DEFORMATION OF ALGEBRAIC CYCLE CLASSES IN CHARACTERISTIC ZERO

SPENCER BLOCH, HÉLÈNE ESNAULT, AND MORITZ KERZ

ABSTRACT. We study a formal deformation problem for rational algebraic cycle classes motivated by Grothendieck's variational Hodge conjecture. We argue that there is a close connection between the existence of a Chow-Künneth decomposition and the existence of expected deformations of cycles. This observation applies in particular to abelian schemes.

1. INTRODUCTION

Let K be a field of characteristic 0. Let Z/K be a smooth projective variety of dimension d and let i be an integer. The following Chow-Künneth property is expected to hold in general as part of conjectures of Grothendieck, Beilinson and Murre, see [Ja, Sec. 5].

$(\mathbf{CK})^i_Z$: There is an idempotent correspondence $\pi^i \in \mathrm{CH}^d(Z \times_K Z)_{\mathbb{Q}}$ such that on $H^*_{\mathrm{dR}}(Z/K)$ the correspondence π^i acts as the projection to $H^i_{\mathrm{dR}}(Z/K)$.

Remark 1.1. An important and well-understood example is the case of an abelian variety A/K. In this case we know $(\mathbf{CK})^i_A$ for all integers i, see for instance [DM].

Set $S = \mathrm{Spec}\, k[[t]]$, where k is a field of characteristic 0. Let X/S be a smooth projective scheme. Let η be the generic point of S, $K = k(\eta)$. Denote by X_n the scheme $X \times_S S_n$ with $S_n = \mathrm{Spec}\, k[t]/(t^n)$.

There is a Chern character ring homomorphism to de Rham cohomology

$$(1.1) \qquad \mathrm{ch} : K_0(X_1) \to H^*_{\mathrm{dR}}(X_1/k).$$

It is an important problem to determine the image of the Chern character. In this note we consider an 'infinitesimal' analogue of this problem along the thickening $X_1 \hookrightarrow X_n$.

By $\nabla : H^*_{\mathrm{dR}}(X/S) \to H^*_{\mathrm{dR}}(X/S)$ we denote derivation along the parameter t with respect to the Gauss-Manin connection and by $H^*_{\mathrm{dR}}(X/S)^{\nabla}$ the kernel of ∇. Solving a formal differential equation we see that the canonical map

$$(1.2) \qquad \Phi : H^*_{\mathrm{dR}}(X/S)^{\nabla} \xrightarrow{\sim} H^*_{\mathrm{dR}}(X_1/k)$$

is an isomorphism [Kt, Prop. 8.9].

We denote by $F^r H^i_{\mathrm{dR}} \subset H^i_{\mathrm{dR}}$ the Hodge filtration on de Rham cohomology (of smooth projective schemes). Our main theorem says:

Theorem 1.2. *Assume that for the scheme X/S as above the property $(\mathbf{CK})^i_{X_\eta}$ holds for all even $i \in \mathbb{Z}$. Then for $\xi_1 \in K_0(X_1)_{\mathbb{Q}}$ the following are equivalent:*

(i) $\Phi^{-1} \circ \mathrm{ch}(\xi_1) \in \bigoplus_i H^{2i}_{\mathrm{dR}}(X/S)^{\nabla} \cap F^i H^{2i}_{\mathrm{dR}}(X/S)$,

(ii) *there is an element $\hat{\xi} \in (\varprojlim_n K_0(X_n)) \otimes \mathbb{Q}$ such that*

$$\mathrm{ch}(\hat{\xi}|_{X_1}) = \mathrm{ch}(\xi_1) \in H^*_{\mathrm{dR}}(X_1/k).$$

A preliminary version of Theorem 1.2 for cohomological Chow groups is shown in Section 5, see Theorem 5.2. The central new ingredient of our proof is to study a ring of correspondences for the non-reduced scheme X_n. Property $(\mathbf{CK})^i_{X_\eta}$ will guarantee that there are enough such correspondences in order to kill the influence of absolute differential forms of k on the deformation behavior. In fact for $\Omega^1_{k/\mathbb{Q}} = 0$ the whole deformation problem is much easier, see Remark 1.3. The proof of Theorem 1.2 is completed via a Chern character isomorphism relying on Zariski descent for algebraic K-theory, see Section 6.

Remark 1.3. In case k is algebraic over $\mathbb{Q}$ we also deduce without assuming $(\mathbf{CK})^i_{X_\eta}$ that conditions (i) and (ii) are equivalent and they are also equivalent to:

(ii') *there is an element $\hat{\xi} \in (\varprojlim_n K_0(X_n)) \otimes \mathbb{Q}$ such that $\hat{\xi}|_{X_1} = \xi_1$.*

See related work [GG], [PaRa], [Mo1]. However, our methods do not show that for general fields k condition (ii') is equivalent to (ii) and we do not see a good reason to expect this.

Theorem 1.2 is motivated by a conjecture of Grothendieck [G, p. 103], which is today called the variational Hodge conjecture. See Appendix A for his original global formulation. In this appendix it is shown that the latter is equivalent to the following "infinitesimal" conjecture.

Conjecture 1.4 (Infinitesimal Hodge)**.** *Statement* (i) *of Theorem 1.2 is equivalent to*

(iii) *there is an element $\xi \in K_0(X)_{\mathbb{Q}}$ such that*

$$\mathrm{ch}(\xi|_{X_1}) = \mathrm{ch}(\xi_1) \in H^*_{\mathrm{dR}}(X_1/k).$$

One shows directly that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) without assuming $(\mathbf{CK})^i_{X_\eta}$ for i even. Conjecture 1.4 is particularly interesting for abelian schemes. Indeed it is known [Ab],[An, Sec. 6] that

Conj. 1.4 for abelian schemes $X/S \implies$ Hodge conj. for abelian varieties.

As $(\mathbf{CK})^i_{X_\eta}$ is known for abelian varieties, see Remark 1.1, one can speculate about what is needed to deduce Conjecture 1.4 for abelian schemes from Theorem 1.2. In order to accomplish this one would have to solve an algebraization problem, namely one has to consider the question how far the map

$$(1.3) \qquad\qquad K_0(X) \to \varprojlim_n K_0(X_n)$$

is from being surjective (after tensoring with $\mathbb{Q}$). Recall that for line bundles the corresponding map

$$\mathrm{Pic}(X) \to \varprojlim_n \mathrm{Pic}(X_n)$$

is an isomorphism by formal existence [EGA3, III.5].

By considering a trivial deformation of an abelian surface we show in Appendix B that the map (1.3) cannot be surjective in general. For abelian schemes the counterexample leaves the following algebraization question open.

Question 1.5. Let X/S be an abelian scheme and $\ell > 1$ an integer. Is the map

$$K_0(X)^{\Psi^{\ell^2} - [\ell]^*} \to \varprojlim_n K_0(X_n)^{\Psi^{\ell^2} - [\ell]^*}$$

surjective after tensoring with $\mathbb{Q}$?

Here ψ^ℓ is the ℓ-th Adams operation [FL] and $[\ell] : X \to X$ is multiplication by ℓ. The upper index notation means that we take the kernel of the corresponding endomorphism.

From Theorem 1.2 we deduce:

Corollary 1.6. *A positive answer to Question 1.5 would imply the Hodge conjecture for abelian varieties.*

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2. Milnor K-theory and differential forms

Let k be a field of characteristic 0 and write $S_n = \operatorname{Spec} k[t]/(t^n)$. Let $S = \operatorname{Spec} k[[t]]$, and let $X \to S$ be a smooth, separated scheme of finite type. Write $X_n = X \times_S S_n$. Write $\Omega^r_{X_n}$, $Z^r_{X_n}$ resp. $B^r_{X_n}$ for the Zariski sheaf of absolute n-forms, closed absolute n-forms resp. exact absolute n-forms on X_n. Let $\mathcal{K}^M_*$ be the Milnor K-sheaf with respect to Zariski topology as studied in [Ke].

Lemma 2.1. *There is an exact sequence of Zariski sheaves*

$$(2.1) \qquad\qquad 0 \to \Omega^{r-1}_{X_1} \xrightarrow{a} Z^r_{X_n} \xrightarrow{f} Z^r_{X_{n-1}}$$

where $a(\eta) = t^{n-1}d\eta + (n-1)t^{n-2}dt \wedge \eta$.

Proof. Note that a is well-defined because $t^n = t^{n-1}dt = 0$ on X_n. Since

$$(2.2) \qquad \ker(\Omega^r_{X_n} \to \Omega^r_{X_{n-1}}) = t^{n-1}\Omega^r_{X_n} + t^{n-2}dt \wedge \Omega^{r-1}_{X_n}$$

the assertion is clear. $\qquad\qquad\square$

Lemma 2.2. *There is an exact sequence of Zariski sheaves*

$$(2.3) \qquad\qquad \Omega^{r-1}_{X_1} \xrightarrow{b} \mathcal{K}^M_{r,X_n} \to \mathcal{K}^M_{r,X_{n-1}} \to 0.$$

Here $b(x\, d\log(y_1) \wedge \cdots \wedge d\log y_{r-1}) = \{1 + xt^{n-1}, y_1, \ldots, y_{r-1}\}$.

Proof. We can present $\Omega^1_{X_1}$ with an exact sequence of $\mathcal{O}_{X_1}$-modules

$$(2.4) \qquad 0 \to \mathcal{R} \to \mathcal{O}_{X_1} \otimes_{\mathbb{Z}} \mathcal{O}^\times_{X_1} \xrightarrow{u \otimes v \mapsto u\,dv/v} \Omega^1_{X_1} \to 0.$$

Here $\mathcal{R}$ is the sub-$\mathcal{O}_{X_1}$-module under left multiplication with generators of the form $a \otimes a + b \otimes b - (a+b) \otimes (a+b)$. It follows that $\Omega^{r-1}_{X_1} = \bigwedge^{r-1}_{\mathcal{O}_{X_1}} \Omega^1_{X_1}$ has a presentation of the form

$$(2.5) \qquad \mathcal{R} \otimes_{\mathbb{Z}} \bigwedge^{r-2}_{\mathbb{Z}} \mathcal{O}^\times_{X_1} \to \mathcal{O}_{X_1} \otimes_{\mathbb{Z}} \bigwedge^{r-1}_{\mathbb{Z}} \mathcal{O}^\times_{X_1} \to \Omega^{r-1}_{X_1} \to 0.$$

For $r = 2$ the map

$$(2.6) \qquad \Omega^1_{X_1} \cong t^{n-1}\Omega^1_{X_n} \to \mathcal{K}^{\mathrm{M}}_{2,X_n}; \quad cda/a \mapsto \{1 + ct^{n-1}, a\}.$$

is well-defined. It boils down to showing

$$\{1 + at^{n-1}, a\} + \{1 + bt^{n-1}, b\} - \{1 + (a+b)t^{n-1}, a+b\} = 0$$

in $\mathcal{K}_{2,X_n}$ for $a, b, a+b$ in $\mathcal{O}^{\times}_{X_1}$. See [B2, Sec. 2] for more details. Then from the presentation (2.5) we deduce that b is well-defined. The exactness of (2.3) is straightforward. □

Proposition 2.3. *The square*

$$(2.7) \qquad \begin{array}{ccc} \mathcal{K}^{\mathrm{M}}_{r,X_n} & \longrightarrow & \mathcal{K}^{\mathrm{M}}_{r,X_1} \\ \downarrow{\scriptstyle d\log} & & \downarrow{\scriptstyle d\log} \\ Z^r_{X_n} & \longrightarrow & Z^r_{X_1} \end{array}$$

is cartesian and there is a morphism between short exact sequences

$$(2.8) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^{r-1}_{X_1} & \overset{b}{\longrightarrow} & \mathcal{K}^{\mathrm{M}}_{r,X_n} & \longrightarrow & \mathcal{K}^{\mathrm{M}}_{r,X_{n-1}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle d\log} & & \downarrow{\scriptstyle d\log} & & \\ 0 & \longrightarrow & \Omega^{r-1}_{X_1} & \overset{a}{\longrightarrow} & Z^r_{X_n} & \longrightarrow & Z^r_{X_{n-1}} & \longrightarrow & 0 \end{array}$$

Proof. We have

$$(2.9) \qquad \mathcal{K}^{\mathrm{M}}_{r,X_1} \times_{Z^r_{X_1}} Z^r_{X_n} = \mathcal{K}^{\mathrm{M}}_{r,X_1} \times_{Z^r_{X_1}} Z^r_{X_{n-1}} \times_{Z^r_{X_{n-1}}} Z^r_{X_n}$$

In order to prove the first statement by induction we are thus reduced to proving that the diagram

$$(2.10) \qquad \begin{array}{ccc} \mathcal{K}^{\mathrm{M}}_{r,X_n} & \longrightarrow & \mathcal{K}^{\mathrm{M}}_{r,X_{n-1}} \\ \downarrow{\scriptstyle d\log} & & \downarrow{\scriptstyle d\log} \\ Z^r_{X_n} & \longrightarrow & Z^r_{X_{n-1}} \end{array}$$

is cartesian. Plugging in Lemmas 2.1 and 2.2 yields

$$(2.11) \qquad \begin{array}{ccccccc} & \Omega^{r-1}_{X_1} & \longrightarrow & \mathcal{K}^{\mathrm{M}}_{r,X_n} & \longrightarrow & \mathcal{K}^{\mathrm{M}}_{r,X_{n-1}} & \longrightarrow & 0 \\ & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle d\log} & & \downarrow{\scriptstyle d\log} & & \\ 0 \longrightarrow & \Omega^{r-1}_{X_1} & \longrightarrow & Z^r_{X_n} & \overset{f}{\longrightarrow} & Z^r_{X_{n-1}} & & \end{array}$$

It follows that (2.10) is cartesian and that the upper row in (2.8) is exact as claimed. The exactness of the lower row of (2.8), i.e. the surjectivity of f in (2.11), follows from the commutative diagram of short exact sequences

$$(2.12) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & B^r_{X_n} & \longrightarrow & Z^r_{X_n} & \longrightarrow & \mathcal{H}^r_{X_n} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & B^r_{X_{n-1}} & \longrightarrow & Z^r_{X_{n-1}} & \longrightarrow & \mathcal{H}^r_{X_{n-1}} & \longrightarrow & 0 \end{array}$$

were $\mathcal{H}^r$ is the de Rham cohomology sheaf. Indeed the right vertical map is an isomorphism by Lemma 2.4 and the left vertical map is obviously surjective. □

Lemma 2.4. *With notation as above, the map of complexes $\Omega_{X_n}^* \to \Omega_{X_1}^*$ is a quasi-isomorphism.*

Proof. The assertion is local so we may assume $X_n = \operatorname{Spec} A_n$ is affine. The algebra $A_1 = A_n/tA_n$ is therefore smooth over k, so by the infinitesimal criterion for smoothness [EGA4, (17.1.1)], there exists a splitting of the surjection $A_n \to A_1$. It follows that we may write $X_n \cong X_1 \times_k S_n$. This implies that there is an isomorphism between differential graded algebras $\Omega_{X_n}^* \cong \Omega_{X_1}^* \otimes_{\mathbb{Q}} \Omega_{S_n}^*$. Since there is a short exact sequence

$$0 \to tk[t]/t^n k[t] \xrightarrow{d} \Omega_{S_n}^1 \to \Omega_{k/\mathbb{Q}}^1 \to 0,$$

we deduce that $\Omega_{A_n}^* \to \Omega_{A_0}^*$ is a quasi-isomorphism as claimed. $\square$

3. Local cohomology

Let X, S, X_n, S_n and k be as in Section 2. One of the central techniques for proving our main Theorem 1.2 will be to study the coniveau complex for Milnor K-sheaves of X_n. A general reference for the coniveau complex is [Ha, Ch. IV].

Definition 3.1. For an arbitrary Zariski sheaf of abelian groups $\mathcal{F}$ on X_1 let us consider the coniveau complex of Zariski sheaves $\mathcal{C}(F)$ defined as
(3.1)
$$\bigoplus_{x \in X_1^{(0)}} i_{x,*} H_x^0(X_1, \mathcal{F}) \to \bigoplus_{x \in X_1^{(1)}} i_{x,*} H_x^1(X_1, \mathcal{F}) \to \bigoplus_{x \in X_1^{(2)}} i_{x,*} H_x^2(X_1, \mathcal{F}) \to \cdots .$$

where the left group is put into cohomological degree 0 and where $i_x : x \to X$ is the natural monomorphism. There is a canonical augmentation $\mathcal{F} \to \mathcal{C}^0(\mathcal{F})$.

By $\mathrm{C}(\mathcal{F})$ we denote the complex of global sections $\Gamma(X, \mathcal{C}(\mathcal{F}))$.

Definition 3.2. An abelian sheaf $\mathcal{F}$ on X_1 is Cohen-Macaulay (CM for short) if for every scheme point $x \in X_1$ we have $H_x^i(X_1, \mathcal{F}) = 0$ for $i \neq \operatorname{codim}(x)$.

A basic observation about CM-sheaves is that they give rise to exact coniveau complexes. This follows directly from the degeneration of the coniveau spectral sequence for these sheaves, see [Ha, Prop. IV.2.6].

Proposition 3.3. *Let $\mathcal{F}$ be a CM abelian Zariski sheaf. Then $\mathcal{C}(\mathcal{F})$ is an acyclic resolution of $\mathcal{F}$. In particular one has $H^*(X, \mathcal{F}) \cong H^*(\mathrm{C}(\mathcal{F}))$.*

The aim of this section is to show:

Theorem 3.4. *The sheaves $\mathcal{K}_{r,X_n}^{\mathrm{M}}, \Omega_{X_n}^i, \mathcal{H}_{X_1}^r, Z_{X_n}^r, B_{X_n}^r$ are CM for all $n \geq 1$.*

We prove the theorem in a series of propositions.

Proposition 3.5. *Let $r, n \geq 0$ be integers. The sheaves $\mathcal{F} = \mathcal{K}_{r,X_n}^{\mathrm{M}}$ and $\mathcal{F} = \Omega_{X_n}^r$ are CM.*

Proof. For the sheaf $\mathcal{F} = \Omega_{X_1}^r$ see [Ha, p. 239]. For the sheaf $\mathcal{F} = \mathcal{K}_{r,X_1}^{\mathrm{M}}$ see [Ke]. We prove that $\mathcal{F} = \mathcal{K}_{r,X_n}^{\mathrm{M}}$ is CM by induction on n. The case of the sheaf $\mathcal{F} = \Omega_{X_n}^r$ works similarly. For $n > 1$ we get from the exact sequence (2.8) an exact sequence

$$H_x^i(X_1, \Omega_{X_1}^r) \to H_x^i(X_1, \mathcal{K}_{r,X_n}^{\mathrm{M}}) \to H_x^i(X_1, \mathcal{K}_{r,X_{n-1}}^{\mathrm{M}}).$$

For $i \neq \operatorname{codim}(x)$ we already know that the groups on the left and the right side vanish, so does the group in the middle. $\square$

Next we study the sheaf $Z^r_{X_1}$.

Proposition 3.6. *For any $r \geq 0$ and $x \in X_1$ the map*

$$H^c_x(X_1, B^r_{X_1}) \to H^c_x(X_1, Z^r_{X_1})$$

induced by the inclusion $B^r_{X_1} \subset Z^r_{X_1}$ is surjective for $c \neq \mathrm{codim}(x)$ and injective for $c \neq \mathrm{codim}(x) + 1$.

Proof. The short exact sequence of sheaves

$$0 \to B^r_{X_1} \to Z^r_{X_1} \to, \mathcal{H}^r_{X_1} \to 0$$

induces a long exact cohomology sequence

$$\cdots \to H^{c-1}_x(X_1, \mathcal{H}^r_{X_1}) \to H^c_x(X_1, B^r_{X_1}) \to H^c_x(X_1, Z^r_{X_1}) \to H^c_x(X_1, \mathcal{H}^r_{X_1}) \to \cdots$$

On the other hand, we known from [BO] that the sheaves $\mathcal{H}^r_{X_1}$ are CM. This shows the proposition. $\square$

Proposition 3.7. *The sheaves $B^*_{X_1}, Z^*_{X_1}$ are CM.*

Proof. Using the exact sequence

$$0 \to Z^r_{X_1} \to \Omega^r_{X_1} \to B^{r+1}_{X_1} \to 0$$

and Proposition 3.5 we get

$$(3.2) \quad H^c_x(X_1, Z^r_{X_1}) = H^{c-1}_x(X_1, B^{r+1}_{X_1}) \quad \text{for } c \notin \{\mathrm{codim}(x), \mathrm{codim}(x) + 1\}.$$

Combining Proposition 3.6 and (3.2) we get for $c > \mathrm{codim}(x)$ the identifications

$$H^c_x(X_1, Z^r_{X_1}) \leftarrow H^c_x(X_1, B^r_{X_1}) = H^{c+1}_x(X_1, Z^{r-1}_{X_1})$$
$$= H^{c+1}_x(X_1, B^{r-1}_{X_1}) = H^{c+2}_x(X_1, Z^{r-2}_{X_1}) = \cdots = 0$$

where the first arrow surjective and for $c < \mathrm{codim}(x)$ the identifications

$$H^c_x(X_1, B^r_{X_1}) = H^c_x(X_1, Z^r_{X_1}) = H^{c-1}_x(X_1, B^{r+1}_{X_1}) = H^{c-1}_x(X_1, Z^{r+1}_{X_1})$$
$$= H^{c-2}_x(X_1, B^{r+2}_{X_1}) = H^{c-2}_x(X_1, Z^{r+2}_{X_1}) = \cdots = 0.$$

$\square$

Corollary 3.8. *For all $n \geq 1$ the sheaves $B^*_{X_n}, Z^*_{X_n}$ are CM.*

Proof. The sheaf $Z^*_{X_n}$ is CM by Propositions 2.3, 3.5 and 3.7. Using Lemma 2.4 and that $B^r_{X_n}$ trivially surjects onto $B^r_{X_{n-1}}$, we get a commutative diagram of short exact sequences

$$(3.3) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^{r-1}_{X_1} & \longrightarrow & B^r_{X_n} & \longrightarrow & B^r_{X_{n-1}} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^{r-1}_{X_1} & \longrightarrow & Z^r_{X_n} & \longrightarrow & Z^r_{X_{n-1}} & \longrightarrow & 0 \end{array}$$

which implies that $B^*_{X_n}$ is CM. $\square$

4. Transfer map

For any algebraic scheme V, it is natural to define the (cohomological) Chow groups

$$(4.1) \qquad \mathrm{CH}^p(V) := H^p(V, \mathcal{K}_p^{\mathrm{M}}).$$

In this way, the graded object $\mathrm{CH}^*(V)$ is automatically a ring, contravariant in V. For regular excellent V the cohomology group $\mathrm{CH}^i(V)$ coincides with the usual Chow group of codimension i cylces on V by [Ke]. We would hope there should exist covariant transfer maps $\mathrm{CH}^*(V) \xrightarrow{f_*} \mathrm{CH}^*(W)$ of graded degree $-d$ for $f : V \to W$ smooth and proper with fiber dimension d. One might further hope for f proper and $d = \dim V - \dim W$ that there should exist a functorial map of coniveau complexes $\mathrm{C}(\mathcal{K}_{V,r}^{\mathrm{M}}) \xrightarrow{f_*} \mathrm{C}(\mathcal{K}_{W,r-d}^{\mathrm{M}})[-d]$ such that when $\mathcal{K}_{W,r-d}^{\mathrm{M}}$ is CM one could define a covariant transfer via $H^p(V, \mathcal{K}_p^{\mathrm{M}}) \to H^p(\mathrm{C}(\mathcal{K}_{p,V}^{\mathrm{M}})) \xrightarrow{f_*} H^{p-d}(\mathrm{C}(\mathcal{K}_{W,p-d}^{\mathrm{M}})) \cong H^{p-d}(W, \mathcal{K}_{W,p-d}^{\mathrm{M}})$.

In what follows we use results from the previous section, together with work of Rost [R] and Grothendieck [Ha], to define a transfer

$$f_* : \mathrm{CH}^i(X_n) \to \mathrm{CH}^{i-d}(Y_n),$$

for $f : X_n \to Y_n$ a smooth proper morphism of relative dimension d between smooth schemes over S_n. This suffices to define a calculus of correspondences on $\mathrm{CH}^*(X_n)$, which is what we will need.

We use the fiber square in Proposition 2.7 to 'glue' the constructions of Rost and Grothendieck. From Propositions 2.7 and 3.5 and from Corollary 3.8 we obtain:

Lemma 4.1. *There is a Cartesian square of complexes*

$$
\begin{array}{ccc}
\mathcal{C}(\mathcal{K}_{r,X_n}^{\mathrm{M}}) & \longrightarrow & \mathcal{C}(\mathcal{K}_{r,X_1}^{\mathrm{M}}) \\
\downarrow & & \downarrow \\
\mathcal{C}(Z_{X_n}^r) & \longrightarrow & \mathcal{C}(Z_{X_1}^r)
\end{array}
$$

Furthermore we get:

Lemma 4.2. *We have a left-exact sequence*

$$0 \to \mathcal{C}(Z_{X_n}^r) \to \mathcal{C}(\Omega_{X_n}^r) \xrightarrow{d} \mathcal{C}(\Omega_{X_n}^{r+1}).$$

Proof. A short-exact sequence of CM sheaves yields a short-exact sequence of coniveau complexes. Applying this to the sequence

$$0 \to Z_{X_n}^r \to \Omega_{X_n}^r \to B_{X_n}^{r+1} \to 0,$$

we reduce to showing that the map $\mathcal{C}(B_{X_n}^{r+1}) \to \mathcal{C}(\Omega_{X_n}^{r+1})$ is injective. By the same logic, we know that $\mathcal{C}(Z_{X_n}^{r+1}) \hookrightarrow \mathcal{C}(\Omega_{X_n}^{r+1})$, so it suffices to show $\mathcal{C}(B_{X_n}^{r+1}) \hookrightarrow \mathcal{C}(Z_{X_n}^{r+1})$. By Lemma 2.4 we have an exact sequence

$$(4.2) \qquad 0 \to B_{X_n}^{r+1} \to Z_{X_n}^{r+1} \to \mathcal{H}_{X_1}^{r+1} \to 0,$$

where $\mathcal{H}_{X_1}^{r+1}$ is CM. We conclude that $\mathcal{C}(B_{X_n}^{r+1}) \hookrightarrow \mathcal{C}(Z_{X_n}^{r+1})$, proving the lemma. $\qquad \square$

Let now $f : X_n \to Y_n$ be as above a smooth proper map of relative dimension d between smooth schemes over S_n. Rost constructs a morphism between complexes [R] which, via the Gysin isomorphisms, is a transfer

$$(4.3) \qquad f_*\mathcal{C}(\mathcal{K}^{\mathrm{M}}_{r,X_1}) \xrightarrow{f_*} \mathcal{C}(\mathcal{K}^{\mathrm{M}}_{r-d,Y_1})[-d].$$

Grothendieck and Hartshorne construct in [Ha] a morphism between complexes of $\mathcal{O}_{Y_n}$-modules

$$(4.4) \qquad f_*\mathcal{C}(\Omega^d_{X_n/Y_n}) \xrightarrow{\mathrm{Tr}_f} \mathcal{C}(\mathcal{O}_{Y_n})[-d].$$

Indeed, using the notation of [Ha] the structure sheaf $\mathcal{O}_{Y_n}$ is pointwise dualizing, so its coniveau complex $\mathcal{C}(\mathcal{O}_{Y_n})$ is a residual complex and

$$f^!\mathcal{C}(\mathcal{O}_{Y_n}) = \mathcal{C}(\Omega^d_{X_n/Y_n})[d].$$

Now we consider the composite morphism

$$(4.5) \qquad f_* : f_*\mathcal{C}(\Omega^r_{X_n}) \to \Omega^{r-d}_{Y_n} \otimes_{\mathcal{O}_{Y_n}} f_*\mathcal{C}(\Omega^d_{X_n/Y_n}) \xrightarrow{\mathrm{Tr}_f}$$
$$\Omega^{r-d}_{Y_n} \otimes_{\mathcal{O}_{Y_n}} \mathcal{C}(\mathcal{O}_{Y_n})[-d] \xrightarrow{\sim} \mathcal{C}(\Omega^{r-d}_{Y_n})[-d]$$

where for the first arrow we use the projection $\Omega^r_{X_n} \to f^*\Omega^{r-d}_{Y_n} \otimes_{\mathcal{O}_{X_n}} \Omega^d_{X_n/Y_n}$ followed by the projection formula. Note that the composite map f_* in (4.5) is compatible with the differential.

One directly shows that the transfer map (4.5) for $n = 1$ is compatible with the transfer map (4.3) with respect to the $d\log$ map. The transfer (4.5) induces thanks to Lemma 4.2 a transfer

$$(4.6) \qquad \mathrm{C}(Z^r_{X_n}) \xrightarrow{f_*} \mathrm{C}(Z^{r-d}_{Y_n})[-d].$$

So we get a commutative diagram of exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{C}(\mathcal{K}^{\mathrm{M}}_{r,X_n}) & \longrightarrow & \mathrm{C}(\mathcal{K}^{\mathrm{M}}_{r,X_1}) \oplus \mathrm{C}(Z^r_{X_n}) & \longrightarrow & \mathrm{C}(Z^r_{X_1}) \\
& & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
0 & \longrightarrow & \mathrm{C}(\mathcal{K}^{\mathrm{M}}_{r-d,Y_n})[-d] & \longrightarrow & \mathrm{C}(\mathcal{K}^{\mathrm{M}}_{r-d,Y_1})[-d] \oplus \mathrm{C}(Z^{r-d}_{Y_n})[-d] & \longrightarrow & \mathrm{C}(Z^{r-d}_{Y_1})[-d]
\end{array}
$$

The left vertical transfer map is defined by this diagram. The transfer map

$$(4.7) \qquad f_* : \mathrm{CH}^i(X_n) \cong H^i(\mathrm{C}(\mathcal{K}^{\mathrm{M}}_{i,X_n})) \to H^{i-d}(\mathrm{C}(\mathcal{K}^{\mathrm{M}}_{i-d,Y_n})) \cong \mathrm{CH}^{i-d}(Y_n),$$

obtained using the above construction, Proposition 3.3 and Theorem 3.4, satisfies the usual properties, for example it is compatible with smooth base change, and we use such properties without further mentioning.

For the remaining part of this section let d be the dimension of the smooth, equidimensional scheme X_n/S_n. If one follows the above construction of the transfer carefully, one can deduce:

Proposition 4.3.

(i) *Composition of correspondences makes* $\mathrm{CH}^d(X_n \times_{S_n} X_n)$ *a ring with unity for any* $n \geq 1$.

(ii) *For* $n \geq 2$ *this ring acts canonically on the long exact cohomology sequence*

$$\cdots \to H^c(X_1, \Omega^{r-1}_{X_1}) \to H^c(X_n; \mathcal{K}^{\mathrm{M}}_r) \to H^c(X_{n-1}; \mathcal{K}^{\mathrm{M}}_r) \to H^{c+1}(X_1, \Omega^{r-1}_{X_1}) \to \cdots$$

associated to (2.8) *for any integer* $r \geq 0$.

(iii) $\ker[\mathrm{CH}^d(X_n \times_{S_n} X_n) \to \mathrm{CH}^d(X_1 \times_k X_1)]$ *is a nilpotent ideal.*

Proposition 4.4. *Fix a positive integer i and assume condition $(\mathbf{CK})^{2i}_{X_\eta}$ from the introduction. Then there exists an inverse system of correspondences $(\pi_n^{2i})_{n \geq 1}$ with $\pi_n^{2i} \in \mathrm{CH}^d(X_n \times_{S_n} X_n)_{\mathbb{Q}}$ such that the following properties hold:*

(i) *each π_n^{2i} is idempotent,*

(ii) *on $H^*_{\mathrm{dR}}(X_n/S_n)$ the correspondence π_n^{2i} acts as the projection to $H^{2i}(X_n/S_n)$.*

Proof. Consider $\pi_\eta^{2i} \in \mathrm{CH}^d(X_\eta \times_K X_\eta)_{\mathbb{Q}}$ as in property $(\mathbf{CK})^{2i}_{X_\eta}$. The element π_1^{2i} is defined as the specialization of π_η^{2i} to the reduced closed fiber X_1. Recall that the specialization map

$$\mathrm{CH}^d(X_\eta \times_K X_\eta) \to \mathrm{CH}^d(X_1 \times_k X_1)$$

is a ring homomorphism of correspondences [F, Sec. 20.3].

As a consequence, π_1^{2i} satisfies properties (i) and (ii) of Proposition 4.4 for $n = 1$. We claim that we can lift this element to an inverse system $(\pi_n^{2i})_{n \geq 1}$ with the requested properties.

We can extend π_η^{2i} to an element $\pi^{2i} \in \mathrm{CH}^d(X \times_S X)_{\mathbb{Q}}$. By the Gersten conjecture for the Milnor K-sheaf of the regular scheme $X \times_S X$ [Ke] we get the left isomorphism in the following diagram

$$\iota_n : \mathrm{CH}^d(X \times_S X) \cong H^d(X \times_S X, \mathcal{K}^{\mathrm{M}}_{d, X \times_S X}) \longrightarrow$$
$$H^d(X_n \times_{S_n} X_n, \mathcal{K}^{\mathrm{M}}_{d, X_n \times_{S_n} X_n}) = \mathrm{CH}^d(X_n \times_{S_n} X_n).$$

Now consider the inverse system of correspondences $\tilde{\pi}_n^{2i} = \iota_n(\pi^{2i})$. They satisfy property (ii) of the proposition and furthermore $\pi_1^{2i} = \tilde{\pi}_1^{2i}$. We will apply a transformation to these correspondences which additionally makes them idempotent.

For an element α of an arbitrary (not necessarily commutative) unital ring and an integer $s \geq 1$ set

$$f_s(\alpha) = \sum_{0 \leq j \leq s} \binom{2s}{j} \alpha^{2s-j}(1 - \alpha)^j.$$

From the argument in the proof of [B, Prop. III.2.10] and from Proposition 4.3(iii) it follows that for s large, depending on n, the element $\pi_n^{2i} = f_s(\tilde{\pi}_n^{2i})$ is idempotent and independent of $s \gg 0$. Observe that $f_s(\pi_1^{2i}) = \pi_1^{2i}$ for all $s \geq 1$, because π_1^{2i} is idempotent. The elements $(\pi_n^{2i})_{n \geq 1}$ form an inverse system of idempotent correspondences, finishing the proof of Proposition 4.4. $\square$

In the next section we use the action of a Künneth type correspondence on cohomology of absolute differential forms as described in the following lemma.

Proposition 4.5. *Consider for given $i \geq 0$ a correspondence $\pi^i \in \mathrm{CH}^d(X_1 \times_k X_1)_{\mathbb{Q}}$ which acts on $H^*_{\mathrm{dR}}(X_1/k)$ as the projection to $H^i_{\mathrm{dR}}(X_1/k)$. Then*

(i) *the action of the correspondence $(\pi^i)^{r+1}$ on $H^c(X_1, \Omega^r_{X_1})$ vanishes for all $c + r < i$.*

(ii) *the action of $(\pi^i)^r$ on*

$$\ker[H^c(X_1, \Omega^r_{X_1}) \to H^c(X_1, \Omega^r_{X_1/k})]$$

vanishes for $c + r = i$.

Proof. The correspondence π^i being algebraic respects the Hodge filtration on de Rham cohomology and therefore acts on its graded pieces, which are Hodge cohomology groups according to Hodge theory. Thus π^i acts trivially on $H^c(X_1, \Omega^r_{X_1/k})$ for $c + r < i$. To pass from relative differential forms to absolute differential forms we use the filtration

$$L^s \Omega^r_{X_1} = \text{im}[\Omega^s_k \otimes_k \Omega^{r-s}_{X_1} \to \Omega^r_{X_1}] \quad (s = 0, \ldots, r).$$

Recall that

$$L^s/L^{s+1}\Omega^r_{X_1} = \Omega^{r-s}_{X_1/k} \otimes_k \Omega^s_k$$

and that π^i acts on the system of morphisms
(4.8)
$$H^c(X_1, \Omega^r_k \otimes_k \mathcal{O}_{X_1}) \to H^c(X_1, L^{r-1}\Omega^r_{X_1}) \to H^c(X_1, L^{r-2}\Omega^r_{X_1}) \to \cdots \to H^c(X_1, \Omega^r_{X_1}).$$

The filtration on cohomology

$$L^s = \text{im}[H^c(X_1, L^s\Omega^r_{X_1}) \to H^c(X_1, \Omega^r_{X_1})]$$

has graded pieces L^s/L^{s+1} which are subquotients of

$$H^c(X_1, \Omega^{r-s}_{X_1/k}) \otimes_k \Omega^s_k.$$

On the latter groups the action of π^i vanishes for $c + r - s < i$ as we have seen above.

This shows (i), while (ii) follows from the additional observation that

$$L^1 = \ker[H^c(X_1, \Omega^r_{X_1}) \to H^c(X_1, \Omega^r_{X_1/k})].$$

$\square$

5. Deformation of Chow groups

Let the notation be as in Section 2. We start this section with a basic lemma from [B1] about the comparison of an obstruction map with a Kodaira-Spencer map.

Lemma 5.1. *The diagram*

$$
\begin{array}{ccccc}
H^i(X_{n+1}, Z^i_{X_{n+1}}) & \longrightarrow & H^i(X_n, Z^i_{X_n}) & \xrightarrow{\text{Ob}} & H^{i+1}(X_1, \Omega^{i-1}_{X_1}) \\
\downarrow & & \downarrow & & \downarrow \\
H^{2i}_{\text{dR}}(X_{n+1}/S_{n+1})^\nabla \cap F^i & \longrightarrow & H^{2i}_{\text{dR}}(X_n/S_n)^\nabla \cap F^i & \xrightarrow{\text{KS}} & H^{i+1}(X_1, \Omega^{i-1}_{X_1/k})
\end{array}
$$

is commutative with exact rows.

Here $\nabla \in \text{End}_k(H^*_{\text{dR}}(X_n/S_n))$ is the Gauss-Manin connection and F^i is the Hodge filtration. The upper row is part of the long exact cohomology sequence associated to (2.8) and KS is induced by the Kodaira-Spencer map [B1, (4.1)].

We can now state the version of our main theorem for (cohomological) Chow groups. Let

$$\text{cl} : \text{CH}^i(X_n) \to H^{2i}_{\text{dR}}(X_n/S_n)^\nabla \subset H^{2i}_{\text{dR}}(X_n/S_n)$$

be the de Rham cycle class map, which is induced by the morphism of complexes

$$d\log : \mathcal{K}^M_i[-i] \to \Omega^\bullet_{X_n/S_n}.$$

The restriction map

$$\Phi : H_{\mathrm{dR}}^{2i}(X/S)^{\nabla} \xrightarrow{\sim} H_{\mathrm{dR}}^{2i}(X_1/k)$$

is an isomorphism by [Kt, Prop. 8.9].

Theorem 5.2. *Assume that for X/S as above and for a fixed i the property* $(\mathbf{CK})_{X_\eta}^{2i}$ *explained in the introduction holds. Then for $\xi_1 \in \mathrm{CH}^i(X_1)_{\mathbb{Q}}$ the following are equivalent:*

(i) $\Phi^{-1} \circ \mathrm{cl}(\xi_1) \in H_{\mathrm{dR}}^{2i}(X/S)^{\nabla} \cap F^i H_{\mathrm{dR}}^{2i}(X/S)$,

(ii) *there is an element $\hat{\xi} \in (\varprojlim_n \mathrm{CH}^i(X_n))_{\mathbb{Q}}$ such that*

$$\mathrm{cl}(\hat{\xi}|_{X_1}) = \mathrm{cl}(\xi_1) \in H_{\mathrm{dR}}^{2i}(X_1/k). \tag{5.1}$$

Remark 5.3. For k algebraic over $\mathbb{Q}$ it has been known to the experts for a long time, (see [GG], and [PaRa] for more recent work in the case of hypersurface sections), that for an element $\xi_1 \in \mathrm{CH}^i(X_1)$ (note that we can use integral coefficients here) condition (i) of the theorem is equivalent to:

(ii') there is an element $\hat{\xi} \in \varprojlim_n \mathrm{CH}^i(X_n)$ such that $\xi|_{X_1} = \xi_1$.

Proof of Theorem 5.2. The implication (ii) $\Rightarrow$ (i) is clear. So consider (i) $\Rightarrow$ (ii).

Claim 5.4. The map

$$(\varprojlim_n \mathrm{CH}^i(X_n))_{\mathbb{Q}} \to \varprojlim_n \mathrm{CH}^i(X_n)_{\mathbb{Q}}$$

is surjective.

Proof. From the short exact sequence (2.8) we get for $n > 1$ a commutative diagram with exact sequences

$$
\begin{array}{ccccccc}
H^i(X_1, \Omega_{X_1}^{i-1}) & \longrightarrow & \mathrm{CH}^i(X_n) & \longrightarrow & \mathrm{CH}^i(X_{n-1}) & \xrightarrow{\mathrm{Ob}} & H^{i+1}(X_1, \Omega_{X_1}^{i-1}) \\
\| & & \downarrow & & \downarrow & & \| \\
H^i(X_1, \Omega_{X_1}^{i-1}) & \longrightarrow & \mathrm{CH}^i(X_n)_{\mathbb{Q}} & \longrightarrow & \mathrm{CH}^i(X_{n-1})_{\mathbb{Q}} & \xrightarrow{\mathrm{Ob}} & H^{i+1}(X_1, \Omega_{X_1}^{i-1})
\end{array}
$$

A diagram chase implies that the map

$$\mathrm{CH}^i(X_n) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{CH}^i(X_{n-1}) \times_{\mathrm{CH}^i(X_{n-1})_{\mathbb{Q}}} \mathrm{CH}^i(X_n)_{\mathbb{Q}} \tag{5.2}$$

is surjective. From this Claim 5.4 follows easily. $\qquad\square$

By Claim 5.4 it is enough to construct a pro-system $\hat{\xi} \in \varprojlim_n \mathrm{CH}^i(X_n)_{\mathbb{Q}}$ satisfying (5.1). We will do this successively.

Choose correspondences π_n^{2i} as in Proposition 4.4. We claim that there exists an element

$$\hat{\xi} = (\hat{\xi}_n)_{n \geq 1} \in \varprojlim_n \mathrm{CH}^i(X_n)_{\mathbb{Q}}$$

such that

$$\hat{\xi}_1 = \pi_1^{2i} \cdot \xi_1 \quad \text{and} \quad \pi_n^{2i} \cdot \hat{\xi}_n = \hat{\xi}_n \quad \text{for all } n \geq 1. \tag{5.3}$$

Indeed, assume we have already constructed $(\hat{\xi}_m)_{1 \leq m \leq n-1}$ with property (5.3). From Proposition 4.3 we know that π_n^{2i} acts on the following diagram with exact row and column

$$
\begin{array}{ccc}
& & K \\
& & \downarrow \\
\mathrm{CH}^i(X_n)_{\mathbb{Q}} \longrightarrow \mathrm{CH}^i(X_{n-1})_{\mathbb{Q}} \overset{\mathrm{Ob}}{\longrightarrow} & H^{i+1}(X_1, \Omega_{X_1}^{i-1}) \\
& & \downarrow \sigma \\
& & H^{i+1}(X_1, \Omega_{X_1/k}^{i-1})
\end{array}
$$

where K is defined as the kernel of σ. By Lemma 5.1 we have $\mathrm{Ob}(\hat{\xi}_{n-1}) \in K$. From the latter and Proposition 4.5 we deduce the third equality in

$$(5.4) \qquad \mathrm{Ob}(\hat{\xi}_{n-1}) = \mathrm{Ob}((\pi_{n-1}^{2i})^{i-1} \cdot \hat{\xi}_{n-1}) = (\pi_1^{2i})^{i-1} \cdot \mathrm{Ob}(\hat{\xi}_{n-1}) = 0.$$

Because the obstruction vanishes we can find $\hat{\xi}'_n \in \mathrm{CH}^i(X_n)_{\mathbb{Q}}$ with $\hat{\xi}'_n|_{X_{n-1}} = \hat{\xi}_{n-1}$. To finish the construction we set $\hat{\xi}_n = \pi_n^{2i} \cdot \hat{\xi}'_n$. $\qquad \square$

6. Motivic complex and Chern character

We begin this section by proving that the canonical map from Milnor K-theory to Quillen K-theory induces an isomorphism on certain relative K-groups. Then we study a Chern character isomorphism using higher algebraic K-theory and motivic cohomology. For both results the techniques are standard, so we only sketch the proofs.

We consider a pro-system of pairs of rings of the form $(A_\bullet, A)$ with $A_\bullet = A[t]/(t^n)$ and we assume that A is essentially smooth over k with $\mathrm{char}(k) = 0$. By $K_*(R', R)$ we denote the relative Quillen K-groups of a homomorphism between rings $R' \to R$.

Proposition 6.1. *For A as above the canonical homomorphism*

$$(6.1) \qquad \ker[K_*^{\mathrm{M}}(A_\bullet) \to K_*^{\mathrm{M}}(A)] \overset{\sim}{\to} K_*(A_\bullet, A)$$

is a pro-isomorphism.

Proof. By Goodwillie's theorem [Go] there is an isomorphism

$$(6.2) \qquad K_{i+1}(A_n, A) \overset{\sim}{\to} \mathrm{HC}_i(A_n, A)$$

for any $n \geq 1$. There is a canonical homomorphism

$$(6.3) \qquad e_n : \mathrm{HC}_i(A_n) \to \Omega_{A_n}^i / B_{A_n}^i \oplus Z_{A_n}^{i-2}/B_{A_n}^{i-2} \oplus Z_{A_n}^{i-4}/B_{A_n}^{i-4} \oplus \cdots,$$

see [W, 9.8]. By the Hochschild-Kostant-Rosenberg theorem [W, Thm. 9.4.7] resp. a pro-version of it (see [Mo2] for a general discussion), one sees that the corresponding maps on Hochschild homology

$$e : \mathrm{HH}_*(A) \to \Omega_A^* \quad \text{resp.} \quad e : \mathrm{HH}_*(A_\bullet) \to \Omega_{A_\bullet}^*$$

induce an isomorphism resp. a pro-isomorphism. Then by a short argument with mixed complexes [W, 9.8.13] one deduces that the map e_1 and the pro-system of maps $e_\bullet$ from (6.3) induce an isomorphism resp. a pro-isomorphism.

Finally, using Lemma 2.4 we see that we get pro-isomorphisms

$$(6.4) \qquad \mathrm{HC}_i(A_\bullet, A) \xrightarrow{e_\bullet} \ker[\Omega^i_{A_\bullet}/B^i_{A_\bullet} \to \Omega^i_A/B^i_A] \xrightarrow{d} \ker[Z^{i+1}_{A_\bullet} \to Z^{i+1}_A].$$

Following the steps of this construction carefully shows that the composition of (6.1), (6.2) and (6.4) is equal to the $d\log$ map, which is an isomorphism by the cartesian square (2.7). $\qquad \square$

More general results in the direction of Proposition 6.1 can be found in [Mo1] and [Mo2].

By classical techniques one constructs a Chern character ring homomorphism

$$(6.5) \qquad \mathrm{ch} : K_0(X_n)_\mathbb{Q} \to \bigoplus_{i \geq 0} \mathrm{CH}^i(X_n)_\mathbb{Q},$$

where we use the notation of Section 2. The Chern character is characterized by the property that the composite morphism

$$H^1(X_1, \mathcal{O}^\times_{X_n}) \cong \mathrm{Pic}(X_n) \to K_0(X_n)_\mathbb{Q} \xrightarrow{\mathrm{ch}} \bigoplus_{i \geq 0} \mathrm{CH}^i(X_n)_\mathbb{Q} \to \mathrm{CH}^1(X_n)_\mathbb{Q}$$

is induced by the canonical isomorphisms $H^1(X_1, \mathcal{O}^\times_{X_n}) \cong \mathrm{CH}^1(X_n)$.

Using Proposition 6.1 and Zariski descent for algebraic K-theory [TT, Sec. 10] we will show in this section that (6.5) induces a pro-isomorphism with respect to n, see Theorem 6.2. The pro-isomorphism (6.5) together with Theorem 5.2 immediately imply Theorem 1.2.

Let $\mathbb{Z}_{X_1}(r)$ be the weight r motivic complex of Zariski sheaves on the smooth variety X_1/k constructed by Suslin-Voevodsky, see [MVW]. We define a motivic complex $\mathbb{Z}_{X_n}(r)$ of the scheme X_n by the homotopy Cartesian square

$$(6.6) \qquad \begin{array}{ccc} \mathbb{Z}_{X_n}(r) & \longrightarrow & \mathbb{Z}_{X_1}(r) \\ \downarrow & & \downarrow \\ \mathcal{K}^{\mathrm{M}}_{r,X_n}[-r] & \longrightarrow & \mathcal{K}^{\mathrm{M}}_{r,X_1}[-r] \end{array}$$

The reader should be warned that the complex $\mathbb{Z}_{X_n}(r)$ for fixed n cannot be 'the correct' motivic complex of X_n, but as a pro-system in n we get the 'right' motivic theory. In fact from comparison with algebraic K-theory we expect the 'proper' homotopy fiber of the upper row of (6.6) to have non-trivial cohomology sheaves in each degree in the interval $[1, r]$ and not only in degree r.

There is a Chern character homomorphism from (higher) algebraic K-theory to the cohomology of the motivic complex $\mathbb{Z}_{X_\bullet}(r)$. The technique of the construction of the higher Chern character is explained in [Gil]. We recall the construction.

The universal Chern character

$$\mathrm{ch} \in \bigoplus_{r \geq 0} H^{2r}(\mathrm{BGL}_k, \mathbb{Q}_{X_1}(r))$$

induces morphisms

$$(6.7) \qquad \mathrm{ch}_r : K_{X_\bullet} \to \mathfrak{K}\mathbb{Q}_{X_\bullet}(r)[2r]$$

in the homotopy category of pro-spectra in the sense of [Isa]. Here $K(X_n)$ is the non-connective K-theory spectrum of X_n [TT, Sec. 6] and $\mathfrak{K}$ is the Eilenberg–MacLane functor.

Moreover, ch induces morphisms of Zariski descent spectral sequences [TT, Sec. 10]

$$^{K}E_2^{i,j}(X_\bullet) = H^i(X_1, \mathcal{K}_{-j,X_\bullet}) \Rightarrow K_{-i-j,X_\bullet}$$
$$^{\mathrm{mot}}E_2^{i,j}(X_\bullet) = H^i(X_1, \mathcal{H}^j(\mathbb{Z}_{X_\bullet}(r))) \Rightarrow H^{i+j}(X_1, \mathbb{Z}_{X_\bullet}(r))$$

of the form

(6.8) $$\mathrm{ch}_r : {^{K}E_2^{i,j}}(X_\bullet)_\mathbb{Q} \to {^{\mathrm{mot}}E_2^{i,j+2r}}(X_\bullet)_\mathbb{Q}.$$

For any $r \geq 0$ there is a similar Chern character of relative theories

(6.9) $$\mathrm{ch}_r : {^{K}E_2^{i,j}}(X_\bullet, X_1) \to {^{\mathrm{mot}}E_2^{i,j+2r}}(X_\bullet, X_1).$$

which is a pro-isomorphism for $r = -j$ by Proposition 6.1 and vanishes otherwise. It is well-known [B3] that the Chern character induces an isomorphism

(6.10) $$\mathrm{ch} : K_i(X_1)_\mathbb{Q} \xrightarrow{\sim} \bigoplus_{r \geq 0} H^{2r-i}(X_1, \mathbb{Q}_{X_1}(r))$$

for any $i \in \mathbb{Z}$. Combining isomorphisms (6.9) and (6.10) we get the required isomorphism between pro-groups:

Theorem 6.2. *For any smooth scheme X/S, which is separated and of finite type, there is a pro-isomorphism*

(6.11) $$\mathrm{ch} : K_i(X_\bullet)_\mathbb{Q} \xrightarrow{\sim} \bigoplus_{r \geq 0} H^{2r-i}(X_1, \mathbb{Q}_{X_\bullet}(r)).$$

Observing that for any $n \geq 1$ there is a canonical isomorphism

(6.12) $$H^{2r}(X_1, \mathbb{Z}_{X_n}(r)) \xrightarrow{\sim} \mathrm{CH}^r(X_n)$$

we see that the pro-isomorphism (6.11) comprises the pro-isomorphism (6.5).

Appendix A. Two versions of Grothendieck's conjecture

In the introduction we stated as Conjecture 1.4 a local version of Grothendieck's principle of the parallel transport of cycles, which we will refer to as the *infinitesimal Hodge conjecture* in the following. His original formulation, today called *variational Hodge conjecture*, is more global and we show in this appendix that the two formulations are equivalent.

Let k be a field of characteristic 0. Let $f : \mathcal{X} \to \mathcal{S}$ be a smooth projective morphism, where $\mathcal{S}/k$ is a smooth variety. Fix a point $s \in \mathcal{S}$ and let $\mathcal{X}_s$ be the fiber over s.

Grothendieck's original conjecture [G, p. 103] can now be stated as follows.

Conjecture A.1 (Variational Hodge). *For $\xi_s \in K_0(\mathcal{X}_s)_\mathbb{Q}$ the following are equivalent:*

(i) $\mathrm{ch}(\xi_s) \in H^*_{\mathrm{dR}}(\mathcal{X}_s/s)$ *lifts to an element in* $H^*_{\mathrm{dR}}(\mathcal{X}/k)$,
(ii) *there is an element $\xi \in K_0(\mathcal{X})_\mathbb{Q}$ with $\mathrm{ch}(\xi|_{\mathcal{X}_s}) = \mathrm{ch}(\xi_s)$.*

Proposition A.2. *The variational Hodge conjecture (Conjecture A.1) for all $k, \mathcal{X}, \mathcal{S}$ as above is equivalent to the infinitesimal Hodge conjecture (Conjecture 1.4) for all k, X as in the introduction.*

Remark A.3. The same proof shows that the variational Hodge conjecture for abelian schemes $\mathcal{X}/\mathcal{S}$ is equivalent to the infinitesimal Hodge conjecture for abelian schemes X/S.

Proof. Infinitesimal Hodge $\Longrightarrow$ Variational Hodge:

By induction on $\dim(\mathcal{S})$ we will reduce to $\dim(\mathcal{S}) = 1$. In order to do this observe first that we can replace without loss of generality $\mathcal{S}$ by a dense open subscheme containing s and $\mathcal{X}$ by the corresponding pullback.

Now let ξ_s satisfy Conjecture A.1(i) and assume without loss of generality that $\mathrm{codim}(s) > 0$. Choose a smooth hypersurface $\mathcal{S}' \subset \mathcal{S}$ containing s, which exists after possibly replacing $\mathcal{S}$ by a dense open subscheme, and set $\mathcal{X}' = \mathcal{X} \times_{\mathcal{S}} \mathcal{S}'$. By the induction assumption there is $\xi' \in K_0(\mathcal{X}')_{\mathbb{Q}}$ with $\mathrm{ch}(\xi'|_{\mathcal{X}_s}) = \mathrm{ch}(\xi_s)$.

Let s' be the generic point of $\mathcal{S}'$ and choose an extension of fields $k \subset k' \subset k(s')$ such that the second inclusion is finite and such that there exists a lift $k' \to \mathcal{O}_{\mathcal{S},s'}$. This lift gives rise to a curve $\mathcal{S}''/k'$ mapping to $\mathcal{S}$ as schemes over k such that s' is contained in the image. Now one applies the one-dimensional case of Conjecture A.1 to the family

$$\mathcal{X}'' = \mathcal{X} \times_{\mathcal{S}} \mathcal{S}'' \to \mathcal{S}'' \text{ with class } \xi'|_{s'} \in K_0(\mathcal{X}_{s'})_{\mathbb{Q}}$$

to get a lifted class $\xi'' \in K_0(\mathcal{X}'')_{\mathbb{Q}}$. Finally, $\mathcal{X}'' \subset \mathcal{X}$ is an inverse limit of open immersions of regular schemes, so we can extend ξ'' to a class $\xi \in K_0(\mathcal{X})_{\mathbb{Q}}$, which will then satisfy the requested Conjecture A.1(ii).

Now we assume $\dim(\mathcal{S}) = 1$. Without loss of generality $k = k(s)$. Using Deligne's partie fixe [D, 4.1] we can assume without loss of generality that the lift α of $\mathrm{ch}(\xi_s)$ in Conjecture A.1(i) lies in the image of

$$\bigoplus_i H^{2i}(\mathcal{X}, \Omega_{\mathcal{X}/k}^{\geq i}) \to H_{\mathrm{dR}}^*(\mathcal{X}/k).$$

The completion of $\mathcal{O}_{\mathcal{S},s}$ along the maximal ideal is isomorphic to $k[[t]]$. So define $X = \mathcal{X} \times_{\mathcal{S}} S$, where $S = \mathrm{Spec}\, k[[t]]$ and apply Conjecture 1.4 to the class $\xi_1 = \xi_s$ and the flat lift $\alpha|_X$ of $\mathrm{ch}(\xi_1)$ to get a lifted class $\dot{\xi} \in K_0(X)_{\mathbb{Q}}$.

Denote by S^{h} the spectrum of the henselization of $\mathcal{O}_{\mathcal{S},s}$ and by X^{h} the pullback $\mathcal{X} \times_{\mathcal{S}} S^{\mathrm{h}}$. By Artin approximation [Ar] there is a class $\ddot{\xi} \in K_0(X^{\mathrm{h}})_{\mathbb{Q}}$ with $\dot{\xi}|_{\mathcal{X}_s} = \ddot{\xi}|_{\mathcal{X}_s}$. By a standard transfer argument we get from $\ddot{\xi}$ a class $\xi \in K_0(\mathcal{X})_{\mathbb{Q}}$ with the requested property of Conjecture A.1(ii).

Variational Hodge $\Longrightarrow$ Infinitesimal Hodge:

Let ξ_1 satisfy property (i) of Theorem 1.2. The idea is roughly the following:

(1) reduce to a situation where $X \to S$ 'extends' to a morphism between complex varieties $\mathcal{X} \to \mathcal{S}$,

(2) use complex Hodge theory in order to show that $\mathrm{ch}(\xi_1)$ extends as a de Rham cohomology class to $H_{\mathrm{dR}}^*(\mathcal{X}/\mathbb{C})$ so that we can apply Conjecture A.1.

By a simple reduction we can assume without loss of generality that k contains $\mathbb{C}$. Then as step (1) we find a local subring $R \subset k[[t]]$ with maximal ideal m which is essentially of finite type over $\mathbb{Q}$ and such that X descends to a projective smooth scheme X_R over R and such that ξ_1 descends to a class in $K_0(X_R \otimes R/m)_{\mathbb{Q}}$. Such an R exists by the techniques of [EGA4, Sec. IV.8].

By resolution of singularities we can assume without loss of generalities that R is regular. Choose a subfield $k' \subset R$ such that the field extension $k' \subset R/m$ is finite and such that k' is algebraically closed in R. Now we can extend

$$\mathrm{Spec}\,(X_R \otimes_{k'} \mathbb{C}) \to \mathrm{Spec}\,(R \otimes_{k'} \mathbb{C})$$

to a smooth projective morphism $f : \mathcal{X} \to \mathcal{S}$ of smooth varieties over $\mathbb{C}$. There is a canonical morphism $\gamma : S \to \mathcal{S}$ over $\mathbb{C}$. In other words we get a cartesian square

$$
\begin{array}{ccc}
X & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow f \\
S & \xrightarrow{\gamma} & \mathcal{S}
\end{array}
$$

The map γ maps the closed point of S to a closed point $s \in \mathcal{S}$ and the generic point of S to the generic point of $\mathcal{S}$. There is an induced class $\xi_s \in K_0(\mathcal{X}_s)_\mathbb{Q}$, which γ pulls back to our originally given class $\xi_1 \in K_0(X_1)_\mathbb{Q}$.

We claim (step (2)) that the de Rham class $\mathrm{ch}(\xi_s) \in H^*_{\mathrm{dR}}(\mathcal{X}_s/\mathbb{C})$ extends to $H^*_{\mathrm{dR}}(\mathcal{X}/\mathbb{C})$ after possibly replacing $\mathcal{S}$ by an étale neighborhood of s. This will allow us to apply Conjecture A.1 to obtain a class $\dot{\xi} \in K_0(\mathcal{X})_\mathbb{Q}$, so that the requested class in $K_0(X)_\mathbb{Q}$ from Conjecture 1.4(iii) is given by $\xi = \gamma^*(\dot{\xi})$. This will finish the proof of Proposition A.2.

To show the claim let $f^\mathbb{C} : \mathcal{X}(\mathbb{C}) \to \mathcal{S}(\mathbb{C})$ be the induced map of complex manifolds and consider the local system $L = R^* f^\mathbb{C}_* \mathbb{Q}$ on $\mathcal{S}(\mathbb{C})$ (we omit any Tate twists). We think of L as an étale manifold over $\mathcal{S}(\mathbb{C})$. In this sense let L_0 be the connected component of the Betti Chern character class $\mathrm{ch}(\xi_s) \in H^*_{\mathrm{B}}(\mathcal{X}_s(\mathbb{C}))$ in the unramified complex space

$$L \cap \bigoplus_{i \geq 0} R^{2i} f^\mathbb{C}_* (\Omega^{\geq i}_{\mathcal{X}/\mathcal{S}})$$

over $\mathcal{S}(\mathbb{C})$.

By [CDK] we know that L_0 is finite over $\mathcal{S}(\mathbb{C})$ and therefore given by a finite unramified scheme over $\mathcal{S}$, which we denote by the same letter. After replacing $\mathcal{S}$ by an étale neighborhood of s we can therefore assume that $L_0 \to \mathcal{S}$ is a closed immersion.

Clearly, $\gamma^*(L_0)$ contains the locus where $\Phi^{-1} \circ \mathrm{ch}(\xi_1)$ lies in the Hodge filtration

$$\bigoplus_{i \geq 0} F^i H^{2i}_{\mathrm{dR}}(X/S)$$

(the map Φ is defined in (1.2)). By our assumption on ξ_1 this locus is all of S. So we get that $L_0 \to \mathcal{S}$ is an isomorphism, since γ has dense image. This means that the monodromy action of $\pi_1(\mathcal{S}(\mathbb{C}), s)$ on $H^*_{\mathrm{B}}(\mathcal{X}_s(\mathbb{C}))$ fixes $\mathrm{ch}(\xi_s)$. By the degeneration of the Leray spectral sequence [D, 4.1] the cohomology class $\mathrm{ch}(\xi_s) \in H^*_{\mathrm{dR}}(\mathcal{X}_s/\mathbb{C})$ extends to $H^*_{\mathrm{dR}}(\mathcal{X}/\mathbb{C})$, proving the claim.

$\square$

APPENDIX B. A COUNTEREXAMPLE TO ALGEBRAIZATION

In this section, we show that algebraization of K_0-classes of vector bundles does not hold in general, i.e. the map (1.3) is usually 'far' from beeing an isomorphism. For a precise statement see Proposition B.5. The idea is to consider a

'pro-0-cycle' on the trivial deformation over $\mathbb{C}[[t]]$ of a smooth projective variety $Y/\mathbb{C}$ whith $p_g > 0$. Roughly speaking we construct such a pro-0-cycle whose top Chern class in absolute Hodge cohomology 'jumps' around so much in the pro-system that it cannot come from absolute differential forms on $Y \otimes_{\mathbb{C}} \mathbb{C}[[t]]$.

We start the discussion by certain elementary observations about absolute differential forms. One defines a weight function on differential forms $\tau \in \Omega^2_{\mathbb{C}/\mathbb{Q}}$ by

$$w(\tau) := \min\{n \mid \tau = \sum_{i=1}^{n} a_i db_i \wedge dc_i, \ a_i, b_i, c_i \in \mathbb{C}.\}$$

The function w is subadditive in the sense that

$$w(\tau_1 + \ldots + \tau_p) \leq \sum w(\tau_i).$$

Lemma B.1. *Let* $\tau = \sum_{i=1}^{n} db_i \wedge dc_i$ *and assume all the* b_i, c_i *are algebraically independent elements in* $\mathbb{C}$. *Then* $w(\tau) = n$.

Proof. Clearly $w(\tau) \leq n$. If $w(\tau) < n$ we can write

$$\text{(B.1)} \qquad \tau = \sum_{j=1}^{n-1} \alpha_j d\beta_j \wedge d\gamma_j.$$

The n-fold wedge $\wedge^n \tau = \tau \wedge \cdots \wedge \tau \in \Omega^n_{\mathbb{C}}$ is equal to $n! db_1 \wedge dc_1 \wedge db_2 \wedge \cdots \wedge dc_n$, which is non-zero in $\Omega^{2n}_{\mathbb{C}}$ as the a_i, b_j are algebraically independent. On the other hand, if (B.1) holds, then $\wedge^n \tau = 0$ in $\Omega^{2n}_{\mathbb{C}}$, a contradiction. $\qquad \square$

Let $R = \mathbb{C}[[t]]$ and write $R_n = R/t^n R$.

The choice of a parameter t yields a natural splitting

$$\text{(B.2)} \qquad \Omega^i_{\mathbb{C}} \otimes_{\mathbb{C}} R_n \to \Omega^i_{R_n} \to \Omega^i_{\mathbb{C}} \otimes_{\mathbb{C}} R_n$$

which is compatible in the pro-system in n. Thus it defines a homomorphism

$$\text{(B.3)} \qquad \Omega^2_R \to \varprojlim_n \Omega^2_{\mathbb{C}} \otimes_{\mathbb{C}} R_n =: \Omega^2_{\mathbb{C}} \widehat{\otimes} R$$

$$\gamma = \sum_{i=1}^{N} f_i(t) dg_i(t) \wedge dh_i(t) \mapsto \tilde{\gamma} = \sum_{p=1}^{\infty} t^p \tau_p,$$

$$f_i, g_i, h_i \in R, \ \tau_p \in \Omega^2_{\mathbb{C}}.$$

Lemma B.2. *One has* $w(\tau_p) \leq N\binom{p+2}{p}$.

Proof. Suppose first $N = 1$ and $\omega = f dg \wedge dh$. Write $f = \sum_{j=0}^{\infty} f^{(j)} t^j$ and similarly for g, h. Then

$$\text{(B.4)} \qquad f dg \wedge dh = \sum_p t^p \sum_{i+j+k=p} f^{(i)} dg^{(j)} \wedge dh^{(k)}.$$

the inner sum has $\binom{p+2}{p}$ terms, and the result follows in this case simply by the definition of w. For N general, we conclude by subadditivity of w. $\qquad \square$

Lemma B.3. *Let*

$$\tilde{\eta} = \sum_{p=0}^{\infty} t^p \eta_p \in \Omega^2_{\mathbb{C}} \widehat{\otimes} \mathbb{C}[[t]].$$

Assume

$$\limsup_{p \to \infty} \frac{w(\eta_p)}{p^2} = \infty.$$

Then $\tilde{\eta}$ does not lift via (B.3) *to an element in* $\Omega^2_{\mathbb{C}[[t]]}$.

Proof. Immediate from Lemma B.2.   □

Remark B.4. All the above lemmas of this section immediately generalize to differential forms of any even degree.

Let $Y/\mathbb{Q}$ be a smooth projective variety. We write Y_A for the base change by a ring $A/\mathbb{Q}$.

Proposition B.5. *Assume that $p_g = \dim_{\mathbb{Q}} H^0(Y, \Omega^{\dim Y}_Y) > 0$ and that the dimension of Y is even. Then the map*

$$K_0(Y_R)_{\mathbb{Q}} \to (\varprojlim_n K_0(Y \times_{\mathbb{C}} R_n)) \otimes \mathbb{Q}$$

is not surjective.

For simplicity of notation we restrict to $\dim Y = 2$ for the rest of this section. The proof of the general case of Proposition B.5 works exactly the same way. For A a ring over $\overline{\mathbb{Q}}$, we have a second Chern character in abolute Hodge cohomology

$$\mathrm{ch}_2 : K_0(Y_A) \to H^2(Y_A, \Omega^2_{Y_A}).$$

Using the Künneth decomposition for differential forms and the resulting projection

$$H^2(Y, \Omega^2_{Y_A}) \to H^2(Y, \mathcal{O}_Y) \otimes \Omega^2_A$$

one defines by composition

(B.5) $$\widetilde{\mathrm{ch}}_2 : K_0(Y_A) \to H^2(Y_A, \Omega^2_{Y_A}) \to H^2(Y, \mathcal{O}_Y) \otimes \Omega^2_A.$$

Taking above A to be R_n and composing with the projection in (B.2), one obtains the homomorphism

(B.6) $$\overline{\mathrm{ch}}_2 : K_0(Y_{R_n}) \to H^2(Y, \mathcal{O}_Y) \otimes_{\overline{\mathbb{Q}}} \Omega^2_{\mathbb{C}} \otimes_{\mathbb{C}} R_n.$$

For the following discussion we choose a point $z \in Y(\overline{\mathbb{Q}})$ and generators t_1, t_2 of the maximal ideal of $\mathcal{O}_{Y,z}$. Write $X = \mathrm{Spec}\, \mathcal{O}_{Y,z}$ and $U = X \setminus z$.

This choice gives rise to an element $\rho \in H^2(Y, \mathcal{O}_Y)$ by the following construction. In fact for later reference we explain the construction after performing a base change to $\mathbb{C}$. Consider the covering $\mathcal{U} = (U_i)_{i=1,2}$ of $U_{\mathbb{C}}$ with $U_i = X_{\mathbb{C}} \setminus V(t_i)$. Now ρ is the image of the Čech cocycle $\frac{1}{t_1 t_2} \in \check{H}^1(\mathcal{U}, \mathcal{O}_{U_{\mathbb{C}}})$ under the composite map

(B.7) $$H^r_n : \check{H}^1(\mathcal{U}, \Omega^r_{U_{R_n}}) \to H^1(U_{\mathbb{C}}, \Omega^r_{U_{R_n}}) \to H^2_z(X_{\mathbb{C}}, \Omega^r_{X_{R_n}}) \xleftarrow{\sim}$$
$$H^2_z(Y_{\mathbb{C}}, \Omega^r_{Y_{R_n}}) \to H^2(Y_{\mathbb{C}}, \Omega^r_{Y_{R_n}})$$

for $n = 1$ and $r = 0$.

Lemma B.6. *Assume $p_g = \dim_{\overline{\mathbb{Q}}} H^0(Y, \Omega^2) > 0$. Then a generic choice of z gives rise to non-vanishing $\rho \in H^2(Y, \mathcal{O}_Y)$.*

Proof. Choose a non-vanishing $\omega \in H^0(Y, \Omega^2_Y)$. Duality theory [Ha] shows that $\omega \cup \rho \in H^2(Y, \Omega^2_Y) \cong \overline{\mathbb{Q}}$ does not vanish if ω does not vanish at z.   □

Lemma B.7. *For $n \geq 1$, the image of*

$$(B.8) \quad \ker\big(K_0(Y_{R_{n+1}}) \to K_0(Y_{R_n})\big) \cap \mathrm{im}\big(K_0(Y_R) \to K_0(Y_{R_{n+1}})\big) \xrightarrow{\overline{ch_2}}$$
$$H^2(Y, \mathcal{O}_Y) \otimes_{\overline{\mathbb{Q}}} \Omega^2_{\mathbb{C}} \otimes_{\mathbb{C}} R_{n+1}$$

contains any element of the form $\rho \otimes t^n(da_1 \wedge db_1 + \cdots + da_p \wedge db_p)$ with $a_i, b_i \in \mathbb{C}$.

Proof. It suffices to show that any element of the form $\rho \otimes t^n da \wedge db$ lies in the image of the map (B.8).

According to Grothendieck–Riemann–Roch [F, Ex. 15.2.15] one has:

Claim B.8.

   (i) For any R_{n+1}-point $x \in Y(R_{n+1})$ there is a canonical pushforward $\mathbb{Z} = K_0(R_{n+1}) \to K_0(Y_{R_{n+1}})$. We denote the image of 1 under this map by $[x]$.

   (ii) Assume that x as in (i) lifts the point z. Then $\mathrm{ch}_2([x])$ is equal to

$$H^2_{n+1}(d\log(t_1 - x^*(t_1)) \wedge d\log(t_2 - x^*(t_2))),$$

   where the map H^2_{n+1} is as defined in (B.7).

For $a, b \in \mathbb{C}$ we consider two R_{n+1}-points $x, y \in Y(R_{n+1})$ specializing to z which are defined by

$$(B.9) \qquad\qquad\qquad x : t_1 \mapsto at^{n-1}, \; t_2 \mapsto bt;$$
$$(B.10) \qquad\qquad\qquad y : t_1 \mapsto 0; \; t_2 \mapsto bt.$$

Observe that the pullbacks of $[x], [y]$ to $K_0(X_{R_n})$ coincide. It is clear that x and y extend to R-points of Y. So $[x] - [y]$ lies in the group on the left side of (B.8). The following claim shows that

$$\overline{\mathrm{ch}}_2([x] - [y]) = \rho \otimes t^n da \wedge db$$

finishing the proof of Lemma B.7. □

Claim B.9.

$$(B.11) \qquad\qquad\qquad \overline{\mathrm{ch}}_2([x]) = \rho \otimes t^n da \wedge db$$
$$(B.12) \qquad\qquad\qquad \overline{\mathrm{ch}}_2([y]) = 0.$$

Proof. We give the proof for x, the case of y works similarly. In the Čech cohomology group $\check{H}^2(\mathcal{U}, \mathcal{O}_U) \otimes_{\overline{\mathbb{Q}}} \Omega^2_{\mathbb{C}} \otimes_{\mathbb{C}} R_{n+1}$ we have

$$d\log(t_1 - x^*(t_1)) \wedge d\log(t_2 - x^*(t_2)) = \frac{dx^*(t_1) \wedge dx^*(t_2)}{(t_1 - x^*(t_1))(t_2 - x^*(t_2))}$$
$$= \frac{t^n da \wedge db}{(t_1 - x^*(t_1))(t_2 - x^*(t_2))} = \frac{t^n da \wedge db}{t_1 t_2}.$$

So (B.11) follows from Claim B.8(ii) and the definition of ρ. □

Let $K = \mathbb{C}((t))$.

Lemma B.10. *One has $K_0(Y_R) \cong K_0(Y_K)$.*

Proof. The boundary $\partial : K_1(Y_K) \to K_0(Y_k)$ is surjective: as Y_R admits a morphism $Y_R \to Y_k$, one applies the formula $x = \partial(x_R \cdot [t])$ where $x_R \in K_0(Y_R)$ is the pullback of $x \in K_0(Y_k)$ via the projection $Y_R \to X_k$ and $[t] \in K_1(K)$ is the class of the unit $t \in K^\times$. It follows that $K_0(Y_R) \hookrightarrow K_0(Y_K)$. For surjectivity, it suffices to note that a coherent sheaf on Y_K can be extended to a coherent sheaf on Y_R, and, as Y_R is regular, it can be resolved by locally free sheaves. $\square$

Proof of Proposition B.5. Recall that for simplicity of notation we assume $\dim Y = 2$. The diagram

$$
(B.13) \qquad
\begin{array}{ccc}
K_0(Y_R) & \xrightarrow{\;(3)\;} & \varprojlim_n K_0(Y \times_{\mathbb{C}} R_n) \\
\big\downarrow{\scriptstyle \widetilde{\mathrm{ch}}_2} & & \big\downarrow{\scriptstyle \overline{\mathrm{ch}}_2} \\
H^2(Y, \mathcal{O}_Y) \otimes_{\overline{\mathbb{Q}}} \Omega_R^2 & \xrightarrow{\;(1)\;} & H^2(Y, \mathcal{O}_Y) \otimes_{\overline{\mathbb{Q}}} \Omega_{\mathbb{C}}^2 \widehat{\otimes} R
\end{array}
$$

commutes. By Lemma B.7, the image of $\overline{\mathrm{ch}}_2$ contains all elements of the form $\rho \otimes \sum_{n=1}^{\infty} t^n \tau_n$ where $\tau_n = \sum_{i=1}^{p(n)} da_i^{(n)} \wedge db_i^{(n)}$. Here all the $a_i^{(n)}$ and $b_i^{(n)}$ are chosen algebraically independent, and we choose a sequence $\{p(n)\}$ such that $\limsup_n \frac{p(n)^2}{n} = \infty$. It follows from Lemma B.3 that $\rho \otimes (\sum_{n=1}^{\infty} t^n \tau_n)$ does not lie in the image of (1) if $\rho \neq 0$, so the map labeled (3) cannot be surjective in this case. Note that by Lemma B.6 a generic choice of the point $z \in Y(\overline{\mathbb{Q}})$ gives rise to non-vanishing ρ. $\square$

Remark B.11. Of course there are also odd-dimensional varieties X, for which algebraization fails. Take for example $X = Y \times_{\overline{\mathbb{Q}}} \mathbb{P}^1$ with Y as in Proposition B.5. In fact any smooth projective $X/\overline{\mathbb{Q}}$ which maps surjectively onto such a Y does not satisfy algebraization in the sense of Proposition B.5.

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5765 S. Blackstone Ave., Chicago, IL 60637, USA

FU Berlin, Mathematik, Arnimallee 3, 14195 Berlin, Germany

Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

E-mail address: spencer_bloch@yahoo.com

E-mail address: esnault@math.fu-berlin.de

E-mail address: moritz.kerz@mathematik.uni-regensburg.de