/W)$ with $\mathcal{E} \cong \mathbb{Q} \otimes E$. This E is locally free by construction, and so the theorem is true for \mathcal{E} .

Next we prove the theorem in the case (5). By the argument in the previous paragraph, we may assume that \mathcal{E} is irreducible. Using Proposition 4.1, we take a lattice E of \mathcal{E} such that the restriction (E_X, ∇) of E to $\text{Crys}(X/k) = \text{MIC}(X)^{\text{an}}$ is μ -semistable. By assumption, $\text{rank } E_X \leq p$. Hence (E_X, ∇) is contained in $\text{MIC}_{p-1}(X)$. So, by [OV07], there exists a Higgs module $(H, \theta) \in \text{HIG}_{p-1}(X)$ such that $C^{-1}(H, \theta) = (E_X, \nabla)$, where C^{-1} is the inverse Cartier transform. By Proposition 3.1, E_X has vanishing Chern classes. From this and the μ -semistability of (E_X, ∇) , we see that (H, θ) is μ -semistable Higgs module with vanishing Chern classes, by [Lan15, Lem. 2, Cor. 1]. Then, by [Lan15, Thm. 11], H is locally free. Hence so is E_X , and then E is locally free. \square

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