NON-LIFTABILITY OF AUTOMORPHISM GROUPS OF A K3 SURFACE IN POSITIVE CHARACTERISTIC

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Abstract. We show that a characteristic 0 model $X_R \to \operatorname{Spec} R$, with Picard number 1 over a geometric generic point, of a K3 surface in characteristic $p \geq 3$, essentially kills all automorphisms (Theorem 5.1). We show that there is an explicitely constructed automorphism on a supersingular K3 surface in characteristic 3, which has positive entropy, the logarithm of a Salem number of degree 22 (Theorem 6.4). In particular it does not lift to characteristic 0. In addition, we show that in any large characteristic, there is an automorphism of a supersingular K3 which has positive entropy and does not lift to characteristic 0 (Theorem 7.5).

1. Introduction

Let X be a K3 surface over an algebraically closed field k of characteristic $p > 0$. A classical theorem [Del81] asserts that the formal universal deformation space $\hat{S}$ of X is unobstructed, and is formally smooth of dimension 20 over $W(k)$, the ring of Witt vectors of k. Moreover, the closed formal subscheme of $\hat{S}$ parametrizing the locus $\hat{\Sigma}(X, L)$ over which a given line bundle L on X lifts, is a hypersurface, flat over $W(k)$. The aim of our article is to understand conditions for automorphisms of X to be or not to be liftable to a proper model $X_R \to \operatorname{Spec} R$ of X, where R is a discrete valuation ring such that $\operatorname{Spec} R \to \hat{S}$ dominates $\operatorname{Spec} W(k)$. Said in words, we study conditions on automorphisms of X to lift to characteristic 0, or not. One motivation for this study is the observation that the crystalline classes of graphs of automorphisms on a positive characteristic K3 surface obey the Fontaine-Mazur p-adic variational Hodge conjecture as expressed in [BlEsKe14, Conj. 1.2] (see Remark 6.5).

Our main results are Theorem 5.1, Theorem 6.4 and Theorem 7.5. Simplified versions are Theorems 1.2, 1.3, 1.4 explained below. For the discussion in the introduction, we assume for simplicity that $p \geq 3$.

Recall one has a natural injective specialization homomomorphism

$$\iota : \operatorname{Aut}^e (X_{\bar{K}}/\bar{K}) \to \operatorname{Aut} (X/k)$$

where K is the field of fractions of R and $\operatorname{Aut}^e(X_{\bar{K}}/\bar{K}) \subset \operatorname{Aut}(X_{\bar{K}}/\bar{K})$ is the subgroup of automorphisms which lift to some proper model $X_R \to \operatorname{Spec}\, R$. We say $f \in \operatorname{Aut} (X/k)$ is

Date: February 5th, 2015.

The first author is supported by the Einstein program and the ERC Advanced Grant 226257. The second author is supported by JSPS Grant-in-Aid (S) No 25220701, JSPS Grant-in-Aid (S) No 22224001, JSPS Grant-in-Aid (B) No 22340009, and by KIAS Scholar Program.

not geometrically liftable to characteristic 0 if it is not in the image of ι (see Section 2 for details).

In the complex case, for any K3 surface M and any given line bundles L_i $(1 \le i \le d \le 19)$ on M for which the L_i, $i = 1, \dots, d$ are part of a $\mathbb{Z}$-basis of $\mathrm{Pic}\,(M)$, there is a smooth proper small deformation

$$\psi : (\mathcal{M}, \mathcal{L}_1, \dots, \mathcal{L}_d) \to \Delta$$

of $(M, L_1, \dots, L_d)$, where Δ is the analytic disc, such that $\mathrm{Pic}\,(\mathcal{M}_t) = \langle (\mathcal{L}_1)_t, \dots, (\mathcal{L}_d)_t \rangle \simeq \mathbb{Z}^d$ for a point $t \in \Delta$, generic in the complex analytic sense. The proof goes via a study of the period map. In particular, if $d = 1$ and L_1 is ample primitive, then $\psi : \mathcal{M} \to \Delta$ is also projective and $\mathrm{Pic}\,(\mathcal{M}_t) = \mathbb{Z}(\mathcal{L}_1)_t$ for generic t. As a consequence, the specialization homomorphism has very small image even if $\mathrm{Aut}\,(M)$ is very large. In other words, interesting automorphisms disappear on the generic fiber $\mathcal{M}_t$ (see eg. [Og03]).

Our first aim is to show the analogous results Theorems 1.1, 1.2 on liftings from characteristic p to characteristic 0 (see Theorems 4.1, 4.3, 5.1 for more precise statements):

Theorem 1.1. *If $p \ge 3$, there is a discrete valuation ring R, finite over the ring of Witt vectors $W(k)$, together with a projective model $X_R \to \mathrm{Spec}\, R$, such that the Picard rank of $X_{\bar{K}}$ is 1.*

The proof is given in [LieOls11, App. A]. The proof in *loc.cit.* relies on [Ogu79], [Ogu83] and on the properties of the stack parametrizing deformations of a K3 surface together with line bundles. In Section 4, we sketch another proof, relying on [Del81]. Theorem 4.3 has its own interest.

As an immediate but remarkable consequence of Theorem 1.1, we obtain:

Theorem 1.2. *If $p \ge 3$, there is a projective model $X_R \to \mathrm{Spec}\, R$ such that no subgroup $G \subset \mathrm{Aut}\,(X)$, except for $G = \{\mathrm{id}_X\}$ is geometrically liftable to $X_R \to \mathrm{Spec}\, R$, unless $\mathrm{Pic}\,(X) = \mathbb{Z} \cdot H$ with self-intersection number $(H^2) = 2$.*

See also Theorem 5.1 2) for the exceptional case. We also note that when X is not supersingular, Theorem 1.2 is in sharp contrast to the model constructed by Lieblich and Maulik [LieMau11], to show the Kamawata-Morrison Cone Conjecture for K3 surfaces in positive characteristic. We prove Theorem 1.2 in Section 5.

The second aim of our article is to show the richness of automophisms of supersingular K3 surfaces of Artin invariant 1, in view of the non-liftability problem. Supersingular K3 surfaces of Artin invariant 1 are unique up to isomorphisms, for each field k of characteristic $p > 0$. They are the most special K3 surfaces (see Section 6 for a brief review). We denote them by $X(p)$. Recently, several interesting aspects of automorphisms of $X(p)$ for various p were studied ([DK09], [DK09-2], [Sh13], [KS12] and references therein). The notion of entropy is classical in the complex case, and is of topological nature (see e.g. [Og14] and references therein); it has been introduced in [EsnSri13] in positive characteristic. The positivity of entropy is a numerical measure of complexity or richness of automorphisms in any characteristic (see Section 2.3). We note here that an automorphism of positive entropy is necessarily of infinite order, but it is a stronger constraint as there are many automorphisms of infinite order with null-entropy.

Our next main results are Theorems 1.3 and 1.4, showing the richness of automorphisms of supersingular K3 surfaces of Artin invariant 1:

Theorem 1.3. *There is an* $f \in \mathrm{Aut}\,(X(3))$ *of positive entropy such that for all* $n \in \mathbb{Z}\setminus\{0\}$, f^n *is not geometrically liftable to characteristic zero. The entropy of* f *is the logarithm of a Salem number of degree* 22.

This is the first explicit example of an automorphism of positive entropy which can never be lifted to characteristic zero. In characteristic 2, (non-explicit) examples have been constructed in [BC13] (see [EOY14, Thm. 4.2] for a slight clarification). See Theorem 6.4 for the precise statement and Section 3 for the definition of Salem numbers.

Recall that there are non-projective complex K3 surfaces with an automorphism for which the entropy is the logarithm of a Salem number of degree 22 ([Mc02]). However, the entropy of an automorphism of a projective K3 surface over a field of characteristic zero is either zero or the logarithm of Salem number of degree ≤ 20 ([Mc02], [Mc13]). In particular, the Salem number we construct in Theorem 1.3 can not be the Salem number associated to the entropy of a projective K3 surface in characteristic 0.

Our construction is entirely based on a result of Kondo and Shimada [KS12] and is mildly supported by a `Mathematica` computation. It would be nicer if one could find a more conceptual reason for the existence.

Finally we show:

Theorem 1.4. *For* p *large, there is an automorphism of* $X(p)$*, of positive entropy, which is not geometrically liftable to characteristic* 0.

See also Theorem 7.5 for a more general statement. Our construction is based on Jang's result [Jan14, Thm. 3.3] together with a result on the Mordell-Weil groups of elliptic fibrations due to Shioda [Sh90]. In the proof, one shows a way to construct an automorphism of positive entropy out of those of null-entropy. This might have an interest on its own. [1]
We prove Theorem 1.3, Theorem 1.4 in Sections 6, 7.

Ackowledgement.

We thank Takao Iohara for kindly accepting to check again our `Mathematica` computation using `Maxima`. We thank Gabriel Fürstenheim for asking us for concrete non-liftable automorphisms, which prompted the starting point of our work. We thank Luc Illusie and Arthur Ogus for a friendly and interesting discussion on the articles [Del81] and [Ogu79]. We thank Matthias Schütt for carefully reading the first version, Junmeyong Jang for mentioning [Jan14] to us, which enabled us to find Theorem 1.4, and Olivier Benoist for mentioning [LieOls11] to us. Finally we thank the referee for a friendly and careful reading of our article which enabled us to improve it.

[1]Since the construction presented in this article has been written, using Shioda's work on the existence of different elliptic structures with rich Néron-Severi groups on the surfaces $X(p)$, it is shown in [EOY14, Thm. 1.1] that for almost all p, $X(p)$ carries an automorphism, the entropy of which is the logarithm of a Salem number of degree 22.

2. Lifting automorphisms of K3 surfaces: notations and formulation of the problems

We introduce some notations and formulate the main questions addressed in our article.

2.1. **Models and Lifts.** Let M be a proper variety defined over a perfect characteristic $p > 0$ field k, and R be a discrete valuation ring (in the sequel abbreviated as DVR) with residue field k and field of fractions $K = \mathrm{Frac}\,(R)$. A *model of* M *over* $\mathrm{Spec}\,R$ is a proper flat morphism of schemes $M_R \to \mathrm{Spec}\,R$ lifting $M \to \mathrm{Spec}\,k$. If M is smooth, then a lift $M_R \to \mathrm{Spec}\,R$ is a model if and only if $M_R \to \mathrm{Spec}\,R$ is smooth. We call a model $M_R \to \mathrm{Spec}\,R$ a *lift to characteristic zero* if K is of characteristic zero. If $M_R \to \mathrm{Spec}\,R$ is a model of X, and $L \supset K$ is any field extension, we say that $M_L = M_R \otimes_R L$ is a *lift* of M to L. If K has characteristic 0, we say M_L is a lift of M *to characteristic* 0. We call a lift $M_R \to \mathrm{Spec}\,R$ to characteristic zero a *projective model* of M if $M_R \to \mathrm{Spec}\,R$ is projective.

2.2. **Automorphisms.** Let M and S be schemes and $\varphi : M \to S$ be a morphism. We denote by $\mathrm{Aut}\,(M/S)$ the *group of automorphisms of* M over S. When φ is flat projective, $\mathrm{Aut}\,(M/S)$ is the group of S-points of a group scheme representing the $\mathrm{Aut}(-/S)$-functor, but we will just use the abstract group $\mathrm{Aut}\,(M/S)$. If S is the spectrum of a ring R, we also write $\mathrm{Aut}(M/R)$. Finally for $S = \mathrm{Spec}\,k$, we write $\mathrm{Aut}(M)$ instead of $\mathrm{Aut}(M/k)$ if there is no danger of confusion.

2.3. **K3 surfaces and automorphisms.** A *K3 surface* V *over a field* F is a smooth projective geometrically irreducible 2-dimensional variety defined over F such that $H^1(V, \mathcal{O}_V) = 0$ and the dualizing sheaf ω_V is trivial, i.e., $\omega_V \simeq \mathcal{O}_V$. As a smooth proper scheme of dimension 2 over a field is necessarily projective, we stick to the notion of 'projective' surface rather than 'algebraic' surface.

Thoughout this article, X *is a K3 surface over an algebraically closed field* k *of characteristic* $p > 0$. A model $X_R \to \mathrm{Spec}\,R$ of a K3 surface has the property that X_K/K is a $K3$ surface.

The *Néron-Severi group* $\mathrm{NS}\,(X)$ of X is isomorphic to $\mathrm{Pic}\,(X)$, and it is a free $\mathbb{Z}$-module of finite rank. The rank is called the *Picard number* of X and denoted by $\rho(X)$. We have $1 \le \rho(X) \le 22$. The Hodge index theorem implies that the Néron-Severi group $\mathrm{NS}\,(X)$ is an even hyperbolic lattice with respect to the intersection from $(*, **)$, i.e., $(*, **) \in \mathrm{Sym}^2(\mathrm{NS}(X)^\vee)$, of signature $(1, \rho(X)-1)$ on $\mathrm{NS}\,(X) \otimes_{\mathbb{Z}} \mathbb{R}$. In addition, $(x^2) := (x, x) \in 2\mathbb{Z}$ for all $x \in \mathrm{NS}\,(X)$, as by the Riemann-Roch theorem $x^2 = 2(\chi(X, x) - \chi(X, \mathcal{O}_X))$. We denote the group of isometries of $(\mathrm{NS}\,(X), (*, **))$ by $\mathrm{O}\,(\mathrm{NS}\,(X))$.

The action by pull-back of line bundles $L \mapsto f^*L$ defines a contravariant representation

$$\mathrm{Aut}\,(X) \to \mathrm{O}\,(\mathrm{NS}\,(X)).$$

Let $f \in \mathrm{Aut}\,(X)$. The *spectral radius* of $f^* \in \mathrm{O}\,(\mathrm{NS}\,(X))$, denoted by $\mathrm{sp}\,(f)$, is the maximum of the absolute values of eigenvalues of $f^* \otimes \mathrm{id}_{\mathbb{C}}|_{\mathrm{NS}\,(X) \otimes \mathbb{C}}$. Here and hereafter for a complex number $\alpha = a + b\sqrt{-1}$ $(a, b \in \mathbb{R})$ the absolute value $|\alpha|$ of α is the non-negative real number

$$|\alpha| = \sqrt{a^2 + b^2}.$$

One has $\det f^* = \pm 1$. Thus, $\mathrm{sp}\,(f) \geq 1$. One defines f to be of *positive entropy* (resp. of *null-entropy*) if $\mathrm{sp}\,(f) > 1$ (resp. $\mathrm{sp}\,(f) = 1$). The entropy of f is defined by

$$h(f) = \log \mathrm{sp}\,(f)\ .$$

This definition is in fact equivalent to the one obtained by first defining the entropy as the natural logarithm of the maximum of absolute values of the eigenvalues of f^* acting of the ℓ-adic cohomology ring $\oplus_i H^i(X, \mathbb{Q}_\ell)$, with respect to all complex embeddings of the eigenvalues, as it is shown in [EsnSri13] that regardless of the choice of the complex embedding, this maximum is taken on the Néron-Severi group. This is also consistent with the notion of entropy for complex projective K3 surfaces. Note that if f is of positive entropy, then f is of infinite order, while the converse is not true in general. Let $G \subset \mathrm{Aut}\,(X)$ be a subgroup. We call G *of null-entropy* (resp. *of positive entropy*) if all the elements of G are of null-entropy (resp. some element of G is of positive entropy).

2.4. **Specializations.** Let $X_R \to \mathrm{Spec}\, R$ be a smooth proper morphism. Recall ([SGA6, X, App.]) that one has a *specialization homomorphism* $sp : \mathrm{Pic}(X_{\bar{K}}) \to \mathrm{Pic}(X)$ *on the Picard group*, which is defined as follows: any $\mathcal{L}_{\bar{K}} \in \mathrm{Pic}(X_{\bar{K}})$ is defined over a finite extension $L \supset K$, $L \subset \bar{K}$, so $\mathcal{L}_{\bar{K}} = \mathcal{L}_L \otimes_L \bar{K}$. Let $R_L \subset L$ be the ring of integers. The restriction homomorphism $\mathrm{Pic}(X_{R_L}) \to \mathrm{Pic}(X_L)$ is an isomorphism as X_{R_L} is smooth. So $\mathcal{L}_L = \mathcal{L}_{R_L} \otimes L$. Then the specialization of $\mathcal{L}_{\bar{K}}$ is $\mathcal{L}_{R_L} \otimes k$.

The specialization factors through the Néron-Severi group $sp_{NS} : NS(X_{\bar{K}}) \to NS(X_k)$ and through the Néron-severi group modulo torsion $sp_{NS/\mathrm{torsion}} : NS(X_{\bar{K}})/\mathrm{torsion} \to NS(X_k)/\mathrm{torsion}$. Then $sp_{NS/\mathrm{torsion}}$ is injective, as sp is compatible with the injections $NS(X_{\bar{K}})/\mathrm{torsion} \to H^2(X_{\bar{K}}, \mathbb{Q}_\ell(1))$, $NS(X)/\mathrm{torsion} \to H^2(X, \mathbb{Q}_\ell(1))$ defined by the Chern class and the specialization $H^2(X_{\bar{K}}, \mathbb{Q}_\ell(1)) \to H^2(X, \mathbb{Q}_\ell(1))$ on ℓ-adic cohomology, which is an isomorphism by the smooth proper base change theorem ([SGA4.5, V, Thm. 3.1]).

Let $X_R \to \mathrm{Spec}\, R$ be a model of a K3 surface X. Then one has a *restriction homomorphism* $\mathrm{Aut}(X_R/R) \to \mathrm{Aut}(X)$. Let us define the subset $\mathrm{Aut}^e(X_{\bar{K}}/\bar{K}) \subset \mathrm{Aut}(X_{\bar{K}}/\bar{K})$ consisting of those automorphisms which lift to some model $X_R \to \mathrm{Spec}\ R$. (Here e stands for extendable). It is clearly a subgroup, where the group law is define after finite base change. Then the restriction homomorphism yields a *specialization homomorphism*

$$\iota : \mathrm{Aut}^e(X_{\bar{K}}/\bar{K}) \to \mathrm{Aut}(X/k).$$

Moreover, sp is equivariant under ι. In addition, as automorphisms are recognized on the associated formal scheme, and $H^0(X, T_{X/k}) = 0$, the specialization homomorphism ι is injective (see [LieMau11, Lem. 2.3]) .

We call f *geometrically liftable* if it is in the image of the specialization homomorphism ι. One similarly defines geometric liftability of a subgroup $G \subset \mathrm{Aut}(X)$.

Remark 2.1. It is natural to ask whether the subgroup $\mathrm{Aut}^e(X_{\bar{K}}/\bar{K}) \subset \mathrm{Aut}(X_{\bar{K}}/\bar{K})$ is a strict subgroup. We give an explicit example for which $\mathrm{Aut}^e(X_{\bar{K}}/\bar{K}) \neq \mathrm{Aut}(X_{\bar{K}}/\bar{K})$ in Theorem 6.2. Any $f \in \mathrm{Aut}\,(X_{\bar{K}}/\bar{K})$ extends uniquely to $\tilde{f} \in \mathrm{Bir}\,(X_{R_L}/R_L)$, the group of birational automorphisms, where L, with $K \subset L \subset \bar{K}$, is a field of definition of f. As X is regular, if $\tilde{f}$ is not regular, then there is a 1-dimensional subscheme $C \subset X$ such that $\tilde{f}$ is well defined as a morphism $\tilde{f} : X_{R_L} \setminus C \to X_{R_L}$, but the morphism does not necessarily extend to X_{R_L}. Finally this implies that for any field extension $L' \supset L$, $L' \subset \bar{K}$, the base changed morphism $\tilde{f} \otimes R_{L'} : X_{R_{L'}} \setminus C \to X_{R_{L'}}$ does not extend either. See [LieMat14, Prop. 4.1] for related phenomena.

3. Automorphisms of even hyperbolic lattices

We call a polynomial $P(x) \in \mathbb{Z}[x]$ a *Salem polynomial* if it is irreducible, monic, of even degree $2d \geq 2$ and the complex zeroes of $P(x)$ are of the form $(1 \leq i \leq d-1)$:

$$a > 1 \ , \ 0 < a^{-1} < 1 \ , \ \alpha_i, \overline{\alpha}_i \in S^1 := \{z \in \mathbb{C} \,|\, |z| = 1\} \setminus \{\pm 1\} \ .$$

Proposition 3.1. *Let r be a positive integer and $L = (\mathbb{Z}^r, (*, **) \in \mathrm{Sym}^2(\mathbb{Z}^r)^\vee)$ be a hyperbolic lattice, i.e., the bilinear form $(*, **)$ is non-degenerate of signature $(1, r-1)$. Let $C := \{x \in L \otimes \mathbb{R} \,|\, (x^2) > 0\}$. Then C has exactly two connected components, say C^0 and $-C^0$. Let $f \in \mathrm{Aut}\, L$ and assume that $f(C^0) \subset C^0$. Then, the characteristic polynomial of f is the product of cyclotomic polynomials and at most one Salem polynomial. In particular, when $f(C^0) \subset C^0$, the characteristic polynomial of f is the product of cyclotomic polynomials if and only if f is of null-entropy if and ony if f is quasi-unipotent, i.e., all the eigenvalues of f^n are 1 for some positive integer n.*

Proof. This is well-known and essentially due to McMullen [Mc02]. See also [Og10]. □

One way for an automorphism f of an hyperbolic lattice to perserve C^0 is to fulfill $f(e) = e$ for a non-zero isotropic vector e.

Remark 3.2. For f as in Proposition 3.1, we define (by a slight abuse of notation) f to be of positive entropy (resp. of null entropy) if $\mathrm{sp}\,(f) > 1$ (resp. $\mathrm{sp}\,(f) = 1$). Thus f is of positive entropy (resp. of null entropy) if and only if the characteristic polynomial of f has a Salem factor (then exactly one) (resp. only cyclotomic factors).

Proposition 3.3. *Let L be as in Proposition 3.1 and f be in $\mathrm{Aut}(L)$. Assume that there is $e \in L \setminus \{0\}$ such that $f(e) = e$ with $(e^2) := (e, e) = 0$. Then the characteristic polynomial of f is the product of cyclotomic polynomials.*

Proof. We may assume without loss of generality that e is primitive in the sense that e is a part of $\mathbb{Z}$-basis of L. By the assumption, f acts on the flag $\mathbb{Z}e \subset (\mathbb{Z}e)^\perp$ and hence induces an automorphism $\bar{f}$ of

$$\overline{L} := (\mathbb{Z}e)^\perp / \mathbb{Z}e \ .$$

The bilinear form of L induces a bilinear form of $\overline{L}$ of signature $(0, r-2)$, i.e., $\overline{L}$ is negative definite or $\{0\}$. If $\overline{L} \neq 0$, the eigenvalues of $\bar{f}$ on $\overline{L}$ are of absolute value 1. Here we use the well-known fact that eigenvalues of a real orthogonal matrix are of absolute value 1. Combining this with $f(e) = e$, we find that the eigenvalues of f are of absolute value 1 except perhaps one eigenvalue counted with multiplicities. Note that $\det f = \pm 1$, as an automorphism of free $\mathbb{Z}$-module L of finite positive rank. Hence the last eigenvalue is also of absolute value 1. Since $f(C^0) \subset C^0$ as $f(e) = e$, this implies the result by Proposition 3.1. □

4. Lifting to characteristic 0 K3 surfaces with Picard number one

Let M be a complex projective K3 surface with a primitive ample line bundle H. Recall H is said to be *primitive* if it is a part of $\mathbb{Z}$-basis of the finitely generated free $\mathbb{Z}$-module $\mathrm{Pic}\,(M) = \mathrm{NS}\,(M)$. It is well-known that any generic fiber $\mathcal{X}_t$ of the Kuranishi family $\kappa : (\mathcal{X}, \mathcal{H}) \to \mathcal{K}$ of (X, H) is of Picard number 1. Here 'generic' is to be understood in the complex analytic sense. In particular, restricting κ to the germ of a smooth curve $\mathcal{B} \subset \mathcal{K}$ containing a generic point, the kernel of the Gauß-Manin connection on H^2 on $\mathcal{B}$ is spanned by the first de Rham Chern class of a generator of $\mathrm{NS}\,(\mathcal{X}_t)$.

In [LieOls11, App. A], M. Lieblich and M. Olsson, prove the analogous result on Picard rank 1 lifts to characteristic 0 of characteristic $p \geq 3$ K3 surfaces.

Theorem 4.1. *Let X be a K3 surface defined over an algebraically closed field k of characteristic $p > 0$, where $p > 2$ if X is Artin-supersingular. Then there is a discrete valuation ring R, finite over the ring of Witt vectors $W(k)$, together with a projective model $X_R \to \mathrm{Spec}\, R$, such that the Picard rank of $X_{\bar{K}}$ is 1, where $K = \mathrm{Frac}\,(R)$ and $\bar{K} \supset K$ is an algebraic closure.*

Their proof (in $p \geq 3$) relies on [Ogu79] and [Ogu83] and properties of stacks for pairs of K3 surfaces together with line bundles.

For a line bundle L on X, we denote by $c_1^{\mathrm{Hodge}}(L) \in H^1(X, \Omega^1_X)$ its Hodge Chern class.

Proposition 4.2. *Let X be a K3 surface defined over an algebraically closed field k of characteristic $p > 0$, with $p > 2$ if X is Artin-supersingular. Then there is an ample primitive line bundle L such that $c_1^{\mathrm{Hodge}}(L) \neq 0$.*

Proof. Unfortunately, we can not prove it directly, this is the reason for the restriction on p. If X is not Artin-supersingular, then by [GK00, Prop. 10.3],

$$c_1^{\mathrm{Hodge}} : \mathrm{Pic}(X)/p\mathrm{Pic}(X) \to H^1(X, \Omega^1_{X/k})$$

is injective. Else, due to our assumption, it is Shioda-supersingular, thus by [GK00, Prop. 11.9], c_1^{Hodge} is not identically zero. Once one line bundle M fulfills $c_1^{\mathrm{Hodge}}(M) \neq 0$, then given any ample line bundle H, $0 \neq c_1^{\mathrm{Hodge}}(M) = c_1^{\mathrm{Hodge}}(M + mpH)$ for any integer m, and for m large, $M + pmH$ is ample. The $\mathbb{Z}$-module $\mathbb{Q} \cdot (M + pmH) \cap NS(X)$ of rank 1 has a generator L such $(M + mH) = aL$ where $a \in \mathbb{N} \setminus \{0\}$. Then L is primitive, ample, and fulfills $c_1^{\mathrm{Hodge}}(L) \neq 0$.

□

In fact, one can prove Theorem 4.1 by using [Del81] and Proposition 4.2 only. Indeed, for L as in Proposition 4.2, one shows ([Ogu79, Prop. 2.2] and [LieMau11, Lem. 4.3]) that the formal hypersurface $\Sigma(X, L) \subset \hat{S} = \mathrm{Spf}\ W[[t_1, \ldots, t_{20}]]$, which is defined as the solution to the deformation functor of the pair (X, L), while $\hat{S}$ is defined as the solution of the deformation functor of X, is formally smooth.

Let $\hat{X}$ be the formal universal K3 surface over $\hat{S}$. Set $\hat{Y} = \hat{X} \times_{\hat{S}} \Sigma(X, L)$ and let $\mathcal{L}$ be the formal universal line bundle on $\hat{Y}$ lifting L on X. Then the de Rham class $c_1^{DR}(\mathcal{L}) \in H^2_{DR}(\hat{Y}/\Sigma(X, L))$ is non-zero as its restriction in $H^1(X, \Omega^1_X)$ is non-zero.

Then one shows, using the injectivity of the crystalline Chern class map $\mathrm{Pic}(X) \to H^2_{DR}(\hat{X}_W/W)$ ([Del81, 2.10])

Theorem 4.3. *The kernel of the Gauß-Manin connection*

$$\nabla : H^2(\Omega^{\geq 1}_{\hat{Y}/\hat{\Sigma}}) \to \Omega^1_{\hat{\Sigma}/W} \otimes H^2_{DR}(\hat{Y}/\hat{\Sigma})$$

is spanned by $c_1^{DR}(\mathcal{L})$ *over* W.

From Theorem 4.3 one easily deduces Theorem 4.1.

Remark 4.4. Given X as in Theorem 4.1, and $L \in \mathrm{Pic}(X)$ as in Proposition 4.2, then $X_R \to \mathrm{Spec}\, R$ of Theorem 4.1 is constructed in such a way that L lifts to X_R.

5. No lifting of automorphisms

In this section, applying Theorem 4.1, we construct a projective model $X_R \to \mathrm{Spec}\, R$ of a K3 surface X, with $K = \mathrm{Frac}(R)$ of characteristic zero, for which almost all automorphisms of X are not geometrically liftable.

Theorem 5.1. *Let X be a K3 surface defined over an algebraic closed field k of characteristic $p > 0$, where $p > 2$ if X is Artin-supersingular.*

1) *Assume that either the Picard number of X is ≥ 2 or that $\mathrm{Pic}\,(X) = \mathbb{Z} \cdot H$ and $H^2 \neq 2$. Then there is a DVR R, finite over $W(k)$, together with a projective model $X_R \to \mathrm{Spec}\, R$ of $X \to \mathrm{Spec}\, k$ such that no subgroup $G \subset \mathrm{Aut}\,(X)$, except for $G = \{\mathrm{id}_X\}$, is geometrically liftable to $X_R \to \mathrm{Spec}\, R$;*
2) *Assume that $\mathrm{Pic}\,(X) = \mathbb{Z} \cdot H$ and $(H^2) = 2$. Then, for any projective model $X_R \to \mathrm{Spec}\, R$ with R finite over $W(k)$, $\mathrm{Aut}^e(X_R/R) = \mathrm{Aut}(X_R/R)$, the specialization homomorphism $\iota : \mathrm{Aut}(X_{\bar{K}}) \to \mathrm{Aut}(X)$ is an isomorphism, and $\mathrm{Aut}(X) = \mathbb{Z}/2$.*

Proof. First we prove 2). By replacing H by $-H$ if necessary, we may assume that H is ample. By Proposition 4.2 or, if $p \neq 2$, simply by $H^2(X, \Omega^2_{X/k}) \ni c_1^{\mathrm{Hodge}}(H)^{\cup 2} =$ residue class of 2 in k, which is thus non-zero, $c_1^{\mathrm{Hodge}}(H) \neq 0$ and H extends to a line bundle H_R for any projective lift $X_R \to \mathrm{Spec}\, R$ to characteristic 0. By [SD74], $h^0(X, H) = 3$, $h^i(X, H) = 0$ ($i \geq 1$), and H is globally generated. Strictly speaking, $p \neq 2$ is assumed in [SD74]. However, since H is an ample generator, $h^0(H) \geq 3$ by the Riemann-Roch theorem. Moreover, any element in $|H|$ is irreducible and reduced. (Indeed, if $C + D \in |H|$ for some non-zero effective divisors C, D, with possibly $C = D$, then $C \in |nH|$ and $D \in |mH|$ for some positive integers n, m by $\mathrm{Pic}\,(X) = \mathbb{Z} \cdot H$. However, then $2 \leq n + m = 1$, a contradiction.) So, we can apply [SD74, Prop. 2.6, Thm. 3.1], which is characteristic free, to our X, to conclude that $|H|$ defines a finite surjective morphism $\varphi : X \to \mathbb{P}^2_k$ of degree 2. This is also separable even for $p = 2$, as X has no non-zero vector field by [RS81]. We denote the covering involution by $\iota \in \mathrm{Aut}\,(X)$. Since $h^i(X, H) = 0$ ($i \geq 1$), it follows that $H^0(X_R, H_R)$ is a rank 3 free module over R, which satisfies base change. It thus defines a finite surjective morphism $\varphi_R : X_R \to \mathbb{P}(H^0(X_R, H_R)^\vee) \cong \mathbb{P}^2_R$, of degree 2, the specialization of which over $\mathrm{Spec}\, k$ is $\varphi : X \to \mathbb{P}^2_k$. We denote by $\iota \in \mathrm{Aut}\,(X_R/R)$ the covering involution of φ_R, which exists as φ_R is finite. Then ι_R specializes to ι. Hence, in the exceptional case $\mathrm{Pic}\,(X) = \mathbb{Z} \cdot H$ with $(H^2) = 2$, the involution $\iota \in \mathrm{Aut}\,(X)$ lifts as an automorphism to any projective lift to characteristic 0, in particular, ι is geometrically liftable to any projective lift to characteristic zero. More precisely, $\mathrm{Pic}\,(X_{\bar{K}}) = \mathbb{Z} \cdot H_{\bar{K}}$, $(H_{\bar{K}}^2) = 2$ and $\mathrm{Aut}\,(X_{\bar{K}}) = \langle \iota_{\bar{K}} \rangle \simeq \mathbb{Z}/2$, where $\bar{K}$ is an algebraic closure of $K = \mathrm{Frac}\,(R)$ and $\iota_{\bar{K}}$ is the covering involution of the morphism $\varphi_{\bar{K}} : X_{\bar{K}} \to \mathbb{P}^2_{\bar{K}}$ given by $|H_{\bar{K}}|$. This finishes the proof of 2). $\square$

Proof. We prove 1). The following lemma ought to be well-known to the experts:

Lemma 5.2. *Let Z be a complex projective K3 surface, with Picard group* $\mathrm{Pic}\,(Z)$ *generated by an ample class H. Assume that $(H^2) \neq 2$. Then* $\mathrm{Aut}\,(Z) = \{\mathrm{id}_Z\}$.

We do not know whether Lemma 5.2 holds also for a K3 surface X with $\rho(X) = 1$ in characteristic $p \geq 3$. However, if X is a K3 surface defined over $\bar{\mathbb{F}}_p$, with $p \geq 3$, the positive solution to the Tate conjecture ([MPe13], also [Ben14] with references therein) implies that $\rho(X)$ is even ([Ar74]).

Proof. We give a proof for the reader's convenience.

Since $H^0(Z, T_Z) = 0$ and $\mathrm{Aut}\,(Z)$ preserves the ample generator H, it follows that $\mathrm{Aut}\,(Z)$ is of dimension 0 and also a closed algebraic subgroupscheme of the affine groupscheme $\mathrm{Aut}\,(\mathbb{P}(H^0(Z, mH)^\vee))$ for large $m > 0$. It follows that $\mathrm{Aut}\,(Z)$ is a finite group.

Let $T(Z)$ be the transcendental lattice of Z, that is, the orthogonal complement of $\mathrm{NS}\,(Z)$ in $H^2(Z, \mathbb{Z}(1))$. Thus the representation $\mathrm{Aut}\,(Z) \to \mathrm{O}\,(\mathrm{NS}\,(Z)) \times \mathrm{O}\,(T(Z))$ has image in $\{1\} \times \mathrm{O}\,(T(Z))$. On the other hand, since Z is projective and $\mathrm{Aut}\,(Z)$ is a finite group, by Nikulin [Ni79], the image of the natural map $G \to \mathrm{O}\,(T(Z))$ is a cyclic group of finite order N, where N is exactly the order of the image of the representation $\mathrm{Aut}\,(Z) \to \mathrm{GL}\,(H^0(Z, \Omega_Z^2)) = \mathbb{C}\omega_Z$, and the Euler function $\varphi(N) := [\mathbb{Q}(e^{2\pi i/N}) : \mathbb{Q}]$ of N is a divisor of the rank of the transcendental lattice $T(Z)$. In our case, $T(Z)$ is of rank $21 = 22 - 1$. In particular, it is an odd number. Hence $N = 1$ or 2. As $(\mathrm{NS}(X) \oplus T(Z)) \otimes_{\mathbb{Z}} \mathbb{Q} = H^2(Z, \mathbb{Q}(1))$ and $\mathrm{Aut}\,(Z)$ stabilizes the lattice $\mathrm{NS}(Z) \oplus T(Z) \subset H^2(Z, \mathbb{Q}(1))$, the image of the representation $\mathrm{Aut}\,(Z) \to GL(H^2(Z, \mathbb{Q}(1)))$ is cyclic of order 1 or 2, as well as the image of the representation $G \to GL(H^2(Z, \mathbb{Z}(1))$. On the other hand, by the global Torelli theorem for complex projective K3 surfaces ([PS71]), the action of $\mathrm{Aut}\,(Z)$ on $H^2(Z, \mathbb{Z}(1))$ is faithful. Thus $\mathrm{Aut}\,(Z)$ is either $\{\mathrm{id}_Z\}$ or cyclic of order 2.

So far, we did not use $(H^2) \neq 2$. Set $2d = (H^2) \geq 2$. Then by [Ni79-2, Cor. 1.6.2], we have

$$\mathbb{Z}/2d \simeq \mathrm{NS}\,(Z)^*/\mathrm{NS}\,(Z) \simeq T(Z)^*/T(Z)\ ,$$

as $H^2(Z, \mathbb{Z})$ is free and unimodular and $\mathrm{NS}\,(Z)$ is primitive in $H^2(Z, \mathbb{Z})$. Here $(-)^*$ means $\mathrm{Hom}_{\mathbb{Z}}((-), \mathbb{Z})$ and $T(Z) = (\mathbb{Z} \cdot H)^\perp$.

As $\iota^* H$ is ample, and H is the unique ample generator of $\mathrm{Pic}\,(Z)$, one concludes that $\iota^*(H) = H$, thus $\iota^* = \mathrm{id}$ on $\mathrm{NS}\,(Z)^*/\mathrm{NS}\,(Z)$. Hence $\iota^* = \mathrm{id}$ on $T(Z)^*/T(Z)$ as well. On the other hand, the involution ι satisfies $\iota^*\omega_Z = -\omega_Z$ by Nikulin's result above. Thus $\iota^* = -\mathrm{id}$ on $T(Z)^*/T(Z)$. Thus $\mathrm{id} = -\mathrm{id}$ on $\mathbb{Z}/2d$. Hence $2d = 2$ as claimed. $\square$

Lemma 5.3. *Let X be as in Theorem 5.1. Assume that* $\mathrm{Pic}(X) = \mathrm{NS}\,(X)$ *is not isomorphic to $\mathbb{Z} \cdot H$ with self-intersection number $(H^2) = 2$. Then, there is an ample primitive line bundle L such that $c_1^{\mathrm{Hodge}}(L) \neq 0$ and $(L^2) \neq 2$.*

Proof. By Proposition 4.2, there is an ample primitive line bundle L_0 such that $c_1^{\rm Hodge}(L_0) \neq 0$. If $(L_0^2) \neq 2$, then we may take $L = L_0$. In particular, if $\rho(X) = 1$, then we are done, as we exclude the case ${\rm Pic}\,(X) = \mathbb{Z} \cdot L$ with $(L^2) = 2$.

So, we may assume without loss of generality that $\rho(X) \geq 2$ and $(L_0^2) = 2$. Since L_0 is primitive and $\rho(X) \geq 2$, we can choose a line bundle M such that $\{L_0, M\}$ is part of a $\mathbb{Z}$-basis of ${\rm Pic}\,(X)$. Replacing M by $M + nL_0$ with large integer n, we may further assume, without loss of generality, that M is also ample. Note here that $\mathbb{Z}\langle L_0, M\rangle = \mathbb{Z}\langle L, M + nL_0\rangle$ and L_0 and M remain part of free $\mathbb{Z}$-basis under this replacement. Now consider $L = pM + L_0$. Then L is ample, as M and L_0 are ample. L is also primitive, as L_0 and M form part of free $\mathbb{Z}$-basis of $\mathbb{Z}$-module ${\rm Pic}\,(X)$. Moreover,

$$(L^2) = p^2(M^2) + 2p(M.L_0) + (L_0^2) > 2\ ,$$

by $(M^2) > 0$, $(M.L_0) > 0$ and $(L_0^2) > 0$ by the ampleness. Thus $(L^2) \neq 2$. Moreover,

$$c_1^{\rm Hodge}(L) = pc_1^{\rm Hodge}(M) + c_1^{\rm Hodge}(L_0) = 0 + c_1^{\rm Hodge}(L_0) = c_1^{\rm Hodge}(L_0) \neq 0\ ,$$

in the k-vector space $H^1(X, \Omega_X^1)$. So, $L = pM + L_0$ satisfies all the requirements. □

Now we are ready to finish the proof of Theorem 5.1 1). We take the model $X_R \to {\rm Spec}\,R$ of Theorem 4.1, Remark 4.4, applied to the ample primitive line bundle L in Lemma 5.3. Assume that G is geometrically liftable to $X_R \to {\rm Spec}\,R$. Then G has to stabilize ${\rm Pic}(X_{\bar{K}})$, thus it fixes the polarisation $\mathcal{L}_{\bar{K}}$. Hence G is a finite group. So we may assume that there is an abstract field isomorphism $\bar{K} \to \mathbb{C}$ and $X_{\bar{K}}$ is a complex projective K3 surface, say Z, with Picard group ${\rm Pic}\,(Z)$ generated by an ample class H_Z with $(H_Z^2) \neq 2$, and G is now a group of automorphisms of Z. Then $G = \{{\rm id}_X\}$ by Lemma 5.2.

□

6. Non-liftable automorphism of positive entropy

The aim of this section is to construct an example of an automorphism of a supersingular K3 surface over an algebraically closed field k of characteristic $p \geq 3$, which is not geometrically liftable to any projective model. As far as we are aware of, this is the first such example. Our construction is based on the work by Kondo-Shimada [KS12], and is (mildly) computer supported. The characteristic p will be equal to 3.

Recall that $\det {\rm NS}\,(X) = -p^{2\sigma_0}$ for a supersingular K3 surface defined over k. The value σ_0 is called the *Artin invariant* of X. Artin [Ar74] proved that $1 \leq \sigma_0 \leq 10$ and Ogus [Ogu79] proved the uniqueness of a supersingular K3 surface over k with $\sigma_0 = 1$, up to isomorphisms. Then this K3 surface is isomorphic to the Kummer K3 surface ${\rm Km}\,(E \times_k E)$ associated to the product abelian surface $E \times_k E$ of any supersingular elliptic curve E/k. Tate and Shioda ([Sh75]) proved that the Fermat quartic K3 surface is supersingular if and ony if $p \equiv 3$ mod 4. There are several other descriptions of supersingular K3 surfaces of Artin invariant 1 (see e.g. [Sh13]).

From now until the end of this section, X is a supersingular K3 surface, defined over k of characteristic 3, *with Artin invariant* 1. As remarked above, X is isomorphic to the Fermat quartic K3 surface. We denote by $q : X \hookrightarrow \mathbb{P}^3$ the projective embedding and set $H = q^*\mathcal{O}_{\mathbb{P}^3}(1)$.

In [KS12], Kondo and Shimada prove the following statements, which are crucial for our construction:

(i) X has two globally generated line bundles L_i ($i = 1, 2$) of degree 2 ($\mathcal{L}_{m_i}$ in their notation [KS12, p. 19]). They are not ample. The linear system associated to $H^0(X, L_i)$ induces a well defined morphism $\varphi_i : X \to \mathbb{P}^2$ which is generically finite $2:1$, but not finite. The Galois involution of the function field extension extends as an automorphism $\tau_i \in \mathrm{Aut}\,(X/\mathbb{P}^2)$. Indeed, it clearly extends as an automorphism in $\mathrm{Aut}(Y_i/\mathbb{P}^2)$ where $\varphi_i : X \to Y_i \to \mathbb{P}^2$ is the Stein factorization, and on the other hand, X is the minimal desingularization of the surface Y_i.

(ii) Let $\mathrm{Aut}\,(X, H)$ be the automorphism group of X induced by the projective linear automorphisms of $\mathbb{P}^3$ under q. It is known that $\mathrm{Aut}\,(X, H)$ is a finite group but of huge order (see. e.g. [DK09], [Mu88]). In $\mathrm{Aut}\,(X, H)$, there is a special element $\tau \in \mathrm{Aut}\,(X, H)$ of order 28 ([KS12, Ex. 3.4]).

They prove the following beautiful description of the automorphism group of X:

Theorem 6.1. $\mathrm{Aut}\,(X) = \langle \tau_1, \tau_2, \mathrm{Aut}\,(X, H)\rangle$.

In the course of the proof, they work with an explicit $\mathbb{Z}$-basis $\mathcal{B}$ of $\mathrm{NS}\,(X)$, consisting of 22 lines among the 112 lines on X ([SSL10, Lem. 6.3]) and compute the (right, hence covariant) representation of $(\tau_i)_*|_{\mathrm{NS}\,(X)}$, $\tau_*|_{\mathrm{NS}\,(X)}$ on $\mathrm{NS}\,(X)$. They actually write explicitely the matrices of $(\tau_i)_*|_{\mathrm{NS}\,(X)}$ and $\tau_*|_{\mathrm{NS}\,(X)}$ in the basis $\mathcal{B}$. We denote them by A_1, A_2, T respectively, of which explicit forms are in Tables 5.4, 5.5, 3.3 in [KS12]. These forms are important for the proof of Theorem 6.1.

Suppose one has a model $X_R \to \mathrm{Spec}\, R$ on which L_i lift to $L_{i,R}$. Then, as $H^0(X_R, L_{i,R}) \otimes_R k = H^0(X, L_i)$, one has base change $Y_{i,R} \otimes_R k = Y_i$ for the Stein factorization $\varphi_{i,R} : X_R \to Y_{i,R} \to \mathbb{P}^2_R$ of the well defined morphism $\varphi_{i,R} : X_R \to \mathbb{P}^2_R$ associated to $H^0(X_R, L_{i,R})$. The Galois involution of the function field extension induced by $\varphi_{i,R}$ extends as an automorphism $\tau_{Y_{i,R}} \in \mathrm{Aut}(Y_{i,R}/\mathbb{P}^2_R)$, which induces a birational automorphism $\tau_{i,R} \in \mathrm{Bir}(X_R/\mathbb{P}^2_R)$. One says that τ_i *lifts to* X_R if $\tau_{i,R} \in \mathrm{Aut}(X_R/\mathbb{P}^2_R) \subset \mathrm{Bir}(X_R/\mathbb{P}^2_R)$.

Theorem 6.2. *There is a projective model* $X_R \to \mathrm{Spec}\ R$ *of* $X = X(3)$ *with* R *of characteristic* 0 *such that* $\mathrm{Aut}^e(X_{\bar{K}}) \neq \mathrm{Aut}\,(X_{\bar{K}})$. *Here* $K = \mathrm{Frac}(R)$ *and* $\bar{K}$ *is an algebraic closure of* K.

Proof. Let H be an ample line bundle on X. By [LieOls11, App. A], there is a model $X_R \to \mathrm{Spec}\ R$ such that L_1, L_2 and H lift. This property is compatible with further base change $R \subset R_L$ where $L \supset K$ is a field extension with $L \subset \bar{K}$. This model is projective. Let us denote by f_i the restriction of $\tau_{i,R}$ to $X_{\bar{K}}$. Then $f_i \in \mathrm{Aut}(X_{\bar{K}})$ as $X_{\bar{K}}$ is a minimal smooth projective surface. It suffices to show that one of f_i ($i = 1, 2$) is not in $\mathrm{Aut}^e(X_{\bar{K}})$. Assume to the contrary that both f_i ($i = 1, 2$) are in $\mathrm{Aut}^e(X_{\bar{K}})$. Then $f_i^*\omega = -\omega$, where ω is a non-zero global 2-form of $X_{\bar{K}}$. Then $(f_1 \circ f_2)^*\omega = \omega$. Thus $(f_1 \circ f_2)^*$ has one eigenvalue equal to one on de Rham cohomology $H^2_{DR}(X_{\bar{K}}/\bar{K})$, thus, by the comparison theorem, on ℓ-adic cohomology $H^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell(1))$ as well, thus by ([SGA4.5, V, Thm. 3.1]), on $H^2_{\text{ét}}(X, \mathbb{Q}_\ell(1))$ as well.

On the other hand, using the explicit forms of A_1 and A_2 in Tables 5.4, 5.5 in [KS12], and `Mathematica (all we need here are Dot command, CharacteristicPolynomial command,` `Factor command`), we find that the characteristic polynomial of $(\tau_2 \circ \tau_1)^*|_{\mathrm{NS}\,(X)} = \tau_{1*} \circ \tau_{2*}|_{NS(X)}$, i.e., of $A_1 A_2$, is

$$(1+x+x^2)(1-11x+10x^2-9x^3+9x^4-10x^5+15x^6-23x^7+19x^8-14x^9+14x^{10}-14x^{11}+19x^{12} -23x^{13}+15x^{14}-10x^{15}+9x^{16}-9x^{17}+10x^{18}-11x^{19}+x^{20})\ ,$$

of which 1 is not a zero, a contradiction. □

Remarks 6.3. On the model of Theorem 6.2, one of the τ_i does not lift, which means that a small modification occurs on the special fiber. In fact one can say the following. Given a projective model $X_R \to \mathrm{Spec}\ R$, with ample line bundle L_R, then τ_i always lifts to an automorphism τ_i^0 of $X_R^i := X_R \setminus \Sigma_i$, where Σ_i is the exceptional locus of $X \to Y_i$. So one can always define the line bundle $L_{i,R} = (\tau_i^0)^* L_R|_{X_R^i} \in \mathrm{Pic}(X_R^i) = \mathrm{Pic}(X_R)$. Then τ_i lifts to X_R if and only if $L_{i,R}$ is ample, which is equivalent to $L_{i,R} \otimes_R k = \tau_i^*(L_R \otimes_R K)$.

The next theorem gives the (first) explicit example of a positive entropy automorphism of a K3 surface which is not geometrically liftable to characteristic zero.

Theorem 6.4. *The automorphism* $f := \tau_1 \circ \tau \circ \tau_2 \circ \tau \in \mathrm{Aut}\,(X)$

(1) *is not geometrically liftable to any projective model* $X_R \to \mathrm{Spec}\, R$ *of characteristic zero as well as its power* f^n $(n \in \mathbb{Z} \setminus \{0\})$*;*
(2) *has positive entropy* $h(f)$ *equal to the logarithm of a Salem number* a *of degree* 22*;*
(3) $h(f)$ *is not the entropy of any automorphism on any projective K3 surface in characteristic* 0*;*
(4) *numerically*
$$a = 26.9943\ldots\ ,\ h(f) = \log 26.9943\ldots\ .$$

Proof. Using the explicit forms of A_1, A_2, T in Tables 5.4, 5.5, 3.3 in [KS12], and `Mathematica (all we need here are Dot command, CharacteristicPolynomial command, Factor command, and NSolve command)`, we find that the characteristic polynomial P of $f_*|_{\mathrm{NS}\,(X)})$, i.e., of $A_1 T A_2 T$, is

$$1-27x+4x^3+3x^4+24x^5+15x^6-7x^7+x^8-14x^9-2x^{10}-5x^{11}-2x^{12} -14x^{13}+x^{14}-7x^{15}+15x^{16}+24x^{17}+3x^{18}+4x^{19}-27x^{21}+x^{22}\ .$$

This is an irreducible Salem polynomial of degree 22. The fact that this is irreducible is checked by `Factor command`. Then this is a Salem polynomial by Proposition 3.1. Indeed, it is either a Salem polynomial or a cyclotomic polynomial of degree 22. But `NSolve command` shows that one of the zeroes of the above polynomial is approximately 26.9943. Hence it is a Salem polynomial of degree 22 with Salem number approximately 26.9943. This shows (2).

By [Smy14, Lem. 2], if λ is a Salem number of degree d, then λ^n is a Salem number of the same degree d for all $n \in \mathbb{N} \setminus \{0\}$. Applying this to the eigenvalue λ of f or of f^{-1} which is a Salem number, one concludes that f^n, for all $n \in \mathbb{Z} \setminus \{0\}$, has entropy the logarithm of a Salem number of degree 22.

As in characteristic 0, an automorphism g always stabilizes $NS(X) \subset H^2(X_{\bar{K}}, \mathbb{Q}_\ell(1))$ and the Picard rank is at most 20, the logarithm of the absolute value of a root of P can not be the entropy of g. This shows (3).

If $f^n,\ n \in \mathbb{Z}$ was lifting to an automorphism g on $X_R \to \operatorname{Spec} R$, then the specialization $\iota : NS(X_{\bar{K}}) \hookrightarrow NS(X)$ (see Section 2.3) would be g equivariant, thus, as the Picard rank of $X_{\bar{K}}$ is at most 20, the minimal polynomial of g could not have degree 22. This shows (1) and finishes the proof. □

Remarks 6.5. 1) As discussed in [BlEsKe14, Conj. 1.2], one expects that the rational crystalline cycle class of an algebraic cycle, expressed as a de Rham class on a model in characteristic 0, is the cycle class of an algebraic cycle on the model, if and only if it is in the right level of the Hodge filtration. For the cycle class c of the graph of an automorphism f on a K3 surface, the conjecture is verified (as written in [Ogu79, Cor. 2.5]). Indeed,

$$c \in F^2 H^4_{DR}(X_R \times_R X_R/R) = F^2 H^4(\hat{X}_R \times_R \hat{X}_R/R)$$

if and only if f^* acting on $H^2_{DR}(\hat{X}_R/R) = \varprojlim_n H^2_{DR}(X_n/(R/\langle \pi^n \rangle))$ respects the Hodge filtration. Here π is the uniformizer of R and $X_n = X \otimes_R R_n,\ R_n = R/\langle \pi^n \rangle$. Clearly, if f lifts, then c is the cycle class of the graph and lies in $F^2H^4(\hat{X}_R \times_R \hat{X}_R/R)$. Let us now assume that $c \in F^2H^4(\hat{X}_R \times_R \hat{X}_R/R)$. The obstruction to lifting f_n on X_n to f_{n+1} on X_{n+1} lies in $H^1(X_n, f_n^* T_{X_n/R_n} \otimes \pi^n|_{X_n})$ and is identified with the action of f_n^* in $\operatorname{Hom}(H^1(f^*(\Omega^1_{X_n/R_n})), \pi^n \otimes H^2(\mathcal{O}_{X_n}))$, thus dies. One constructs in this way a prosystem of lifts $\varprojlim_n f_n$, thus, a formal scheme $\varprojlim_n \Gamma_n$, where $\Gamma_n \subset X_n \times_{R_n} X_n$ is the graph of f_n, thus, by [EGA3, Chap.III, Thm. 5.4.5], a projective scheme $\Gamma_R \subset X_R \times_R X_R$ which lifts the graph of f and thus defines the lift.

So the test whether or not an automorphism lifts to characteristic 0 is of p-adic nature. On the other hand, the test we develop in Theorem 6.4 relies on the degree of an algebraic integer. It is of course very specific to our situation, nonetheless it is intriguing.

2) The Salem number we define in Theorem 6.4 does not come from the entropy of an automorphism on a projective K3 surface in characteristic 0. One could perhaps speculate that there is a projective model $V_R \to \operatorname{Spec} R$ of a higher dimensional smooth projective variety V_K in characteristic 0, with an automorphism f_R of V_R/R, such that its entropy is reached on the class of a 1-cycle, the support of which, by specialization, lies on the K3 surface considered in Theorem 6.4, as a higher codimensional cycle on $V_R \otimes_R k$. Though we do not have any computation going in this direction, this would just be nice.

7. Lifting of automorphisms of supersingular K3 surfaces of Artin invariant 1 in large characteristic

Throughout this section, k is an algebraically closed field of characteristic $p \geq 3$ and $X = X(p)$ as in Section 6. So $X(p)$ is a supersingular K3 surface of Artin invariant 1, and is uniquely defined up to isomorphism with this property.

Using Ogus' crystalline Torelli theorem [Ogu83], J. Jang [Jan14, Thm.3.3] proved the following theorem:

Theorem 7.1. *The image of the representation of* $\operatorname{Aut}(X)$ *in the linear automorphism of the one dimensional vector space* $k \cdot \omega = H^0(X, \Omega^2_{X/k})$ *is a cyclic group of cardinality* $p+1$.

We denote by h an element in $\operatorname{Aut}(X)$ such that $h^*\omega = \xi_{p+1}\omega$ where ξ_{p+1} is a primitive $(p+1)$-th root of unity in k.

Remark 7.2. Let M be a projective K3 surface over a characteristic 0 field K. Then the image of $\mathrm{Aut}(M)$ in the linear automorphism of the one dimensional vector space $K \cdot \omega = H^0(M, \Omega^2_{M/K})$ is a cyclic group of order ≤ 66 ([Ni79]). In fact Nikulin considered the image of a finite subgroup, but, by the finiteness of the pluri-canonical representation in characteristic 0 (Ueno-Deligne, [Ue75, Thm. 14.10]), the proof extends to the whole automorphism group.

Jang deduces from Theorem 7.1 and Remark 7.2 the following:

Corollary 7.3. *If $p \ge 67$, then h is not geometrically liftable to characteristic* 0.

The aim of this section is to construct an element $\tau \in \mathrm{Aut}(X)$ *of positive entropy* which is not geometrically liftable to characteristic 0. (Recall $X = X(p)$).

Definition 7.4. We define β to be the least common multiple of the natural numbers n such that the value of the Euler function $\varphi(n)$ is smaller or equal to 22.

Theorem 7.5. *If $p + 1 \ge 67\beta$, then there is an automorphism τ in $\mathrm{Aut}(X(p))$, of positive entropy, which is not geometrically liftable to characteristic* 0.

In order to prove the theorem, we first state the following lemmata.

For an endomorphism θ of a free $\mathbb{Z}$-module of finite type, we denote by $\mathrm{tr}(\theta)$ its trace with values in $\mathbb{Z}$.

Lemma 7.6. *Let X be a K3 surface over any field. Let $f \in \mathrm{Aut}\,(X)$ such that*

$$|\mathrm{tr}\,(f^*|_{\mathrm{NS}\,(X)})| \ge 23.$$

Then f is of positive entropy. Conversely, if f is of positive entropy, then there is a positive integer N such that $|\mathrm{tr}\,((f^n)^*|_{\mathrm{NS}\,(X)})| \ge 23$ for all integers n such that $n \ge N$.

Proof. Recall that $\mathrm{NS}\,(X)$ is of rank $\rho \le 22$. Let α_i $(1 \le i \le \rho)$ be the eigenvalues of $f^*|_{\mathrm{NS}\,(X)}$. If f is not of positive entropy, then α_i are cyclotomic integers by Remark 3.2. Hence

$$|\mathrm{tr}\,(f^*|_{\mathrm{NS}\,(X)})| = |\sum_{i=1}^{\rho} \alpha_i| \le \sum_{i=1}^{\rho} |\alpha_i| = \rho < 23\ .$$

Hence f is of positive entropy if $|\mathrm{tr}\,(f^*|_{\mathrm{NS}\,(X)})| \ge 23$. Assume that f is of positive entropy. Then, after renumbering, α_1 is a Salem number $a > 1$, α_2 is $1/a$ and all other α_k are of absolute value 1 by Remark 3.2. Then

$$\mathrm{tr}\,((f^N)^*|_{\mathrm{NS}\,(X)}) = a^N + \frac{1}{a^N} + \sum_{k=3}^{\rho} \alpha_k^N\ ,$$

which is an integer, in particular, real. Hence

$$\mathrm{tr}\,((f^N)^*|_{\mathrm{NS}\,(X)}) = a^N + \frac{1}{a^N} + \sum_{k=3}^{\rho} \mathrm{Re}\,(\alpha_k^N)\ .$$

Since $|\alpha_k^N| = |\alpha_k|^N = 1$, it follows that

$$-18 = \sum_{k=3}^{\rho} -1 \le \sum_{k=3}^{\rho} \mathrm{Re}\,(\alpha_k^N) \le \sum_{k=3}^{\rho} 1 = 18\ .$$

On the other hand, since $a > 1$, it follows that

$$\lim_{N \to \infty} a^N + \frac{1}{a^N} = +\infty \ .$$

Hence, there is N such that $\mathrm{tr}\,((f^n)^*|_{\mathrm{NS}\,(X)}) \geq 23$, hence $|\mathrm{tr}\,((f^n)^*|_{\mathrm{NS}\,(X)})| \geq 23$, for all integers n such that $n \geq N$. □

Lemma 7.7. *Let $X = X(p)$. Then there is $f \in \mathrm{Aut}\,(X)$ such that $|\mathrm{tr}\,(f^*|_{\mathrm{NS}\,(X)})| > 23$, and $f^*\omega = \omega$. In particular, f is of positive entropy.*

Proof. Recall that $X(p) = \mathrm{Km}(E \times_k E)$ where E is a supersingular elliptic curve. The projections $pr_i : E \times_k E \to E,\ i = 1, 2$ descend to elliptic fibrations $\varphi_i : X \to \mathbb{P}^1 \cong E/\langle \pm 1 \rangle$ (thus with section) and with exactly 4 singular fibers of type I_0^*. We choose a zero-section of φ and denote it by O. We identify it with its image $O \subset X$. Let $\mathrm{MW}\,(\varphi_i)$ be the Mordell-Weil group of φ_i. It acts on the generic fiber X_η of φ by translation, thus is an abelian subgroup of $\mathrm{Aut}(X)$.

One has ([Sh90])

$$\text{rank } \mathrm{MW}(\varphi_i) = \text{rank } \mathrm{NS}(X) - 2 - 4 \times 4 = 4.$$

For each $i = 1, 2$, choose $f_i \in \mathrm{MW}\,(\varphi_i)$ such that f_i is of infinite order. The Néron-Severi class e_i of a closed fiber of φ_i is stable under f_i, i.e., $f_i^*(e_i) = e_i$ for $i = 1, 2$, e_1, e_2 are of self-intersection 0, and they are linearly independent in the hyperbolic lattice $\mathrm{NS}\,(X)$. Then, by [Og09, Thm. 3.1], $f_3 := f_1^{n_1} \circ f_2^{n_2} \in \mathrm{Aut}\,(X)$ is of positive entropy for large positive integers n_1, n_2. Note that $f_i^*\omega = \omega$ as f_i is a translation automorphism of an elliptic fibration. Thus $f_3^*\omega = \omega$ as well. Hence $f = f_3^N$ for large N is a required solution. □

Proof of Theorem 7.5. If h is of positive entropy, we are done. So we assume h has null-entropy. This means, the characteristic polynomial of $h^*|_{\mathrm{NS}(X)}$ is the product of cyclotomic polynomials of degree ≤ 22. Set $g := h^\beta$. As rank $\mathrm{NS}(X) = 22$, $g^*|_{\mathrm{NS}(X)}$ is unipotent, so there is a basis of $\mathrm{NS}\,(X) \otimes \mathbb{Q}$ in which $g^*|_{\mathrm{NS}(X)\otimes\mathbb{Q}}$ is represented by the Jordan canonical form:

$$g^*|_{\mathrm{NS}(X)\otimes\mathbb{Q}} = J := J(r_1, 1) \oplus \cdots \oplus J(r_s, 1) \ .$$

Set $s = \max\,\{r_i\}_{i=1}^s - 1$ and choose f as in Lemma 7.7.

If $s = 0$, then $g^*|_{\mathrm{NS}(X)} = \mathrm{Id}$. Hence $(f \circ g)^*|_{\mathrm{NS}(X)} = f^*|_{\mathrm{NS}(X)}$ and $(f \circ g)^*\omega = \xi_m\omega$ with some $m \geq 67$. Thus $f \circ g \in \mathrm{Aut}(X)$ is of positive entropy and not geometrically liftable to characteristic 0.

Assume $s > 0$.

In the same basis, $f^*|_{\mathrm{NS}(X)\otimes\mathbb{Q}}$ is represented by a matrix $A = (A_{ij})$ where A_{ij} is $r_i \times r_j$ matrix located at the (i, j)-block. Then

$$\mathrm{tr}\,((f \circ g^N)^*|_{\mathrm{NS}\,(X)}) = \mathrm{tr}\,(J^N A) = \sum_{i=1}^{s} \mathrm{tr}\,(J(r_i, 1)^N A_{ii}).$$

We want to estimate each summand of the formula above. For instance, if $r_1 = 4$, then

$$J_4 := J(4,1) = I_4 + R_4 \ , \ R_4 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ ,$$

and $R^4 = 0$. Thus, by the binomial expansion

$$J_4^N = I_4 + N \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{N(N-1)}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{N(N-1)(N-2)}{6} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ .$$

Hence, for a 4×4-matrix $A_4 := (a_{ij})$ of rational entries, we have

$$\mathrm{tr}\,(J_4^N A_4) = \mathrm{tr}\,(A_4) + N(a_{21} + a_{32} + a_{43}) + \frac{N(N-1)}{2}(a_{31} + a_{42}) + \frac{N(N-1)(N-2)}{6} a_{41} \ ,$$

which is a polynomial of N of degree $\le 3 = 4 - 1$, with rational coefficients depending on A_4 (and independent of N). For exactly the same reason, from the expansion of $J(r,1)^N$ as above, we find that $\mathrm{tr}\,((f \circ g^N)^*|_{\mathrm{NS}\,(X)})$ is of the form

$$\mathrm{tr}\,((f \circ g^N)^*|_{\mathrm{NS}\,(X)}) = a_s N^s + a_{s-1} N^{s-1} + \cdots + a_1 N + \mathrm{tr}\,A \ .$$

Here a_k are rational numbers depending only on A (and independent of N). So, if it happened that $|\mathrm{tr}\,((f \circ g^{k_n})^*|_{\mathrm{NS}\,(X)})| \le 22$ for some sequence of positive integers

$$k_1 < k_2 < k_3 < \cdots < k_n < \cdots \ \to +\infty \ ,$$

one would have

$$a_s = a_{s-1} = \cdots = a_1 = 0 \ ,$$

and hence

$$|\mathrm{tr}\,((f \circ g^N)^*|_{\mathrm{NS}\,(X)})| = |\mathrm{tr}\,(A)|$$

for all positive integer N. But

$$|\mathrm{tr}\,(A)| = |\mathrm{tr}\,(f^*|_{\mathrm{NS}\,(X)}))| \ge 23 \ ,$$

by our choice of f, a contradiction to $|\mathrm{tr}\,((f \circ g^{k_n})^*|_{\mathrm{NS}\,(X)})| \le 22$. Hence, there are only finitely many positive integers ℓ such that $|\mathrm{tr}\,((f \circ g^{\ell})^*|_{\mathrm{NS}\,(X)}))| \le 22$. Thus, there is a positive integer M such that $|\mathrm{tr}\,((f \circ g^N)^*|_{\mathrm{NS}\,(X)}))| \ge 23$ for all integers N such that $N \ge M$.

Let us choose N such that $N \ge M$ and $(N, m) = 1$. Define $\tau := f \circ g^N$. Then $|\mathrm{tr}(\tau^*|_{\mathrm{NS}(X)})| \ge 23$, and $\tau^* \omega = \xi_m^N \omega$. Note ξ_m^N is a primitive m-th root of unity with $m \ge 67$. Thus $\tau \in \mathrm{Aut}(X)$ is of positive entropy and not geometrically liftable to characteristic 0. □

Remark 7.8. In [EOY14] we show that various elliptic structures on $X(p)$, for $p \ge 11$ and $p \ne 13$, with a rich Mordell-Weil group force the existence of autmorphisms with entropy the logarithm of a degree 22 Salem number.

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