Hélène Esnault
Eckart Viehweg

Lectures on Vanishing Theorems

1992

Hélène Esnault, Eckart Viehweg
Fachbereich 6, Mathematik
Universität-Gesamthochschule Essen
D-45117 Essen, Germany

esnault@uni-essen.de

viehweg@uni-essen.de

ISBN 3-7643-2822-3 (Basel)

ISBN 0-8176-2822-3 (Boston)

We cordially thank Birkhäuser-Verlag for their permission to make this book available on the web. The page layout might be slightly different from the printed version.

Acknowledgement

These notes grew out of the DMV-seminar on algebraic geometry (Schloß Reisensburg, October 13 - 19, 1991). We thank the DMV (German Mathematical Society) for giving us the opportunity to organize this seminar and to present the theory of vanishing theorems to a group of younger mathematicians. We thank all the participants for their interest, for their useful comments and for the nice atmosphere during the seminar.

Table of Contents

Introduction

K. Kodaira's vanishing theorem, saying that the inverse of an ample invertible sheaf on a projective complex manifold X has no cohomology below the dimension of X and its generalization, due to Y. Akizuki and S. Nakano, have been proven originally by methods from differential geometry ([39] and [1]).

Even if, due to J.P. Serre's GAGA-theorems [56] and base change for field extensions the algebraic analogue was obtained for projective manifolds over a field k of characteristic $p = 0$, for a long time no algebraic proof was known and no generalization to $p > 0$, except for certain lower dimensional manifolds. Worse, counterexamples due to M. Raynaud [52] showed that in characteristic $p > 0$ some additional assumptions were needed.

This was the state of the art until P. Deligne and L. Illusie [12] proved the degeneration of the Hodge to de Rham spectral sequence for projective manifolds X defined over a field k of characteristic $p > 0$ and liftable to the second Witt vectors $W_2(k)$.

Standard degeneration arguments allow to deduce the degeneration of the Hodge to de Rham spectral sequence in characteristic zero, as well, a result which again could only be obtained by analytic and differential geometric methods beforehand. As a corollary of their methods M. Raynaud (loc. cit.) gave an easy proof of Kodaira vanishing in all characteristics, provided that X lifts to $W_2(k)$.

Short time before [12] was written the two authors studied in [20] the relations between logarithmic de Rham complexes and vanishing theorems on complex algebraic manifolds and showed that quite generally vanishing theorems follow from the degeneration of certain Hodge to de Rham type spectral sequences. The interplay between topological and algebraic vanishing theorems thereby obtained is also reflected in J. Kollár's work [41] and in the vanishing theorems M. Saito obtained as an application of his theory of mixed Hodge modules (see [54]).

It is obvious that the combination of [12] and [20] give another algebraic approach to vanishing theorems and it is one of the aims of these lecture notes to present it in all details. Of course, after the Deligne-Illusie-Raynaud proof of the original Kodaira and Akizuki-Nakano vanishing theorems, the main motivation to present the methods of [20] along with those of [12] is that they imply as well some of the known generalizations.

Generalizations have been found by D. Mumford [49], H. Grauert and O. Riemenschneider [25], C.P. Ramanujam [51] (in whose paper the method of coverings already appears), Y. Miyaoka [45] (the first who works with integral parts of $\mathbb{Q}$ divisors, in the surface case), by Y. Kawamata [36] and the second author [63]. All results mentioned replace the condition "ample" in Kodaira's result by weaker conditions. For Akizuki-Nakano type theorems A. Sommese (see for example [57]) got some improvement, as well as F. Bogomolov and A. Sommese (as explained in [6] and [57]) who showed the vanishing of the global sections in certain cases.

Many of the applications of vanishing theorems of Kodaira type rely on the surjectivity of the adjunction map

$$H^b(X, \mathcal{L} \otimes \omega_X(B)) \longrightarrow H^b(B, \mathcal{L} \otimes \omega_B)$$

where B is a divisor and $\mathcal{L}$ is ample or is belonging to the class of invertible sheaves considered in the generalizations.

J. Kollár [40], building up on partial results by Tankeev, studied the adjunction map directly and gave criteria for $\mathcal{L}$ and B which imply the surjectivity.

This list of generalizations is probably not complete and its composition is evidently influenced by the fact that all the results mentioned and some slight improvements have been obtained in [20] and [22] as corollaries of two vanishing theorems for sheaves of differential forms with values in "integral parts of $\mathbb{Q}$-divisors", one for the cohomology groups and one for restriction maps between cohomology groups.

In these notes we present the algebraic proof of Deligne and Illusie [12] for the degeneration of the Hodge to de Rham spectral sequence (Lecture 10). Beforehand, in Lectures 8 and 9, we worked out the properties of liftings of schemes and of the Frobenius morphism to the second Witt vectors [12] and the properties of the Cartier operator [34] needed in the proof. Even if some of the elegance of the original arguments is lost thereby, we avoid using the derived category. The necessary facts about hypercohomology and spectral sequences are shortly recalled in the appendix, at the end of these notes.

During the first seven lectures we take the degeneration of the Hodge to de Rham spectral sequence for granted and we develop the interplay between cyclic coverings, logarithmic de Rham complexes and vanishing theorems (Lectures 2 - 4).

We try to stay as much in the algebraic language as possible. Lectures 5 and 6 contain the geometric interpretation of the vanishing theorems obtained, i.e. the generalizations mentioned above. Due to the use of H. Hironaka's embedded resolution of singularities, most of those require the assumption that the manifolds considered are defined over a field of characteristic zero.

Raynaud's elegant proof of the Kodaira-Akizuki-Nakano vanishing theorem is reproduced in Lecture 11, together with some generalization. However, due to the non-availability of desingularizations in characterisitic p, those generalizations seem to be useless for applications in geometry over fields of characteristic $p > 0$.

In characteristic zero the generalized vanishing theorems for integral parts of $\mathbb{Q}$-divisors and J. Kollár's vanishing for restriction maps turned out to be powerful tools in higher dimensional algebraic geometry. Some examples, indicating "how to use vanishing theorems" are contained in the second half of Lecture 6, where we discuss higher direct images and the interpretation of vanishing theorems on non-compact manifolds, and in Lecture 7. Of course, this list is determined by our own taste and restricted by our lazyness. In particular, the applications of vanishing theorems in the birational classification theory and in the minimal model program is left out. The reader is invited to consult the survey's of S. Mori [46] and of Y. Kawamata, K. Matsuda and M. Matsuki [38].

There are, of course, more subjects belonging to the circle of ideas presented in these notes which we left aside:

- L. Illusie's generalizations of [12] to variations of Hodge structures [32].
- J.-P. Demailly's analytic approach to generalized vanishing theorems [13].
- M. Saito's results on "mixed Hodge modules and vanishing theorems" [54], related to J. Kollár's program [41].
- The work of I. Reider, who used unstability of rank two vector bundles (see [6]) to show that certain invertible sheaves on surfaces are generated by global sections [53] (see however (7.23)).
- Vanishing theorems for vector bundles.
- Generalizations of the vanishing theorems for integral parts of $\mathbb{Q}$-divisors ([2], [3], [42], [43] and [44]).

However, we had the feeling that we could not pass by the generic vanishing theorems of M. Green and R. Lazarsfeld [26]. The general picture of "vanishing theorems" would be incomplete without mentioning this recent development. We include in Lectures 12 and 13 just the very first results in this direction. In particular, the more explicit description and geometric interpretation of the "bad locus in $\mathrm{Pic}^0(X)$ ", contained in A. Beauville's paper [5] and Green and Lazarsfeld's second paper [27] on this subject is missing. During the preparation of these notes C. Simpson [58] found a quite complete description of such "degeneration loci".

The first Lecture takes possible proofs of Kodaira's vanishing theorem as a pretext to introduce some of the key words and methods, which will reappear throughout these lecture notes and to give a more technical introduction to its subject.

Methods and results due to P. Deligne and Deligne-Illusie have inspired and influenced our work. We cordially thank L. Illusie for his interest and several conversations helping us to understand [12].

§ 1 Kodaira's vanishing theorem, a general discussion

Let X be a projective manifold defined over an algebraically closed field k and let $\mathcal{L}$ be an invertible sheaf on X. By explicit calculations of the Čech-cohomology of the projective space one obtains:

1.1. Theorem (J. P. Serre [55]). *If $\mathcal{L}$ is ample and $\mathcal{F}$ a coherent sheaf, then there is some $\nu_0 \in \mathbb{N}$ such that*

$$H^b(X, \mathcal{F} \otimes \mathcal{L}^\nu) = 0 \quad \textit{for} \quad b > 0 \quad \textit{and} \quad \nu \geq \nu_0$$

In particular, for $\mathcal{F} = \mathcal{O}_X$, one obtains the vanishing of $H^b(X, \mathcal{L}^\nu)$ for $b > 0$ and ν sufficiently large.

If $\operatorname{char}(k) = 0$, then "$\nu$ sufficiently large" can be made more precise. For example, it is enough to choose ν such that $\mathcal{A} = \mathcal{L}^\nu \otimes \omega_X^{-1}$ is ample, where $\omega_X = \Omega_X^n$ is the canonical sheaf of X, and to use:

1.2. Theorem (K. Kodaira [39]). *Let X be a complex projective manifold and $\mathcal{A}$ be an ample invertible sheaf. Then*

$$a) \quad H^b(X, \omega_X \otimes \mathcal{A}) = 0 \qquad \textit{for } b > 0$$

$$b) \quad H^{b'}(X, \mathcal{A}^{-1}) = 0 \qquad \textit{for } b' < n = \dim\ X.$$

Of course it follows from Serre-duality that a) and b) are equivalent. Moreover, since every algebraic variety in characteristic 0 is defined over a subfield of $\mathbb{C}$, one can use flat base change to extend (1.2) to manifolds X defined over any algebraically closed field of characteristic zero.

1.3. Theorem (Y. Akizuki, S. Nakano [1]). *Under the assumptions made in (1.2), let Ω_X^a denote the sheaf of a-differential forms. Then*

$$a) \quad H^b(X, \Omega_X^a \otimes \mathcal{A}) = 0 \qquad \textit{for } a + b > n$$

$$b) \quad H^{b'}(X, \Omega_X^{a'} \otimes \mathcal{A}^{-1}) = 0 \qquad \textit{for } a' + b' < n.$$

For a long time, the only proofs known for (1.2) and (1.3) used methods of complex analytic differential geometry, until in 1986 P. Deligne and L. Illusie found an elegant algebraic approach to prove (1.2) as well as (1.3), using characteristic p methods. About one year earlier, trying to understand several generalizations of (1.2), the two authors obtained (1.2) and (1.3) as a direct consequence of the decomposition of the de Rham-cohomology $H^k(Y,\mathbb{C})$ into a direct sum

$$\bigoplus_{b+a=k} H^b(Y,\Omega_Y^a)$$

or, equivalently, of the degeneration of the "Hodge to de Rham" spectral sequence, both applied to cyclic covers $\pi : Y \longrightarrow X$.

As a guide-line to the first part of our lectures, let us sketch two possible proofs of (1.2) along this line.

1. Proof: With Hodge decomposition for non-compact manifolds and topological vanishing: For sufficiently large N one can find a non-singular primedivisor H such that $\mathcal{A}^N = \mathcal{O}_X(H)$. Let $s \in H^0(X,\mathcal{A}^N)$ be the corresponding section. We can regard s as a rational function, if we fix some divisor A with $\mathcal{A} = \mathcal{O}_X(A)$ and take

$$s \in \mathbb{C}(X) \quad \text{with} \quad (s) + N \cdot A = H.$$

The field $L = \mathbb{C}(X)(\sqrt[N]{s})$ depends only on H. Let $\pi : Y \longrightarrow X$ be the covering obtained by taking the normalization of X in L (see (3.5) for another construction).

An easy calculation (3.13) shows that Y is non-singular as well as $D = (\pi^* H)_{red}$ and that $\pi : Y \longrightarrow X$ is unramified outside of D. One has

$$\pi^*\Omega_X^a(\log\ H) = \Omega_Y^a(\log\ D)$$

where $\Omega_X^a(\log\ H)$ denotes the sheaf of a-differential forms with logarithmic poles along H (see (2.1)). Moreover

$$\pi_*\mathcal{O}_Y = \bigoplus_{i=0}^{N-1} \mathcal{A}^{-i} \qquad \text{and}$$

$$\pi_*\Omega_Y^n(\log\ D) = \bigoplus_{i=0}^{N-1} \Omega_X^n(\log\ H) \otimes \mathcal{A}^{-i} = \bigoplus_{i=0}^{N-1} \Omega_X^n \otimes \mathcal{A}^{N-i}$$

Deligne [11] has shown that

$$H^k(Y-D,\mathbb{C}) \cong \bigoplus_{b+a=k} H^b(Y,\Omega_Y^a(\log\ D)).$$

Since $X-H$ is affine, the same holds true for $Y-D$ and hence $H^k(Y-D,\mathbb{C}) = 0$ for $k > n$. Altogether one obtains for $b > 0$

$$0 = H^b(Y,\Omega_Y^n(\log\ D)) = \bigoplus_{i=0}^{N-1} H^b(X,\Omega_X^n \otimes \mathcal{A}^{N-i}).$$

□

In fact, a similar argument shows as well that

$$H^b(X, \Omega_X^a(\log\ H) \otimes \mathcal{A}^{-1}) = 0$$

for $a+b > n$. We can deduce (1.3) from this statement by induction on dim X using the residue sequence (as will be explained in (6.4)).

The two ingredients of the first proof can be interpretated in a different way. First of all, since the de Rham complex on $Y - D$ is a resolution of the constant sheaf one can use GAGA [56] and Serre's vanishing to obtain the topological vanishing used above. Secondly, the decomposition of the de Rham cohomology of Y into the direct sum of (a, b)-forms, implies that the differential

$$d : \Omega_Y^a \longrightarrow \Omega_Y^{a+1}$$

induces the zero map

$$d : H^b(Y, \Omega_Y^a) \longrightarrow H^b(Y, \Omega_Y^{a+1}).$$

Using this one can give another proof of (1.2):

2. Proof: Closedness of global (p, q) forms and Serre's vanishing theorem: Let us return to the covering $\pi : Y \to X$ constructed in the first proof. The Galois-group G of $\mathbb{C}(Y)$ over $\mathbb{C}(X)$ is cyclic of order N. A generator σ of G acts on Y and D and hence on the sheaves $\pi_*\Omega_Y^a$ and $\pi_*\Omega_Y^a(\log\ D)$. Both sheaves decompose in a direct sum of sheaves of eigenvectors of σ and, if we choose the N-th root of unity carefully, the i-th summand of

$$\pi_*\Omega_Y^a(\log\ D) = \Omega_X^a(\log\ H) \otimes \pi_*\mathcal{O}_Y = \bigoplus_{i=0}^{N-1} \Omega_X^a(\log\ H) \otimes \mathcal{A}^{-i}$$

consists of eigenvectors with eigenvalue e^i. For $e^i \neq 1$ the eigenvectors of $\pi_*\Omega_Y^a$ and of $\pi_*\Omega_Y^a(\log\ D)$ coincide, the difference of both sheaves is just living in the invariant parts Ω_X^a and $\Omega_X^a(\log\ H)$. Moreover, the differential

$$d : \mathcal{O}_Y \longrightarrow \Omega_Y^1$$

is compatible with the G-action and we obtain a $\mathbb{C}$-linear map (in fact a connection)

$$\nabla_i : \mathcal{A}^{-i} \longrightarrow \Omega_X^1(\log\ H) \otimes \mathcal{A}^{-i}.$$

Both properties follow from local calculations. Let us show first, that

$$\pi_*\Omega_Y^a = \Omega_X^a \oplus \bigoplus_{i=1}^{N-1} \Omega_X^a(\log\ H) \otimes \mathcal{A}^{-i} .$$

Since H is non-singular one can choose local parameters $x_1, \ldots, x_n$ such that H is defined by $x_1 = 0$. Then

$$y_1 = \sqrt[N]{x_1} \quad \text{and} \quad x_2, \ldots, x_n$$

are local parameters on Y. The local generators

$$N \cdot \frac{dx_1}{x_1}, dx_2, \ldots, dx_n \quad \text{of} \quad \Omega^1_X(\log \ H)$$

lift to local generators

$$\frac{dy_1}{y_1}, dx_2, \ldots, dx_n \quad \text{of} \quad \Omega^1_Y(\log \ D).$$

The a-form

$$\phi = s \cdot \frac{dy_1}{y_1} \wedge dx_2 \wedge \ldots \wedge dx_a$$

(for example) is an eigenvector with eigenvalue e^i if and only if the same holds true for s, i.e. if $s \in \mathcal{O}_X \cdot y_1^i$. If ϕ has no poles, s must be divisible by y_1. This condition is automatically satisfied as long as $i > 0$. For $i = 0$ it implies that s must be divisible by $y_1^N = x_1$.

The map ∇_i can be described locally as well. If

$$s = t \cdot y_1^i \in \mathcal{O}_X \cdot y_1^i$$

then on Y one has

$$ds = y_1^i \cdot dt + t \cdot dy_1^i$$

and therefore d respects the eigenspaces and ∇_i is given by

$$\nabla_i(s) = (dt + \frac{i}{N} \cdot t\frac{dx_1}{x_1}) \cdot y_1^i.$$

If $Res : \Omega^1_X(\log \ H) \longrightarrow \mathcal{O}_H$ denotes the residue map, one obtains in addition that

$$(Res \ \otimes id_{\mathcal{A}^{-1}}) \circ \nabla_1 : \mathcal{A}^{-1} \longrightarrow \mathcal{O}_H \otimes \mathcal{A}^{-1}$$

is the $\mathcal{O}_X$-linear map

$$s \longmapsto \frac{1}{N} s \mid_H .$$

Since $d : H^b(Y, \mathcal{O}_Y) \longrightarrow H^b(Y, \Omega^1_Y)$ is the zero map, the direct summand

$$\nabla_1 : H^b(X, \mathcal{A}^{-1}) \longrightarrow H^b(X, \Omega^1_X(\log \ D) \otimes \mathcal{A}^{-1})$$

is the zero map as well as the restriction map

$$N \cdot (Res \ \otimes id_{\mathcal{A}^{-1}}) \circ \nabla_1 : H^b(X, \mathcal{A}^{-1}) \longrightarrow H^b(H, \mathcal{O}_H \otimes \mathcal{A}^{-1}).$$

Hence, for all b we have a surjection

$$H^b(X, \mathcal{A}^{-N-1}) = H^b(X, \mathcal{O}_X(-H) \otimes \mathcal{A}^{-1}) \longrightarrow H^b(X, \mathcal{A}^{-1}).$$

Using Serre duality and (1.1) however, $H^b(X, \mathcal{A}^{-N-1}) = 0$ for $b < n$ and N sufficiently large.

□

Again, the proof of (1.2) gives a little bit more:

If $\mathcal{A}$ is an invertible sheaf such that $\mathcal{A}^N = \mathcal{O}_X(H)$ for a non-singular divisor H, then the restriction map

$$H^b(X, \mathcal{A}^{-1}) \longrightarrow H^b(H, \mathcal{O}_H \otimes \mathcal{A}^{-1})$$

is zero.

This statement is a special case of J. Kollár's vanishing theorem ([40], see (5.6,a)).

The main theme of the first part of these notes will be to extend the methods sketched above to a more general situation:
If one allows Y to be any cyclic cover of X whose ramification divisor is a normal crossing divisor, one obtains vanishing theorems for the cohomology (or for the restriction maps in cohomology) of a larger class of locally free sheaves.
Or, taking a more axiomatic point of view, one can consider locally free sheaves $\mathcal{E}$ with logarithmic connections

$$\nabla : \mathcal{E} \longrightarrow \Omega^1_X(\log\ H) \otimes \mathcal{E}$$

and ask which proporties of ∇ and H force cohomology groups of $\mathcal{E}$ to vanish. The resulting "vanishing theorems for integral parts of $\mathbb{Q}$-divisors" (5.1) and (6.2) will imply several generalizations of the Kodaira-Nakano vanishing theorem (see Lectures 5 and 6), especially those obtained by Mumford, Grauert and Riemenschneider, Sommese, Bogomolov, Kawamata, Kollár ......

However, the approach presented above is using (beside of algebraic methods) the Hodge theory of projective manifolds, more precisely the degeneration of the Hodge to de Rham spectral sequence

$$E_1^{ab} = H^b(Y, \Omega^a_Y(\log\ D)) \Longrightarrow \mathbb{H}^{a+b}(Y, \Omega^\bullet_Y(\log\ D))$$

again a result which for a long time could only be deduced from complex analytic differential geometry.

Both, the vanishing theorems and the degeneration of the Hodge to de Rham spectral sequence do not hold true for manifolds defined over a field of characteristic $p > 0$. However, if Y and D both lift to the ring of the second Witt-vectors (especially if they can be lifted to characteristic 0) and if $p \geq \dim\ X$, P. Deligne and L. Illusie were able to prove the degeneration (see [12]). In fact, contrary to characteristic zero, they show that the degeneration is induced by some local splitting:

If F_k and F_Y are the absolute Frobenius morphisms one obtains the geometric Frobenius by

$$\begin{tikzcd} Y \arrow[r, "F"] \arrow[dr] & Y' = Y \times_{\mathrm{Spec}\ k} \mathrm{Spec}\ k \arrow[r, "\sigma"] \arrow[d] & Y \arrow[d] \\ & \mathrm{Spec}\ k \arrow[r, "F_k"] & \mathrm{Spec}\ k \end{tikzcd}$$

with $F_Y = \sigma \circ F$. If we write $D' = (\sigma^* D)_{red}$ then, roughly speaking, they show that $F_*(\Omega^\bullet_Y(\log\ D))$ is quasi-isomorphic to the complex

$$\bigoplus_a \Omega^a_{Y'}(\log\ D')[-a]$$

with $\Omega^a_{Y'}(\log\ D')$ in degree a and with trivial differentials.

By base change for σ one obtains

$$\dim\ \mathbb{H}^k(Y, \Omega^\bullet_Y(\log\ D)) = \sum_{a+b=k} \dim\ H^b(Y', \Omega^a_{Y'}(\log\ D'))$$

$$= \sum_{a+b=k} \dim\ H^b(Y, \Omega^a_Y(\log\ D)).$$

Base change again allows to lift this result to characteristic 0.

Adding this algebraic proof, which can be found in Lectures 8 - 10, to the proof of (1.2) and its generalizations (Lectures 2 - 6) one obtains algebraic proofs of most of the vanishing theorems mentioned.

However, based on ideas of M. Raynaud, Deligne and Illusie give in [12] a short and elegant argument for (1.3) in characteristic p (and, by base change, in general):
By Serre's vanishing theorem one has for some $m \gg 0$

$$H^b(Y, \Omega^a_Y \otimes \mathcal{A}^{-p^\nu}) = 0 \quad \text{for} \quad \nu \geq (m+1)$$

and $a + b < n$, where $\mathcal{A}$ is ample on Y. One argues by descending induction on m:
As

$$\mathcal{A}^{p^{(m+1)}} = F^*(\mathcal{A}'^{p^m}) \ \text{ for } \ \mathcal{A}' = \sigma^* \mathcal{A}$$

and as $\Omega_Y^\bullet$ is a $\mathcal{O}_{Y'}$ complex, $\Omega_Y^\bullet \otimes \mathcal{A}^{-p^{(m+1)}}$ is a complex of $\mathcal{O}_{Y'}$ sheaves with

$$\mathbb{H}^k(Y, \Omega_Y^\bullet \otimes \mathcal{A}^{-p^{m+1}}) = 0 \quad \text{for} \quad k < n.$$

However one has

$$F_*(\Omega_Y^\bullet \otimes \mathcal{A}^{-p^{m+1}}) = \bigoplus_a \Omega_{Y'}^a \otimes \mathcal{A}'^{-p^m}[-a]$$

and

$$0 = H^b(Y', \Omega_{Y'}^a \otimes \mathcal{A}'^{-p^m}) = H^b(Y, \Omega_Y^a \otimes \mathcal{A}^{-p^m})$$

for $a + b < n$.

Unfortunately this type of argument does not allow to weaken the assumptions made in (1.2) or (1.3). In order to deduce the generalized vanishing theorems from the degeneration of the Hodge to de Rham spectral sequence in characteristic 0 we have to use H. Hironaka's theory of embedded resolution of singularities, at present a serious obstruction for carrying over arguments from characteristic 0 to characteristic p. Even the Grauert-Riemenschneider vanishing theorem (replace "ample" in (1.2) by "semi-ample of maximal Iitaka dimension") has no known analogue in characteristic p (see §11).

M. Green and R. Lazarsfeld observed, that "ample" in (1.2) can sometimes be replaced by "numerically trivial and sufficiently general". To be more precise, they showed that $H^b(X, \mathcal{N}^{-1}) = 0$ for a general element $\mathcal{N} \in Pic^0(X)$ if b is smaller than the dimension of the image of X under its Albanese map

$$\alpha : X \longrightarrow \mathrm{Alb}(X).$$

By Hodge-duality (for Hodge theory with values in unitary rank one bundles) $H^b(X, \mathcal{N}^{-1})$ can be identified with $H^0(X, \Omega_X^b \otimes \mathcal{N})$. If $H^b(X, \mathcal{N}^{-1}) \neq 0$ for all $\mathcal{N} \in Pic^0(X)$ the deformation theory for cohomology groups, developed by Green and Lazarsfeld, implies that for all $\omega \in H^0(X, \Omega_X^1)$ the wedge product

$$H^0(X, \Omega_X^b \otimes \mathcal{N}) \longrightarrow H^0(X, \Omega_X^{b+1} \otimes \mathcal{N})$$

is non-trivial. This however implies that the image of X under the Albanese map, or equivalently the subsheaf of Ω_X^1 generated by global sections is small. For example, if

$$S^b(X) = \{\mathcal{N} \in Pic^0(X);\ H^b(X, \mathcal{N}^{-1}) \neq 0\},$$

then the first result of Green and Lazarsfeld says that

$$\mathrm{codim}_{Pic^0(X)}(S^b(X)) \geq \dim(\alpha(X)) - b.$$

It is only this part of their results we include in these notes, together with some straightforward generalizations due to H. Dunio [14] (see Lectures 12 and 13). The more detailed description of $S^b(X)$, due to Beauville [5], Green-Lazarsfeld [27] and C. Simpson [58] is just mentioned, without proof, at the end of Lecture 13.

§ 2 Logarithmic de Rham complexes

In this lecture we want to start with the definition and simple properties of the sheaf of (algebraic) logarithmic differential forms and of sheaves with logarithmic integrable connections, developed in [10]. The main examples of those will arise from cyclic covers (see Lecture 3). Even if we stay in the algebraic language, the reader is invited (see 2.11) to compare the statements and constructions with the analytic case.

Throughout this lecture X will be an algebraic manifold, defined over an algebraically closed field k, and $D = \sum_{j=1}^{r} D_j$ a reduced normal crossing divisor, i.e. a divisor with non-singular components D_j intersecting each other transversally.

We write $\tau : U = X - D \longrightarrow X$ and

$$\Omega_X^a(*D) = \varinjlim_{\nu} \Omega_X^a(\nu \cdot D) = \tau_* \Omega_U^a.$$

Of course $(\Omega_X^\bullet(*D), d)$ is a complex.

2.1. Definition. $\Omega_X^a(\log\ D)$ denotes the subsheaf of $\Omega_X^a(*D)$ of *differential forms with logarithmic poles* along D, i.e.: if $V \subseteq X$ is open, then

$$\Gamma(V, \Omega_X^a(\log\ D)) =$$

$$\{\ \alpha \in \Gamma(V, \Omega_X^a(*D));\ \alpha \text{ and } d\alpha \text{ have simple poles along } D\}.$$

2.2. Properties.
a)

$$(\Omega_X^\bullet(\log\ D), d) \hookrightarrow (\Omega_X^\bullet(*D), d).$$

is a subcomplex.
b)

$$\Omega_X^a(\log\ D) = \bigwedge^a \Omega_X^1(\log\ D)$$

c) $\Omega_X^a(\log\ D)$ is locally free. More precisely:
For $p \in X$, let us say with $p \in D_j$ for $j = 1, \ldots, s$ and $p \notin D_j$ for $j = s+1, \ldots, r$, choose local parameters $f_1, \ldots, f_n$ in p such that D_j is defined by $f_j = 0$ for $j = 1, \ldots, s$. Let us write

$$\delta_j = \begin{cases} \frac{df_j}{f_j} & \text{if} \quad j \le s \\ df_j & \text{if} \quad j > s \end{cases}$$

and for $I = \{j_1, \ldots, j_a\} \subset \{1, \ldots, n\}$ with $j_1 < j_2 \ldots < j_a$

$$\delta_I = \delta_{j_1} \wedge \ldots \wedge \delta_{j_a}.$$

Then $\{\delta_I;\ \sharp I = a\}$ is a free system of generators for $\Omega^a_X(\log\ D)$.

PROOF: (see [10], II, 3.1 - 3.7). a) is obvious and b) follows from the explicite form of the generators given in c).
Since δ_j is closed, δ_I is a local section of $\Omega^a_X(\log\ D)$. By the Leibniz rule the $\mathcal{O}_X$-module Ω spanned by the δ_I is contained in $\Omega^a_X(\log\ D)$. Ω is locally free and, in order to show that $\Omega = \Omega^a_X(\log\ D)$ it is enough to consider the case $s = 1$. Each local section $\alpha \in \Omega^a_X(*D)$ can be written as

$$\alpha = \alpha_1 + \alpha_2 \wedge \frac{df_1}{f_1},$$

where α_1 and α_2 lie in $\Omega^a_X(*D)$ and $\Omega^{a-1}_X(*D)$ and where both are in the subsheaves generated over $\mathcal{O}(*D)$ by wedge products of $df_2, \ldots, df_n$.
$\alpha \in \Omega^a_X(\log\ D)$ implies that

$$f_1 \cdot \alpha = f_1 \cdot \alpha_1 + \alpha_2 \wedge df_1\ \in \Omega^a_X \quad \text{and} \quad f_1 d\alpha = f_1 d\alpha_1 + d\alpha_2 \wedge df_1 \in \Omega^{a+1}_X.$$

Hence α_2 as well as $f_1\alpha_1$ are without poles. Since

$$d(f_1\alpha_1) = df_1 \wedge \alpha_1 + f_1 d\alpha_1 = df_1 \wedge \alpha_1 + f_1 d\alpha - d\alpha_2 \wedge df_1$$

the form $df_1 \wedge \alpha_1$ has no poles which implies $\alpha_1 \in \Omega^a_X$.

□

Using the notation from (2.2,c) we define

$$\alpha : \Omega^1_X(\log\ D) \longrightarrow \bigoplus_{j=1}^{s} \mathcal{O}_{D_j}$$

by

$$\alpha(\sum_{j=1}^{n} a_j \delta_j) = \bigoplus_{j=1}^{s} a_j|_{D_j}.$$

For $a \geq 1$ we have correspondingly a map

$$\beta_1 : \Omega^a_X(\log\ D) \longrightarrow \Omega^{a-1}_{D_1}(\log\ (D - D_1)|_{D_1})$$

given by:
If φ is a local section of $\Omega^a_X(\log\ D)$, we can write

$$\varphi = \varphi_1 + \varphi_2 \wedge \frac{df_1}{f_1}$$

where φ_1 lies in the span of the δ_I with $1 \notin I$ and

$$\varphi_2 = \sum_{1 \in I} a_I \delta_{I-\{1\}}.$$

Then

$$\beta_1(\varphi) = \beta_1(\varphi_2 \wedge \frac{df_1}{f_1}) = \sum a_I \delta_{I-\{1\}}|_{D_1}.$$

Of course, β_i will denote the corresponding map for the i-th component. Finally, the natural restriction of differential forms gives

$$\gamma_1 : \Omega^a_X(\log\ (D - D_1)) \longrightarrow \Omega^a_{D_1}(\log\ (D - D_1)|_{D_1}).$$

Since the sheaf on the left hand side is generated by

$$\{f_1 \cdot \delta_I;\ 1 \in I\}\ \cup\ \{\delta_I;\ 1 \notin I\}$$

we can describe γ_1 by

$$\gamma_1(\sum_{1 \in I} f_1 a_I \delta_I + \sum_{1 \notin I} a_I \delta_I) = \sum_{1 \notin I} a_I \delta_I|_{D_1}.$$

Obviously one has

2.3. Properties. One has three exact sequences:
a)

$$0 \to \Omega^1_X \longrightarrow \Omega^1_X(\log\ D) \xrightarrow{\alpha} \bigoplus_{j=1}^{r} \mathcal{O}_{D_j} \to 0.$$

b)

$$0 \to \Omega^a_X(\log\ (D - D_1)) \longrightarrow \Omega^a_X(\log\ D) \xrightarrow{\beta_1} \Omega^{a-1}_{D_1}(\log\ (D - D_1)|_{D_1}) \to 0.$$

c)

$$0 \to \Omega^a_X(\log\ D)(-D_1) \longrightarrow \Omega^a_X(\log\ (D - D_1)) \xrightarrow{\gamma_1} \Omega^a_{D_1}(\log\ (D - D_1)|_{D_1}) \to 0.$$

By (2.2,b) $(\Omega^\bullet_X(\log\ D), d)$ is a complex. It is the most simple example of a logarithmic de Rham complex.

2.4. Definition. Let $\mathcal{E}$ be a locally free coherent sheaf on X and let

$$\nabla : \mathcal{E} \longrightarrow \Omega^1_X(\log\ D) \otimes \mathcal{E}$$

be a k-linear map satisfying

$$\nabla(f \cdot e) = f \cdot \nabla(e) + df \otimes e.$$

One defines
$$\nabla_a : \Omega^a_X(\log\ D) \otimes \mathcal{E} \longrightarrow \Omega^{a+1}_X(\log\ D) \otimes \mathcal{E}$$
by the rule
$$\nabla_a(\omega \otimes e) = d\omega \otimes e + (-1)^a \cdot \omega \wedge \nabla(e).$$
We assume that $\nabla_{a+1} \circ \nabla_a = 0$ for all a. Such ∇ will be called an *integrable logarithmic connection along* D, or just a connection. The complex
$$(\Omega^\bullet_X(\log\ D) \otimes \mathcal{E}, \nabla_\bullet)$$
is called the *logarithmic de Rham complex* of $(\mathcal{E}, \nabla)$.

2.5. Definition. For an integrable logarithmic connection
$$\nabla : \mathcal{E} \longrightarrow \Omega^1_X(\log\ D) \otimes \mathcal{E}$$
we define the residue map along D_1 to be the composed map
$$Res_{D_1}(\nabla) : \mathcal{E} \xrightarrow{\nabla} \Omega^1_X(\log\ D) \otimes \mathcal{E} \xrightarrow{\beta'_1 = \beta_1 \otimes id_{\mathcal{E}}} \mathcal{O}_{D_1} \otimes \mathcal{E}.$$

2.6. Lemma.
a) $Res_{D_1}(\nabla)$ *is* $\mathcal{O}_X$*-linear and it factors through*
$$\mathcal{E} \xrightarrow{restr.} \mathcal{O}_{D_1} \otimes \mathcal{E} \longrightarrow \mathcal{O}_{D_1} \otimes \mathcal{E}$$
where restr. the restriction of $\mathcal{E}$ *to* D_1*. By abuse of notations we will call the second map* $Res_{D_1}(\nabla)$ *again.*
b) One has a commutative diagram
$$\begin{array}{ccc}
\Omega^a_X(\log\ (D - D_1)) \otimes \mathcal{E} & \xrightarrow{(\nabla_a)\circ(incl.)} & \Omega^{a+1}_X(\log\ D) \otimes \mathcal{E} \\
\Big\downarrow{\scriptstyle \gamma_1 \otimes id_{\mathcal{E}}} & & \Big\downarrow{\scriptstyle \beta_1 \otimes id_{\mathcal{E}} = \beta'_1} \\
\Omega^a_{D_1}(\log\ (D - D_1)\mid_{D_1}) \otimes \mathcal{E} & \xrightarrow{((-1)^a \cdot id) \otimes Res_{D_1}(\nabla)} & \Omega^a_{D_1}(\log\ (D - D_1)\mid_{D_1}) \otimes \mathcal{E}
\end{array}$$

Proof: a) We have
$$\nabla(g \cdot e) = g \cdot \nabla(e) + dg \otimes e \quad \text{and} \quad \beta'_1(\nabla(g \cdot e)) = g \cdot \beta'_1(\nabla(e)).$$
If f_1 divides g then $g \cdot \beta'_1(\nabla(e)) = 0$.
b) For $\omega \in \Omega^a_X(\log\ (D - D_1))$ and $e \in \mathcal{E}$ we have
$$\begin{aligned}
\beta'_1(\nabla_a(\omega \otimes e)) &= \beta'_1(d\omega \otimes e + (-1)^a \cdot \omega \wedge \nabla(e)) \\
&= \beta'_1((-1)^a \cdot \omega \wedge \nabla(e)).
\end{aligned}$$

If $\omega = f_1 \cdot a_I \cdot \delta_I$ for $1 \in I$, then

$$(-1)^a \omega \wedge \nabla(e) \in \Omega_X^{a+1}(\log\ D)(-D_1)$$

and $\beta_1'(\nabla_a(\omega \otimes\ e)) = 0$. On the other hand, $\gamma_1(\omega) \otimes e = 0$ by definition.
If $\omega = a_I \delta_I$ for $1 \notin I$, then

$$\gamma_1(\omega) \otimes e = a_I \cdot \delta_I\ |_{D_1} \otimes e$$

and

$$\beta_1'((-1)^a \omega \wedge \nabla(e)) = (-1)^a \omega|_{D_1} \otimes\ Res_{D_1}(\nabla)(e).$$

□

2.7. Lemma. *Let*

$$B = \sum_{j=1}^{r} \mu_j D_i$$

be any divisor and $(\nabla, \mathcal{E})$ *as in (2.4). Then* ∇ *induces a connection* ∇^B *with logarithmic poles on*

$$\mathcal{E} \otimes \mathcal{O}_X(B) = \mathcal{E}(B)$$

and the residues satisfy

$$Res_{D_j}(\nabla^B) = Res_{D_j}(\nabla) - \mu_j \cdot id_{D_j}.$$

Proof: A local section of $\mathcal{E}(B)$ is of the form

$$\sigma = \prod_{j=1}^{s} f_j^{-\mu_j} \cdot e$$

and

$$\nabla^B(\sigma) = \prod_{j=1}^{s} f_j^{-\mu_j} \nabla(e) + d(\prod_{j=1}^{s} f_j^{-\mu_j}) \otimes e =$$

$$= \prod_{j=1}^{s} f_j^{-\mu_j} \nabla(e) + \sum_{k=1}^{s} (\prod_{j=1}^{s} f_j^{-\mu_j}) \cdot (-\mu_k) \frac{df_k}{f_k} \otimes e.$$

Hence $\nabla^B : \mathcal{E}(B) \longrightarrow \Omega_X^1(\log\ D) \otimes \mathcal{E}(B)$ is well defined. One obtains

$$Res_{D_1}(\nabla^B(\sigma)) = \prod_{j=1}^{s} f_j^{-\mu_j}\ Res_{D_1}(\nabla(e)) + \prod_{j=1}^{s} f_j^{-\mu_j}(-\mu_1) \otimes e\ |_{D_1}\ .$$

□

2.8. Definition. a) We say that $(\nabla, \mathcal{E})$ satisfies the condition $(*)$ if for all divisors

$$B = \sum_{j=1}^{r} \mu_j D_j \geq D$$

and all $j = 1 \ldots r$ one has an isomorphism of sheaves

$$Res_{D_j}(\nabla^B) = Res_{D_j}(\nabla) - \mu_j \cdot id_{D_j} : \mathcal{E} \mid_{D_j} \longrightarrow \mathcal{E} \mid_{D_j} .$$

b) We say that $(\nabla, \mathcal{E})$ satisfies the condition (!) if for all divisors

$$B = \sum_{j=1}^{r} -\nu_j D_j \leq 0$$

and all $j = 1, \ldots, r$

$$Res_{D_j}(\nabla^B) = Res_{D_j}(\nabla) + \nu_j \cdot \ id_{D_j} : \mathcal{E} \mid_{D_j} \longrightarrow \mathcal{E} \mid_{D_j}$$

is an isomorphism of sheaves.

In other words, $(*)$ means that no $\mu_j \in \mathbb{Z}$, $\mu_j \geq 1$, is an eigenvalue of $Res_{D_j}(\nabla)$ and (!) means the same for $\mu_j \in \mathbb{Z}$, $\mu_j \leq 0$. We will see later, that $(*)$ and (!) are only of interest if char $(k) = 0$.

2.9. Properties.
a) Assume that $(\mathcal{E}, \nabla)$ satisfies $(*)$ and that $B = \sum \mu_j D_j \geq 0$. Then the natural map

$$(\Omega^\bullet_X(\log\ D) \otimes \mathcal{E}, \nabla_\bullet) \longrightarrow (\Omega^\bullet_X(\log\ D) \otimes \mathcal{E}(B), \nabla^B_\bullet)$$

between the logarithmic de Rham complexes is a quasi-isomorphism.
b) Assume that $(\mathcal{E}, \nabla)$ satisfies (!) and that $B = \sum -\mu_j D_j \leq 0$. Then the natural map

$$(\Omega^\bullet_X(\log\ D) \otimes \mathcal{E}(B), \nabla^B_\bullet) \longrightarrow (\Omega^\bullet_X(\log\ D) \otimes \mathcal{E}, \nabla_\bullet)$$

between the logarithmic de Rham complexes is a quasi-isomorphism.

(2.9) follows from the definition of $(*)$ and (!) and from:

2.10. Lemma. *For $(\mathcal{E}, \nabla)$ as in (2.4) assume that*

$$Res_{D_1}(\nabla) : \mathcal{E} \mid_{D_1} \longrightarrow \mathcal{E} \mid_{D_1}$$

is an isomorphism. Then the inclusion of complexes

$$(\Omega^\bullet_X(\log\ D) \otimes \mathcal{E}(-D_1), \nabla^{-D_1}_\bullet) \longrightarrow (\Omega^\bullet_X(\log\ D) \otimes \mathcal{E}, \nabla_\bullet)$$

is an quasi-isomorphism.

PROOF: Consider the complexes $\mathcal{E}^{(\nu)}$:

$$\mathcal{E}(-D_1) \longrightarrow \Omega^1_X(\log\ D)\otimes\mathcal{E}(-D_1) \longrightarrow \ldots \longrightarrow \Omega^{\nu-1}_X(\log\ D)\otimes\mathcal{E}(-D_1) \longrightarrow$$
$$\longrightarrow \Omega^{\nu}_X(\log\ (D-D_1))\otimes\mathcal{E} \longrightarrow \Omega^{\nu+1}_X(\log\ D)\otimes\mathcal{E} \longrightarrow \ldots \longrightarrow \Omega^{n}_X(\log\ D)\otimes\mathcal{E}$$

We have an inclusion

$$\mathcal{E}^{(\nu+1)} \longrightarrow \mathcal{E}^{(\nu)}$$

and, by (2.6,b) the quotient is the complex

$$0 \longrightarrow \Omega^{\nu}_{D_1}(\log\,(D-D_1)|_{D_1})\otimes\mathcal{E} \xrightarrow{(-1)^{\nu}\otimes Res_{D_1}(\nabla)} \Omega^{\nu}_{D_1}(\log\,(D-D_1)|_{D_1})\otimes\mathcal{E} \longrightarrow 0$$

Since the quotient has no cohomology all the $\mathcal{E}^{(\nu)}$ are quasi-isomorphic, especially $\mathcal{E}^{(0)}$ and $\mathcal{E}^{(n)}$, as claimed.

□

2.11. The analytic case

At this point it might be helpful to consider the analytic case for a moment: $\mathcal{E}$ is a locally free sheaf over the sheaf of analytic functions $\mathcal{O}_X$,

$$\nabla : \mathcal{E} \longrightarrow \Omega^1_X(\log\ D)\otimes\mathcal{E}$$

is a holomorphic and integrable connection. Then $\ker(\nabla\,|_U) = V$ is a local constant system. If $(*)$ holds true, i.e. if the residues of ∇ along the D_j do not have strictly positive integers as eigenvalues, then (see [10], II, 3.13 and 3.14)

$$(\Omega^{\bullet}_X(\log\ D)\otimes\mathcal{E},\nabla_{\bullet})$$

is quasi-isomorphic to $R\tau_*V$. By Poincaré-Verdier duality (see [20], Appendix A) the natural map

$$\tau_!V^{\vee} \longrightarrow (\Omega^{\bullet}_X(\log\ D)\otimes\mathcal{E}^{\vee}(-D),\nabla^{\vee}_{\bullet})$$

is a quasi-isomorphism. Hence (!) implies that the natural map

$$\tau_!V \longrightarrow (\Omega^{\bullet}_X(\log\ D)\otimes\mathcal{E},\ \nabla_{\bullet})$$

is a quasi-isomorphism as well. In particular, topological properties of U give vanishing theorems for

$$\mathbb{H}^l(X,\Omega^{\bullet}_X(\log\ D)\otimes\mathcal{E})$$

and for some l. More precisely, if we choose $r(U)$ to be the smallest number that satisfies:

> For all local constant systems V on U one has $H^l(U,V) = 0$ for $l > n + r(U)$,

then one gets:

2.12. Corollary.
a) If $(\mathcal{E},\nabla)$ satisfies $(*)$, then for $l > n + r(U)$

$$\mathbb{H}^l(X,\Omega^{\bullet}_X(\log\ D)\otimes\mathcal{E}) = H^l(U,V) = 0.$$

b) If $(\mathcal{E},\nabla)$ satisfies (!), then for $l < n - r(U)$

$$\mathbb{H}^l(X,\Omega^{\bullet}_X(\log\ D)\otimes\mathcal{E}) = H^l_c(U,V) = 0.$$

By GAGA (see [56]), (2.12) remains true if we consider the complex of algebraic differential forms over the complex projective manifold X, even if the number $r(U)$ is defined in the analytic topology.

(2.12) is of special interest if both, $(*)$ and $(!)$, are satisfied, i.e. if none of the eigenvalues of $Res_{D_j}(\nabla)$ is an integer. Examples of such connections can be obtained, analytically or algebraically, by cyclic covers.

If U is affine (or a Stein manifold) one has $r(U) = 0$. For U affine there is no need to use GAGA and analytic arguments. Considering blowing ups and the Leray spectral sequence one can obtain (2.12) for algebraic sheaves from:

2.13. Corollary. *Let X be a projective manifold defined over the algebraically closed field k. Let B be an effective ample divisor, $D = B_{red}$ a normal crossing divisor and $(\mathcal{E}, \nabla)$ a logarithmic connection with poles along D (as in (2.4)).*
a) *If $(\mathcal{E}, \nabla)$ satisfies $(*)$, then for $l > n$*

$$\mathbb{H}^l(X, \Omega^\bullet_X(\log\ D) \otimes \mathcal{E}) = 0.$$

b) *If $(\mathcal{E}, \nabla)$ satisfies (!), then for $l < n$*

$$\mathbb{H}^l(X, \Omega^\bullet_X(\log\ D) \otimes \mathcal{E}) = 0.$$

PROOF: (2.9) allows to replace $\mathcal{E}$ by $\mathcal{E}(N \cdot B)$ in case a) or by $\mathcal{E}(-N \cdot B)$ in case b) for $N > 0$. By Serre's vanishing theorem (1.1) we can assume that

$$H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{E}) = 0$$

for $a + b = l$. The Hodge to de Rham spectral sequence (see (A.25)) implies (2.13).

□

§ 3 Integral parts of $\mathbb{Q}$-divisors and coverings

Over complex manifolds the Riemann Hilbert correspondence obtained by Deligne [10] is an equivalence between logarithmic connections $(\mathcal{E}, \nabla)$ and representations of the fundamental group $\pi_1(X - D)$. For applications in algebraic geometry the most simple representations, i.e. those who factor through cyclic quotient groups of $\pi_1(X - D)$, turn out to be useful. The induced invertible sheaves and connections can be constructed directly as summands of the structure sheaves of cyclic coverings. Those constructions remain valid for all algebraically closed fields.

Let X be an algebraic manifold defined over the algebraically closed field k.

3.1. Notation. a) Let us write $\mathrm{Div}(X)$ for the group of divisors on X and

$$\mathrm{Div}_{\mathbb{Q}}(X) = \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Hence a $\mathbb{Q}$-*divisor* $\Delta \in \mathrm{Div}_{\mathbb{Q}}(X)$ is a sum

$$\Delta = \sum_{j=1}^{r} \alpha_j D_j$$

of irreducible prime divisors D_j with coefficients $\alpha_j \in \mathbb{Q}$.
b) For $\Delta \in \mathrm{Div}_{\mathbb{Q}}(X)$ we write

$$[\Delta] = \sum_{j=1}^{r} [\alpha_j] \cdot D_j$$

where for $\alpha \in \mathbb{Q}$, $[\alpha]$ denotes the integral part of α, defined as the only integer such that

$$[\alpha] \leq \alpha < [\alpha] + 1.$$

$[\Delta]$ will be called the integral part of Δ.
c) For an invertible sheaf $\mathcal{L}$, an effective divisor

$$D = \sum_{j=1}^{r} \alpha_j Dj$$

and a positive natural number N, assume that $\mathcal{L}^N = \mathcal{O}_X(D)$. Then we will write for $i \in \mathbb{N}$

$$\mathcal{L}^{(i,D)} = \mathcal{L}^i(-[\frac{i}{N}D]) = \mathcal{L}^i \otimes \mathcal{O}_X(-[\frac{i}{N} \cdot D]).$$

Usually N and D will be fixed and we just write $\mathcal{L}^{(i)}$ instead of $\mathcal{L}^{(i,D)}$.
d) If

$$D = \sum_{j=1}^{r} \alpha_j D_j$$

is a normal crossing divisor, we will write, for simplictiy,

$$\Omega_X^a(\log\ D) \text{ instead of } \Omega_X^a(\log\ D_{red}).$$

In spite of their strange definition the sheaves $\mathcal{L}^{(i)}$ will turn out to be related to cyclic covers in a quite natural way. We will need this to prove:

3.2. Theorem. *Let X be a projective manifold,*

$$D = \sum_{j=1}^{r} \alpha_j D_j$$

be an effective normal crossing divisor, $\mathcal{L}$ an invertible sheaf and $N \in \mathbb{N} - \{0\}$ prime to $\mathrm{char}(k)$*, such that $\mathcal{L}^N = \mathcal{O}_X(D)$. Then for $i = 0, \ldots, N-1$ the sheaf ${\mathcal{L}^{(i)}}^{-1}$ has an integrable logarithmic connection*

$$\nabla^{(i)} : {\mathcal{L}^{(i)}}^{-1} \longrightarrow \Omega^1_X(\log\ D^{(i)}) \otimes {\mathcal{L}^{(i)}}^{-1}$$

$$\textit{with poles along} \quad D^{(i)} = \sum_{\substack{j=1 \\ \frac{i \cdot \alpha_j}{N} \notin \mathbb{Z}}}^{r} D_j,$$

satisfying:
a) *The residue of $\nabla^{(i)}$ along D_j is given by multiplication with*

$$(i \cdot \alpha_j - N \cdot [\frac{i \cdot \alpha_j}{N}]) \cdot N^{-1} \ \in k.$$

b) *Assume that either* $\mathrm{char}(k) = 0$*, or, if* $\mathrm{char}(k) = p \neq 0$*, that X and D admit a lifting to $W_2(k)$ (see (8.11)) and that $p \geq \dim\ X$. Then the spectral sequence*

$$E_1^{ab} = H^b(X, \Omega^a_X(\log\ D^{(i)}) \otimes {\mathcal{L}^{(i)}}^{-1}) \Longrightarrow \mathbb{H}^{a+b}(X, \Omega^\bullet_X(\log\ D^{(i)}) \otimes {\mathcal{L}^{(i)}}^{-1})$$

associated to the logarithmic de Rham complex

$$(\Omega^\bullet_X(\log\ D^{(i)}) \otimes {\mathcal{L}^{(i)}}^{-1}, \nabla^{(i)}_\bullet)$$

degenerates in E_1.
c) Let A and B be reduced divisors (both having the lifting property (8.11) if $\mathrm{char}(k) = p \neq 0$*) such that B, A and $D^{(i)}$ have pairwise no commom components and such that $A + B + D^{(i)}$ is a normal crossing divisor. Then $\nabla^{(i)}$ induces a logarithmic connection*

$$\mathcal{O}_X(-B) \otimes {\mathcal{L}^{(i)}}^{-1} \longrightarrow \Omega^1_X(\log\ (A + B + D^{(i)}))(-B) \otimes {\mathcal{L}^{(i)}}^{-1}$$

and under the assumptions of b) the spectral sequence

$$E_1^{ab} = H^b(X, \Omega^a_X(\log\ (A + B + D^{(i)}))(-B) \otimes {\mathcal{L}^{(i)}}^{-1}) \Longrightarrow$$
$$\mathbb{H}^{a+b}(X, \Omega^\bullet_X(\log\ (A + B + D^{(i)}))(-B) \otimes {\mathcal{L}^{(i)}}^{-1})$$

degenerates in E_1 as well.

3.3. Remarks. a) In (3.2), whenever one likes, one can assume that $i = 1$. In fact, one just has to replace $\mathcal{L}$ by $\mathcal{L}' = \mathcal{L}^i$ and D by $D' = i \cdot D$ .Then

$$\mathcal{L}'^N = \mathcal{O}_X(i \cdot D) = \mathcal{O}_X(D')$$

and

$$\mathcal{L}'^{(1,D')} = \mathcal{L}'(-[\frac{D'}{N}]) = \mathcal{L}^i(-[\frac{i}{N}D]).$$

b) Next, one can always assume that $0 < \alpha_j < N$. In fact, if $\alpha_1 \geq N$, then

$$\mathcal{L}' = \mathcal{L}(-D_1) \quad \text{and} \quad D' = D - N \cdot D_1$$

give the same sheaves as $\mathcal{L}$ and D:

$$\mathcal{L}'^{(i,D')} = \mathcal{L}^i(-i \cdot D_1 - [\frac{i}{N} \cdot D']) = \mathcal{L}^i(-[\frac{i}{N} \cdot D]).$$

c) In particular, for $i = 1$ and $0 < a_j < N$ we have

$$\mathcal{L}^{(1)} = \mathcal{L} \quad \text{and} \quad D^{(1)} = D.$$

Nevertheless, in the proof of (3.2) we stay with the notation, as started.
d) Finally, for $i \geq N$ one has

$$\mathcal{L}^{(i,D)} = \mathcal{L}^i(-[\frac{i}{N} \cdot D]) = \mathcal{L}^{i-N}(-[\frac{i-N}{N} \cdot D]) = \mathcal{L}^{(i-N,D)}.$$

The "$\mathcal{L}^{(i)}$" are the most natural notation for "integral parts of $\mathbb{Q}$- divisors" if one wants to underline their relations with coverings. In the literature one finds other equivalent notations, more adapted to the applications one has in mind:

3.4. Remarks.
a) Sometimes the integral part $[\Delta]$ is denoted by $\lfloor \Delta \rfloor$.
b) One can also consider the round up $\{\Delta\} = \lceil \Delta \rceil$ given by

$$\{\Delta\} = -[-\Delta]$$

or the fractional part of Δ given by

$$< \Delta >= \Delta - [\Delta].$$

c) For $\mathcal{L}$, N and D as in (3.1,c) one can write

$$\mathcal{L} = \mathcal{O}_X(C)$$

for some divisor C. Then

$$\Delta = C - \frac{1}{N} \cdot D \;\in \mathrm{Div}_{\mathbb{Q}}(X)$$

has the property that $N \cdot \Delta$ is a divisor linear equivalent to zero. One has

$$\mathcal{L}^{(i,D)} = \mathcal{O}_X(i \cdot C - [\frac{i}{N} \cdot D]) = \mathcal{O}_X(-[-i \cdot \Delta]) = \mathcal{O}_X(\{i \cdot \Delta\}).$$

d) On the other hand, for $\Delta \in \mathrm{Div}_Q(X)$ and $N > 0$ assume that $N \cdot \Delta$ is a divisor linear equivalent to zero. Then one can choose a divisor C such that $C - \Delta$ is effective. For $\mathcal{L} = \mathcal{O}_X(C)$ and $D = N \cdot C - N \cdot \Delta \;\in\; \mathrm{Div}(X)$ one has

$$\mathcal{L}^N = \mathcal{O}_X(N \cdot C) = \mathcal{O}_X(D)$$

and

$$\mathcal{L}^{(i,D)} = \mathcal{O}_X(i \cdot C - [\frac{i}{N}D]) =$$

$$\mathcal{O}_X(-[-i \cdot C + \frac{i}{N} \cdot D]) = \mathcal{O}_X(\{i \cdot \Delta\}).$$

e) Altogether, (3.2) is equivalent to:

For $\Delta \in \mathrm{Div}_{\mathbb{Q}}(X)$ such that $N \cdot \Delta$ is a divisor linear equivalent to zero, assume that $< \Delta >$ is supported in D and that D is a normal crossing divisor. Then $\mathcal{O}_X(\{\Delta\})$ has a logarithmic integrable connection with poles along D which satisfies a residue condition similar to (3.2,a) and the E_1-degeneration.

We leave the exact formulation and the translation as an exercise.

3.5. Cyclic covers. Let $\mathcal{L}, N$ and

$$D = \sum_{j=1}^{r} \alpha_j D_j$$

be as in (3.1,c) and let $s \in H^0(X, \mathcal{L}^N)$ be a section whose zero divisor is D. The dual of

$$s : \mathcal{O}_X \longrightarrow \mathcal{L}^N \text{ , i.e. } s^\vee : \mathcal{L}^{-N} \longrightarrow \mathcal{O}_X,$$

defines a $\mathcal{O}_X$-algebra structure on

$$\mathcal{A}' = \bigoplus_{i=0}^{N-1} \mathcal{L}^{-i}.$$

In fact,

$$\mathcal{A}' = \bigoplus_{i=0}^{\infty} \mathcal{L}^{-i}/I$$

where I is the ideal-sheaf generated locally by

$$\{s^\vee(l) - l, \ l \text{ local section of } \mathcal{L}^{-N}\}.$$

Let

$$Y' = \mathbf{Spec}_X(\mathcal{A}') \xrightarrow{\pi'} X$$

be the spectrum of the $\mathcal{O}_X$-algebra $\mathcal{A}'$, as defined in [30], page 128, for example.
Let $\pi : Y \to X$ be the finite morphism obtained by normalizing $Y' \to X$. To be more precise, if Y' is reducible, Y will be the disjoint union of the normalizations of the components of Y' in their function fields. We will call Y *the cyclic cover obtained by taking the n-th root out of s* (or *out of D*, if $\mathcal{L}$ is fixed).

Obviously one has:

3.6. Claim. Y is uniquely determined by:
a) $\pi : Y \to X$ is finite.
b) Y is normal.
c) There is a morphism $\phi : \mathcal{A}' \to \pi_* \mathcal{O}_Y$ of $\mathcal{O}_X$-algebras, isomorphic over some dense open subscheme of X.

3.7. Notations. For D, N and $\mathcal{L}$ as in (3.1,c) let us write

$$\mathcal{A} = \bigoplus_{i=0}^{N-1} \mathcal{L}^{(i)^{-1}}.$$

The inclusion

$$\mathcal{L}^{-i} \longrightarrow \mathcal{L}^{(i)^{-1}} = \mathcal{L}^{-i}([\frac{i}{N} \cdot D])$$

gives a morphism of $\mathcal{O}_X$-modules

$$\phi : \mathcal{A}' \longrightarrow \mathcal{A}.$$

3.8. Claim. $\mathcal{A}$ has a structure of an $\mathcal{O}_X$-algebra, such that ϕ is a homomorphism of algebras.

Proof: The multiplication in $\mathcal{A}'$ is nothing but the multiplication

$$\mathcal{L}^{-i} \times \mathcal{L}^{-j} \longrightarrow \mathcal{L}^{-i-j}$$

$$\text{composed with} \quad s^{\vee} : \mathcal{L}^{-i-j} \longrightarrow \mathcal{L}^{-i-j+N} \ ,$$

in case that $i + j \geq N$. For $i, j \geq 0$ one has

$$[\frac{i}{N} \cdot D] + [\frac{j}{N} \cdot D] \leq [\frac{i+j}{N} \cdot D]$$

and, for $i + j \geq N$, one has

$$\mathcal{L}^{(i+j)} = \mathcal{L}^{i+j}(-[\frac{i+j}{N} \cdot D]) = \mathcal{L}^{i+j-N}(-[\frac{i+j-N}{N} \cdot D]) = \mathcal{L}^{(i+j-N)}.$$

This implies that the multiplication of sections

$$\mathcal{L}^{(i)^{-1}} \times \mathcal{L}^{(j)^{-1}} \longrightarrow \mathcal{L}^{-i-j}([\frac{i}{N} D] + [\frac{j}{N} D]) \longrightarrow \mathcal{L}^{(i+j)^{-1}}$$

is well defined, and that for $i + j \geq N$ the right hand side is nothing but $\mathcal{L}^{(i+j-N)^{-1}}$.

□

3.9.

Assume that N is prime to char(k), e a fixed primitive N-th root of unit and $G =< \sigma >$ the cyclic group of order N. Then G acts on $\mathcal{A}$ by $\mathcal{O}_X$-algebra homomorphisms defined by:

$$\sigma(l) = e^i \cdot l \quad \text{for a local section} \quad l \quad \text{of} \quad \mathcal{L}^{(i)^{-1}} \subset \mathcal{A}.$$

Obviously the invariants under this G-action are

$$\mathcal{A}^G = \mathcal{O}_X.$$

3.10. Claim. Assume that N is prime to char(k). Then

$$\mathcal{A} = \pi_* \mathcal{O}_Y \quad \text{or (equivalently)} \quad Y = \textbf{Spec}(\mathcal{A}) \ .$$

3.11. Corollary (see [16]). *The cyclic group G acts on Y and on $\pi_*\mathcal{O}_Y$. One has $Y/G = X$ and the decomposition*

$$\pi_* \mathcal{O}_Y = \bigoplus_{i=0}^{N-1} \mathcal{L}^{(i)^{-1}}$$

is the decomposition in eigenspaces.

PROOF OF 3.10.: For any open subvariety X_0 in X with $\operatorname{codim}_X(X - X_0) \geq 2$ and for $Y_0 = \pi^{-1}(X_0)$ consider the induced morphisms

$$\begin{array}{ccc} Y_0 & \xrightarrow{\iota'} & Y \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi} \\ X_0 & \xrightarrow{\iota} & X \end{array}$$

Since Y is normal one has $\iota'_* \mathcal{O}_{Y_0} = \mathcal{O}_Y$ and $\pi_* \mathcal{O}_Y = \iota_* \pi_{0_*} \mathcal{O}_{Y_0}$. Since $\mathcal{A}$ is locally free, (3.10) follows from

$$\pi_{0_*} \mathcal{O}_{Y_0} = \mathcal{A}|_{X_0}.$$

Especially we may choose $X_0 = X - \operatorname{Sing}(D_{red})$ and, by abuse of notations, assume from now on that D_{red} is non-singular.

As remarked in (3.6) the equality of $\mathcal{A}$ and $\pi_* \mathcal{O}_Y$ follows from:

3.12. Claim. **Spec** $(\mathcal{A}) \longrightarrow X$ is finite and **Spec**$(\mathcal{A})$ is normal.

PROOF: (3.12) is a local statement and to prove it we may assume that $X = \operatorname{Spec} B$ and that D consists of just one component, say $D = \alpha_1 \cdot D_1$. Let us fix isomorphisms $\mathcal{L}^i \simeq \mathcal{O}_X$ for all i and assume that D_1 is the zero set of $f_1 \in B$. For some unit $u \in B^*$ the section $s \in H^0(X, \mathcal{L}^N) \simeq B$ is identified with $f = u \cdot f_1^{\alpha_1}$. For completeness, we allow D (or α_1) to be zero.
The $\mathcal{O}_X$-algebra $\mathcal{A}'$ is given by the B-algebra

$$H^0(X, \mathcal{A}') = \bigoplus_{i=0}^{N-1} H^0(X, \mathcal{L}^{-i})$$

which can be identified with the quotient of the ring of polynomials

$$A' = B[t]/_{t^N - f} = \bigoplus_{i=0}^{N-1} B \cdot t^i.$$

In this language

$$A = \bigoplus_{i=0}^{N-1} B \cdot t^i \cdot f_1^{-[\frac{i}{N}\alpha_1]} = \bigoplus_{i=0}^{N-1} H^0(X, \mathcal{L}^{(i)^{-1}}) = H^0(X, \mathcal{A})$$

and $\phi : \mathcal{A}' \to \mathcal{A}$ induces the natural inclusion $A' \hookrightarrow A$.

Hence (3.12) follows from the first part of the following claim.

□

3.13. Claim. Using the notations introduced above, assume that N is prime to char(k). Then one has
a) Spec A is non-singular and π : Spec $A \longrightarrow$ Spec B is finite.
b) If $\alpha_1 = 0$, then Spec $A \longrightarrow$ Spec B is non-ramified (hence étale).
c) if α_1 is prime to N, we have a defining equation $g \in A$ for $\Delta_1 = (\pi^* D_1)_{red}$ with

$$g^N = u^a \cdot f_1 \quad \text{for some} \quad a \in \mathbb{N}.$$

d) If Γ is a divisor in Spec B such that $D + \Gamma$ has normal crossings, then $\pi^*(D + \Gamma)$ has normal crossings as well.

Proof: Let us first consider the case $\alpha_1 = 0$. Then

$$A' = A = B[t]/_{t^N - u}$$

for $u \in B^*$. A is non-singular, as follows, for example, from the Jacobi-criterion, and A is unramified over B. Hence

$$\text{Spec } A \longrightarrow \text{Spec } B$$

is étale in this case and a), b) and d) are obvious.

If $\alpha_1 = 1$, then again

$$A' = A = B[t]/_{t^N - u \cdot f_1}.$$

For $p \in$ Spec B, choose $f_2, \ldots, f_n$ such that $f_1 \cdot u, f_2, \ldots, f_n$ is a local parameter-system in p. Then $t, f_2, \ldots, f_n$ will be a local parameter system, for $q = \pi^{-1}(p)$.

Similar, if α_1 is prime to N, and if c) holds true, g and $f_2, \ldots, f_n$ will be a local parameter system in q and, composing both steps, Spec A will always be non-singular and d) holds true.

Let us consider the ring

$$R = B[t_0, t_1]/_{t_0^N - u, t_1^N - f_1}.$$

Identifying t with $t_0 \cdot t_1^{\alpha_1}$ we obtain A' as a subring of R. Spec R is non-singular over p and Spec $R \longrightarrow$ Spec B is finite.

The group $H = <\sigma_0> \oplus <\sigma_1>$ with $ord\ (\sigma_0) = ord\ (\sigma_1) = N$ operates on R by

$$\sigma_\nu(t_\mu) = \begin{cases} t_\mu & \text{if } \nu \neq \mu \\ e \cdot t_\mu & \text{if } \nu = \mu \end{cases}$$

Let H' be the kernel of the map $\gamma : H \longrightarrow G = <\sigma>$ given by $\gamma(\sigma_0) = \sigma$ and $\gamma(\sigma_1) = \sigma^{\alpha_1}$. The quotient

$$\text{Spec } (R)/_{H'} = \text{Spec } R^{H'}$$

is normal and finite over Spec B.
One has $(\sigma_0^\mu, \sigma_1^\nu) \in H'$, if and only if $\mu + \nu\alpha_1 \equiv 0 \bmod N$. Hence $R^{H'}$ is generated by monomials $t_0^a \cdot t_1^b$ where $a, b \in \{0, \ldots, N-1\}$ satisfy:

$(*) \qquad a\mu + b\nu \equiv 0 \bmod N$ for all (μ, ν) with $\mu + \alpha_1\nu \equiv 0 \bmod N$.

Obviously, $(*)$ holds true for (a, b) if $b \equiv a \cdot \alpha_1 \bmod N$. On the other hand, choosing ν to be a unit in $\mathbb{Z}/_N$, $(*)$ implies that $b \equiv a \cdot \alpha_1 \bmod N$.

Hence, for all (a, b) satisfying $(*)$ we find some k with $b = a \cdot \alpha_1 + k \cdot N$. Since $a, b \in \{0, \ldots, N-1\}$ we have

$$\frac{a \cdot \alpha_1}{N} \geq -k = \frac{a \cdot \alpha_1}{N} - \frac{b}{N} > \frac{a \cdot \alpha_1}{N} - 1$$

or $k = -[\frac{a \cdot \alpha_1}{N}]$.

Therefore one obtains

$$R^{H'} = \bigoplus_{a=0}^{N-1} t_0^a \cdot t_1^{a\cdot\alpha_1 - N\cdot[\frac{a\cdot\alpha_1}{N}]} \cdot B = \bigoplus_{a=0}^{N-1} (t_0 \cdot t_1^{\alpha_1})^a \cdot f_1^{-[\frac{a\cdot\alpha_1}{N}]} \cdot B$$

$$\text{and hence} \qquad R^{H'} = \bigoplus_{a=0}^{N-1} t^a \cdot f_1^{-[\frac{a\cdot\alpha_1}{N}]} \cdot B = A.$$

If α_1 is prime to N, we can find $a \in \{0, \ldots, N-1\}$ with $a \cdot \alpha_1 = 1 + l \cdot N$ for $l \in \mathbb{Z}$. Then

$$a \cdot \alpha_1 - N \cdot [\frac{a \cdot \alpha_1}{N}] = 1$$

and $g = t^a \cdot f_1^{-[\frac{a \cdot \alpha_1}{N}]}$ satisfies

$$g^N = u^a \cdot f_1^{a \cdot \alpha_1 - N[\frac{a \cdot \alpha_1}{N}]} = u^a \cdot f_1.$$

$\square$

3.14. Remarks.
a) If Y is irreducible, for example if D is reduced, the local calculation shows Y is nothing but the normalisation of X in $k(X)(\sqrt[N]{f})$, where f is a rational function giving the section s.
b) $\pi' : Y' \longrightarrow X$ can be as well described in the following way (see [30], p. 128-129):
Let $\mathbf{V}(\mathcal{L}^{-\alpha}) = \mathrm{Spec}\ (\bigoplus_{i=0}^{\infty} \mathcal{L}^{-\alpha})$ be the geometric rank one vector bundle associated to $\mathcal{L}^{-\alpha}$. The geometric sections of $\mathbf{V}(\mathcal{L}^{-\alpha}) \longrightarrow X$ correspond to $H^0(X, \mathcal{L}^{\alpha})$. Hence s gives a section σ of $\mathbf{V}(\mathcal{L}^{-N})$ over X. We have a natural map

$$\tau : \mathbf{V}(\mathcal{L}^{-1}) \longrightarrow \mathbf{V}(\mathcal{L}^{-N})$$

and $Y' = \tau^{-1}(\sigma(X))$.

The local computation in (3.13) gives a little bit more information than asked for in (3.12):

3.15. Lemma. *Keeping the notations and assumptions from (3.5) assume that N is prime to $\mathrm{char}(k)$. Then one has*
a) Y is reducible, if and only if for some $\mu > 1$, dividing N, there is a section s' in $H^0(X, \mathcal{L}^{\frac{N}{\mu}})$ with $s = s'^{\mu}$.
b) $\pi : Y \to X$ is étale over $X - D_{red}$ and Y is non-singular over $X - \mathrm{Sing}(D_{red})$.
c) For $\Delta_j = (\pi^* D_j)_{red}$ we have

$$\pi^* D = \sum_{j=1}^{r} \frac{N \cdot \alpha_j}{gcd(N, \alpha_j)} \cdot \Delta_j.$$

d) If Y is irreducible then the components of Δ_j have over D_j the ramification index

$$e_j = \frac{N}{gcd(N, \alpha_j)}.$$

PROOF: For a) we can consider the open set $\mathrm{Spec}\ B \subset X - D_{red}$. Hence $\mathrm{Spec}\ B[t]/_{t^N - u}$ is in Y dense and open. Y is reducible if and only if $t^N - u$ is reducible in $B[t]$, which is equivalent to the existence of some $u' \in B$ with $u = u'^{\mu}$.
b) has been obtained in (3.13) part a) and b).
For c) and d) we may assume that $D = \alpha_1 \cdot D_1$ and, splitting the covering in

two steps, that either N divides α_1 or that N is prime to α_1.
In the first case, we can as well choose α_1 to be zero (by 3.3,b) and c) as well as d) follow from part b).
If α_1 is prime to N,then $\pi^*D = e_1 \cdot \alpha_1 \cdot \Delta_1$. Since π^*D is the zero locus of $f = t^N$, N divides $e_1 \cdot \alpha_1$. On the other hand, since e_1 divides deg $(Y/X) = N$, one has $e_1 = N$ in this case.

□

3.16. Lemma. *Keeping the notations from (3.5) assume that N is prime to* char(k) *and that D_{red} is non-singular. Then one has:*
a) (Hurwitz's formula) $\pi^*\Omega^b_X(\log\ D) = \Omega^b_Y(\log\ (\pi^*D))$.
b) The differential d on Y induces a logarithmic integrable connection

$$\pi_*(d) : \bigoplus_{i=0}^{N-1} \mathcal{L}^{(i)^{-1}} \longrightarrow \pi_*\Omega^1_Y(\log\ (\pi^*D)) = \bigoplus_{i=0}^{N-1} \Omega^1_X(\log\ D) \otimes \mathcal{L}^{(i)^{-1}},$$

compatible with the direct sum decomposition.
c) If $\nabla^{(i)} : \mathcal{L}^{(i)^{-1}} \longrightarrow \Omega^1_X(\log\ D) \otimes \mathcal{L}^{(i)^{-1}}$ *denotes the i-th component of* $\pi_*(d)$ *then* $\nabla^{(i)}$ *is a logarithmic integrable connection with residue*

$$Res_{D_j}(\nabla^{(i)}) = (\frac{i \cdot \alpha_j}{N} - [\frac{i \cdot \alpha_j}{N}]) \cdot id_{\mathcal{O}_{D_j}}.$$

d) One has

$$\pi_*(\Omega^b_Y) = \bigoplus_{i=0}^{N-1} \Omega^b_X(\log\ D^{(i)}) \otimes \mathcal{L}^{(i)^{-1}} \quad for \quad D^{(i)} = \sum_{\substack{j=1 \\ \frac{i \cdot \alpha_j}{N} \in \mathbb{Q} - \mathbb{Z}}}^{r} D_j.$$

e) The differential

$$\pi_*(d) : \pi_*\mathcal{O}_Y = \bigoplus_{i=0}^{N-1} \mathcal{L}^{(i)^{-1}} \longrightarrow \pi_*(\Omega^1_Y) = \bigoplus_{i=0}^{N-1} \Omega^1_X(\log\ D^{(i)}) \otimes \mathcal{L}^{(i)^{-1}}$$

decomposes into a direct sum of

$$\nabla^{(i)} : \mathcal{L}^{(i)^{-1}} \longrightarrow \Omega^1_X(\log\ D^{(i)}) \otimes \mathcal{L}^{(i)^{-1}}.$$

PROOF: Again we can argue locally and assume that $X =$ Spec B and $D = \alpha_1 D_1$ as in (3.12).
If $\alpha_1 = 0$, or if N divides α_1, then f_1 is a defining equation for $\Delta_1 = (\pi^*D_1)_{red}$ and the generators for $\Omega^b_X(\log\ D)$ are generators for $\Omega^b_Y(\log\ \pi^*D)$ as well.
For α_1 prime to N, we have by (3.13,c) a defining equation g for $\Delta_1 = (\pi^*D_1)_{red}$ with $g^N = u^a \cdot f_1$. Hence

$$N \cdot \frac{dg}{g} = \frac{df_1}{f_1} + a \cdot \frac{du}{u}$$

and, since $N \in k^*$ and $a \cdot \frac{du}{u} \in \Omega^1_X$, one finds that $\frac{df_1}{f_1}$ and $\pi^*\Omega^1_X$ generate $\Omega^1_Y(\log\ \pi^*D)$.

We can split π in two coverings of degree $N \cdot \gcd(N, \alpha_1)^{-1}$ and $\gcd(N, \alpha_1)$. Hence we obtain a) for $b = 1$. The general case follows.

The group G acts on $\pi_*\Omega^b_Y$ and $\pi_*\Omega^b_Y(\log\ \pi^*D))$ compatibly with the inclusion, and the action on the second sheaf is given by $id \otimes \sigma$ if one writes

$$\pi_*\Omega^b_Y(\log\ (\pi^*D)) = \Omega^b_X(\log\ D) \otimes \pi_*\mathcal{O}_Y.$$

Let l be a local section of $\Omega^b_X(\log\ D) \otimes \mathcal{L}^{(i)^{-1}}$ written as

$$l = \phi \cdot g_i \quad \text{for} \quad \phi \in \Omega^b_X(\log\ D) \quad \text{and} \quad g_i = t^i \cdot f_1^{-[\frac{i \cdot \alpha_1}{N}]}.$$

Since

$$g_i^N = u^i \cdot f_1^{i \cdot \alpha_1 - N \cdot [\frac{i\alpha_1}{N}]}$$

has a zero along Δ_1 if and only if

$$\frac{i \cdot \alpha_1}{N} \notin \mathbb{Z},$$

we find that l lies in Ω^b_Y in this case.
On the other hand, if g_i is a unit, l lies in Ω^b_Y if and only if ϕ has no pole along D and we obtain d).

We have

$$N\frac{dg_i}{g_i} = i \cdot \frac{du}{u} + (i \cdot \alpha_1 - N[\frac{i \cdot \alpha_1}{N}])\frac{df_1}{f_1}$$

or

$$dg_i = (\frac{i}{N}\frac{du}{u} + (\frac{i}{N}\alpha_1 - [\frac{i \cdot \alpha_1}{N}])\frac{df_1}{f_1}) \cdot g_i.$$

Hence,

$$d(g_i \cdot \phi) \in \Omega^{b+1}_X(\log\ D) \otimes \mathcal{L}^{(i)^{-1}},$$

and (π_*d) respects the direct sum decomposition. Obviously, the Leibniz rule for d implies that (π_*d) as well as the components $\nabla^{(i)}$ are connections and b) and e) hold true.

Finally, for c), let $\phi \in \mathcal{O}_X$. Then by the calculations given above, we find

$$Res_{D_1}(\nabla^{(i)})(g_i \cdot \phi) = (\frac{i}{N}\alpha_1 - [\frac{i \cdot \alpha_1}{N}])g_i \cdot \phi\,|_{D_1}\ .$$

□

PROOF OF 3.2,A:
If X is projective and

$$D = \sum_{j=1}^{r} \alpha_j D_j$$

a normal crossing divisor we found the connection

$$\nabla^{(i)} : \mathcal{L}^{(i)^{-1}} \longrightarrow \Omega^1_X(\log\ D^{(i)}) \otimes \mathcal{L}^{(i)^{-1}}$$

with the residues as given in (3.2,a) over the open submanifold $X - \mathrm{Sing}(D_{red})$. Of course, $\nabla^{(i)}$ extends to X since

$$\mathrm{codim}_X(\mathrm{Sing}(D_{red})) \geq 2.$$

□

Over a field k of characteristic zero, to prove the E_1-degeneration, as stated in (3.2,b) or (3.2,c) one can apply the degeneration of the logarithmic Hodge to de Rham spectral sequence (see (10.23) for example) to some desingularization of Y. We will sketch this approach in (3.22). One can as well reduce (3.2,b) to the more familiar degeneration of the Hodge spectral sequence

$$E_1^{ab} = H^b(T, \Omega^a_T) \Longrightarrow \mathbb{H}^{a+b}(T, \Omega^\bullet_T)$$

for projective manifolds T by using the following covering Lemma, due to Y. Kawamata [35]:

3.17. Lemma. *Keeping the notations from (3.5) assume that N is prime to* $\mathrm{char}(k)$ *and that D is a normal crossing divisor. Then there exists a manifold T and a finite morphism*

$$\delta : T \longrightarrow Y$$

such that:
a) The degree of δ divides a power of N.
b) If A and B are reduced divisors such that $D + A + B$ has at most normal crossings and if $A + B$ has no common component with D, then we can choose T such that $(\pi \circ \delta)^*(D + A + B)$ is a normal crossing divisor and $(\pi \circ \delta)^*A$ as well as $(\pi \circ \delta)^*B$ are reduced.

PROOF OF (3.2) IN CHARACTERISTIC ZERO, ASSUMING THE E_1 DEGENERATION OF THE HODGE TO DE RHAM SPECTRAL SEQUENCE:
Let $X_0 = X - \mathrm{Sing}(D_{red})$, $Y_0 = \pi^{-1}(X_0)$ and $T_0 = \delta^{-1}(Y_0)$.
$\delta_*\Omega^\bullet_{T_0}$ contains $\Omega^\bullet_{Y_0}$ as direct summand. Since $(\pi \circ \delta)$ is flat $(\pi \circ \delta)_*\Omega^\bullet_T$ will contain

$$\bigoplus_{i=0}^{N-1} \Omega^\bullet_X(\log\ D^{(i)}) \otimes \mathcal{L}^{(i)^{-1}}$$

as a direct summand. The E_1-degeneration of the spectral sequence

$$E_1^{ab} = H^b(T, \Omega^a_T) \Longrightarrow \mathbb{H}^{a+b}(T, \Omega^\bullet_T)$$

implies (3.2,b) for each $i \in \{0, \ldots, N-1\}$. Finally, if A and B are the divisors considered in (3.2,c), $A' = (\pi \circ \delta)^* A$ and $B' = (\pi \circ \delta)^* B$,

$$\Omega^\bullet_X(\log\ (A + B + D^{(i)}))(-B) \otimes \mathcal{L}^{(i)^{-1}}$$

is a direct summand of

$$(\pi \circ \delta)_* \Omega^\bullet_T(\log\ (A' + B'))(-B')$$

and we can use the E_1-degeneration of

$$E_1^{ab} = H^b(T, \Omega^a_T(\log\ (A' + B'))(-B')) \Longrightarrow \mathbb{H}^{a+b}(T, \Omega^\bullet_T(\log\ (A' + B'))(-B')).$$

□

3.18. Remarks.
a) If $A = B = 0$ the degeneration of the spectral sequence, used to get (3.2,b), follows from classical Hodge theory. In general, i.e. for (3.2,c), one has to use the Hodge theory for open manifolds developed by Deligne [11].
In these lectures (see (10.23)) we will reproduce the algebraic proof of Deligne and Illusie for the degeneration.
b) If char $(k) \neq 0$ and if $X, \mathcal{L}$ and D admit a lifting to $W_2(k)$ (see (8.11)), then the manifold T constructed in (3.17) will again admit a lifting to $W_2(k)$. Hence the proof of (3.2,b and c) given above shows as well:

Assuming the degeneration of the Hodge to de Rham spectral sequence (proved in (10.21)) theorem (3.2) holds true under the additional assumption that $\mathcal{L}$ lifts to $W_2(k)$ as well.

c) Using (3.2,a) we will give a direct proof of (3.2,b and c) at the end of §10, without using (3.17), for a field k of characteristic $p \neq 0$. By reduction to characteristic p one obtains a second proof of (3.2) in characteristic zero.
d) In Lectures 4 - 7, we will assume (3.2) to hold true.

To prove (3.17) we need:

3.19. Lemma (Kawamata [35]). *Let X be a quasi-projective manifold, let*

$$D = \sum_{j=1}^{r} D_j$$

be a reduced normal crossing divisor, and let

$$N_1, \ldots, N_r \ \in \mathbb{N} - \{0\}$$

be prime to char(k). *Then there exists a projective manifold* Z *and a finite morphism* $\tau : Z \to X$ *such that:*
a) For $j = 1, \ldots r$ *one has* $\tau^* D_j = N_j \cdot (\tau^* D_j)_{red}$.
b) $\tau^*(D)$ *is a normal crossing divisor.*
c) The degree of τ *divides some power of* $\prod_{j=1}^r N_j$.
d) If X *and* D *satisfy the lifting property (8.11) the same holds true for* Z.

PROOF: If we replace the condition that $D = \sum_{j=1}^r D_j$ is the decomposition of D into irreducible (non-singular) components by the condition that $D = \sum_{j=1}^r D_j$ for non-singular divisors $D_1 \ldots D_r$ we can construct Z by induction and hence assume that $N_1 = N$ and $N_2 = \ldots = N_r = 1$.

Let $\mathcal{A}$ be an ample invertible sheaf such that $\mathcal{A}^N(-D_1)$ is generated by its global sections. Choose $n = \dim\ X$ general divisors $H_1, \ldots, H_n$ with

$$\mathcal{O}_X(H_i) = \mathcal{A}^N(-D_1).$$

The divisor $D + \sum_{i=1}^n H_i$ will be a reduced normal crossing divisor. Let

$$\tau_i : Z_i \longrightarrow X$$

be the cyclic cover obtained by taking the N-th root out of $H_i + D_1$. Then Z_i satisfies the properties a), c) and d) asked for in (3.19) but, Z_i might have singularities over $H_i \cap D_1$ and $\tau_i^*(D)$ might have non-normal crossings over $H_i \cap D_1$. Let Z be the normalization of

$$Z_1 \times_X Z_2 \times_X \ldots \times_X Z_n.$$

Z can inductively be constructed as well in the following way:
Let $Z^{(\nu)}$ be the normalization of $Z_1 \times_X \ldots \times_X Z_\nu$ and $\tau^{(\nu)} : Z^{(\nu)} \to X$ the induced morphism. Then, outside of the singular locus of $Z^{(\nu)}$, the cover $Z^{(\nu+1)}$ is obtained from $Z^{(\nu)}$ by taking the N-th root out of

$${\tau^{(\nu)}}^*(H_{\nu+1} + D_1) = {\tau^{(\nu)}}^*(H_{\nu+1}) + N \cdot ({\tau^{(\nu)}}^* D_1)_{red}.$$

This is the same as taking the N-th root out of ${\tau^{(\nu)}}^*(H_{\nu+1})$ by (3.2,b) and (3.10). Since this divisor has no singularities, we find by (3.15,b) that the singularities of $Z^{(\nu+1)}$ lie over the singularities of $Z^{(\nu)}$, hence inductively over $H_1 \cap D_1$. However, as Z is independent of the numbering of the H_i, the singularities of Z are lying over

$$\bigcap_{i=1}^n (H_i \cap D_1) = (\bigcap_{i=1}^n H_i) \cap D_1 = \emptyset.$$

$\square$

Proof of (3.17): Let $\tau : Z \to X$ be the covering constructed in (3.19) for

$$D_{red} = \sum_{j=1}^{r} D_j$$

and $N = N_1 = \ldots = N_r$. Let T be the normalization of $Z \times_X Y$. Then T is obtained again by taking the N-th root out of $\tau^* D$. Since $\tau^* D = N \cdot D'$ for some divisor D' on Z, we can use (3.3,b), (3.10) and (3.15,b) to show that T is étale over Z.

For part c), we apply the same construction to the manifold Z, given for the divisor $D + A + B$, where the prescribed multiplicities for the components of A and B are one.

□

Generalizations and variants in the analytic case

(3.17) is a special case of the more general covering lemma of Kawamata:

3.20. Lemma. *Let X be a projective manifold,* $\mathrm{char}(k) = 0$ *and let $\pi : Y \to X$ be a finite cover such that the ramification locus $D = \Delta(Y/X)$ in X has normal crossings. Then there exists a manifold T and a finite morphism $\delta : T \to Y$. Moreover, one can assume that $\pi \circ \delta : T \to X$ is a Galois cover.*

For the proof see [35]. As shown in [63] (3.16) can be generalized as well:

3.21. Lemma. *(Generalized Hurwitz's formula) For $\pi : Y \to X$ as in* (3.20) *let $\delta : Z \to Y$ be a desingularization such that $(\pi \circ \delta)^* D = D'$ is a normal crossing divisor. Then one has an inclusion*

$$\delta^* \pi^* \Omega^a_X(\log \ D) \longrightarrow \Omega^a_Z(\log \ D')$$

giving an isomorphism over the open subscheme U in Z where $(\pi \circ \delta) \mid_Z$ is finite.

If Y in (3.20) is normal, it has at most quotient singularities (see (3.24) for a slightly different argument). In particular, Y has rational singularities (see [62] or (5.13)), i.e.:

$$R^b \delta_* \mathcal{O}_Z = 0 \ \text{ for } \ b > 0.$$

One can even show (see [17]):

3.22. Lemma. *For Y normal and $\pi : Y \to X$, $\delta : Z \to Y$ as in (3.21) and $\tau = \pi \circ \delta$ one has:*

$$R^b \tau_* \Omega^a_Z(\log \ D') = \begin{cases} \Omega^a_X(\log \ D) \otimes \bigoplus_{i=0}^{N-1} \mathcal{L}^{(i)^{-1}} & \textit{for } b = 0 \\ 0 & \textit{for } b > 0 \end{cases}$$

For $b = 0$ this statement follows directly from (3.21). For $b > 0$, however, the only way we know to get (3.22) is to use GAGA and the independence from the choosen compactification of the mixed Hodge structure of the open manifold $Z - D'$ (see Deligne [10]).

Using (3.21) and (3.22) one finds again (see [20]):
The degeneration of the spectral sequence

$$E_1^{ab} = H^b(Z, \Omega_Z^a(\log\ D')) \Longrightarrow H^{a+b}(Z, \Omega_Z^\bullet(\log\ D'))$$

implies (3.2,b).

Let us end this section with the following

3.23. Corollary. *Under the assumptions of 3.2 assume that $k = \mathbb{C}$. Then*

$$\dim\ (H^b(X, \Omega_X^a(\log\ D^{(i)}) \otimes \mathcal{L}^{(i)^{-1}})) = \dim\ (H^a(X, \Omega_X^b(\log\ D^{(N-i)}) \otimes \mathcal{L}^{(N-i)^{-1}})).$$

Proof: By GAGA we can assume that we consider the analytic sheaf of differentials. The Hodge duality on the covering T constructed in (3.17) is given by conjugation. Since under conjugation e^i goes to e^{N-i} for a primitive N-th root of unity, we obtain (3.23).

□

Let us end this section by showing that the cyclic cover Y constructed in (3.5) has at most quotient singularities. Slightly more generally one has the following lemma which, as mentioned above, also follows from (3.20).

3.24. Lemma. *Let X be a quasi-projective manifold, Y a normal variety and let $\pi : Y \longrightarrow X$ be a separable finite cover. Assume that the ramification divisor*

$$D = \sum_{j=1}^{m} D_j = \Delta(Y/X)$$

of π in X is a normal crossing divisor and that for all j and all components B_j^i of $\pi^{-1}(D_j)$ the ramification index $e(B_j^i)$ is prime to char k.
Then Y has at most quotient singularities, i.e. each point $y \in Y$ has a neighbourhood of the form T/G where T is nonsingular and G a finite group acting on T.

Proof: One can assume that X is affine. For $j = 1, \cdots, m$ define

$$n_j = lcm\{e(B_j^i);\ B_j^i \text{ component of } \pi^{-1}(D_j)\}.$$

Let $\tau : Z \longrightarrow X$ be the cyclic cover obtained by taking sucessively the n_j-th root out of D_j. In other terms, Z is the normalization of the fibered product of the different coverings of X obtained by taking the n_j root out of D_j or, equivalently, τ is the composition of

$$Z = Z_m \xrightarrow{\tau_m} Z_{m-1} \xrightarrow{\tau_{m-1}} \cdots \longrightarrow Z_1 \xrightarrow{\tau_1} Z_0 = X$$

where $\tau_j : Z_j \longrightarrow Z_{j-1}$ is the cover obtained by taking the n_j-th root out of $(\tau_1 \circ \tau_2 \circ \cdots \circ \tau_{j-1})^*(D_j)$. By (3.15,b) Z_j is non singular. Z is Galois over X with Galois group

$$G = \prod_{j=1}^{m} \mathbb{Z}/n_j \cdot \mathbb{Z}.$$

Let T be the normalization of $Z \times_X Y$ and $\delta : T \longrightarrow Y$ the induced morphism. Each component T_0 of T is Galois over Y with a subgroup of G as Galois group.

The morphism $\delta_0 = \delta|_{T_0}$ is obtained by taking sucessively the $\frac{n_j}{\alpha_j}$-th root out of

$$\pi^{-1}(D_j) = \sum_{i=1}^{r_j} \frac{e(B_j^i)}{\alpha_j} \cdot B_j^i$$

for

$$\alpha_j = gcd\{e(B_j^i);\ B_j^i \text{ component of } \pi^{-1}(D_j)\}.$$

By (3.15) all components of $\delta^{-1}(B_j^i)$ have ramification index

$$\frac{\frac{n_j}{\alpha_j}}{gcd\{\frac{n_j}{\alpha_j}, \frac{e(B_j^i)}{\alpha_j}\}} = \frac{n_j}{e(B_j^i)}$$

over Y. Hence they are ramified over X with order n_j. In other terms, the induced morphism $T_0 \longrightarrow Z$ is unramified and T_0 is a non-singular Galois cover of Y.

□

§ 4 Vanishing theorems, the formal set-up.

Theorem 3.2 , whose proof has been reduced to the E_1-degeneration of a Hodge to de Rham spectral-sequences, implies immediately several vanishing theorems for the cohomology of the sheaves $\mathcal{L}^{(i)}$.
To underline that in fact the whole information needed is hidden in (3.2) and (2.9) we consider in this lecture a more general situation and we state the assumptions explicitly, which are needed to obtain the vanishing of certain cohomology groups.

(4.2) and (4.8) are of special interest for applications whereas the other variants can been skipped at the first reading.

4.1. Assumptions. Let X be a projective manifold defined over an algebraically closed field k and let

$$D = \sum_{j=1}^{r} D_j$$

be a reduced normal crossing divisor. Let $\mathcal{E}$ be a locally free sheaf on X of finite rank and let

$$\nabla : \mathcal{E} \longrightarrow \Omega^1_X(\log\ D) \otimes \mathcal{E}$$

be an integrable connection with logarithmic poles along D.
We will assume in the sequel that ∇ *satisfies the* E_1*-degeneration* i.e. that *the Hodge to de Rham spectral sequence (A.25)*

$$E_1^{ab} = H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{E}) \Longrightarrow \mathbb{H}^{a+b}(X, \Omega^\bullet_X(\log\ D) \otimes \mathcal{E})$$

degenerates in E_1.

4.2. Lemma (Vanishing for restriction maps I). *Assume that* ∇ *satisfies the condition (!) of (2.8), i.e. that for all* $\mu \in \mathbb{N}$ *and for* $j = 1, \ldots, r$ *the map*

$$Res_{D_j}(\nabla) + \mu \cdot id_{\mathcal{O}_{D_j}} : \mathcal{E}\,|_{D_j} \longrightarrow \mathcal{E}\,|_{D_j}$$

is an isomorphism. Assume that ∇ *satisfies the* E_1*-degeneration (4.1). Then for all effective divisors*

$$D' = \sum_{j=1}^{r} \mu_j D_j$$

and all b *the natural map*

$$H^b(X, \mathcal{O}_X(-D') \otimes \mathcal{E}) \longrightarrow H^b(X, \mathcal{E})$$

is surjective.

Proof: By (2.9,b) the map

$$\Omega^\bullet_X(\log\ D) \otimes \mathcal{E}(-D') \longrightarrow \Omega^\bullet_X(\log\ D) \otimes \mathcal{E}$$

is a quasi-isomorphism and hence induces an isomorphism of the hypercohomology groups. Let us consider the exact sequences of complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^{\bullet \geq 1}_X(\log\ D) \otimes \mathcal{E} & \longrightarrow & \Omega^\bullet_X(\log\ D) \otimes \mathcal{E} & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Omega^{\bullet \geq 1}_X(\log\ D) \otimes \mathcal{E}(-D') & \longrightarrow & \Omega^\bullet_X(\log\ D) \otimes \mathcal{E}(-D') & \longrightarrow & \mathcal{E}(-D') & \longrightarrow & 0.
\end{array}$$

By assumption, the spectral sequence for $\Omega^\bullet_X(\log\ D) \otimes \mathcal{E}$ degenerates in E_1, which implies that the morphism α in the following diagram is surjective (see (A.25)).

$$\begin{array}{ccc}
\mathbb{H}^b(X, \Omega^\bullet_X(\log\ D) \otimes \mathcal{E}) & \xrightarrow{\alpha} & H^b(X, \mathcal{E}) \\
\uparrow{=} & & \uparrow{\beta} \\
\mathbb{H}^b(X, \Omega^\bullet_X(\log\ D) \otimes \mathcal{E}(-D')) & \longrightarrow & H^b(X, \mathcal{E}(-D'))
\end{array}$$

Hence β is surjective as well.

$\square$

4.3. Variant. *If in (4.2)*

$$D' = \sum_{j=1}^{s} \mu_j D_j \geq 0 \quad for \quad s \leq r,$$

then it is enough to assume that for $j = 1, \ldots, s$ *and for* $0 \leq \mu \leq \mu_j - 1$

$$Res_{D_j}(\nabla) + \mu \cdot id_{\mathcal{O}_{D_j}}$$

is an isomorphism.

PROOF: By (2.10) this is enough to give the quasi-isomorphism

$$\Omega^\bullet_X(\log\ D) \otimes \mathcal{E}(-D') \longrightarrow \Omega^\bullet_X(\log\ D) \otimes \mathcal{E}$$

needed in the proof of (4.2).

□

4.4. Lemma (Dual version of (4.2) and (4.3)). *Assume that*

$$\nabla : \mathcal{E} \longrightarrow \Omega^1_X(\log\ D) \otimes \mathcal{E}$$

satisfies the E_1*-degeneration and that for* $j = 1, \ldots, s$ *and* $1 \leq \mu \leq \mu_j$

$$Res_{D_j}(\nabla) - \mu \cdot id_{\mathcal{O}_{D_j}}$$

is an isomorphism (for example, if ∇ *satisfies the condition* $(*)$ *from (2.8,a)). Then for*

$$D' = \sum_{j=1}^{s} \mu_j D_j$$

and all b *the map*

$$H^b(X, \omega_X(D) \otimes \mathcal{E}) \longrightarrow H^b(X, \omega_X(D + D') \otimes \mathcal{E})$$

is injective.

PROOF: Consider the diagram

$$\begin{array}{ccc} H^b(X, \omega_X(D + D') \otimes \mathcal{E}) & \longrightarrow & \mathbb{H}^{n+b}(X, \Omega^\bullet_X(\log\ D) \otimes \mathcal{E}(D')) \\ \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \gamma} \\ H^b(X, \omega_X(D) \otimes \mathcal{E}) & \xrightarrow{\alpha} & \mathbb{H}^{n+b}(X, \Omega^\bullet_X(\log\ D) \otimes \mathcal{E}). \end{array}$$

α is injective by the E_1-degeneration (see (A.25)) , γ is an isomorphism by (2.10) and hence β is injective.

□

The lemma (4.2) or its variant (4.3) implies that for all b the natural restriction maps

$$H^b(X, \mathcal{E}) \longrightarrow H^b(D', \mathcal{O}_{D'} \otimes \mathcal{E})$$

are the zero maps. For higher differential forms this remains true, if D' is a non-singular divisor:

4.5. Lemma (Vanishing for restriction maps II). *Assume that*

$$\nabla : \mathcal{E} \longrightarrow \Omega^1_X(\log\ D) \otimes \mathcal{E}$$

satisfies E_1-degeneration. Let D' be a non-singular subdivisor of D and assume that for all components D_j of D' the map $Res_{D_j}(\nabla)$ is an isomorphism. (For example this follows from condition (!) in (2.8,b)).
Then the restriction (see (2.3))

$$H^b(X, \Omega^a_X(\log\ (D - D')) \otimes \mathcal{E}) \longrightarrow H^b(D', \Omega^a_{D'}(\log\ (D - D')\ |_{D'}) \otimes \mathcal{E})$$

is zero for all a and b.

PROOF: As we have seen in (2.6,b) the restriction map factors through

$$H^b(\nabla_a) : H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{E}) \longrightarrow H^b(X, \Omega^{a+1}_X(\log\ D) \otimes \mathcal{E})$$

provided $Res_{D_j}(\nabla)$ is an isomorphism on the different components D_j of D'. By E_1-degeneration, $H^b(\nabla_a)$ is the zero map (see (A.25)).

□

Before we are able to state the global vanishing for $\mathcal{E}$ or $\Omega^a_X(\log\ D) \otimes \mathcal{E}$ we need some more notations.

4.6. Definition. Let $U \subset X$ be an open subscheme and let B be an effective divisor with $B_{red} = X - U$. Then we define the *(coherent) cohomological dimension* of (X, B) to be the least integer α such that for all coherent sheaves $\mathcal{F}$ and all $k > \alpha$ one finds some $\nu_0 > 0$ with $H^k(X, \mathcal{F}(\nu \cdot B)) = 0$ for all multiples ν of ν_0. Finally, for the reduced divisor $D = X - U$, we write

$$\begin{aligned} cd(X, D) = \mathrm{Min}\{\ \alpha\ ;\ & \text{there exists some effective divisor } B \text{ with } B_{red} = D, \\ & \text{such that } \alpha \text{ is the cohomological dimension of } (X, B)\}. \end{aligned}$$

4.7. Examples.
a) For $D = X - U$ the embedding $\iota : U \to X$ is affine and for a coherent sheaf $\mathcal{G}$ on X we have

$$H^b(U, \mathcal{G}\ |_U) = H^b(X, \iota_*(\mathcal{G}\ |_U)) = \varinjlim_{\alpha \in \mathbb{N}} H^b(X, \mathcal{G} \otimes \mathcal{O}_X(\alpha \cdot B)),$$

where B is any effective divisor with $B_{red} = D$. In particular, if $b > cd(X, D)$ we find

$$H^b(U, \mathcal{G}\ |_U) = 0$$

b) By Serre duality one obtains as well that for $b < n - cd(X, D)$ we can find $B > 0$ such that for a locally free sheaf $\mathcal{G}$ and all multiples ν of some $\nu_0 > 0$ one has

$$\dim H^b(X, \mathcal{G} \otimes \mathcal{O}_X(-\nu \cdot B)) = 0.$$

c) If D is the support of an effective ample divisor, then Serre's vanishing theorem (see (1.1)) implies $cd(X, D) = 0$. We are mostly interested in this case, hopefully an excuse for the clumsy definition given in (4.6).

4.8. Lemma (Vanishing for cohomology groups).
Assume that X is projective and that

$$\nabla : \mathcal{E} \longrightarrow \Omega^1_X(\log\ D) \otimes \mathcal{E}$$

satisfies the E_1-degeneration (see (4.1)).
a) If ∇ satisfies the condition $(*)$ of (2.8) and if $a + b > n + cd(X, D)$, then

$$H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{E}) = 0.$$

b) If ∇ satisfies the condition $(!)$ of (2.8) and if $a + b < n - cd(X, D)$, then

$$H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{E}) = 0.$$

Proof: Let us choose $\alpha \in \mathbb{Z}$ with $\alpha \geq 0$ in case a) and with $\alpha \leq 0$ in case b). For $B \geq D$, (2.9) tells us that

$$\Omega^\bullet_X(\log\ D) \otimes \mathcal{E} \quad \text{and} \quad \Omega^\bullet_X(\log\ D) \otimes \mathcal{E}(\alpha \cdot B)$$

are quasi-isomorphic. In both cases we have a spectral sequence

$$E_1^{ab} = H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{E}(\alpha \cdot B)) \Longrightarrow$$
$$\Longrightarrow \mathbb{H}^{a+b}(X, \Omega^\bullet_X(\log\ D) \otimes \mathcal{E}(\alpha \cdot B)) = \mathbb{H}^{a+b}(X, \Omega^\bullet_X(\log\ D) \otimes \mathcal{E}).$$

By assumption this spectral sequence degenerates for $\alpha = 0$ and, for arbitrary α we have (see (A.16))

$$\sum_{a+b=l} \dim\ H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{E}) = \dim\ \mathbb{H}^l(X, \Omega^\bullet_X(\log\ D) \otimes \mathcal{E})$$
$$\leq \sum_{a+b=l} \dim\ H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{E}(\alpha \cdot B)).$$

By definition of $cd(X, D)$ we can choose B such that the right hand side is zero for $l > n + cd(X, D)$ and all $\alpha > 0$ in case a), or $l < n - cd(X, D)$ and all $\alpha < 0$ in case b).

$\square$

The same argument shows:

4.9. Variant. *In (4.8) we can replace a) and b) by:*
c) Let D_* and $D_!$ be effective divisors, both smaller than D, and assume that

i) For all components D_j of D_* and all $\mu \in \mathbb{N} - \{0\}$

$$Res_{D_j}(\nabla) - \mu \cdot id_{\mathcal{O}_{D_j}}$$

is an isomorphism.

ii) For all components D_j of $D_!$ and all $\mu \in \mathbb{N}$

$$Res_{D_j}(\nabla) + \mu \cdot id_{\mathcal{O}_{D_j}}$$

is an isomorphism.

Then

$$H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{E}) = 0$$

for $a + b > n + cd(X, D_*)$ and for $a + b < n - cd(X, D_!)$.

The analytic case

As we have seen in the proof of (4.8) the condition $(*)$ implies that

$$\mathbb{H}^l(X, \Omega^\bullet_X(\log\ D) \otimes \mathcal{E}) = 0 \quad \text{for} \quad l > n + cd(X, D).$$

For $k = \mathbb{C}$, this is not the best possible result. In fact, as mentioned in (2.12), $(*)$ implies that over $\mathbb{C}$

$$\mathbb{H}^l(X, \Omega^\bullet_X(\log\ D) \otimes \mathcal{E}) = 0 \quad \text{for} \quad l > n + r(X - D)$$

where $r(X - D)$ is the least number α such that $H^l(X - D, V) = 0$ for all locally constant systems V on $X - D$ and $l > n + \alpha$.

(2.12) and the E_1-degeneration asked for in (4.8) and (4.9) imply immediately that "$cd(\ )$" in 4.8 and 4.9 can be replaced by "$r(\ )$".
As we will see, $r(X - D)$ might be smaller than $cd(X, D)$. For the results which follow we only know, at present, proofs by analytic methods.

4.10. Definition. Let U be an algebraic irreducible variety and $g : U \to W$ a morphism. Then define

$$r(g) = \text{Max}\{ \ \dim\ \Gamma - \dim\ g(\Gamma) - \text{codim}\ \Gamma; \ \Gamma \text{ irreducible closed subvariety of } U \ \}$$

4.11. Properties.

a)

$$r(g) = \text{Max}\{ \ \dim\ (\text{generic fibre of } g \mid_\Gamma) - \text{codim}\ \Gamma; \ \Gamma \text{ irreducible closed subvariety of } U \ \}$$

b) If b denotes the maximal fibre dimension for g, then

$$r(g) \leq \text{Max}\{\dim\ U - \dim\ W; \ \ b - 1\}.$$

c) If $U' \subseteq U$ is open and dense, then $r(g \mid_{U'}) \leq r(g)$.
d) If $\Delta \subseteq U$ is closed then $r(g \mid_\Delta) \leq r(g) + \text{codim}_U\ (\Delta)$.

PROOF: a) and c) are obvious and b) follows from a). For d) one remarks that for $\Gamma \subset \Delta$ one has

$$\mathrm{codim}_{\Delta}\ (\Gamma) = \mathrm{codim}_{U}\ (\Gamma) - \mathrm{codim}_{U}\ (\Delta).$$

□

4.12. Lemma. *(Improvement of 4.8 using analytic methods)*
Let X be a projective manifold defined over an algebraically closed field k of characteristic zero. Assume that

$$\nabla : \mathcal{E} \longrightarrow \Omega^1_X(\log\ D) \otimes \mathcal{E}$$

is an integrable connection satisfying the E_1-degeneration and let

$$g : X - D \longrightarrow W$$

be a proper surjective morphism to an affine variety W.
a) If ∇ satisfies the condition $(*)$ of (2.8) then

$$H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{E}) = 0$$

for $a + b > n + r(g)$.
b) If ∇ satisfies the condition (!) of (2.8) then

$$H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{E}) = 0$$

for $a + b < n - r(g)$.

PROOF: By flat base chance we can replace k by any other field k', such that $X, D, \mathcal{E}$ are defined over k'. Hence, we may assume that $k = \mathbb{C}$.
By GAGA (see [56]) we may assume in (4.12) that all the sheaves and ∇ are analytic. Then, by (2.12) and by the E_1-degeneration it is enough to show:

4.13. Lemma. *Let U be an analytic manifold, W be an affine manifold and $g : U \to W$ be a proper morphism. Then, for all local constant systems V on U and $l > \dim\ (U) + r(g)$ one has $H^l(U, V) = 0$.*

PROOF (SEE [22]): The sheaves $R^a g_* V$ are analytically constructible sheaves ([61]) and their support

$$S_a = \mathrm{Supp}(R^a g_* V)$$

must be a Stein space, hence

$$H^b(W, R^a g_* V) = 0 \quad \text{for} \quad b > \dim\ S_a.$$

However, the general fibre of $g\mid_{g^{-1}(S_a)}$ must have a dimension larger than or equal to $\frac{a}{2}$. Hence

$$2 \cdot (\dim\ g^{-1}(S_a) - \dim\ S_a) \geq a$$

and $H^b(W, R^a g_* V) = 0$ for

$$a + b > n + r(g) \geq 2 \cdot \dim\ g^{-1}(S_a) - \dim\ S_a \geq a + \dim\ S_a.$$

By the Leray spectral sequence (A.27)

$$E_2^{ba} = H^b(W, R^a g_* V) \Longrightarrow H^{a+b}(U, V)$$

one obtains (4.13).

□

4.14. Remark. If W is affine and $g : U \to W$ obtained by blowing up a point, then for X and D as in (4.6) one has $cd(X, D) = \dim\ U - 1$, whereas $r(g) = \dim\ U - 2$.

§ 5 Vanishing theorems for invertible sheaves

In this lecture we will deduce several known generalizations of the Kodaira-Nakano vanishing theorem by applying the vanishing (5.1) obtained for "integral parts of $\mathbb{Q}$-divisors" from (3.2), combined with (4.2). Needless to say that in all those corollaries of (5.1) one loses some information and that it might be more reasonable to try to work with (5.1) or correspondingly with (6.2) directly, whenever it is possible.

Let us remind you, that the proof of (3.2) is not yet complete. The necessary arguments needed to show the E_1-degeneration will only be presented in Lecture 10.

Very quickly we will have to restrict ourselves to characteristic zero. One reason is that the condition (∗) and (!) are too much to ask for in characteristic $p \neq 0$. But more substantially, most of our proofs will start with "blow up B to get a normal crossing divisor", hence with an application of H. Hironaka's theorem on the existence of desingularizations.

Let us start with (4.2). For simplicity, we restrict ourselves to $i = 1$ and $\mathcal{L} = \mathcal{L}^{(1)}$. By (3.3) we are not losing any information.

5.1. Vanishing for restriction maps related to $\mathbb{Q}$-divisors:

Let X be a projective manifold defined over an algebraically closed field k, let $\mathcal{L}$ be an invertible sheaf, $N \in \mathbb{N} - \{0\}$ and let

$$D = \sum_{j=1}^{r} \alpha_j D_j$$

be a normal crossing divisor with $0 < \alpha_j < N$ and $\mathcal{L}^N = \mathcal{O}_X(D)$. Let

$$D' = \sum_{j=1}^{r} \mu_j D_j$$

be an effective divisor. Then one has:

a) *If* char $(k) = 0$ *then for all b the natural morphism*

$$H^b(X, \mathcal{L}^{-1}(-D')) \longrightarrow H^b(X, \mathcal{L}^{-1})$$

is surjective and hence the morphism

$$H^b(X, \omega_X \otimes \mathcal{L}) \longrightarrow H^b(X, \omega_X(D') \otimes \mathcal{L})$$

injective.

b) *If* char $(k) = 0$ *and if* C *is a reduced divisor without common component with* D *such that* $D + C$ *is a normal crossing divisor, then for all* b *the natural morphism*

$$H^b(X, \mathcal{L}^{-1}(-C - D')) \longrightarrow H^b(X, \mathcal{L}^{-1}(-C))$$

is surjective and hence the morphism

$$H^b(X, \omega_X(C) \otimes \mathcal{L}) \longrightarrow H^b(X, \omega_X(D' + C) \otimes \mathcal{L})$$

injective.

c) *If* char $(k) = p \neq 0$, *then a) and b) hold true under the additional assumptions:*

> *i)* X *and* D *(as well as* C*) satisfy the lifting property (8.11) and* dim $(X) \leq p$.
>
> *ii)* N *is prime to* p.
>
> *iii) For all* j *and* $0 \leq \mu \leq \mu_j - 1$ *one has* $\alpha_j + \mu \cdot N \not\equiv 0$ *mod* p.

Proof: By (3.2,c) $\mathcal{O}_X(-C) \otimes \mathcal{L}^{-1}$ has a logarithmic connection ∇ with poles along $C + D_{red}$ satisfying E_1-degeneration, and $Res_{D_j}(\nabla) = \frac{\alpha_j}{N}$. Hence (5.1) follows from (4.2) and (4.3) or (4.4).

□

5.2. Corollary (Kodaira [39], Deligne, Illusie [12]). *Let* X *be a projective manifold and* $\mathcal{L}$ *an invertible sheaf. If* char $(k) = p > 0$, *then assume in addition that* X *and* $\mathcal{L}$ *admit a lifting to* $W_2(k)$ *(8.11) and that* dim $X \leq p$. *Then, if* $\mathcal{L}$ *is ample,*

$$H^b(X, \mathcal{L}^{-1}) = 0 \quad \textit{for} \quad b < n = \dim\ (X)$$

Proof: Choose N, prime to $p = $ char (k), such that

$$H^b(X, \mathcal{L}^{-N-1}) = H^{n-b}(X, \omega_X \otimes \mathcal{L}^{N+1}) = 0$$

for $b < n$, and such that $\mathcal{L}^N$ is generated by global sections. If D is a general section of $\mathcal{L}^N$, then we can apply (5.1) and find

$$H^b(X, \mathcal{L}^{-1}(-D)) \longrightarrow H^b(X, \mathcal{L}^{-1})$$

to be surjective. Since the group on the left hand side is zero, we are done.

□

If char $(k) = p$, then we will see in (11.3) that it is sufficient to assume that X lifts to $W_2(k)$. The condition that $\mathcal{L}$ lifts as well is not necessary.

5.3. Definition. Let X be a projective variety and $\mathcal{L}$ be an invertible sheaf on X. If $H^0(X, \mathcal{L}^\nu) \neq 0$, the sections of $\mathcal{L}$ define a rational map

$$\phi_\nu = \phi_{\mathcal{L}^\nu} : X \longrightarrow \mathbb{P}(H^0(X, \mathcal{L}^\nu)).$$

The *Iitaka-dimension* $\kappa(\mathcal{L})$ of $\mathcal{L}$ is given by

$$\kappa(\mathcal{L}) = \begin{cases} -\infty & \text{if } H^0(X, \mathcal{L}^\nu) = 0 \text{ for all } \nu \\ \mathrm{Max}\{\dim \phi_\nu(X);\ H^0(X, \mathcal{L}^\nu) \neq 0\} & \text{otherwise} \end{cases}$$

5.4. Properties. For X and $\mathcal{L}$ as above one has:

a) $\kappa(\mathcal{L}) \in \{-\infty, 0, 1, \ldots, \dim\ X\}$.

b) If $H^0(X, \mathcal{L}^\nu) \neq 0$ for some $\nu > 0$ then one can find $a, b \in \mathbb{R}$, $a, b > 0$, such that

$$a \cdot \mu^{\kappa(\mathcal{L})} \leq \dim\ H^0(X, \mathcal{L}^{\nu \cdot \mu}) \leq b \cdot \mu^{\kappa(\mathcal{L})} \quad \text{for all} \quad \mu \in \mathbb{N} - \{0\}.$$

c) If $\kappa(\mathcal{L}) \neq -\infty$, then

$$\kappa(\mathcal{L}) = \ \mathrm{tr.deg}\ (\bigoplus_{\mu \geq 0} H^0(X, \mathcal{L}^\mu)) \ - 1.$$

d) One has $\kappa(\mathcal{L}) = \dim\ X$, if and only if for some $\nu > 0$ and some effective divisor C the sheaf $\mathcal{L}^\nu(-C)$ is ample.

e) If $\mathcal{A}$ is very ample and A the zero divisor of a general section of $\mathcal{A}$, then

$$\kappa(\mathcal{L}\mid_A) \geq Min\ \{\kappa(\mathcal{L}), \dim\ A\}.$$

PROOF: a), b) and c) are wellknown and their proof can be found, for example, in [46], §1.
For d) let A be an ample effective divisor. For $n = \dim\ X$ and some $\nu \in \mathbb{N} - \{0\}$ one finds $a, b \in \mathbb{R}$, $a, b > 0$, with $a \cdot \mu^n < \dim\ H^0(X, \mathcal{L}^{\nu \cdot \mu})$ and $\dim\ H^0(X, \mathcal{L}^{\nu \cdot \mu}|_A) < b \cdot \mu^{n-1}$. Hence, the exact sequence

$$0 \longrightarrow H^0(X, \mathcal{L}^{\nu \cdot \mu}(-A)) \longrightarrow H^0(X, \mathcal{L}^{\nu \cdot \mu}) \longrightarrow H^0(A, \mathcal{L}^{\nu \cdot \mu}\mid_A)$$

shows that for some μ we have $\mathcal{O}_X(A)$ as a subsheaf of $\mathcal{L}^{\nu \cdot \mu}$. On the other hand, if $\mathcal{A} \subset \mathcal{L}^\nu$ is ample then

$$n = \kappa(\mathcal{A}) \leq \kappa(\mathcal{L}^\nu) = \kappa(\mathcal{L}).$$

If $\kappa(\mathcal{L}) = n$ in e), then, using d) for example, $\kappa(\mathcal{L} \mid_A) = n - 1$.
If $\kappa(\mathcal{L}) < n$, then $H^0(X, \mathcal{O}_X(-A) \otimes \mathcal{L}^\nu) = 0$ for all ν and b) implies

$$\kappa(\mathcal{L} \mid_A) \geq \kappa(\mathcal{L}).$$

□

For our purposes we can take (5.4,b) as definition of $\kappa(\mathcal{L})$, and we only need to know (5.4,d) and (5.4,e).

5.5. Definition. An invertible sheaf $\mathcal{L}$ on X is called

a) *semi-ample*, if for some $\mu > 0$ the sheaf $\mathcal{L}^\mu$ is generated by global sections.

b) *numerically effective (nef)* if for all curves C in X one has

$$\deg\ (\mathcal{L} \mid_C) \geq 0.$$

The proof of (5.2) can be modified to give in characteristic zero a stronger statement:

5.6. Corollary. *Let X be a projective manifold defined over a field k of characteristic zero and let $\mathcal{L}$ be an invertible sheaf.*

a) (Kollár [40])
If $\mathcal{L}$ is semi-ample and B an effective divisor with $H^0(X, \mathcal{L}^\nu(-B)) \neq 0$ for some $\nu > 0$, then the natural maps

$$H^b(X, \mathcal{L}^{-1}(-B)) \longrightarrow H^b(X, \mathcal{L}^{-1})$$

are surjective for all b, or, equivalently, the adjunction map

$$H^b(X, \mathcal{L} \otimes \omega_X(B)) \longrightarrow H^b(B, \mathcal{L} \otimes \omega_B)$$

is surjective for all b.

b) (Grauert-Riemenschneider [25])
If $\mathcal{L}$ is semi-ample and $\kappa(\mathcal{L}) = n = \dim\ X$, then

$$H^b(X, \mathcal{L}^{-1}) = 0 \quad \textit{for} \quad b < n.$$

Proof: Obviously a) and b) are compatible with blowing ups $\tau : X' \to X$. In fact, using the Leray spectral sequence (A.27) we just have to remember that

$$R^b \tau_* \tau^* \mathcal{L} = \mathcal{L} \otimes R^b \tau_* \mathcal{O}_{X'} = \begin{cases} \mathcal{L} & \text{for} \quad b = 0 \\ 0 & \text{for} \quad b \neq 0 \end{cases}$$

(See (5.13) for a generalization).
Hence we may assume in a) that $\mathcal{L}^\nu = \mathcal{O}_X(B + C)$ for an effective normal crossing divisor $B + C$. We can choose some μ, with

$$[\frac{B+C}{\mu}] = 0$$

and such that $\mathcal{L}^\mu$ is generated by its global sections. If D_1 is a general divisor of $\mathcal{L}^\mu$, i.e. the zero set of a general $s \in H^0(X, \mathcal{L}^\mu)$, then $D = D_1 + B + C$ has normal crossings and $[\frac{D}{\mu}] = 0$. Hence, for $N = \nu + \mu$ and $D' = B$ the assumptions of (5.1,a) hold true and we obtain a).

For b), let us choose some divisor C and some ν such that $\mathcal{L}^\nu(-C) = \mathcal{A}$ is ample. Replacing $\mathcal{A}$ by some multiple, we may assume by Serre's vanishing theorem (1.1) that

$$H^b(X, \mathcal{L}^{-1} \otimes \mathcal{A}^{-1}) = H^{n-b}(X, \omega_X \otimes \mathcal{L} \otimes \mathcal{A}) = 0$$

for $b < n$, and that $\mathcal{A} = \mathcal{O}_X(B)$ for some divisor B. By a)

$$H^b(X, \mathcal{O}_X(-B) \otimes \mathcal{L}^{-1}) \longrightarrow H^b(X, \mathcal{L}^{-1})$$

is surjective. One obtains b), since the left hand side is zero.

□

It is not difficult to modify both parts of this proof to include in b) the case that $\mathcal{L}$ is nef and $\kappa(\mathcal{L}) = \dim\ X$. Moreover, considering very ample divisors on X and using induction on $\dim(X)$, one can as well remove the assumption "$\kappa(\mathcal{L}) = \dim(X)$" and obtain the vanishing for $b < \kappa(\mathcal{L})$.
We leave the details to the reader. Those techniques will appear in (5.12) anyway, when we prove a more general statement.

5.7. Lemma. *For an invertible sheaf $\mathcal{L}$ on a projective manifold X the following two conditions are equivalent:*

a) *$\mathcal{L}$ is numerically effective.*

b) *For an ample sheaf $\mathcal{A}$ and all $\nu > 0$ the sheaf $\mathcal{L}^\nu \otimes \mathcal{A}$ is ample.*

PROOF: By Seshadri's criterion $\mathcal{A}'$ is ample if and only if for some $\epsilon > 0$ and all curves C in X

$$\deg\ (\mathcal{A}' \mid_C) \geq \epsilon \cdot m(C)$$

where $m(C)$ is the maximal multiplicity of points on C.

□

5.8. Lemma. *For $X, \mathcal{L}$ as in (5.7), assume that $\mathcal{L}$ is numerically effective (and, if* $\text{char}\ (k) = p \neq 0$, *that X and $\mathcal{L}$ satisfy the lifting property (8.11) and that* $\dim\ X \leq p$*). Then one has:*

a) $\kappa(\mathcal{L}) = n = \dim\ X$, *if and only if* $c_1(\mathcal{L})^n > 0$ *(where* $c_1(\mathcal{L})$ *is the Chern class of* $\mathcal{L}$*).*

b) *For* $b \geq 0$ *and for all invertible sheaves* $\mathcal{F}$ *one has a constant* $c_b > 0$ *with*

$$\dim\ H^b(X, \mathcal{F} \otimes \mathcal{L}^\nu) \leq c_b \cdot \nu^{n-b} \quad \textit{for all} \quad \nu \in \mathbb{N}.$$

PROOF: a) follows from b) and from the Hirzebruch-Riemann-Roch theorem which tells us that $\chi(X, \mathcal{L}^\nu)$ is a polynomial of deg n with highest coefficient

$$\frac{1}{n!} \cdot c_1(\mathcal{L})^n.$$

For b) we assume by induction on dim X, that it holds true for all hypersurfaces H in X. We can choose an H, which satisfies

$$H^b(X, \mathcal{O}_X(H) \otimes \mathcal{L}^\nu \otimes \mathcal{F}) = 0.$$

In fact, we just have to choose H such that $\mathcal{F} \otimes \omega_X^{-1} \otimes \mathcal{O}_X(H)$ is ample. Then by (5.7)

$$\mathcal{F} \otimes \omega_X^{-1} \otimes \mathcal{O}_X(H) \otimes \mathcal{L}^\nu$$

will be ample for all $\nu \geq 0$ and the vanishing required holds true by (5.2). From the exact sequence

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{L}^\nu \longrightarrow \mathcal{F} \otimes \mathcal{L}^\nu \otimes \mathcal{O}_X(H) \longrightarrow \mathcal{F} \otimes \mathcal{O}_H(H) \otimes \mathcal{L}^\nu \longrightarrow 0$$

we obtain an isomorphism

$$H^{b-1}(H, \mathcal{F} \otimes \mathcal{O}_H(H) \otimes \mathcal{L}^\nu) \simeq H^b(X, \mathcal{F} \otimes \mathcal{L}^\nu)$$

for $b > 1$ and a surjection

$$H^0(H, \mathcal{F} \otimes \mathcal{O}_H(H) \otimes \mathcal{L}^\nu) \longrightarrow H^1(X, \mathcal{F} \otimes \mathcal{L}^\nu).$$

By induction we find c_b for $b \geq 1$ and, since $H^0(X, \mathcal{F} \otimes \mathcal{L}^\nu)$ is bounded above by a polynomial of deg ν, for $b = 0$ as well.

□

Even if $\mathcal{L}$ is nef, there is in general no numerical characterisation of $\kappa(\mathcal{L})$. For example, there are numerically effective invertible sheaves $\mathcal{L}$ with $\kappa(\mathcal{L}) = -\infty$. Following Kawamata [37], one defines:

5.9. Definition. Let $\mathcal{L}$ be a numerically effective invertible sheaf. Then the *numerical Iitaka-dimension* is defined as

$$\nu(\mathcal{L}) = \text{Min}\ \{\nu \in \mathbb{N} - \{0\};\ \ c_1(\mathcal{L})^\nu \text{ numerically trivial }\} - 1.$$

5.10. Properties. Let $X, \mathcal{L}$ be as in (5.8). Then one has:

a) $\nu(\mathcal{L}) \geq \kappa(\mathcal{L})$.

b) If $\mathcal{L}$ is semi-ample then

$$\nu(\mathcal{L}) = \kappa(\mathcal{L}).$$

PROOF: If $\nu(\mathcal{L}) = \dim\ X$ or $\kappa(\mathcal{L}) = \dim\ X$, then (5.8,a) gives

$$\nu(\mathcal{L}) = \kappa(\mathcal{L}) = n = \dim\ X.$$

Hence we can assume both to be smaller than n. By (5.4,e) we have for a general hyperplane section H of X

$$\kappa(\mathcal{L}\mid_H) \geq \kappa(\mathcal{L})$$

and obviously

$$\nu(\mathcal{L}\mid_H) = \nu(\mathcal{L}).$$

By induction on $\dim\ (X)$ one obtains a).
For b) we may assume that $\mathcal{L} = \tau^*\mathcal{M}$ for a morphism $\tau : X \to Z$ with $\mathcal{M}$ ample and with $\dim\ (Z) = \kappa(\mathcal{L})$. Then

$$c_1(\mathcal{L})^\nu = \tau^* c_1(\mathcal{M})^\nu = 0$$

if and only if $\nu > \dim\ Z$.

□

The following lemma, due to Y. Kawamata [37], is more difficult to prove, and we postpone its proof to the end of this lecture.

5.11. Lemma. *For an invertible sheaf $\mathcal{N}$ on a projective manifold X, defined over a field k of characteristic zero, the following two conditions are equivalent:*
a) *$\mathcal{N}$ is numerically effective and $\nu(\mathcal{N}) = \kappa(\mathcal{N})$*
b) *There exist a blowing up $\tau : Z \to X$, some $\mu_0 \in \mathbb{N} - \{0\}$ and an effective divisor C on Z such that*

$$\tau^*\mathcal{N}^\mu \otimes \mathcal{O}_Z(-C)$$

is semi-ample for all $\mu \in \mathbb{N} - \{0\}$ divisible by μ_0.

5.12. Corollary. *Let X be a projective manifold defined over a field k of characteristic zero, let $\mathcal{L}$ be an invertible sheaf on X, let*

$$D = \sum_{j=1}^{r} \alpha_j D_j$$

be a normal crossing divisor and $N \in \mathbb{N}$. Assume that

$$0 < \alpha_j < N \quad \text{for} \quad j = 1, \ldots, r.$$

Then one has:
a) *If $\mathcal{L}^N(-D)$ is semi-ample and B an effective divisor such that*

$$H^0(X, (\mathcal{L}^N(-D))^\nu \otimes \mathcal{O}_X(-B)) \neq 0$$

for some $\nu > 0$, then for all b the maps

$$H^b(X, \mathcal{L}^{-1}(-B)) \longrightarrow H^b(X, \mathcal{L}^{-1})$$

are surjective.
b) *In a) one can replace "semi-ample" by the assumption that $\mathcal{L}^N(-D)$ is numerically effective and*

$$\kappa(\mathcal{L}^N(-D)) = \nu(\mathcal{L}^N(-D)).$$

c) (Kawamata [36] - Viehweg [63])
If $\mathcal{L}^N(-D)$ is numerically effective and

$$c_1(\mathcal{L}^N(-D))^n > 0,$$

then

$$H^b(X, \mathcal{L}^{-1}) = 0 \quad \text{for} \quad b < n.$$

d) (Kawamata [36] - Viehweg [63])
If $\mathcal{L}^N(-D)$ is numerically effective, then

$$H^b(X, \mathcal{L}^{-1}) = 0 \quad \text{for} \quad b < \kappa(\mathcal{L}).$$

e) *Part d) remains true if one replaces $\kappa(\mathcal{L})$ by $\kappa(\mathcal{L} \otimes \mathcal{N}^{-1})$ for a numerically effective invertible sheaf $\mathcal{N}$.*

Again, the assumptions are compatible with blowing ups, except for "$0 < \alpha_j < N$". We need:

5.13. Claim. Let $\tau : X' \to X$ be a proper birational morphism and $\mathcal{M} = \tau^*\mathcal{L}$. Assume that $\Delta = \tau^* D$ has normal crossings. Then for

$$\mathcal{M}^{(i)} = \mathcal{M}(-[\frac{i \cdot \Delta}{N}])$$

one has

$$R^b \tau_* \mathcal{M}^{(i)^{-1}} = \begin{cases} \mathcal{L}^{(i)^{-1}} & \text{for} \quad b = 0 \\ 0 & \text{for} \quad b \neq 0. \end{cases}$$

PROOF: We may assume that X is affine and that $\mathcal{L} = \mathcal{O}_X$. For $b = 0$ claim (5.13) follows from the inequality

$$[\frac{i \cdot \Delta}{N}] = [\frac{i \cdot \tau^* D}{N}] \geq \tau^*[\frac{i \cdot D}{N}].$$

In general, for $b \geq 0$, (5.13) follows from the rationality of the singularities of the cyclic covers Y and Y' obtained by taking the N-th rooth out of D and Δ. In fact, let Y'' be a desingularization of Y' and let

$$\begin{array}{ccccc} Y'' & \xrightarrow{\sigma} & Y' & \xrightarrow{\delta} & Y \\ & & {\scriptstyle\pi'}\downarrow & & {\scriptstyle\pi}\downarrow \\ & & X' & \xrightarrow{\tau} & X \end{array}$$

be the induced morphisms. If Y' has rational singularities, one has by definition $R^a\sigma_*\mathcal{O}_{Y''} = 0$ for $a > 0$. Hence, if Y has rational singularities as well,

$$R^a(\delta \circ \sigma)_*\mathcal{O}_{Y''} = R^a\delta_*\mathcal{O}_{Y'} = 0$$

and

$$R^a\tau_*(\pi'_*\mathcal{O}_{Y'}) = 0,$$

which implies (5.13).

By (3.24) we know that Y and Y' have quotient singularities. This implies that Y and Y' have rational singularities (see for example [62]). Let us recall the proof:

Let Y be any normal variety with quotient singularities and $\varphi : Z \longrightarrow Y$ the corresponding Galois cover with Z non singular. Let $\delta : Y' \longrightarrow Y$ be a desingularization such that $D' = \delta^*(\Delta(Z/Y))$ is a normal crossing divisor, where $\Delta(Z/Y)$ denotes the set of ramified points in Y. If Z' is the normalization of Y' in the function field of Z, (3.24) tells us that Z' has at most quotient singularities. Let finally $\gamma : Z'' \longrightarrow Z'$ be a desingularization. Altogether we obtain

$$\begin{array}{ccccc} Z'' & \xrightarrow{\gamma} & Z' & \xrightarrow{\delta'} & Z \\ & & {\scriptstyle\varphi'}\downarrow & & {\scriptstyle\varphi}\downarrow \\ & & Y' & \xrightarrow{\delta} & Y \end{array}$$

where Z'', Z and Y' are nonsingular.

Let us assume that for all quotient singularities and for all a with $a_0 > a > 0$ we know that the a-th higher direct image of the structure sheaf of the desingularization is zero. Then the Leray spectral sequence gives an injection

$$R^{a_0}\delta'_*\mathcal{O}_{Z'} = R^{a_0}\delta'_*(\gamma_*\mathcal{O}_{Z''}) \hookrightarrow R^{a_0}(\delta' \circ \gamma)_*\mathcal{O}_{Z''}.$$

Since $\delta' \circ \gamma$ is a birational proper morphism of non singular varieties

$$R^{a_0}(\delta' \circ \gamma)_*\mathcal{O}_{Z''} = 0.$$

Since the finite morphisms φ and φ' have no higher direct images one obtains

$$R^{a_0}\delta_*(\varphi'_*\mathcal{O}_{Z'}) = \varphi_*(R^{a_0}\delta'_*\mathcal{O}_{Z'}) = 0.$$

However, $\mathcal{O}_{Y'}$ is a direct summand of $\varphi'_*\mathcal{O}_{Z'}$ and hence $R^{a_0}\delta_*\mathcal{O}_{Y'} = 0$.

□

Proof of 5.12: let us first reduce b) to a):
Applying (5.13) and replacing $\mathcal{M}$ by $\mathcal{M}^{(1)}$, we can assume that the morphism $\tau : Z \to X$ in (5.11,b), applied to $\mathcal{N} = \mathcal{L}^N(-D)$, is an isomorphism and that for the divisor C in (5.11,b) $D+C$ is a normal crossing divisor. $\mathcal{L}^{N\cdot\mu}(-\mu\cdot D-C)$ is semi-ample for all μ divisible by μ_0. Choosing μ large enough, the multiplicities of $\mu\cdot D + C$ will be bounded above by $N\cdot\mu$. Moreover, we can assume that

$$H^0(X, \mathcal{L}^{N\cdot\mu}(-\mu\cdot D - C)) \neq 0$$

and hence, replacing μ again by some multiple, that

$$H^0(X, (\mathcal{L}^{N\cdot\mu}(-\mu\cdot D - C))^\nu \otimes \mathcal{O}_X(-B)) \neq 0$$

for some $\nu > 0$. Hence a) implies b).

To prove a), let us write

$$\mathcal{L}^{\nu\cdot N}(-\nu\cdot D) = \mathcal{O}_X(B + B')$$

or

$$\mathcal{L}^{\nu\cdot N} = \mathcal{O}_X(\nu\cdot D + B + B')$$

Blowing up, again, we can assume $D + B + B'$ to be a normal crossing divisor. For μ sufficiently large, we can assume that $(\mathcal{L}^N(-D))^\mu$ is generated by global sections. If H is zero set of a general section, then

$$\mathcal{L}^{(\nu+\mu)\cdot N} = \mathcal{O}_X(H + (\nu+\mu)\cdot D + B + B').$$

If μ is large enough, the multiplicities of the components of

$$D' = H + (\nu+\mu)\cdot D + B + B'$$

are smaller than $(\nu+\mu)\cdot N$ and, applying (5.1,a) for $N' = (\nu+\mu)\cdot N$ and $\mathcal{L}^{N'} = \mathcal{O}_X(D')$, we obtain (5.12,a).

Let us remark next, that d), under the additional condition that $\kappa(\mathcal{L}) = n$, implies c):
In fact, $c_1(\mathcal{L}^N(-D))^n > 0$ implies by (5.8,a) that

$$n = \kappa(\mathcal{L}^N(-D)) \leq \kappa(\mathcal{L}^N) = \kappa(\mathcal{L}).$$

To prove d), for $\kappa(\mathcal{L}) = n$, we can apply (5.4,d). Hence we find a divisor $C > 0$ and $\mu > 0$ with $\mathcal{L}^{\mu}(-C)$ ample. Then by (5.7)

$$\mathcal{L}^{N\cdot\nu+\mu}(-\nu \cdot D - C)$$

is ample for all ν and, by Serre's vanishing theorem (1.1)

$$H^b(X, \mathcal{L}^{-1} \otimes (\mathcal{L}^{-N\cdot\nu-\mu}(\nu \cdot D + C))^{\eta}) = 0 \quad \text{for} \quad b < n$$

and for η sufficiently large. As in the proof of (5.6) or by (5.13) this condition is compatible with blowing ups. Hence we may assume $D + C$ to be a normal crossing divisor and, choosing ν large enough, we may again assume that the multiplicities of $D' = \nu \cdot D + C$ are smaller than $N' = N \cdot \nu + \mu$. Replacing D' and N' by some high multiple we can assume in addition that $\mathcal{L}^{N'}(-D')$ is generated by global section and that

$$H^b(X, \mathcal{L}^{-N'-1}(D')) = 0 \quad \text{for} \quad b < n.$$

For an effective divisor B with $\mathcal{O}_X(B) = \mathcal{L}'^N(-D')$ we can apply a) and find

$$0 = H^0(X, \mathcal{L}^{-1}(-B)) \longrightarrow H^0(X, \mathcal{L}^{-1})$$

to be surjective.
For $\kappa(\mathcal{L}) < \dim \ X$ part d) is finally reduced to the case $\kappa(\mathcal{L}) = \dim \ X$ by induction:
Let H be a general hyperplane such that

$$H^b(X, \mathcal{O}_X(-H) \otimes \mathcal{L}^{-1}) = 0 \quad \text{for} \quad b < n.$$

The exact sequence

$$0 \longrightarrow \mathcal{O}_X(-H) \otimes \mathcal{L}^{-1} \longrightarrow \mathcal{L}^{-1} \longrightarrow \mathcal{L}^{-1} \mid_H \longrightarrow 0$$

give isomorphisms

$$H^b(X, \mathcal{L}^{-1}) \simeq H^b(H, \mathcal{L}^{-1} \mid_H)$$

for $b < n - 1$. Since $\kappa(\mathcal{L}) \leq n - 1$ we have $\kappa(\mathcal{L} \mid_H) \geq \kappa(\mathcal{L})$ and both groups vanish for $b < \kappa(\mathcal{L})$ by induction on $\dim X$.
e) follows by the same argument: If $\kappa(\mathcal{L} \otimes \mathcal{N}^{-1}) = \dim X$, then (5.4,c) and (5.7) imply that $\kappa(\mathcal{L}) = \dim X$ as well. For $\kappa(\mathcal{L} \otimes \mathcal{N}^{-1}) < \dim X$ again

$$\kappa(\mathcal{L}|_H \otimes \mathcal{N}^{-1}|_H) \geq \kappa(\mathcal{L} \otimes \mathcal{N}^{-1})$$

and by induction one obtains e).

□

Let us end this section by proving Kawamata's lemma (5.11) which was needed to reduce (5.12,b) to (5.12,a):

PROOF OF (5.11): Let us assume b). Since $\tau^*\mathcal{N}^\mu$ is nef, if and only if $\mathcal{N}$ is nef, and since $\nu(\mathcal{N}) = \nu(\tau^*\mathcal{N})$, we can assume that τ is an isomorphism. Moreover, we can assume $\mu_0 = 1$.
"$\mathcal{N}^\mu(-C)$ semi-ample for all $\mu > 0$" implies that $\mathcal{N}$ is nef. One obtains from (5.10)

$$\nu(\mathcal{N}) \geq \kappa(\mathcal{N}) \geq \kappa(\mathcal{N}^\mu(-C)) = \nu(\mathcal{N}^\mu(-C)).$$

For $\nu = \nu(\mathcal{N})$ the leading term in μ of

$$c_1(\mathcal{N}^\mu(-C))^\nu = (\mu \cdot c_1(\mathcal{N}) - C)^\nu$$

is $\mu^\nu \cdot c_1(\mathcal{N})^\nu$. Since this term intersects $H_1 \cdot \ldots \cdot H_{n-\nu}$ strictly positively, for general hyperplanes $H_1, \ldots, H_{n-\nu}$, we find $\nu \leq \nu(\mathcal{N}^\mu(-C))$.

To show the other direction, let $\phi_{\mu_0} : X \to Y$ be the rational map

$$X \longrightarrow \phi_{\mu_0}(X) = Y \subset \mathbb{P}(H^0(X, \mathcal{N}^{\mu_0})).$$

We can and we will assume that ϕ_{μ_0} has a connected general fibre, that dim $(Y) = \kappa(\mathcal{N})$ and, blowing X up if necessary, that ϕ_{μ_0} is a morphism. For some effective divisor D we have

$$\mathcal{N}^{\mu_0}(-D) = \phi_{\mu_0}^*\mathcal{L}, \quad \text{for} \quad \mathcal{L} \quad \text{ample on} \quad Y.$$

If F is a general fibre of ϕ_{μ_0}, then $D\,|_F$ is nef.

5.14. Claim. $D\,|_F$ is zero.

Assuming (5.14) we can blow up Y and X and assume that $D = \phi_{\mu_0}^*\Delta$ for some divisor Δ on Y. For example, blowing up Y one can assume that ϕ_{μ_0} factors over a flat morphism $\phi' : X' \to Y$ and that $\mathcal{N}$ is the pullback of a sheaf $\mathcal{N}'^{\mu_0}(-D') = \phi'^*\mathcal{L}$ for some semi-ample sheaf $\mathcal{L}$ and by (5.14) $D' \subseteq \phi'^*\Delta$ for some divisor Δ on Y. Since $\mathcal{N}'$ is numerically effective $D' {\cdot} C \geq 0$ for all curves C in X' contained in a fibre of ϕ'. This is only possible if $D' = \phi'^*\Delta$. Let us denote by $\tau : X \to Y$ the morphism obtained. We have $\mathcal{N}^{\mu_0} = \tau^*\mathcal{M}$ for some sheaf $\mathcal{M}$ on Y. Of course $\kappa(\mathcal{M}) = \dim Y$ and (5.12,b) holds true for $\mathcal{M}$ on Y, i.e. $\mathcal{M}^\mu(-\Gamma)$ is ample for some divisor $\Gamma > 0$ and all $\mu >> 0$. Then

$$\mathcal{N}^{\mu \cdot \mu_0}(-\tau^*\Gamma)$$

is semi-ample for all $\mu >> 0$.

□

PROOF OF (5.14): We may assume that $\mu_0 = 1$. For $\phi = \phi_1$, one has

$$c_1(\mathcal{N}) = c_1(\phi^*\mathcal{L}) + D = \phi^* c_1(\mathcal{L}) + D.$$

D is effective, hence $c_1(\mathcal{N})^{\nu_1} \cdot c_1(\phi^*\mathcal{L})^{\nu_2} \cdot D$ are semi-positive cycles, i.e. for $n = \nu_1 + \nu_2 + 1 + r$ one has

$$H_1 \cdot \ldots \cdot H_r \cdot c_1(\mathcal{N})^{\nu_1} \cdot c_1(\phi^*\mathcal{L})^{\nu_2} \cdot D \geq 0.$$

By definition

$$0 \equiv c_1(\mathcal{N})^{\nu+1} = c_1(\mathcal{N})^{\nu} \cdot (c_1(\phi^*\mathcal{L}) + D) \quad \text{for} \quad \nu = \nu(\mathcal{N}).$$

Since $c_1(\phi^*\mathcal{L})$ is also represented by an effective divisor, this is only possible if

$$0 \equiv c_1(\mathcal{N})^{\nu} \cdot c_1(\phi^*\mathcal{L}) = c_1(\mathcal{N})^{\nu-1} \cdot c_1(\phi^*\mathcal{L}) \cdot (c_1(\phi^*\mathcal{L}) + D).$$

The same argument shows that $c_1(\phi^*\mathcal{L})^2 \cdot c_1(\mathcal{N})^{\nu-1} \equiv 0$ and after ν steps we get

$$c_1(\phi^*\mathcal{L})^{\nu} \cdot c_1(\phi^*\mathcal{L}) + c_1(\phi^*\mathcal{L})^{\nu} \cdot D \equiv 0$$

and hence

$$c_1(\phi^*\mathcal{L})^{\nu} \cdot D = F \cdot D = 0.$$

$\square$

§ 6 Differential forms and higher direct images

The title of this lecture is a little bit misleading. We want to apply the vanishing theorems for differential forms with values in invertible sheaves of integral parts of $\mathbb{Q}$-divisors (which follow directly from (3.2), (4.8) and (4.13)) to some more concrete situations.

For invertible sheaves themselves one obtains thereby different proofs of (5.2), (5.6,b), (5.12,c) and (5.12,d) but, as far as we can see, nothing more. For $\Omega^a_X \otimes \mathcal{L}^{-1}$ we obtain the Kodaira-Nakano vanishing theorem and some generalizations. Finally we consider the vanishing for higher direct images, which can be reduced, as usually, to the global vanishing theorems. As a straightforward application one obtains vanishing theorems for certain non compact manifolds.

In Lecture 5 we could at least point out some of the intermediate steps which remain true in characteristic $p \neq 0$. However, since $(*)$ and (!) only make sense in characteristic 0, we can as well assume throughout this chapter:

6.1. Assumptions. X is a projective manifold defined over an algebraically closed field k of characteristic zero and $\mathcal{L}$ is an invertible sheaf on X.

Global vanishing theorems in characteristic $p > 0$ will appear, as far as it is possible, in Lecture 11.

6.2. Global vanishing theorem for integral parts of $\mathbb{Q}$-divisors.

For $X, \mathcal{L}$ as in (6.1) let

$$D = \sum_{j=1}^{r} \alpha_j D_i$$

be a normal crossing divisor, $N \in \mathbb{N}$ with $0 < a_j < N$ for $j = 1 \ldots r$ and $\mathcal{L}^N = \mathcal{O}_X(D)$. Then one has:
a)
$$H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{L}^{-1}) = 0,$$
for $a + b < n - cd(X, D)$ and for $a + b > n + cd(X, D)$.
b) *Let A and B be reduced divisors such that $D + A + B$ has normal crossings and such that A, B and D have pairwise no common component. Then*
$$H^b(X, \Omega^a_X(\log\ (A + B + D))(-B) \otimes \mathcal{L}^{-1}) = 0$$
for $a + b < n - cd(X, D + B)$ and for $a + b > n + cd(X, D + A)$.
c) *If there exists a proper morphism*
$$g : X - D \longrightarrow W$$
for an affine variety W, then one can replace $cd(X, D)$ by $r(g)$ in a) (see (4.10)).

Proof: By 3.2 $\mathcal{L}^{-1}(-B)$ has a logarithmic integrable connection ∇ with poles along $A + B + D$, such that the E_1-degeneration holds true. Moreover, $Res_{D_j}(\nabla) \notin \mathbb{Z}$ for $j = 1, \ldots, r$. For a component A_j of A we have $Res_{A_j}(\nabla) = 0$, and for a component B_j of B we have $Res_{B_j}(\nabla) = 1$. Hence a) follows from (4.8), b) from (4.9) and finally c) from (4.13).

□

6.3. Corollary. *For $X, \mathcal{L}$ as in (6.1), assume that $\mathcal{L}^N = \mathcal{O}_X(D)$ for a normal crossing divisor $D = \sum_{j=1}^r \alpha_j D_i$ with $0 < \alpha_j < N$ and assume that there exists an ample effective divisor B with $B_{red} = D_{red}$. Then $H^b(X, \mathcal{L}^{-1}) = 0$ for $b < n$.*

Proof: Apply (6.2,a), and (4.7,c).

□

2^ND^ proof of (5.12,c), (5.12,d) and (5.12.e).: As we have seen in Lecture 5, it is enough to proof (5.12,d) for $\kappa(\mathcal{L}^N(-D)) = \dim\ X$. Moreover, we may blow up, whenever we like.

We can write (replacing N and D by some multiple)
$$\mathcal{L}^N(-D) = \mathcal{A}(\Gamma)$$
for some effective divisor Γ and some ample sheaf $\mathcal{A}$. Blowing up, we can replace everything by some high multiple and subtract some effective divisor E from the pullback of $\mathcal{A}$ such that the sheaf obtained remains ample. Hence one can assume $D + \Gamma$ to be a normal crossing divisor. Since $\mathcal{L}^N(-D)$ is numerically effective, we can replace $\mathcal{A}$ by $\mathcal{A} \otimes \mathcal{L}^N(-D)$ and repeating this we can assume

that the multiplicities of $D + \Gamma$ are bounded by N. Finally, replacing again everything by some multiple we are reduced to the case that $\mathcal{A}$ is very ample. Writing

$$\mathcal{L}^N = \mathcal{O}_X(D') \quad \text{for} \quad D' = D + \Gamma + H,$$

H a general divisor for $\mathcal{A}$, we can apply (6.3).

□

6.4. Corollary (Akizuki-Kodaira-Nakano [1]). *For $X, \mathcal{L}$ as in (6.1) assume that $\mathcal{L}$ is ample. Then*

$$H^b(X, \Omega_X^a \otimes \mathcal{L}^{-1}) = 0 \quad \textit{for} \quad a + b < n.$$

Moreover, if $A + B$ is a reduced normal crossing divisor, the same holds true for

$$H^b(X, \Omega_X^a(\log\ (A + B))(-B) \otimes \mathcal{L}^{-1}).$$

Proof: We can write $\mathcal{L}^N = \mathcal{O}_X(D)$ for a non-singular divisor D and we may even assume that $D + A + B$ is a reduced normal crossing divisor. Moreover, for N large enough, $D + B$ and $D + A$ will both be ample and

$$cd(X, D + B) = cd(D + A) = 0.$$

By (2.3,b) one has an exact sequence

$$\begin{aligned} \ldots \longrightarrow H^{b-1}(D, \Omega_D^{a-1}(\log\ (A + B)\ |_D)(-B\ |_D) \otimes \mathcal{L}^{-1}) \longrightarrow \\ \longrightarrow H^b(X, \Omega_X^a(\log\ (A + B))(-B) \otimes \mathcal{L}^{-1}) \longrightarrow \\ \longrightarrow H^b(X, \Omega_X^a(\log\ (A + B + D))(-B) \otimes \mathcal{L}^{-1}) \longrightarrow \ldots \end{aligned}$$

By induction on dim X we can assume that the first group is zero for $a + b < n + 1$ and by (6.2,b) the last group is zero for $a + b \neq n$.

□

If one tries to weaken "$\mathcal{L}$ ample" in this proof, one has to replace it by some condition compatible with the induction step.

6.5. Definition. An invertible sheaf $\mathcal{L}$ is called *l-ample* if the following two conditions hold true:

a) $\mathcal{L}^N$ is generated by global sections, for some $N \in \mathbb{N} - \{0\}$, and hence $\phi_N : X \longrightarrow \mathbb{P}(H^0(X, \mathcal{L}^N))$ a morphism.

b) For N as in a) $\ l \geq \mathrm{Max}\{\dim\ \phi_N^{-1}(z); z \in \phi_N(X)\}$.

6.6. Corollary (A. Sommese [57], generalized in [22]).
For $X, \mathcal{L}$ as in (6.1) assume $\mathcal{L}$ to be l-ample. Then

$$H^b(X, \Omega_X^a \otimes \mathcal{L}^{-1}) = 0$$

for $a + b < Min\ \{\kappa(\mathcal{L}),\ \dim\ X - l + 1\}$.

PROOF: Using the notation from (6.5), we have seen in (4.11,b) that

$$r(\phi_N) \leq \text{Max } \{\dim\ X - \kappa(\mathcal{L}),\ l-1\ \}.$$

Hence (6.6) is a special case of the following more technical statement .

□

6.7. Corollary. *For $X, \mathcal{L}$ as in (6.1) let $\tau : X \to Y$ be a morphism and let E be an effective normal crossing divisor with $\tau^{-1}(\tau(X-E)) = X - E$. If for some ample sheaf $\mathcal{A}$ on Y and for some $\nu > 0$ one has $\mathcal{L}^\nu = \tau^*\mathcal{A}$, then*

$$H^b(X, \Omega^a_X(\log\ E) \otimes \mathcal{L}^{-1}) = 0$$

for $a + b < \dim\ X - r(\tau\ |_{X-E})$.

PROOF: If I is the ideal sheaf of $\tau(E)$, then $\mathcal{A}^\mu \otimes I$ will be generated by global sections for some $\mu \gg 0$. Hence, we may assume that $\mathcal{L}^N(-E)$ is generated by global section for $N = \nu \cdot \mu$. Moreover, we can assume that N is larger than the multiplicities of the components of E. If D is the divisor of a general section of $\mathcal{L}^N(-E)$, then $D + E$ is a normal crossing divisor. We have for

$$\tau : D \longrightarrow \tau(D),\ \mathcal{L}\ |_D \quad \text{and} \quad E\ |_D$$

the same assumptions as those asked for in 6.7. Moreover, by (4.11,d) we have

$$r(\tau\ |_{X-E}) + 1 \geq r(\tau\ |_{D-E}).$$

By induction on dim X we may assume that

$$H^{b-1}(D, \Omega^{a-1}_D(\log\ E\ |_D) \otimes \mathcal{L}^{-1}) = 0$$

for

$$a + b < n - r(\tau\ |_{X-E})\ \leq\ n + 1 - r(\tau\ |_{D-E}).$$

The exact sequence (see (2.3,b))

$$0 \to \Omega^a_X(\log\ E) \otimes \mathcal{L}^{-1} \to \Omega^a_X(\log\ (D+E)) \otimes \mathcal{L}^{-1} \to \Omega^{a-1}_D(\log\ E\ |_D) \otimes \mathcal{L}^{-1} \to 0$$

implies that for those a, b the map

$$H^b(X, \Omega^a_X(\log\ E) \otimes \mathcal{L}^{-1}) \longrightarrow H^b(X, \Omega^a_X(\log\ (D+E)) \otimes \mathcal{L}^{-1})$$

is injective. However, since

$$r(\tau\ |_{X-E}) \geq r(\tau\ |_{X-(D+E)})$$

(6.2,c) tells us that

$$H^b(X, \Omega^a_X(\log\ (D+E)) \otimes \mathcal{L}^{-1}) = 0$$

for $a + b < \dim\ X - r(\tau\ |_{X-E})$. □

6.8. Remarks.
a) The reader will have noticed that (6.7) does not use the full strength of (6.2). Several other applications and extensions of (6.2) can be found in the literature, (see for example [2] [3] [43] [44]).
b) Sometimes it is nicer to use the dual version of (6.7). Since

$$\bigwedge^n \Omega^1_X(\log\ E) = \omega_X \otimes \mathcal{O}(E_{red})$$

we find the dual of $\Omega^a_X(\log\ E)$ to be

$$\omega_X^{-1} \otimes \Omega^{n-a}_X(\log\ E)(-E_{red})$$

and by Serre duality (6.7) is equivalent to the vanishing of

$$H^b(X, \Omega^a_X(\log\ E)(-E_{red}) \otimes \mathcal{L})$$

for

$$a + b > \dim\ X + r(\tau\ |_{X-E}).$$

c) For (6.7) we used the invariant $r(g)$ and lemma (4.12), the latter being proved by analytic methods. However, playing around with de Rham complexes and their hypercohomology, one should be able to find an algebraic analogue of those arguments.

In [63] the second author used the Hodge duality (as in (3.23)) to reduce vanishing for $H^b(X, \mathcal{L}^{-1})$ to the Bogomolov-Sommese vanishing theorem. The latter fits nicely into the scheme explained in this lecture, (see the proof of (13.10,a)).

6.9. Corollary (F. Bogomolov, A. Sommese). *For $X, \mathcal{L}$ as in (6.1) and for a normal crossing divisor B one has*

$$H^0(X, \Omega^a_X(\log\ B) \otimes \mathcal{L}^{-1}) = 0$$

for $a < \kappa(\mathcal{L})$.

PROOF: (6.9) is compatible with blowing ups and we can assume that

$$\phi_N : X \longrightarrow \mathbb{P}(H^0(X, \mathcal{L}^N))$$

is a morphism. For N large enough, we can choose D such that $\phi_N\ |_{X-D}$ has equidimensional fibres of dimension $n - \kappa(\mathcal{L})$ and such that $\mathcal{L}^N = \mathcal{O}_X(D)$. Moreover we may assume $B + D$ to be a normal crossing divisor. By (6.2,c)

$$H^0(X, \Omega^a_X(\log\ (B + D)) \otimes \mathcal{L}^{(1)^{-1}}) = 0$$

for $a < \kappa(\mathcal{L})$. As $\Omega^a_X(\log\ B) \otimes \mathcal{L}^{-1}$ is a subsheaf of $\Omega^a_X(\log\ (B + D)) \otimes \mathcal{L}^{(1)^{-1}}$, one obtains (6.8).

□

Global vanishing theorems always give rise to the vanishing of certain direct image sheaves.

6.10. Notations. Let X be a manifold, defined over an algebraically closed field of characteristic zero and let $f : X \to Z$ be a proper surjective morphism. Let $\mathcal{L}$ be an invertible sheaf on X. We will call $\mathcal{L}$
a) *f-numerically effective* if for all curves C in X with dim $f(C) = 0$ one has deg $(\mathcal{L} \mid_C) \geq 0$
b) *f-semi-ample* if for some $N > 0$ the natural map $f^* f_* \mathcal{L}^N \longrightarrow \mathcal{L}^N$ is surjective.

The relative Grauert-Riemenschneider vanishing theorem says, that for a birational morphism $f : X \to Z$ one has $R^b f_* \omega_X = 0$ for $b > 0$. As a generalization one obtains:

6.11. Corollary.
a) *For $f : X \to Z$ as in (6.10) let $\mathcal{L}$ be an invertible sheaf such that $\mathcal{L}^N(-D)$ is f-numerically effective for a normal crossing divisor*

$$D = \sum_{j=1}^{r} \alpha_j D_j.$$

Then

$$R^b f_*(\mathcal{L}^{(1)} \otimes \omega_X) = 0 \quad \textit{for} \quad b > \dim\ X - \dim\ Z - \kappa(\mathcal{L} \mid_F)$$

where F is a general fibre of f.
b) *In particular, if $f : X \to Z$ is birational and if $D = f^*\Delta$ is a normal crossing divisor for some effective Cartier divisor Δ on Z, then*

$$R^b f_*(\omega_X \otimes \mathcal{O}_X(-[\frac{D}{N}])) = 0 \quad \textit{for} \quad b > 0.$$

Proof: Obviously, since $\mathcal{O}_X(-D) = f^*\mathcal{O}_Z(-\Delta)$ is f numerically effective, b) is a special case of a).
As usual, in a), we can add the assumption $0 < \alpha_j < N$ for $j = 1, \ldots r$, and we will have $\mathcal{L}^{(1)} = \mathcal{L}$.

The statement being local in Z, we can assume Z to be affine or, compactifying X and Z, we can assume Z to be projective. By (5.13) we are allowed to blow X up and we can assume that X is projective as well.
The assumptions made imply that for $\mathcal{A}$ ample invertible on Z

$$f^*\mathcal{A}^\nu \otimes \mathcal{L}^N(-D)$$

will be numerically effective for $\nu >> 0$ and that

$$\kappa(f^*\mathcal{A}^\nu \otimes \mathcal{L}) \geq \kappa(\mathcal{L} \mid_F) + \dim\ (Z).$$

Using Serre's vanishing (1.1) we can assume that for all $c > 0$ and $b \geq 0$

$$H^c(Z, \mathcal{A}^\nu \otimes R^b f_*(\mathcal{L} \otimes \omega_X)) = 0$$

and that $H^0(Z, \mathcal{A}^\nu \otimes R^b f_*(\mathcal{L} \otimes \omega_X))$ generates the sheaf $\mathcal{A}^\nu \otimes R^b f_*(\mathcal{L} \otimes \omega_X)$. By the Leray spectral sequence (A.27) we obtain

$$H^b(X, f^*\mathcal{A}^\nu \otimes \mathcal{L} \otimes \omega_X) = H^0(Z, \mathcal{A}^\nu \otimes R^b f_*(\mathcal{L} \otimes \omega_X))$$

and by (5.12,d) this group is zero for

$$b > \dim\ X - \dim\ (Z) - \kappa(\mathcal{L}\mid_F) \geq \dim\ X - \kappa(f^*\mathcal{A}^\nu \otimes \mathcal{L}).$$

□

In the special case for which $\mathcal{L}^N(-D)$ is f-semi-ample the vanishing of

$$R^b f_*(\mathcal{L} \otimes \omega_X) \quad \text{for} \quad b > \dim\ X - \dim Z$$

follows as well from the next statement, due to J. Kollár, [40].

6.12. Corollary of 5.12,a) (J. Kollár). *In addition to the assumption of (6.11,a) we even assume that $\mathcal{L}^N(-D)$ is f-semi-ample. Then*

$$R^b f_*(\mathcal{L}^{(1)} \otimes \omega_X)$$

has no torsion for $b \geq 0$.

PROOF: As above we can assume X and Z to be projective and $\mathcal{L}^N(-D)$ to be semi-ample. Moreover, we may assume $\mathcal{L}^N(-D)$ to contain $f^*\mathcal{A}$ for a very ample sheaf $\mathcal{A}$ on Z, that $\mathcal{L} = \mathcal{L}^{(1)}$, that

$$H^c(X, R^b f_*(\mathcal{L} \otimes \omega_X)) = 0 \quad \text{for} \quad c > 0$$

and that $R^b f_*(\mathcal{L} \otimes \omega_X)$ is generated by its global sections. If $R^b f_*(\mathcal{L} \otimes \omega_X)$ has torsion for some b, then the map

$$R^b f_*(\mathcal{L} \otimes \omega_X) \longrightarrow R^b f_*(\mathcal{L} \otimes \omega_X) \otimes \mathcal{O}_Z(A)$$

has a non-trivial kernel $\mathcal{K}$ for some effective ample divisor A on Z. We may assume that $\mathcal{O}_Z(A) = \mathcal{A}$. Replacing $\mathcal{L}$ by $\mathcal{L} \otimes f^*\mathcal{A}^\nu$ again, we can assume that $H^0(Z, \mathcal{K}) \neq 0$. For $B = f^*A$,this implies that $H^0(Z, \mathcal{K})$ lies in the kernel of

$$H^b(X, \mathcal{L} \otimes \omega_X) \longrightarrow H^b(X, \mathcal{L} \otimes \omega_X(B))$$

and hence that

$$H^{n-b}(X, \mathcal{L}^{-1}(-B)) \longrightarrow H^{n-b}(X, \mathcal{L}^{-1})$$

is not surjective, contradicting (5.12,a).

□

Some of the vanishing theorems mentioned and a partial degeneration of the Hodge to de Rham spectral sequence remain true for certain non-compact manifolds. One explanation for those results, obtained by I. Bauer and S. Kosarew in [4] and [42] by different methods, is the following lemma.

6.13. Lemma. *Let Z be a projective variety in characteristic zero, $U \subseteq Z$ be an open non-singular subvariety, $\delta : X \to Z$ be a desingularization such that $\iota : U \simeq \delta^{-1}(U) \to X$. Assume that $X - \iota(U) = E$ for a reduced normal crossing divisor E. Then, for $a + b < \dim\ X - \dim\ \delta(E) - 1$ and all invertible sheaves $\mathcal{M}$ on Z one has*

$$H^b(X, \Omega^a_X(\log\ E) \otimes \delta^*\mathcal{M}) = H^b(U, \Omega^a_X \otimes \mathcal{M}\ |_U).$$

Proof: Let $\mathcal{A}$ be an ample invertible sheaf on Z and let E' be an effective exceptional divisor, such that $\mathcal{O}_X(-E')$ is relatively ample for δ.
For fixed $\nu \geq 0$ we can choose $\mathcal{A}$ large enough, such that for all a, b

$$R^b\delta_*(\Omega^a_X(\log\ E) \otimes \mathcal{O}_X(-E - \nu \cdot E')) \otimes \mathcal{A}$$

is generated by global sections and

$$H^c(Z, R^b\delta_*(\Omega^a_X(\log\ E) \otimes \mathcal{O}_X(-E - \nu \cdot E')) \otimes \mathcal{A}) = 0$$

for $c > 0$. Moreover, for $\nu > 0$, we may assume that $\tau^*\mathcal{A}(-\nu \cdot E')$ is ample. Using Serre duality (as explained in (6.8,b)) and the Leray spectral sequence (A.27) one finds

$$H^{n-b}(X, \Omega^{n-a}_X(\log\ E) \otimes \delta^*\mathcal{A}^{-1}(\nu \cdot E'))^* =$$
$$H^b(X, \Omega^a_X(\log\ E)(-E) \otimes \delta^*\mathcal{A}(-\nu \cdot E')) =$$
$$H^0(Z, R^b\delta_*(\Omega^a_X(\log\ E) \otimes \mathcal{O}_X(-E - \nu \cdot E')) \otimes \mathcal{A}).$$

By (6.4), for $\nu > 0$, or by (6.7), for $\nu = 0$, we find

$$R^b\delta_*(\Omega^a_X(\log\ E) \otimes \mathcal{O}_X(-E - \nu \cdot E')) = 0$$

for $a + b > \dim\ X$. For those a and b and for

$$\mathcal{K}_\nu = \mathcal{O}_X / \mathcal{O}_X(-\nu \cdot E')$$

we have

$$R^b\delta_*(\Omega^a_X(\log\ E)(-E) \otimes \mathcal{K}_\nu) = 0.$$

One obtains for $a + b > \dim\ X + \dim\ \tau(E)$ from the Leray spectral sequence (A.27) that

$$H^b(X, \delta^*\mathcal{M}^{-1} \otimes \Omega^a_X(\log\ E)(-E) \otimes \mathcal{K}_\nu) = 0.$$

Hence for all $\nu \geq 0$ and $a + b > \dim\ X + \dim\ \tau(E) + 1$ the map

$$H^b(X, \delta^*\mathcal{M}^{-1} \otimes \Omega^a_X(\log\ E) \otimes \mathcal{O}_X(-E - \nu \cdot E')) \longrightarrow$$

$$\longrightarrow H^b(X, \delta^*\mathcal{M}^{-1} \otimes \Omega_X^a(\log\ E)(-E))$$

is bijective. By Serre duality again,

$$H^b(X, \delta^*\mathcal{M} \otimes \Omega_X^a(\log\ E)) \longrightarrow H^b(X, \delta^*\mathcal{M} \otimes \Omega_X^a(\log\ E) \otimes \mathcal{O}_X(\nu \cdot E'))$$

is an isomorphism for $a + b < \dim\ X - \dim\ \tau(E) - 1$ and, taking the limit for $\nu \in \mathbb{N}$, we obtain (6.13).

□

6.14. Corollary (I. Bauer, S. Kosarew [4]). *Let Z be a projective variety in characteristic zero, $U \subseteq Z$ be an open non-singular subvariety. Then, for $k < n - \dim\ (Z - U) - 1$ one has*

$$\dim \mathbb{H}^k(U, \Omega_U^\bullet) = \sum_{a+b=k} \dim H^b(U, \Omega_U^a).$$

PROOF: We can choose a desingularisation $\delta : X \to Z$ and E as in (6.13). Then we have a natural map of spectral sequences

$$\begin{array}{ccc} E_1^{ab} = H^b(U, \Omega_U^a) & \Longrightarrow & \mathbb{H}^{a+b}(U, \Omega_U^\bullet) \\ \big\uparrow \varphi_{a,b} & & \big\uparrow \varphi \\ E_1'^{ab} = H^b(X, \Omega_X^a(\log\ E)) & \Longrightarrow & \mathbb{H}^{a+b}(X, \Omega_X^\bullet(\log\ E)). \end{array}$$

Since $E_1'^{ab}$ degenerates in E_1 and since $\varphi_{a,b}$ are isomorphisms for

$$a + b < n - \dim\ (Z - U) - 1$$

the second spectral sequence has to degenerate for those a, b.

□

6.15. Corollary (see also I. Kosarew, S. Kosarew [42]).
For Z and U as in (6.14) let $\mathcal{L}$ be an l-ample invertible sheaf on Z. Then

$$H^b(U, \Omega_U^a \otimes \mathcal{L}^{-1}|_U) = 0$$

for

$$a + b < Min\ \{\kappa(\mathcal{L}), \dim\ X - l + 1,\ \dim\ X - \dim\ (Z - U) - 1\ \}$$

PROOF: For X, E as in (6.13) and $\mathcal{M} = \delta^*\mathcal{L}$ we have by (6.7)

$$H^b(X, \mathcal{M}^{-1} \otimes \Omega_X^a(\log\ E)) = 0$$

for $a + b < \dim\ X - r(\tau|_U)$ where

$$\tau : X \xrightarrow{\delta} Z \xrightarrow{\phi_N} \phi_N(Z)$$

is the composition of δ with the map given by global sections of $\mathcal{L}^N$.
However,

$$r(\tau\mid_U) \leq r(\phi_N) \leq \text{Max}\ \{\dim\ X - \kappa(\mathcal{L}), l-1\}.$$

□

6.16. Remark. For $k = \mathbb{C}$, the reason for which certain coherent sheaves $\mathcal{F}$ on Z satisfy $H^b(Z, \mathcal{F} \otimes \mathcal{H}) = 0$ for $\mathcal{H}$ ample, seems to be related to the existence of connections. This point of view, which is exploited in J. Kollár's work on vanishing theorems [40], [41] and extended by M. Saito (see [54] and the references given there), should imply that sheaves arising as natural subquotients of $\mathcal{O}_Z \otimes R^k f_* V$ for a morphism $f : X \to Z$ of manifolds and a locally constant system V, sometimes have vanishing properties as the one stated above.
J. Kollár proved, for example, that for a morphism $f : X \to Z$, where X and Z are projective varieties and X non-singular, one has

$$H^c(Z, R^b f_* \omega_X \otimes \mathcal{H}) = 0$$

for $c > 0$ and $\mathcal{H}$ ample on Z.

Slightly more generally one has

6.17. Corollary (of (5.12,b)). *Let $f : X \to Z$ be a surjective morphism of projective varieties defined over an algebraically closed field of characteristic zero, with X non-singular. Let $\mathcal{L}$ be an invertible sheaf on X,*

$$D = \sum_{j=1}^{r} \alpha_j D_j$$

a normal crossing divisor and $N \in \mathbb{N}$ with

$$0 < \alpha_j < N \quad \text{for} \quad j = 1, \ldots, r.$$

a) *If $\mathcal{L}^N(-D)$ is semi-ample and $\mathcal{K}$ a numerically effective invertible sheaf on Z with $\kappa(\mathcal{K}) = \dim\ Z$, then for $c > 0$ and $b \geq 0$*

$$H^c(Z, \mathcal{K} \otimes R^b f_*(\omega_X \otimes \mathcal{L})) = 0.$$

b) If $\mathcal{L}^N(-D)$ is numerically effective, $\kappa(\mathcal{L}^N(-D)) = \nu(\mathcal{L}^N(-D))$, and if $(\mathcal{L}^N(-D))^\mu$ contains $f^*\mathcal{H}$ for some ample sheaf $\mathcal{H}$ on Z and some $\mu > 0$, then for $c > 0$ and all b

$$H^c(Z, R^b f_*(\omega_X \otimes \mathcal{L})) = 0.$$

Proof: By (5.10,b), replacing $\mathcal{L}$ by $\mathcal{L} \otimes f^*\mathcal{K}$, a) follows from b).
If H is the zero divisor of a general section of $\mathcal{H}^\mu$ for $\mu \gg 0$ and if $B = f^*H$ then B is a non-singular divisor and the assumptions of (5.12,b) hold true. Hence

$$H^b(X, \omega_X \otimes \mathcal{L}) \longrightarrow H^b(X, \omega_X(B) \otimes \mathcal{L})$$

is injective for all b. Since H is in general position we have exact sequences

$$\begin{array}{ccccccc} 0 \longrightarrow & R^b f_*(\mathcal{L}\otimes\omega_X) & \longrightarrow & R^b f_*(\mathcal{L}\otimes\omega_X(B)) & \longrightarrow & R^b f_*(\mathcal{L}\otimes\omega_B) \to 0 \\ & \| & & \| & & \| \\ & R^b f_*(\mathcal{L}\otimes\omega_X) & \longrightarrow & \mathcal{O}_Z(H)\otimes R^b f_*(\mathcal{L}\otimes\omega_X) & \longrightarrow & \mathcal{O}_Z(H)\otimes R^b f_*(\mathcal{L}\otimes\omega_X)\mid_H \end{array}$$

By induction on dim Z we may assume that

$$H^c(H, R^b f_*(\mathcal{L}\otimes\omega_B)) = 0 \quad \text{for} \quad c > 0$$

and, if we choose μ large enough, we find by Serre's vanishing theorem

$$H^c(Z, R^b f_*(\mathcal{L}\otimes\omega_X)) = 0 \quad \text{for} \quad c \geq 2.$$

In the Leray-spectral sequence (see A.27) all the differentials are zero, since $E_2^{ab} \neq 0$ just for $a = 0$ or $a = 1$, and hence the upper line in the following diagram is exact.

$$\begin{array}{ccccc} 0 \to H^1(Z, R^{b-1}f_*(\mathcal{L}\otimes\omega_X)) \to & H^b(X,\mathcal{L}\otimes\omega_X) & \to & H^0(Z, R^b f_*(\mathcal{L}\otimes\omega_X)) \\ & {\scriptstyle\alpha}\downarrow & & \downarrow \\ & H^b(X,\mathcal{L}\otimes\omega_X(B)) & \to & H^0(Z, R^b f_*(\mathcal{L}\otimes\omega_X(B))) \end{array}$$

Since α is injective and since

$$H^b(X,\mathcal{L}\otimes\omega_X(B)) = H^0(Z, R^b f_*(\mathcal{L}\otimes\omega_X(B)))$$

we find $H^1(Z, R^{b-1}f_*(\mathcal{L}\otimes\omega_X)) = 0$ for all b.

□

§ 7 Some applications of vanishing theorems

The vanishing theorems for integral parts of $\mathbb{Q}$-divisors and for numerically effective sheaves (5.12,c) and (5.12,d), as well as (5.6,a) turned out to be useful for applications in higherdimensional complex projective geometry. We will not be able in these notes to include an outline of the Iitaka-Mori classification of threefolds, and the reader interested in this direction is invited to regard S. Mori's beautiful survey [46].

In this lecture we just want to give a flavour as to how one should try to use vanishing theorems to attack certain types of questions. The choice made is obviously influenced by our personal taste.

We will assume in this lecture:

All varieties are defined over an algebraically closed field k of characteristic zero.

7.1. Example: Surfaces of general type. For a projective surface S' of general type, i.e. for a non-singular S' with $\kappa(\omega_{S'}) = \dim S' = 2$, one can blow down exceptional curves $E \simeq \mathbb{P}^1$ with $E \cdot E = -1$ ([30], p. 414). After finitely many steps one obtains a surface S without any exceptional curve, a minimal model of S' ([30], p. 418). S is characterised by

7.2. Claim. ω_S is *nef*.

PROOF: $\kappa(S) \geq 0$ implies that $\omega_S^N = \mathcal{O}_S(D)$ for D effective. A curve C with $\deg\ (\omega_S\ |_C) < 0$ must be a component of D and $C^2 < 0$. However, the adjunction formula gives

$$-2 \leq 2g(C) - 2 = \deg\ (\omega_S\ |_C) + C^2.$$

Hence the only solution is $C^2 = -1$ and $\deg(\omega_S\ |_C) = -1$, which forces C to be exceptional.

□

D. Mumford in his appendix to [65] used the contraction of (-2) curves to show:

7.3. Theorem. *If S is a minimal model and $\kappa(\omega_S) = 2$, then ω_S is semi-ample.*

X. Benveniste and Y. Kawamata ($\dim X = 3$) and Y. Kawamata and V. Shokurov (see [46] for the references) generalised (7.3) to the higher dimensional case. Their ideas, cut back to the surface case, give a simple proof of (7.3).

PROOF OF 7.3 (FROM THE DIPLOM-THESIS OF T. NAKOVICH, ESSEN):
Step 1.: If $p \in S$ does not lie on any curve C with $\deg\ (\omega_S\ |_C) = 0$, then for some $\nu \gg 0$ there is $s \in H^0(S, \omega_S^\nu)$ with $s(p) \neq 0$.

PROOF: Let $\tau : S' \to S$ be the blowing up of p and E the exceptional curve. One has

$$\deg\ (\tau^* \omega_S^\mu(-E)\ |_{C'}) = \mu \cdot \deg\ (\omega_S\ |_C) - E \cdot C'$$

for curves C' in S' with $C = \tau(C') \neq p$. Hence, for some $\mu \gg 0$ the sheaf $\mathcal{L} = \tau^* \omega_S^\mu(-E)$ will be nef. By (5.12,c) we find

$$H^1(S', \mathcal{L}^{-2}) \cong H^1(S', \omega_{S'} \otimes \mathcal{L}^2) = H^1(S', \tau^* \omega_S^{2\mu+1}(-E)) = 0$$

and hence

$$H^0(S', \tau^* \omega_X^{2\mu+1}) \longrightarrow H^0(E, \mathcal{O}_E)$$

is surjective.

□

Step 2.: For $\nu \gg 0$ let

$$D = \sum_{j=1}^{r} \alpha_j C_j$$

be the base locus of ω_S^ν (i.e. $\omega_S^\nu(-D)$ is generated by $H^0(S, \omega_S^\nu)$ outside of a finite number of points). Then D is a normal crossing divisor, $C_j^2 = -2$ for $j = 1, \ldots, r$ and ω_S^ν is generated by $H^0(S, \omega_S^\nu)$ outside of D.

PROOF: By step 1, if for some $p \in S$ there is no section s of ω_S^ν with $s(p) \neq 0$, then p lies on some curve C with deg $(\omega_S \mid_C) = 0$ and necessarily C is contained in the base locus. We know thereby that deg $(\omega_S \mid_{C_j}) = 0$ for the components C_j of D. By the Hodge-index theorem ([30], p. 364) one finds for any reduced subdivisor C of D that $C \cdot C < 0$. If we take $C = C_j$, then the adjunction formula shows that

$$C \cdot C = -2 \quad \text{and} \quad C \simeq \mathbb{P}^1.$$

For $C = (C_1 + C_2)$ we get

$$C_1 \cdot C_2 < 2$$

and C_1 and C_2 intersect transversally.

□

Step 3. For D as in Step 2, $\omega_S^\nu(-D)$ is nef, hence $\omega_S^N(-D)$ for $N \geq \nu$ is nef as well. We can choose $N > \alpha_j$ for $j = 1, \ldots r$. For some $i > 0$

$$D' = [\frac{i \cdot D}{N}] = \sum_{j=1}^{r} [\frac{i \cdot \alpha_j}{N}] \cdot C_j$$

will be reduced and non zero. By (5.12,c) again, we have for $\mathcal{L} = \omega_S$

$$H^1(S, \mathcal{L}^{(i)^{-1}}) \cong H^1(S, \omega_S \otimes \mathcal{L}^{(i)}) = H^1(S, \omega_S^{i+1}(-D')) = 0$$

and

$$H^0(S, \omega_S^{i+1}) \longrightarrow H^0(D', \omega_S^{i+1} \mid_{D'})$$

is surjective. Since the right hand side is nontrivial (in fact its dimension is just the number of connected components of D'), for $i+1$ the base locus does not contain D'. After finitely many steps we are done.

□

The proof of (7.1) is a quite typical example in two respects. First of all, vanishing of H^1 or more general by the surjectivity of the adjunction map in (5.6,a) allows to pull back sections of invertible sheaves on divisors. Secondly it shows again how to play around with integral parts, a method which already appeared in the proof of (5.12).

Corollary (5.12), as stated, has the disadvantage that D has to be a normal crossing divisor. Let us try next to study some weaker conditions.

7.4. Definition. Let X be a normal variety and D be an effective Cartier divisor on X. Let $\tau : X' \to X$ be a blowing up, such that X' is non singular and $D' = \tau^* D$ is a normal crossing divisor. We define:
a)

$$\omega_X\{\frac{-D}{N}\} = \tau_* \omega_{X'}(-[\frac{D'}{N}]).$$

b) $\mathcal{C}_X(D, N) =$ Coker $(\omega_X\{\frac{-D}{N}\} \longrightarrow \omega_X)$ where ω_X is the reflexive hull of ω_{X_0} for $X_0 = X - \text{Sing }(X)$.
c) (see [23])

$$e(D) = \text{Min }\{N > 0; \mathcal{C}_X(D, N) = 0\}.$$

d) If X is compact and $\mathcal{L}$ invertible, $H^0(X, \mathcal{L}) \neq 0$, then

$$e(\mathcal{L}) = \text{Max }\{e(D);\ D \geq 0 \ \text{ and } \ \mathcal{O}_X(D) = \mathcal{L}\}.$$

7.5. Properties (see [23]). Let X and D be as in (7.4).
a) If X has at most rational singularities, then $e(D)$ is finite.
b) If X is non-singular and D a normal crossing divisor then

$$\omega_X\{\frac{-D}{N}\} = \omega_X(-[\frac{D}{N}]).$$

c) $\omega_X\{\frac{-D}{N}\}, \mathcal{C}_X(D, N)$ and $e(D)$ are independent of the blowing up $\tau : X' \to X$ choosen.
d) Let H be a prime Cartier divisor on X, not contained in D, such that H is normal. Then one has a natural inclusion

$$\omega_H\{\frac{-D\mid_H}{N}\} \longrightarrow \omega_X\{\frac{-D}{N}\} \otimes \mathcal{O}_X(H)\mid_H .$$

e) If in d) X and H have rational sigularities, then for $N \geq e(D\mid_H)$, H does not meet the support of $\mathcal{C}_X(D, N)$.

Proof: a) is obvious since for $N \gg 0$

$$\tau_*\omega_{X'}(-[\frac{D}{N}]) = \tau_*\omega_{X'} = \omega_X.$$

Similar to (5.13), part b) can be deduced from (3.24) and from the fact, that quotient singularities are rational singularities. A more direct argument is as follows. We have an inclusion

$$\omega_X\{\frac{-D}{N}\} \longrightarrow \omega_X(-[\frac{D}{N}])$$

and it is enough to prove b) for some blowing up dominating τ. Hence, it is enough to consider the case that τ is a sequence of blowings with non-singular centers. Let us write

$$\omega_{X'} = \tau^* \omega_X \otimes \mathcal{O}_{X'}(\sum_{i=1}^{t} \alpha_i \cdot E_i).$$

For $m_i = \mathrm{codim}_X(\tau(E_i))$ one has $\alpha_i \geq m_i - 1$.

In fact, assume this to hold true for $\tau_1 : X_1 \longrightarrow X$ and

$$\omega_{X_1} = \tau_1^* \omega_X \otimes \mathcal{O}_{X_1}(\sum_{i=1}^{t-1} \alpha_i \cdot E'_i).$$

If $\delta : X' \longrightarrow X_1$ is the blowing up with center S and E_t the exceptional divisor then, for $m = \mathrm{codim}_{X_1}(S)$, one has

$$\omega_{X'} = \delta^* \omega_{X_1} \otimes \mathcal{O}_{X'}((m-1) \cdot E_t)$$

(see [30], p. 188). If $m_t > m$ then S lies on some E'_ν with $m_t - m \leq m_\nu - 1$. Hence

$$\alpha_t \geq m - 1 + \alpha_\nu \geq m_t - 1.$$

On the other hand, assume that $\tau(E_\mu)$ lies on s different components of D, let us say on $D_1, \cdots, D_s$ but not in D_j for $j > s$. Then $m_\mu \geq s$ and, if

$$D = \sum_{j=1}^{r} \alpha_j D_j$$

one has

$$[\sum_{j=1}^{s} \frac{\alpha_j}{N}] \leq \sum_{j=1}^{s} [\frac{\alpha_j}{N}] + s - 1 \leq \sum_{j=1}^{s} [\frac{\alpha_j}{N}] + \alpha_\mu.$$

One obtains

$$[\frac{D'}{N}] \leq \tau^* [\frac{D}{N}] + \sum_{i=1}^{t} \alpha_i \cdot E_i$$

and hence

$$\tau^* \omega_X(-[\frac{D}{N}]) \subset \omega_{X'}(-[\frac{D'}{N}]).$$

c) follows from b). Hence in d) we may assume that D' intersects the proper transform H' on H transversally and, of course, that H' is non-singular. Then

$$[\frac{D'}{N}] \mid_{H'} = [\frac{D' \mid_{H'}}{N}].$$

One has a commutative diagram

$$\begin{array}{ccccc}
\tau_* \omega_{X'}(-[\frac{D'}{N}] + H') & \xrightarrow{\alpha} & \tau_* \omega_{H'}(-[\frac{D'|_{H'}}{N}]) & \longrightarrow & \omega_H \\
\downarrow & & & & \downarrow = \\
\tau_* \omega_{X'}(-[\frac{D'}{N}]) \otimes \mathcal{O}_X(H) & \longrightarrow & \tau_* \omega_{X'}(-[\frac{D'}{N}]) \otimes \mathcal{O}_X(H) \mid_H & \xrightarrow{\gamma} & \omega_H
\end{array}$$

The cokernel of α lies in $R^1\tau_*\omega_{X'}(-[\frac{D'}{N}])$, and (6.11) shows that α is surjective. We obtain therefore a non-trivial morphism

$$\alpha' : \omega_H\{\frac{-D\mid_H}{N}\} \longrightarrow \omega_X\{\frac{-D}{N}\} \otimes \mathcal{O}_X(H)\mid_H .$$

Since $\omega_H\{\frac{-D|_H}{N}\}$ is torsion free d) holds true.
In e) we know that $\omega_H\{\frac{-D|_H}{N}\}$ is isomorphic to ω_H. Hence γ is surjective. Therefore $\omega_X\{\frac{-D}{N}\} \otimes \mathcal{O}_X(H)$ must be isomorphic to $\omega_X \otimes \mathcal{O}_X(H)$ in a neighbourhood of H.

$\square$

7.6. Remark. The diagram used to prove d) gives slightly more. Instead of assuming that $D' = \tau^*D$ it is enough to take any normal crossing divisor D' on X' not containing H'. Then the inclusion

$$\tau_*\omega_{H'}(-[\frac{D'\mid_{H'}}{N}]) \hookrightarrow \tau_*\omega_{X'}(-[\frac{D}{N}]) \otimes \mathcal{O}_X(H)\mid_H$$

exists whenever $H' + D'$ is a normal crossing divisor and $\mathcal{O}_{X'}(-D')$ is τ-numerically effective (see (6.10)).

Up to now, we do not even know that $e(\mathcal{L})$ is finite. This however follows from the first part of the next lemma, since every sheaf $\mathcal{L}$ lies in some ample invertible sheaf.

7.7. Lemma. *Let X be a projective manifold and let $\mathcal{L}$ be an invertible sheaf.*
a) *If $\mathcal{L}$ is very ample and $\nu > 0$, then*

$$e(\mathcal{L}^\nu) \leq \nu \cdot c_1(\mathcal{L})^{\dim X} + 1$$

b) *For $s \in H^0(X, \mathcal{L})$ with zero-locus D assume that for some $p \in X$ the section s has the multiplicity μ i.e.:*

$$s \in m_p^\mu \otimes \mathcal{L} \quad \textit{but} \quad s \notin m_p^{\mu+1} \otimes \mathcal{L}.$$

Then

$$\omega_X\{\frac{-D}{N}\} \longrightarrow \omega_X$$

is an isomorphism in a neighbourhood of p for $N > \mu$.
c) *If under the assumption of* b)

$$\mu' = [\frac{\mu}{N}] - \dim X + 1 \geq 0$$

then $\omega_X\{\frac{-D}{N}\}$ is contained in $m_p^{\mu'} \otimes \omega_X$.

PROOF: a) Let $D \geq 0$ be a divisor, $\mathcal{O}_X(D) = \mathcal{L}^\nu$.

If X is a curve then $[\frac{D}{N}] = 0$ for $N > \deg\ D + 1 = \nu \cdot c_1(\mathcal{L}) + 1$.

In general, let H be the divisor of a general section of $\mathcal{L}$. By induction

$$e(\mathcal{L}^\nu\mid_H) \leq \nu \cdot c_1(\mathcal{L}\mid_H)^{\dim H} + 1 = \nu \cdot c_1(\mathcal{L})^{\dim X} + 1.$$

(7.5,e) tells us that $\mathcal{C}_X(D,N)$ is supported outside of H for

$$N \geq \nu \cdot c_1(\mathcal{L})^{\dim X} + 1$$

and moving H we find $\mathcal{C}_X(D,N) = 0$.

For b) and c) we may assume that the blowing up $\tau : X' \to X$ factors through the blowing up $\varrho : X_p \to X$ of p. For $D_p = \varrho^* D$ and for the exceptional divisor E of ϱ we have

$$\Delta = D_p - \mu \cdot E \geq 0$$

and Δ does not contain E. Assume $N > \mu$. One has

$$\mathcal{O}_E(\Delta\mid_E) = \mathcal{O}_{\mathbf{P}^{n-1}}(\mu)$$

and, by part a), one obtains

$$\omega_E\{\frac{-\Delta\mid_E}{N}\} = \omega_E.$$

From (7.5,e) one knows that

$$\omega_{X_p}\{\frac{-\Delta}{N}\} \longrightarrow \omega_{X_p}$$

is an isomorphism in a neighbourhood of E. Hence

$$\omega_{X_p}(-E) = \omega_{X_p}\{\frac{-\Delta}{N}\} \otimes \mathcal{O}_{X_p}(-E) = \omega_{X_p}\{\frac{-\Delta - N\cdot E}{N}\}$$

is contained in $\omega_{X_p}\{\frac{-D_p}{N}\}$ which implies that $\omega_X = \omega_X\{\frac{-D}{N}\}$ near p.

If $\mu' = [\frac{\mu}{N}] - n + 1 \geq 0$ then

$$\omega_{X_p}\{\frac{-D_p}{N}\} \subset \omega_{X_p}\{\frac{-\mu\cdot E}{N}\} = \omega_{X_p}(-[\frac{\mu}{N}]\cdot E)$$

and

$$\omega_X\{\frac{-D}{N}\} = \varrho_*\omega_{X_p}\{\frac{-D_p}{N}\} \subset \varrho_*\varrho^*\omega_X((n-1-[\frac{\mu}{N}])\cdot E) = m_p^{\mu'} \otimes \omega_X.$$

□

The sheaves $\omega_X\{\frac{-D}{N}\}$ are describing the correction terms needed if one wants to generalize the vanishing theorems (5.12,c) or (5.12,d) to non normal crossing divisors. For example one obtains:

7.8. Proposition. *Let X be a projective manifold, $\mathcal{L}$ be an invertible sheaf and D be a divisor such that $\mathcal{L}^N(-D)$ is numerically effective and*

$$c_1(\mathcal{L}^N(-D))^n > 0$$

for $n = \dim X$. Then

$$H^b(X, \omega_X\{\frac{-D}{N}\} \otimes \mathcal{L}) = 0 \quad \textit{for} \quad b > 0.$$

Proof: This follows from (5.12,c) and (6.11,b) by using the Leray spectral sequence (A.27).

□

7.9. Remark. Demailly proved in [13] an analytic improvement of Kodaira's vanishing theorem. It would be nice to understand the relation of his positivity condition with the one arising from (7.8), i.e. with the condition that

$$\omega_X\{\frac{-D}{N}\} = \omega_X.$$

One of the reasons for the interest in vanishing theorems as (7.8) is implication that certain sheaves are generated by global sections. For example one has:

7.10. Corollary. *Under the assumptions of (7.8) let $\mathcal{H}$ be a very ample sheaf. Then*

$$\mathcal{H}^{\dim X} \otimes \mathcal{L} \otimes \omega_X\{\frac{-D}{N}\}$$

is generated by global sections.

Proof.: For

$$\mathcal{F} = \mathcal{L} \otimes \omega_X\{\frac{-D}{N}\} \otimes \omega_X^{-1}$$

we have

$$H^b(X, \mathcal{F} \otimes \mathcal{H}^\nu \otimes \omega_X) = 0 \quad \text{for} \quad b > 0 \quad \text{and} \quad \nu \geq 0.$$

For general sections $H_1, \ldots, H_n$ of $\mathcal{H}$ passing through a given point p and for

$$Y_r = \bigcap_{i=1}^r H_i$$

we obtain

$$H^b(Y_r, \mathcal{F} \otimes \mathcal{H}^\nu \otimes \omega_{Y_r}) = 0, \quad \text{for} \quad b > 0 \quad \text{and} \quad \nu \geq 0,$$

by regarding the cohomology sequence given by the short exact sequence

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{H}^{\nu} \otimes \omega_{Y_r} \longrightarrow \mathcal{F} \otimes \mathcal{H}^{\nu+1} \otimes \omega_{Y_r} \longrightarrow \mathcal{F} \otimes \mathcal{H}^{\nu} \otimes \omega_{Y_{r+1}} \longrightarrow 0.$$

By induction we may assume that

$$\mathcal{F} \otimes \mathcal{H}^{\dim(Y_{r+1})} \otimes \omega_{Y_{r+1}}$$

is generated by global sections in p, and, using the cohomology sequence again one finds the same for

$$\mathcal{F} \otimes \mathcal{H}^{\dim(Y_r)} \otimes \omega_{Y_r}.$$

$\square$

Let us apply (7.10) for $X = \mathbb{P}^n$ to study the behaviour of zeros of homogeneous polynomials:

7.11. Example: Zeros of polynomials . Let S be a finite set of points in $\mathbb{P}^n$, for $n \geq 2$, and

$$\omega_\mu(S) = \mathrm{Min}\{ \quad d > 0;\ \text{there exists } s \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \text{ with multiplicity at least } \mu \text{ in each } p \in S\}.$$

7.12. Claim. For $\mu' < \mu$ one has

$$\frac{\omega_{\mu'}(s)}{\mu' + n - 1} \leq \frac{\omega_\mu(s)}{\mu}.$$

PROOF.: For $d = \omega_\mu(S)$ we have a section $s \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ with divisor D' such that s has multiplicity at least μ in each $p \in S$. Choose

$$d' = [\frac{d}{\mu}(\mu' + n - 1)].$$

Since d' does not change if we replace d by $\nu \cdot d + 1$ and μ by $\nu \cdot \mu$ for $\nu \gg 0$, we can assume that $D' = D + H$ for a hyperplane H not meeting S.

For $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d' + 1)$, for the divisor $(d' + 1) \cdot D$, and for $N = d$, the assumptions of (7.8) hold true and (7.10) tells us that

$$\mathcal{O}_{\mathbb{P}^n}(n + d' + 1) \otimes \omega_{\mathbb{P}^n}\{\frac{-(d' + 1) \cdot D}{d}\}$$

is globally generated. Since $d' + 1 > \frac{d}{\mu}(\mu' + n - 1)$ and hence $\mu' + n - 1 \leq \frac{(d'+1)\cdot\mu}{d}$ we can apply (7.7,c) and find

$$\mathcal{O}_{\mathbb{P}^n}(n + d' + 1) \otimes \omega_{\mathbb{P}^n}\{\frac{-(d' + 1) \cdot D}{d}\}$$

to be a subsheaf of

$$\mathcal{O}_{\mathbb{P}^n}(d') \otimes \bigotimes_{p \in S} m_p^{\mu'}.$$

$\square$

7.13. Remark. For some generalizations and improvements and for the history of this kind of problem see [18].

Up to this point, the applications discussed are based on the global vanishing theorems for invertible sheaves. J. Kollár's vanishing theorem (5.6,a) for restriction maps or, equivalently, vanishing theorems for the cohomology of higher direct image sheaves ((6.16) and (6.17)) are nice tools to study families of projective varieties:

7.14. Example: Families of varieties over curves .
Let X be a projective manifold, Z a non-singular curve and $f : X \to Z$ a surjective morphism. We call a locally free sheaf $\mathcal{F}$ on Z *semi-positive*, if for some (or equivalently: all) ample invertible sheaf $\mathcal{A}$ on Z and for all $\eta > 0$ the sheaf

$$S^{\eta}(\mathcal{F}) \otimes \mathcal{A}$$

is ample. One has

7.15. Theorem (Fujita [24]). *For $f : X \to Z$ as above $f_*\omega_{X/Z}$ is semi-positive.*

Here $\omega_{X/Z} = \omega_X \otimes f^*\omega_Z^{-1}$ is the dualizing sheaf of X over Z. In [40] J. Kollár used his vanishing theorem (5.6,a) to give a simple proof of (7.15) and of its generalization to higher dimensional Z, obtained beforehand by Y. Kawamata [35]. As usually, one obtains similar results adding the $\mathcal{L}^{(i)}$. For example, using the notations introduced in (7.4) one has:

7.16. Variant. *Assume in addition that $\mathcal{L}$ is an invertible sheaf, D an effective divisor and that $\mathcal{L}^N(-D)$ is semi-ample. Then one has*
a) *The sheaf $f_*(\mathcal{L} \otimes \omega_{X/Z}\{\frac{-D}{N}\})$ is semi-positive.*
b) *If for a general fibre F of f one has $N \geq e(D\mid_F)$, then $f_*(\mathcal{L} \otimes \omega_{X/Z})$ is semi-positive.*

7.17. Corollary. *Under the assumption of (7.16) assume that D contains a smooth fibre of f. Then, if $N \geq e(D\mid_F)$, the sheaf $f_*(\mathcal{L} \otimes \omega_{X/Z})$ is ample.*

Proof of (7.17): Recall that a vectorbundle $\mathcal{F}$ on Z is ample, if and only if $\tau^*\mathcal{F}$ is ample for some finite cover

$$\tau : Z' \longrightarrow Z.$$

Hence, if $\mathcal{L}'$, D', X' and f' are obtained by pullback from the corresponding objects over Z, it is enough to show that

$$\tau^* f_*(\mathcal{L} \otimes \omega_{X/Z}) = f'_*(\mathcal{L}' \otimes \omega_{X'/Z'})$$

is ample. If, for the ramification locus $\Delta(Z'/Z)$ of Z' over Z, the morphism f is smooth in a neighbourhood of $f^{-1}(\Delta(Z'/Z))$ then $f' : X' \longrightarrow Z'$ satisfies

again the assumption made in (7.14).
Choosing Z' to be ramified of order N over the point $p \in Z$ with $f^{-1}(p) \subseteq D$, we can reduce (7.17) to the case for which D contains the N-th multiple of a fibre, say $N \cdot f^{-1}(p)$. We have

$$f_*(\mathcal{L} \otimes \omega_{X/Z}\{-\frac{D}{N}\}) \subset f_*(\mathcal{L} \otimes \omega_{X/Z}) \otimes \mathcal{O}_Z(-p)$$

and this inclusion is an isomorphism over some open set. In fact, by (7.5,e) the assumption $N \geq e(D\mid_F)$ implies that $\omega_{X/Z}$ and $\omega_{X/Z}\{-\frac{D}{N}\}$ are the same in a neighbourhood of a general fibre F. Hence

$$f_*(\mathcal{L} \otimes \omega_{X/Z}) \otimes \mathcal{O}_Z(-p)$$

is semi-positive.

$\square$

Proof of (7.16): As in the proof of (7.17) the assumption made in part b) implies that

$$f_*(\mathcal{L} \otimes \omega_{X/Z}\{\frac{-D}{N}\}) \subset f_*(\mathcal{L} \otimes \omega_{X/Z})$$

is an isomorphism over some non-empty open subvariety and b) follows from a).

By definition of $\omega_X\{\frac{-D}{N}\}$ we can assume D to be a normal crossing divisor. Moreover, we can assume that the multiplicities in D are strictly smaller than N and hence $\mathcal{L} = \mathcal{L}(-[\frac{D}{N}])$. Let $p \in Z$ be a point in general position and $F = f^{-1}(p)$. By (5.12,a) applied to the semiample sheaf $\mathcal{L}(F)$ we have a surjection

$$H^0(X, \mathcal{L}(F) \otimes \omega_X(F)) \longrightarrow H^0(F, \mathcal{L}(F) \otimes \omega_F)$$

and $f_*(\mathcal{L} \otimes \omega_X) \otimes \mathcal{O}_Z(2 \cdot p)$ is generated by global sections in a neighbourhood of p.

7.18. Claim. $\omega_Z(2 \cdot p) \otimes \bigotimes^\eta f_*(\mathcal{L} \otimes \omega_{X/Z})$ is semi-positive for all $\eta > 0$.

Proof: For $\eta = 1$ (7.18) holds true as we just found a trivial subsheaf $\bigoplus^r \mathcal{O}_Z$ of

$$f_*(\mathcal{L} \otimes \omega_X) \otimes \mathcal{O}_Z(2 \cdot p)$$

of full rank.
In general, let $Y' = X \times_Z \ldots \times_Z X$ (η-times) be the fibre product and $\delta : Y \to Y'$ be a desingularization. The induced morphisms $g' : Y' \to Z$ and $g : Y \to Z$ satisfy:

i) Y' is flat and Gorenstein over Z and

$$\omega_{Y'/Z} = \bigotimes_{j=1}^{\eta} pr_j^* \omega_{X/Z}.$$

ii) For

$$\mathcal{M}' = \bigotimes_{j=1}^{\eta} pr_j^* \mathcal{L} \quad \text{and} \quad \mathcal{M} = \delta^* \mathcal{M}'$$

one has an inclusion, surjective at the general point of Z,

$$g_*(\mathcal{M} \otimes \omega_{Y/Z}) \subset g'_*(\mathcal{M}' \otimes \omega_{Y'/Z}) = \bigotimes^{\eta} f_*(\mathcal{L} \otimes \omega_{X/Z})$$

iii) On a general fibre $\widetilde{F}$ of g the divisor

$$\Delta = \sum_{j=1}^{\eta} \delta^* pr_j^* D$$

has normal crossings, $[\frac{\Delta}{N}]|_{\tilde{F}} = 0$, and $\mathcal{M}^N(-\Delta)$ is semi-ample.

In fact, i) is the compatibility of relative dualizing sheaves with pullback, ii) follows from flat base change and the inclusion $\delta_* \omega_Y \subset \omega_{Y'}$ and iii) is obvious. Using those three properties, (7.18) follows from the case "$\eta = 1$" applied to $g : Y \to Z$.

□

Since $S^{\eta}(\ \ )$ is a quotient of $\bigotimes^{\eta}(\ \ )$ and since the quotient of a semi-positive sheaf is again semipositive, one obtains (7.16).

□

7.19. Remarks.
a) If $\dim Z > 1$, then the arguments used in the proof of (7.16,a) show that for all $\eta > 0$ and $\mathcal{H}$ very ample the reflexive hull $\mathcal{G}$ of

$$S^{\eta}(f_*(\mathcal{L} \otimes \omega_{X/Z})) \otimes \mathcal{H}^{\dim(X)+1} \otimes \omega_Z$$

is generated by $H^0(Z, \mathcal{G})$ over some open set. This led the second author to the definition "weakly-positive" (see [64]).
b) One can make (7.17) more explicit and measure the degree of ampleness by giving lower bound for the degree of invertible quotient sheaves of $f_*(\mathcal{L} \otimes \omega_{X/Z})$. Details have been worked out in [23]. These explicit bounds, together with the Kodaira-Spencer map can be used for families of curves over Z to give another proof of the Theorem of Manin saying that the Mordell conjecture holds true for curves over function fields over $\mathbb{C}$.

The vanishing statements for higher direct images (6.11) and its corollaries (7.5,d) and (7.6) are useful to study singularities.

7.20. Example: Deformation of quotient singularities.
Let X be a normal variety and $f : X \to S$ a flat morphism from X to a non-singular curve S.

7.21. Theorem. *Assume that X is normal and that for some $s_0 \in S$ the variety $X_0 = f^{-1}(s_0)$ is a reduced normal surface with quotient singularities. Then the general fibre $f^{-1}(\eta) = X_\eta$ has at most quotient singularities.*

Proof (see [19]):
If Y is a normal surface with rational singularities then Mumford [47] has shown that for $p \in Y$,

$$\text{Spec } (\mathcal{O}_{Y,p}) - p = U$$

has only finitely many non-isomorphic invertible sheaves. Hence for some $N > 0$ one has $\omega_U^N = \mathcal{O}_U$ and for some $N > 0$ the reflexive hull $\omega_X^{[N]}$ of ω_Y^N is invertible. Let

$$\delta : Y' \longrightarrow Y$$

be a desingularization. Since Y has rational singularities we have

$$\delta_* \omega_{Y'} = \omega_Y$$

and

$$\delta^* \omega_Y / torsion \ \subset \omega_{Y'}.$$

We may assume that $\delta^* \omega_Y / torsion = \mathcal{K}$ is invertible and we write

$$\omega_{Y'} = \mathcal{K} \otimes \mathcal{O}_{Y'}(F).$$

With this notation we have for some effective divisor D

$$\delta^* \omega_Y^{[N]} = \mathcal{K}^N \otimes \mathcal{O}_{Y'}(D).$$

The divisors D and F can be used to characterize the quotient singularities among the rational singularities:

7.22. Claim. Y has quotient singularities if and only if $[\frac{D}{N}] \leq F$.

Proof.: If one replaces D and N by some common multiple, the inequality in (7.22) is not affected. The question being local we may hence assume that N is the smallest integer with $\omega_Y^{[N]}$ invertible and $\omega_Y^{[N]} \simeq \mathcal{O}_Y$.

For $\mathcal{K}^{-1} = \mathcal{L}$ one has $\mathcal{L}^N = \mathcal{O}_{Y'}(D)$ and, as in (3.5), we can consider the cyclic cover Z' obtained by taking the N-th root out of D. Let Z be the normalization of Y in $k(Z')$ and

$$\begin{array}{ccc} Z' & \xrightarrow{\delta'} & Z \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow[\delta]{} & Y \end{array}$$

the induced morphism (Z is usually called *the canonical covering* of Y). One has

$$\pi'_* \omega_{Z'} = \bigoplus_{i=0}^{N-1} \omega_{Y'} \otimes \mathcal{L}^{(i)}.$$

In fact, this follows from (3.11) by duality for finite morphisms (see [30], p. 239) or, since

$$\mathcal{L}^{-i}([\frac{i \cdot D}{N}] + D^{(i)}) = \mathcal{L}^{-i}(D - [\frac{(N-i) \cdot D}{N}]) = \mathcal{L}^{(N-i)}$$

from (3.16,d).
Recall that Z has rational singularities if and only if $\delta'_* \omega_{Z'} = \omega_Z$.
Assume that Y has a quotient singularity in the point p. If $\widetilde{U}$ is the universal cover of $Y - p$ then the normalization $\widetilde{Z}$ of Y in $k(\widetilde{U})$ is non singular and, by construction it dominates Z. Hence Z has quotient singularities and

$$\pi_* \delta'_* \omega_{Z'} = \delta_* \pi'_* \omega_{Z'}$$

is reflexive. In particular $\delta_* \omega_{Y'} \otimes \mathcal{L}^{(1)}$ is reflexive. One has

$$\omega_{Y'} \otimes \mathcal{L}^{(1)} = \mathcal{K} \otimes \mathcal{L} \otimes \mathcal{O}_{Y'}(F - [\frac{D}{N}]) = \mathcal{O}_{Y'}(F - [\frac{D}{N}])$$

and the reflexivity of $\delta_* \omega_{Y'} \otimes \mathcal{L}^{(1)}$ is equivalent to $F \geq [\frac{D}{N}]$.

On the other hand, $F \geq [\frac{D}{N}]$ implies that the summand $\delta_* \mathcal{O}_{Y'}(F - [\frac{D}{N}])$ of $\delta_* \pi'_* \omega_{Z'}$ has one section without zero on Y. Hence $\delta'_* \omega_{Z'}$ has a section without zero on $Z - \pi^{-1}(p)$, which implies that $\delta'_* \omega_{Z'}$ is invertible and coincides with ω_Z. So Z has a rational singularity and is Gorenstein. Those singularities are called rational double points, and they are known to be quotient singularities. Therefore Y has a quotient singularity as well.

□

Proof of (7.21): Let $\delta : X' \to X$ be a desingularization. We assume that the proper transform X'_0 of X_0 is non-singular and write $\delta_0 = \delta \mid_{X'_0}$.
By (7.5,d), applied in the case "$D = 0$ ", we have a natural inclusion

$$\delta_{0*} \omega_{X'_0} \longrightarrow \delta_* \omega_{X'} \otimes \mathcal{O}_X(X_0) \mid_{X_0} .$$

Since X_0 has rational singularities one has

$$\delta_{0*} \omega_{X'_0} = \omega_{X_0} = (\omega_X \otimes \mathcal{O}_X(X_0))|_{X_0}.$$

One obtains $\delta_* \omega_{X'} = \omega_X$, at least if one replaces S by a neighbourhood of s_0. Hence X and X_η have at most rational singularities.

Let us choose $N > 1$, such that both, $\omega_{X_\eta}^{[N]}$ and $\omega_{X_0}^{[N]}$ are invertible (It might happen, nevertheless, that $\omega_X^{[N]}$ is not invertible). We may assume that we have choosen X such that

$$\mathcal{K} = \delta^* \omega_X / torsion$$

is an invertible sheaf. Hence

$$\mathcal{K}_0 = \mathcal{K} \otimes \mathcal{O}_{X'}(\delta^* X_0) \mid_{X'_0}$$

is invertible and generated by global sections. Moreover, one has maps

$$\delta_0^* \omega_{X_0} = \delta^*(\omega_X \otimes \mathcal{O}_X(X_0))|_{X'_0} \longrightarrow \mathcal{K}_0$$

and $\mathcal{K}_0$ contains $\delta_0^* \omega_{X_0}/torsion$. Hence both sheaves must be the same.

Blowing up again, we can assume $\mathcal{M} = \delta^* \omega_X^{[N]}/torsion$ to be locally free and isomorphic to $\mathcal{K}^N(D)$ where D is a divisor in the exceptional locus of δ such that $X'_0 + D$ has at most normal crossings.

We can choose an embedding $\omega_X \hookrightarrow \mathcal{O}_X$ such that the zero-set does not contain X_0. If correspondingly $\mathcal{K} = \mathcal{O}_{X'}(-\Delta)$, for some $\Delta \geq 0$ we can choose the inclusion $\omega_X \hookrightarrow \mathcal{O}_X$ such that

$$D' = N \cdot \Delta - D \geq 0.$$

Blowing up we can assume that $D' + X'_0$ is a normal crossing divisor. By definition

$$\mathcal{O}_X(-D') = \mathcal{O}_X(-N \cdot \Delta + D) = \mathcal{M}$$

and $\mathcal{O}_X(-D')$ is δ-numerically effective.

It is our aim to use (7.6) in order to compare the sheaves $\mathcal{K}_0$ and $\delta_0^* \omega_{X_0}^{[N]}$ with $\mathcal{K}$ and with $\mathcal{M}$. Some unpleasant but elementary calculations will show that the inequality (7.22), applied to X_0, gives a similar inequality for the general fibre X_η.

Let us write

$$\omega_{X'_0} = \mathcal{K}_0(F_0) \quad \text{and} \quad \delta_0^* \omega_{X_0}^{[N]} = \mathcal{K}_0^N \otimes \mathcal{O}_{X'_0}(D_0).$$

By (7.22) one has

$$F_0 \geq [\frac{1}{N} D_0].$$

Since

$$\mathcal{K}_0^N \otimes \mathcal{O}_{X'_0}(D \mid_{X'_0}) \simeq (\mathcal{K}^N \otimes \mathcal{O}_{X'}(D) \otimes \mathcal{O}_{X'}(+N \cdot \delta^* X_0)) \mid_{X'_0}$$

is a subsheaf of $\delta_0^*\omega_{X_0}^{[N]}$ one obtains that

$$(*)\qquad F_0 \geq [\frac{1}{N}D\mid_{X_0'}] .$$

On the other hand, since

$$\mathcal{K}_0^{N-1} = \mathcal{O}_{X_0'}(-(N-1)\Delta\mid_{X_0'}) \otimes \mathcal{O}_{X'}((N-1)\delta^* X_0)\mid_{X_0'}$$

and

$$[\frac{N-1}{N}D'] = (N-1)\Delta - D + [\frac{1}{N}D]$$

one has

$$\omega_{X_0'}(-[\frac{(N-1)}{N}D'\mid_{X_0'}]) = \mathcal{K}_0(F_0 - (N-1)\Delta\mid_{X_0'} + D\mid_{X_0'} - [\frac{1}{N}D\mid_{X_0'}]) =$$

$$\mathcal{K}_0^N(D\mid_{X_0'} + F_0 - [\frac{1}{N}D\mid_{X_0'}]) \otimes \mathcal{O}_{X'}(-(N-1)\delta^* X_0)\mid_{X_0'} =$$

$$\mathcal{M}\mid_{X_0'}(F_0 - [\frac{1}{N}D\mid_{X_0'}]) \otimes \mathcal{O}_{X'}(\delta^* X_0)\mid_{X_0'} .$$

By the inequality $(*)$ the sheaf

$$\delta_{0*}(\omega_{X_0'}(-[\frac{(N-1)}{N}D'\mid_{X_0'}]) \otimes \mathcal{O}_{X'}(-\delta^* X_0)\mid_{X_0'})$$

contains $\delta_{0*}(\mathcal{M}\mid_{X_0'})$. If F is the divisor with $\omega_{X'} = \mathcal{K} \otimes \mathcal{O}_{X'}(F)$ we get from (7.6) $\delta_{0*}(\mathcal{M}\mid_{X_0'})$ as a subsheaf of

$$\delta_*\omega_X(-[\frac{(N-1)}{N}D'])\mid_{X_0} = \delta_*(\mathcal{K}\otimes\mathcal{O}_{X'}(F)\otimes\mathcal{O}_{X'}(-(N-1)\Delta + D - [\frac{1}{N}D]))\mid_{X_0}$$

$$= \delta_*(\mathcal{K}^N \otimes \mathcal{O}_{X'}(D + F - [\frac{1}{N}D]))\mid_{X_0} = \delta_*\mathcal{M}(F - [\frac{1}{N}D])\mid_{X_0} .$$

Of course, we have a natural morphism $\delta_*\mathcal{M} \longrightarrow \delta_{0*}(\mathcal{M}\mid_{X_0'})$ and the induced map

$$\delta_*\mathcal{M} \longrightarrow \delta_*\mathcal{M}(F - [\frac{1}{N}D])\mid_{X_0}$$

is surjective outside of the singular locus of X_0. We have natural maps

$$\omega_X^{[N]} \longrightarrow \delta_*\delta^*\omega_X^{[N]} \longrightarrow \delta_*\mathcal{M} \longrightarrow \delta_*\mathcal{M}(F - [\frac{1}{N}D])|_{X_0} \longrightarrow \omega_X^{[N]}|_{X_0}.$$

The sheaf $\delta_*\mathcal{M}(F - [\frac{1}{N}D])$ is torsionfree and, since X_0 is a Cartier divisor, $\delta_*\mathcal{M}(F - [\frac{1}{N}D])|_{X_0}$ has no torsion as well. Therefore

$$\omega_X^{[N]}|_{X_0} = \delta_*\mathcal{M}(F - [\frac{1}{N}D])|_{X_0}.$$

Since $\mathcal{M} = \delta^*\omega_X^{[N]}/torsion$, this is only possible if $F \geq [\frac{1}{N}\cdot D]$. Hence $F_\eta \geq [\frac{D_\eta}{N}]$ where "η" denotes the restriction to the general fibre and the theorem follows from (7.22).

□

7.23. Example: Adjoint linear systems on surfaces.
Studying adjoint linear systems on higher dimensional manifolds, L. Ein and R. Lazarsfeld [15] realized, that (7.7, b and c) and (7.8) can be used to reprove part of I. Reider's theorem [53] and to obtain similar results for threefolds. We cordially thank them for allowing us to add their argument in the surface case to the final version of these notes.

7.24. Theorem (I. Reider, [53]). *Let S be a non-singular projective surface, defined over an algebraically closed field of characteristic zero, let $p \in S$ be a closed point and let $\mathcal{L}$ be a numerically effective invertible sheaf on S. Assume that $c_1(\mathcal{L})^2 > 4$ and that for all curves C with $p \in C \subset S$ one has $c_1(\mathcal{L})\cdot C > 1$. Then there is a section $\sigma \in H^0(S, \mathcal{L} \otimes \omega_S)$ with $\sigma(p) \neq 0$.*

PROOF, FOLLOWING §1 OF [15]:
Let $\mathcal{H}$ be an ample invertible sheaf on S and let m_p be the ideal sheaf of p. For $\nu \gg 0$, one has $H^2(S, \mathcal{L}^\nu \otimes \mathcal{H}^{-1}) = 0$ and by the Riemann-Roch formula one finds $a, b \in \mathbb{N}$ with

$$h^0(S, \mathcal{H}^{-1} \otimes \mathcal{L}^\nu \otimes m_p^{2\cdot\nu}) \geq h^0(S, \mathcal{H}^{-1} \otimes \mathcal{L}^\nu) - h^0(S, \mathcal{O}_S/m_p^{2\cdot\nu})$$

$$\geq \frac{1}{2} \cdot c_1(\mathcal{L})^2 \cdot \nu^2 + a \cdot \nu + b - h^0(S, \mathcal{O}_S/m_p^{2\cdot\nu}).$$

Since

$$h^0(S, \mathcal{O}_S/m_p^{2\cdot\nu}) = \frac{1}{2} \cdot (4 \cdot \nu^2 + 2 \cdot \nu)$$

one finds for $\nu \gg 0$ a section s of $\mathcal{H}^{-1} \otimes \mathcal{L}^\nu$ with multiplicity $\mu_p \geq 2 \cdot \nu$ in p (see (7.7,b). Let

$$D = \Delta + \sum_{i=1}^{r} \nu_i \cdot D_i$$

be the zero-divisor of s, where Δ is an effective divisor not containing p and $p \in D_i$ for $i = 1 \cdots r$. If D' is any effective divisor the $\mathcal{H}^\eta(-D')$ will be ample for $\eta \gg 0$. Replacing D by $\eta \cdot D + D'$ and ν by $\nu \cdot \eta$ for a suitably choosen divisor D', we may assume that $r > 1$ and that $\nu_1 > \nu_2 > \cdots > \nu_r$. Of course we can also assume that μ_p is even.

7.25. Claim. If $\mu_p > 2 \cdot \nu_1$ then (7.24) holds true.

PROOF: Let $N = \frac{\mu_p}{2}$ By the choice of s one has $N \geq \nu$ and

$$\mathcal{L}^N(-D) = \mathcal{L}^{N-\nu} \otimes \mathcal{H}$$

is ample. By (7.8) one has

$$H^1(S, \omega_S\{\frac{-D}{N}\} \otimes \mathcal{L}) = 0.$$

By (7.7,b), or just by definition of $\omega_S\{\frac{-D}{N}\}$, one can find some open neighbourhood U of p such that

$$\omega_S\{\frac{-D}{N}\} \longrightarrow \omega_S$$

is an isomorphism on $U - p$. Moreover, by (7.7,c), the inclusion factors like

$$\omega_S\{\frac{-D}{N}\} \longrightarrow \omega_S \otimes m_p \longrightarrow \omega_S.$$

Let $\mathcal{M}$ be the sheaf $j_*j^*(\omega_S\{\frac{-D}{N}\}\otimes\mathcal{L})$ where $j : S-p \longrightarrow S$ is the inclusion. For some nontrivial skyscraper sheaf $\mathcal{C}$ supported in p, one has an exact sequence

$$0 \longrightarrow \omega_S\{\frac{-D}{N}\} \otimes \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{C} \longrightarrow 0.$$

Hence, $\mathcal{M}$ has a section σ with $\sigma(p) \neq 0$.

□

It remains to consider the case where $\mu_p \leq 2 \cdot \nu_1$. If $\mu_p(D_i)$ denotes the multiplicity of D_i in p, then

$$\mu_p = \sum_{i=1}^{r} \nu_i \cdot \mu_p(D_i).$$

Since $r \geq 2$ this implies that $\mu_p(D_1) = 1$.

Let us take $N = \nu_1$. Again, $N \geq \nu$ and by (7.8)

$$H^1(S, \omega_S\{\frac{-D}{N}\} \otimes \mathcal{L}) = 0.$$

One has an inclusion

$$\omega_S\{\frac{-D}{N}\} \otimes \mathcal{L} \longrightarrow \omega_S \otimes \mathcal{L}(-D_1 - [\frac{\Delta}{N}])$$

whose cokernel is a skyscraper sheaf. Hence

$$H^1(S, \omega_S \otimes \mathcal{L}(-D_1 - [\frac{\Delta}{N}])) = 0$$

and the restriction map

$$H^0(S, \omega_S \otimes \mathcal{L}(-[\frac{\Delta}{N}])) \longrightarrow H^0(D_1, \omega_{D_1} \otimes \mathcal{L}(-D_1 - [\frac{\Delta}{N}])|_{D_1})$$

is surjective.

The right hand side contains a section σ with $\sigma(p) \neq 0$, since

$$\deg(\mathcal{L}(-D_1 - [\frac{\Delta}{N}])|_{D_1}) \geq 2.$$

In fact one has:

$$N \cdot \deg(\mathcal{L}(-D_1 - [\frac{\Delta}{N}])|_{D_1}) = \deg(\mathcal{L}^N(-D)|_{D_1}) + \sum_{i=2}^{r} \nu_i \cdot D_i \cdot D_1 + (\Delta - [\frac{\Delta}{N}]) \cdot D_1$$

$$\geq (N - \nu) \cdot c_1(\mathcal{L}) \cdot D_1 + c_1(\mathcal{H}) \cdot D_1 + \sum_{i=2}^{r} \nu_i \cdot D_i \cdot D_1$$

$$> (N - \nu) \cdot c_1(\mathcal{L}) \cdot D_1 + \sum_{i=2}^{r} \nu_i \cdot \mu_p(D_i) = (N - \nu) \cdot c_1(\mathcal{L}) \cdot D_1 + (\mu_p - \nu_1).$$

Since $c_1(\mathcal{L}) \cdot D_1 \geq 2$ and $\mu_p \geq 2 \cdot \nu$, one obtains

$$N \cdot \deg(\mathcal{L}(-D_1 - [\frac{\Delta}{N}])|_{D_1}) > 2 \cdot N - 2 \cdot \nu + \mu_p - N \geq N.$$

□

7.26. Remark. It is likely that the other parts of I. Reider's theorem [53], i.e. the lower bounds for $c_1(\mathcal{L})^2$ and for $c_1(\mathcal{L}) \cdot C$ which imply that $H^0(S, \omega_S \otimes \mathcal{L})$ separates points and tangent directions, can be obtained in a similar way.

§ 8 Characteristic p methods: Lifting of schemes

Up to this point we did not prove the degeneration of the Hodge spectral sequence used in (3.2). Before doing so in Lecture 10 let us first recall what we want to prove.

8.1.

Let X be a proper smooth variety (or a scheme) over a field k. One introduces the *de Rham cohomology*

$$H^b_{DR}(X/k) := \mathbb{H}^b(X, \Omega^\bullet_{X/k})$$

where $\Omega^\bullet_{X/k}$ is the complex of regular differential forms, defined over k, the so called *de Rham complex.*

In order to compute it, one introduces the *"Hodge to de Rham" spectral sequence* associated to the Hodge filtration $\Omega^{\geq a}_{X/k}$ (see (A.25):

$$E_1^{ab} = H^b(X, \Omega^a_{X/k}) \Longrightarrow H^{a+b}_{DR}(X/k).$$

If $k = \mathbb{C}$, the field of complex numbers, the classical Hodge theory tells us that the Hodge spectral sequence

$$E_{1\ an}^{ab} = H^b(X_{an}, \Omega^a_{X_{an}}) \Longrightarrow \mathbb{H}^{a+b}(X_{an}, \Omega^\bullet_{X_{an}})$$

degenerates in E_1, where $\Omega^\bullet_{X_{an}}$ is the de Rham complex of holomorphic differential forms (see (A.25)).

In fact, one has

$$\mathbb{H}^{a+b}(X_{an}, \Omega^\bullet_{X_{an}}) = H^{a+b}(X_{an}, \mathbb{C})$$

and by Hodge theory

$$\dim\ H^l(X_{an}, \mathbb{C}) = \sum_{a+b=l} \dim\ H^b(X_{an}, \Omega^a_{X_{an}}).$$

As explained in (A.22), this equality is equivalent to the degeneration of $E_{1\ an}$.

As by Serre's GAGA theorems [56],

$$H^b(X_{an}, \Omega^a_{X_{an}}) = H^b(X, \Omega^a_{X/\mathbb{C}}),$$

the Hodge spectral sequence and the "Hodge to de Rham" spectral sequence coincide and therefore the second one degenerates in E_1 as well.

If k is any field of characteristic zero, one obtains the same result by flat base change:

8.2. Theorem. *Let X be a proper smooth variety over a field k of characteristic zero. Then the Hodge to de Rham spectral sequence degenerates in E_1 or, equivalently,*

$$\dim\ H^l_{DR}(X/k) = \sum_{a+b=l} \dim\ H^b(X, \Omega^a_{X/k}).$$

As we have already seen in Lecture 1 and 6, theorem (8.2) implies the Akizuki - Kodaira - Nakano vanishing theorem:

AKNV: *If $\mathcal{L}$ is ample invertible, then*

$$H^b(X, \Omega^a_{X/k} \otimes \mathcal{L}^{-1}) = 0 \quad \textit{for} \quad a + b < \dim\ X$$

(where, of course, char $k = 0$*).*

Mumford [47] has shown that over a field k of characteristic $p > 0$ the E_1 degeneration for the Hodge to de Rham spectral sequence fails and, finally, Raynaud [52] gave a counterexample to AKNV in characteristic $p > 0$.

The aim of this and of the next three lectures is to present Deligne-Illusie's answer to those counterexamples:

8.3. Theorem (Deligne - Illusie [12]). *Let X be a proper smooth variety over a perfect field k of characteristic $p \geq \dim\ X$ lifting to the ring $W_2(k)$ of the second Witt vectors (see (8.11)). Then both, the E_1-degeneration of the Hodge to de Rham spectral sequence and AKNV hold true.*

Actually they prove a slightly stronger version of (8.3), as will be explained later. Unfortunately one cannot derive from their methods the stronger vanishing theorems mentioned in Lecture 5 such as Grauert-Riemenschneider or Kawamata-Viehweg directly. As indicated, the geometric methods of the first part of these Lecture Notes fail as well. It is still an open problem which of those statements remains true under the assumptions of (8.3).

Finally, by standard techniques of reduction to characteristic $p > 0$, Deligne - Illusie show:

8.4. Proposition. *Theorem (8.2) and AKNV over a field k of characteristic zero are consequences of theorem (8.3).*

In the rest of this lecture, we will try to discuss to some extend elementary properties and examples of liftings to $W_2(k)$.

8.5. Liftings of a scheme.
Let S be a scheme defined over $\mathbb{F}_p$ the field with p elements.

8.6. Definition. A *lifting* of S to $\mathbb{Z}/p^2$ is a scheme $\widetilde{S}$, defined and flat over $\mathbb{Z}/p^2$, such that $S = \widetilde{S} \times_{\mathbb{Z}/p^2} \mathbb{F}_p$.

8.7. Properties.
a) S is defined by a nilpotent ideal sheaf (of square zero) in $\widetilde{S}$. In particular the inclusion $S \subset \widetilde{S}$ or, if one prefers, the projection

$$\mathcal{O}_{\widetilde{S}} \longrightarrow \mathcal{O}_S$$

induces the identity on the *underlying topological spaces* $(S)_{top}$ and $(\widetilde{S})_{top}$.
b) From the exact sequence of $\mathbb{Z}/p^2$-modules

$$0 \longrightarrow p \cdot \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

one obtains, since $\mathcal{O}_{\widetilde{S}}$ is flat over $\mathbb{Z}/p^2$, the exact sequence of $\mathcal{O}_{\widetilde{S}}$-modules

$$0 \longrightarrow p \cdot \mathcal{O}_{\widetilde{S}} \longrightarrow \mathcal{O}_{\widetilde{S}} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

and, from the isomorphism of $\mathbb{Z}/p^2$-modules

$$p : \mathbb{Z}/p \longrightarrow p \cdot \mathbb{Z}/p^2,$$

one obtains the isomorphism of $\mathcal{O}_{\widetilde{S}}$-modules

$$p : \mathcal{O}_S \longrightarrow p \cdot \mathcal{O}_{\widetilde{S}}.$$

8.8. Example. Let k be a perfect field of characteristic p and $S = \text{Spec } k$. Then $\widetilde{S}$ exists and is uniquely determined (up to isomorphism) by (8.7,b):

$\widetilde{S} = \text{Spec } W_2(k)$, where $W_2(k)$ is called the *ring of the second Witt vectors* of k.

In concrete terms, $W_2(k) = k \oplus k \cdot p$ as additive group and the multiplication is defined by

$$(x + y \cdot p)(x' + y' \cdot p) = x \cdot x' + (x \cdot y' + x' \cdot y) \cdot p.$$

8.9. Assumptions. Throughout Lectures 8 to 11 S will be a *noetherian scheme over* $\mathbb{F}_p$ *with a lifting* $\widetilde{S}$ *to* $\mathbb{Z}/p^2$.
X will denote a noetherian S-scheme, $D \subset X$ will be a reduced Cartier divisor. X will be supposed to be *smooth over* S, which means that locally X is étale over the affine space $\mathbf{A}^n_S$ over S (here $n = \dim_S X$). D *will be a normal crossing divisor over* S, i.e.:
D is the union of smooth divisors D_i over S and one can choose the previous étale cover such that the coordinates of $\mathbf{A}^n_S$ pull back to a parameter system $(t_1, \ldots, t_n)$ on X, for which D is defined by

$$t_1 \cdot \ldots \cdot t_r, \quad \text{for some} \quad r \leq n.$$

We allow D to be empty.

8.10. Definition. For X smooth and D a normal crossing divisor over S, we define the sheaf $\Omega^1_{X/S}$ (log D) of *one forms with logarithmic poles along* D as the $\mathcal{O}_X$-sheaf generated locally by

$$\frac{dt_i}{t_i} \text{ , for } \ i \leq r \text{ , and by } \ dt_i \text{ , for } \ i > r$$

(where we use the notation from (8.9)). $\Omega^1_{X/S}(\log\ D)$ is locally free of rank n and the definition coincides with the one given in (2.1) for $S = \text{Spec } k$. Finally we define

$$\Omega^a_{X/S}(\log\ D) = \bigwedge^a \Omega^1_{X/S}(\log\ D).$$

8.11. Definition. A *lifting of*

$$D = \sum_{j=1}^{r} D_j \ \subset \ X$$

to $\widetilde{S}$ consists of a scheme $\widetilde{X}$ and subschemes $\widetilde{D}_j$ of $\widetilde{X}$, all defined and flat over $\widetilde{S}$ such that $X = \widetilde{X} \times_{\widetilde{S}} S$ and $D_j = \widetilde{D}_j \times_{\widetilde{S}} S$. We write

$$\widetilde{D} = \sum_{j=r}^{r} \widetilde{D}_j.$$

If k is a perfect field of characteristic p and $S = \text{Spec } k$, we say that (X, D) *admits a lifting to* $W_2(k)$ if liftings $\widetilde{X}$ and $\widetilde{D}_j$ exist over $\widetilde{S} = \text{Spec } W_2(k)$.

If $\mathcal{L}$ is an invertible sheaf on X, we say that X and $\mathcal{L}$ *admit a lifting to* $W_2(k)$ if there is a lifting $\widetilde{X}$ of X over $\widetilde{S} = \text{Spec } W_2(k)$ and an invertible sheaf $\widetilde{\mathcal{L}}$ on $\widetilde{X}$ with $\widetilde{\mathcal{L}}|_X = \mathcal{L}$.

8.12. Remark. Of course, $\widetilde{X}$ is also a lifting of the $\mathbb{F}_p$-scheme X to a scheme $\widetilde{X}$ over $\mathbb{Z}/p^2$. In particular, (8.7.a) remains true and we have

$$(\widetilde{X})_{top} = (X)_{top} \quad \text{and} \quad (\widetilde{D})_{top} = (D)_{top}.$$

One can make (8.7,b) more precise:

8.13. Lemma. *Let X be smooth over S and let $\widetilde{X}$ be a scheme over $\widetilde{S}$ with $\widetilde{X} \times_{\widetilde{S}} S = X$. Then the following conditions are equivalent.*

a) *$\widetilde{X}$ is smooth over $\widetilde{S}$.*

b) *$\widetilde{X}$ is a lifting of X to $\widetilde{S}$.*

c) *There is an exact sequence of $\mathcal{O}_{\widetilde{X}}$-modules*

$$0 \longrightarrow p \cdot \mathcal{O}_{\widetilde{X}} \longrightarrow \mathcal{O}_{\widetilde{X}} \xrightarrow{r} \mathcal{O}_X \longrightarrow 0$$

together with an $\mathcal{O}_{\widetilde{X}}$-isomorphism

$$p : \mathcal{O}_X \longrightarrow p \cdot \mathcal{O}_{\widetilde{X}}$$

satisfying

$$p(x) = p \cdot \widetilde{x} \text{ , for } \widetilde{x} \in \mathcal{O}_{\widetilde{X}} \text{ , and } x = r(\widetilde{x}).$$

d) *If $\widetilde{U} \subseteq \widetilde{X}$ is an open subscheme, U its image in X,*

$$\pi : U \longrightarrow \mathbb{A}^n_S = \mathbf{Spec}\ \mathcal{O}_S[t_1, \ldots, t_n]$$

an étale morphism and if $\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n \in \mathcal{O}_{\widetilde{U}}$ satisfy $r(\widetilde{\varphi}_i) = \varphi_i = \pi^* t_i$, then π extends to an étale morphism

$$\widetilde{\pi} : \widetilde{U} \longrightarrow \mathbb{A}^n_{\widetilde{S}} = \mathbf{Spec}\ \mathcal{O}_{\widetilde{S}}[t_1, \ldots, t_n]$$

with $\widetilde{\pi}^*(t_i) = \widetilde{\varphi}_i$ for $i = 1, \ldots, n$.

e) *For each $a \geq 0$ one has an exact sequence of $\mathcal{O}_{\widetilde{X}}$-modules*

$$0 \longrightarrow p \cdot \Omega^a_{\widetilde{X}/\widetilde{S}} \longrightarrow \Omega^a_{\widetilde{X}/\widetilde{S}} \xrightarrow{r} \Omega^a_{X/S} \longrightarrow 0$$

and an $\mathcal{O}_{\tilde{X}}$*-isomorphism*

$$p : \Omega^a_{X/S} \longrightarrow p \cdot \Omega^a_{\tilde{X}/\tilde{S}}$$

satisfying

$$p(\omega) = p \cdot \widetilde{\omega} \ , \textit{ for } \ \widetilde{\omega} \in \Omega^a_{\tilde{X}/\tilde{S}} \ , \textit{ and } \ \omega = r(\widetilde{\omega}).$$

PROOF: A smooth morphism is flat, and flatness implies c). Obviously d) implies e) and a). Hence the only part to prove is that c) implies d).
Using the notations from d) (for $\widetilde{X} = \widetilde{U}$) we can, of course, define

$$\widetilde{\pi} : \widetilde{X} \longrightarrow \mathbf{A}^n_{\tilde{S}} \ \text{ with } \ \widetilde{\pi}^*(t_i) = \widetilde{\varphi}_i.$$

Given a relation $\sum \widetilde{\lambda}_\nu \widetilde{m}_\nu = 0$ in $\mathcal{O}_{\tilde{X}}$ between different monomials $\widetilde{m}_\nu$ in $\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n$ the exact sequence in c) implies $\widetilde{\lambda}_\nu = p \cdot \widetilde{\mu}_\nu$ for $\widetilde{\mu}_\nu \in \mathcal{O}_{\tilde{X}}$ and the isomorphism in c) shows that one has

$$\sum \mu_\nu \cdot m_\nu = 0 \ \text{ for } \ \mu_\nu = r(\widetilde{\mu}_\nu) \ \text{ and } \ m_\nu = r(\widetilde{m}_\nu).$$

Hence $\mu_\nu = 0$ as well as $\widetilde{\mu}_\nu = 0$.

If $g_1, \ldots, g_r$ are locally independent generators of $\mathcal{O}_X$ as a $\mathcal{O}_{\mathbf{A}^n_S}$-module, and if $\widetilde{g}_1, \ldots, \widetilde{g}_r$ are liftings to $\mathcal{O}_{\tilde{X}}$, then each $\widetilde{x} \in \mathcal{O}_{\tilde{X}}$ verifies

$$x = r(\widetilde{x}) = \sum_{i=1}^{r} \lambda_i g_i$$

for some $\lambda_i \in \mathcal{O}_{\mathbf{A}^n_S}$. If $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_r$ are liftings of $\lambda_1, \ldots, \lambda_r$ to $\mathcal{O}_{\mathbf{A}^n_{\tilde{S}}}$, then

$$\widetilde{x} - \sum_{i=1}^{r} \widetilde{\lambda}_i \widetilde{g}_i \in p \cdot \mathcal{O}_{\tilde{X}}$$

and one can find $\widetilde{\mu}_i \in \mathcal{O}_{\mathbf{A}^n_{\tilde{S}}}$ with

$$\widetilde{x} - \sum_{i=1}^{r} \widetilde{\lambda}_i \widetilde{g}_i = p(\sum_{i=1}^{r} \mu_i g_i) = \sum_{i=1}^{r} p \cdot \widetilde{\mu}_i \widetilde{g}_i,$$

and

$$\widetilde{x} = \sum_{i=1}^{r} (\widetilde{\lambda}_i + p \cdot \widetilde{\mu}_i) \cdot \widetilde{g}_i.$$

In other terms, $\widetilde{g}_1, \ldots \widetilde{g}_r$ are generators of $\mathcal{O}_{\tilde{X}}$ as a $\mathcal{O}_{\mathbf{A}^n_{\tilde{S}}}$-module. They are independent by the same argument which gave the independence of the m_μ

above. $\mathcal{O}_{\tilde{X}}$ as a free $\mathcal{O}_{\mathbf{A}^n_{\tilde{S}}}$-module is flat.

Finally, (locally in X)

$$\Omega^1_{X/S} = \pi^*\Omega^1_{\mathbf{A}^n_S} = \bigoplus_{1=1}^{n} \mathcal{O}_X d\varphi_i$$

and

$$\widetilde{\pi}^*\Omega^1_{\mathbf{A}^n_{\tilde{S}}} = \bigoplus_{i=1}^{n} \mathcal{O}_{\tilde{X}} d\widetilde{\varphi}_i$$

surjects to $\Omega^1_{\tilde{X}/\tilde{S}}$. In fact, if $\widetilde{\omega} \in \Omega^1_{\tilde{X}/\tilde{S}}$,

$$\widetilde{\omega} - \sum_{i=1}^{n} \widetilde{\lambda}_i d\widetilde{\varphi}_i \in \mathrm{im}(\Omega^1_{X/S} \xrightarrow{\cdot p} \Omega^1_{\tilde{X}/\tilde{S}})$$

for some $\widetilde{\lambda}_i \in \mathcal{O}_{\tilde{X}}$ and, as above, one can modify the $\widetilde{\lambda}_i$ to get

$$\widetilde{\omega} = \sum_{i=1}^{n} (\widetilde{\lambda}_i + p \cdot \widetilde{\mu}_i) d\widetilde{\varphi}_i.$$

As $\widetilde{\pi}^*\Omega^1_{\mathbf{A}^n_{\tilde{S}}} \longrightarrow \Omega^1_{\tilde{X}/\tilde{S}}$ is injective as well, $\widetilde{\pi}$ is étale.

□

8.14. Lemma. *Let X be a smooth S-scheme and*

$$D = \sum_{j=1}^{r} D_j$$

be a normal crossing divisor over S. Let $\widetilde{X}$ be a lifting of X to $\widetilde{S}$ and $\widetilde{D}_j \subseteq \widetilde{X}$ subschemes with

$$\widetilde{D}_j \otimes_{\tilde{S}} S = D_j$$

for $j = 1, \ldots, r$. Then the following conditions are equivalent:

a)

$$\widetilde{D} = \sum_{j=1}^{r} \widetilde{D}_j \;\subset\; \widetilde{X}$$

is a lifting of $D \subset X$ to $\widetilde{S}$.

b) *The components of $\widetilde{D}$ are Cartier divisors in $\widetilde{X}$.*

c) *If in (8.13,d) one knows that $D\mid_U$ is the zero-set of $\varphi_1 \cdot \ldots \cdot \varphi_s$, then one can choose $\widetilde{\pi} : \widetilde{U} \longrightarrow \mathbf{A}^n_{\tilde{S}}$, such that $\widetilde{D}_j \mid_{\tilde{U}}$ is the zero set of $\widetilde{\pi}^*(t_j)$.*

PROOF: If $\widetilde{D} \subset \widetilde{X}$ is a lifting of $D \subset X$, then the flatness of $\widetilde{X}$ and $\widetilde{D}_j$ over $\widetilde{S}$ implies that the ideal sheaf $J_{\tilde{D}_j}$ of $\widetilde{D}_j$ is flat over $\widetilde{S}$. We have again an exact sequence

$$0 \longrightarrow p \cdot J_{\tilde{D}_j} \longrightarrow J_{\tilde{D}_j} \longrightarrow J_{D_j} \longrightarrow 0$$

where J_{D_j} is the ideal sheaf of D_j, and an isomorphism

$$p : J_{D_j} \longrightarrow p \cdot J_{\tilde{D}_j}.$$

If $\widetilde{\varphi}_j$ is a lifting of φ_j to $J_{\tilde{D}_j}$, then for any $\widetilde{g} \in J_{\tilde{D}_j}$ one has $g = \lambda \cdot \varphi_j$ and

$$\widetilde{g} - \widetilde{\lambda} \cdot \widetilde{\varphi}_j \in p \cdot I_{\tilde{D}_j}$$

is of the form $p \cdot \widetilde{\mu} \cdot \widetilde{\varphi}_j = p(\mu \cdot \varphi_j)$ for some $\widetilde{\mu} \in \mathcal{O}_{\tilde{X}}$. Hence $\widetilde{\varphi}_j$ is a defining equation for $\widetilde{D}_j$.
By (8.13,d) b) implies c) and obviously c) implies a).

□

8.15. Definition. Using the notations from (8.14,c) and (8.13,d) we define for a lifting $\widetilde{D} \subset \widetilde{X}$ of $D \subset X$ to $\widetilde{S}$ the sheaf

$$\Omega^1_{\tilde{X}/\tilde{S}}(\log\ \widetilde{D})$$

to be the $\mathcal{O}_{\tilde{X}}$-sheaf generated by

$$\frac{d\widetilde{\varphi}_j}{\widetilde{\varphi}_j} \quad \text{for} \quad j = 1, \ldots, s \quad \text{and} \quad d\widetilde{\varphi}_j \quad \text{for} \quad j = s+1, \ldots, n.$$

8.16. Properties.
a) For all a the sheaves

$$\Omega^a_{\tilde{X}/\tilde{S}}(\log\ \widetilde{D}) = \bigwedge^a \Omega^1_{\tilde{X}/\tilde{S}}(\log\ \widetilde{D})$$

are locally free over $\mathcal{O}_{\tilde{X}}$.
b) One has an exact sequence of $\mathcal{O}_{\tilde{X}}$-modules

$$0 \longrightarrow p \cdot \Omega^a_{\tilde{X}/\tilde{S}}(\log\ \widetilde{D}) \longrightarrow \Omega^a_{\tilde{X}/\tilde{S}}(\log\ \widetilde{D}) \longrightarrow \Omega^a_{X/S}(\log\ D) \longrightarrow 0$$

and an $\mathcal{O}_{\tilde{X}}$-isomorphism

$$p : \Omega^a_{X/S}(\log\ D) \longrightarrow p \cdot \Omega^a_{\tilde{X}/\tilde{S}}(\log\ \widetilde{D}).$$

8.17. Proposition. *Let X be a smooth S-scheme.*

a) *Locally in the Zariski topology X has a lifting $\widetilde{X}$ to $\widetilde{S}$.*

b) *If $\widetilde{X}$ is a lifting of X to $\widetilde{S}$, if $\widetilde{X}$ is affine and Y a complete intersection in X, then there exists a lifting $\widetilde{Y}$ of Y to $\widetilde{S}$ and an embedding $\widetilde{Y} \subset \widetilde{X}$.*

c) *In particular, if D is a S-normal crossing divisor on X then locally in the Zariski topology $D \subset X$ has a lifting $\widetilde{D} \subset \widetilde{X}$ to $\widetilde{S}$.*

Proof: Locally X is a complete intersection in an affine space over S. Hence a) follows from b). In b) we may assume that Y is a divisor, let us say the zero set of $\varphi \in \mathcal{O}_X$. We can choose $\widetilde{Y}$ to be the zero set of any lifting $\widetilde{\varphi} \in \mathcal{O}_{\widetilde{X}}$ of φ. In fact, the flatness follows easily from (8.13,c) or from the following argument. Choose

$$\pi : X \longrightarrow \mathbf{A}^n_S = \operatorname{Spec} \mathcal{O}_X[t_1, \ldots, t_n]$$

with $\varphi = \pi^*(t_1)$. By (8.13,d) π extends to an étale map $\widetilde{\pi} : \widetilde{X} \longrightarrow \mathbf{A}^n_{\widetilde{S}}$ with $\widetilde{\varphi} = \widetilde{\pi}^*(t_1)$.

□

8.18. Isomorphisms between liftings

Let in the sequel X be a smooth S-scheme, $D \subset X$ be an S-normal crossing divisor and let, for $i = 1, 2$, $\widetilde{D}^{(i)} \subset \widetilde{X}^{(i)}$ be two liftings of $D \subset X$ to $\widetilde{S}$.

8.19. Notations. A morphism $u : \widetilde{X}^{(1)} \to \widetilde{X}^{(2)}$ is called an *isomorphism of liftings*

$$u : (\widetilde{X}^{(1)}, \widetilde{D}^{(1)}) \longrightarrow (\widetilde{X}^{(2)}, \widetilde{D}^{(2)})$$

if $u \mid_X = id_X$ and if

$$u^*(\mathcal{O}_{\widetilde{X}^{(2)}}(-\widetilde{D}^{(2)})) = \mathcal{O}_{\widetilde{X}^{(1)}}(-\widetilde{D}^{(1)}).$$

8.20. Remark. We have seen in (8.12) that $(\widetilde{X}^{(i)})_{top} = (X)_{top}$ and hence u is the identity on the topological spaces. Henceforth, giving u is the same as giving the morphism

$$u^* : \mathcal{O}_{\widetilde{X}^{(2)}} \longrightarrow \mathcal{O}_{\widetilde{X}^{(1)}}$$

of sheaves of rings on $(X)_{top}$. The assumption $u|_X = id_X$ forces u^* to be an isomorphism.

8.21. Lemma. *Locally in the Zariski topology there exists an isomorphism of liftings*

$$u : (\widetilde{X}^{(1)}, \widetilde{D}^{(1)}) \longrightarrow (\widetilde{X}^{(2)}, \widetilde{D}^{(2)}).$$

Proof: Locally the diagonal $\Delta \subset X \times X$ is a complete intersection and we can lift it to

$$\widetilde{\Delta} \subset \widetilde{X}^{(1)} \times \widetilde{X}^{(2)}.$$

For example, if $\varphi_1, \ldots, \varphi_n$ are local parameters on X and $\widetilde{\varphi}_1^{(i)}, \ldots, \widetilde{\varphi}_n^{(i)}$ liftings in $\mathcal{O}_{\widetilde{X}^{(i)}}$ such that $D^{(i)}$ is the zero locus of $\varphi_1^{(i)} \cdot \ldots \cdot \varphi_s^{(i)}$, then we can choose $\widetilde{\Delta}$ to be defined by

$$\widetilde{\varphi}_j^{(1)} \otimes 1 - 1 \otimes \widetilde{\varphi}_j^{(2)} \quad \text{for} \quad j = 1, \ldots, n.$$

We have isomorphisms of liftings

$$p_1 : \widetilde{\Delta} \longrightarrow \widetilde{X}^{(1)}, \quad p_2 : \widetilde{\Delta} \longrightarrow \widetilde{X}^{(2)}$$

and $u = p_2 \circ p_1^{-1}$ satisfies

$$u^*(\mathcal{O}_{\widetilde{X}^{(2)}}(-\widetilde{D}^{(2)})) = \mathcal{O}_{\widetilde{X}^{(1)}}(-\widetilde{D}^{(1)}).$$

□

Let

$$u, v : (\widetilde{X}^{(1)}, \widetilde{D}^{(1)}) \longrightarrow (\widetilde{X}^{(2)}, \widetilde{D}^{(2)})$$

be two isomorphisms of liftings. For $\widetilde{x} \in \mathcal{O}_{\widetilde{X}^{(2)}}$ one has

$$(u^* - v^*)(p \cdot \widetilde{x}) = p(u^* - v^*)(\widetilde{x}) = p(id - id)(x) = 0$$

therefore $(u^* - v^*)|_{p \cdot \mathcal{O}_{\widetilde{X}^{(2)}}} = 0$. Of course, the map

$$\mathcal{O}_X = \mathcal{O}_{\widetilde{X}^{(2)}} / p \cdot \mathcal{O}_{\widetilde{X}^{(2)}} \longrightarrow \mathcal{O}_X = \mathcal{O}_{\widetilde{X}^{(1)}} / p \cdot \mathcal{O}_{\widetilde{X}^{(1)}}$$

induced by $(u^* - v^*)$ is zero as well, and $(u^* - v^*)$ factors through

$$(u^* - v^*) : \mathcal{O}_X \longrightarrow p \cdot \mathcal{O}_{\widetilde{X}^{(1)}} = p(\mathcal{O}_X).$$

For $x, y \in \mathcal{O}_X$ with liftings $\widetilde{x}, \widetilde{y} \in \mathcal{O}_{\widetilde{X}^{(2)}}$ one has

$$(u^* - v^*)(\widetilde{x} \cdot \widetilde{y}) = u^*(\widetilde{x}) \cdot u^*(\widetilde{y}) - v^*(\widetilde{x}) \cdot v^*(\widetilde{y}) = x \cdot (u^* - v^*)(\widetilde{y}) + y \cdot (u^* - v^*)(\widetilde{x}).$$

Hence $p^{-1} \circ (u^* - v^*) : \mathcal{O}_X \longrightarrow \mathcal{O}_X$ is a derivation and factors through

$$\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/S} \longrightarrow \mathcal{O}_X$$

where we denote the second morphism by $(u^* - v^*)$ again. If $\widetilde{t}_j^{(i)}$ is a local equation for $\widetilde{D}_j^{(i)}$, $i = 1, 2$, with reduction $t_j \in \mathcal{O}_X$, then

$$u^*(\widetilde{t}_j^{(2)}) = \widetilde{t}_j^{(1)} \cdot (1 + p \cdot \lambda)$$

and
$$v^*(\widetilde{t}_j^{(2)}) = \widetilde{t}_j^{(1)}(1 + p \cdot \mu).$$
Hence
$$(u^* - v^*)(dt_j) = t_j \cdot (\lambda - \mu) \in t_j \cdot \mathcal{O}_X$$
and $p^{-1} \circ (u^* - v^*)$ even factors through
$$\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/S}(\log\ D) \longrightarrow \mathcal{O}_X.$$

8.22. Proposition. *Keeping the notations from (8.19) let*
$$u : (\widetilde{X}^{(1)}, \widetilde{D}^{(1)}) \longrightarrow (\widetilde{X}^{(2)}, \widetilde{D}^{(2)})$$
be an isomorphism of liftings. Then
$$\{v : (\widetilde{X}^{(1)}, \widetilde{D}^{(1)}) \longrightarrow (\widetilde{X}^{(2)}, \widetilde{D}^{(2)});\ v \textit{ isomorphism of liftings } \}$$
is described by the affine space
$$u^* + \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}(\log\ D), \mathcal{O}_X).$$

PROOF: It just remains to show that for
$$\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}(\log\ D), \mathcal{O}_X)$$
we can find v. Define
$$v^* : \mathcal{O}_{\widetilde{X}^{(2)}} \longrightarrow \mathcal{O}_{\widetilde{X}^{(1)}}$$
by
$$v^*(\widetilde{x}) = u^*(\widetilde{x}) - p \cdot \varphi(dx).$$
If $\widetilde{t}_j^{(i)}$ is as above an equation of $\widetilde{D}_j^{(i)}$,
$$v^*(\widetilde{t}_j^{(2)}) = \widetilde{t}_j^{(1)} \cdot (1 + p \cdot \lambda) - p \cdot t \cdot \gamma \quad \text{for some} \quad \gamma \in \mathcal{O}_X,$$
or
$$v^*(\widetilde{t}_j^{(2)}) = \widetilde{t}_j^{(1)}(1 + p(\lambda - \gamma))$$
and v^* satisfies the conditions posed in (8.19.).

□

8.23. Proposition. *Let X be an affine scheme, smooth over S and let D be a normal crossing divisor over S. Let $\widetilde{D}^{(i)} \subset \widetilde{X}^{(i)}$ be two liftings of $D \subset X$ to $\widetilde{S}$. Then there exists an isomorphism of liftings*
$$u : (\widetilde{X}^{(1)}, \widetilde{D}^{(1)}) \longrightarrow (\widetilde{X}^{(2)}, \widetilde{D}^{(2)}).$$

Proof: Of course, (8.22) just says that the isomorphisms of liftings over a fixed open set form a "torseur" under the group

$$\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}(\log\ D), \mathcal{O}_X)$$

and, since X is affine and

$$H^1(X, \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/S}(\log\ D), \mathcal{O}_X)) = 0$$

one obtains (8.23). However, to state this in the elementary language used up to now, let us avoid this terminology:

From (8.21) we know that there is an affine open cover $\mathcal{U} = \{X_\alpha\}$ of X and isomorphisms of liftings

$$u_\alpha : (\widetilde{X}^{(1)}_\alpha, \widetilde{D}^{(1)}_\alpha) \longrightarrow (\widetilde{X}^{(2)}_\alpha, \widetilde{D}^{(2)}_\alpha).$$

By (8.22) $p^{-1} \circ (u^*_\alpha - u^*_\beta)$ defines a 1-cocycle with values in the sheaf

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/S}(\log\ D), \mathcal{O}_X).$$

Since

$$H^1(X, \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/S}(\log\ D), \mathcal{O}_X) = 0$$

we find

$$\varphi_\alpha \in \Gamma(X_\alpha, \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/S}(\log\ D), \mathcal{O}_X))$$

in some possibly finer cover $\{X_\alpha\}$ such that

$$p^{-1} \circ (u^*_\alpha - u^*_\beta) = \varphi_\alpha - \varphi_\beta.$$

Hence the isomorphisms of liftings $u^*_\alpha - p \cdot \varphi^*_\alpha$ glue together to $u : \widetilde{X}_1 \to \widetilde{X}_2$. □

§ 9 The Frobenius and its liftings

Everything in this lecture is either elementary or taken from [12].

Let S be a noetherian scheme defined over $\mathbb{F}_p$ and let X be a noetherian S-scheme.

9.1. Definition. The *absolute Frobenius of* S is the endomorphism

$$F_S : S \longrightarrow S$$

defined by the following conditions.
F is the identity on the topological space and

$$F^*_S : \mathcal{O}_S \longrightarrow \mathcal{O}_S$$

is given by $F_S^*(a) = a^p$. In particular for $x \in \mathcal{O}_X$ and $\lambda \in \mathcal{O}_S$ one has

$$F_X^*(\lambda x) = \lambda^p x^p = F_S^*(\lambda) \cdot F_X^*(x),$$

and therefore one has a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{F_X} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow[F_S]{} & S \end{array}$$

For $X' = X \times_{F_S} S$ this allows to factorize F_X:

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[dr, "f"'] & X' \arrow[r, "pr_1"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ & S \arrow[r, "F_S"'] & S \end{tikzcd}$$

with $F_X = pr_1 \circ F$ and $f' = pr_2$. By abuse of notations we write

$$F_S = pr_1 : X' \longrightarrow X.$$

F is called the *relative Frobenius* (relative to S). For

$$x \otimes \lambda \in \mathcal{O}_{X'} = \mathcal{O}_X \otimes_{F_S} \mathcal{O}_S \quad \text{one has} \quad F^*(x \otimes \lambda) = x^p \cdot \lambda$$

and for

$$x \in \mathcal{O}_X \quad \text{one has} \quad F_S^*(x) = x \otimes 1.$$

9.2. Remark. The absolute Frobenius F_S is a morphism of schemes. In fact, $F_S^* : \mathcal{O}_{S,s} \longrightarrow \mathcal{O}_{S,s}$ satisfies

$$F_S^{*^{-1}}(m_{S,s}) = \{x \in \mathcal{O}_{S,s};\ x^p \in m_{S,s}\} = m_{S,s}$$

for any prime ideal $m_{S,s}$, and hence it is a local homomorphism on the local rings. If $S = \mathrm{Spec} A$, then F_S is induced by the p-th power map $A \to A$.

9.3. Properties.
a) Since $F_S : (S)_{top} \to (S)_{top}$ is the identity,

$$F_S : (X')_{top} \to (X)_{top}$$

is an isomorphism of topological spaces, as well as

$$F : (X)_{top} \to (X')_{top}.$$

b) If $t_1, \dots, t_m$ are locally on X, generators of $\mathcal{O}_X$, i.e.

$$\mathcal{O}_X = \mathcal{O}_S[t_1, \dots, t_m]/_{<f_1, \dots, f_s>}$$

and $f = \sum \lambda_{\underline{i}} \cdot \underline{t}^{\underline{i}}$, for $\underline{t}^{\underline{i}} = t_1^{i_1} \cdot \ldots \cdot t_m^{i_m}$, and $\lambda_{\underline{i}} \in \mathcal{O}_S$, then one has

$$F_S^*(f) = \sum \lambda_{\underline{i}}^p \underline{t}^{\underline{i}} .$$

Hence

$$\mathcal{O}_{X'} = \mathcal{O}_X \otimes_{F_S} \mathcal{O}_S = \mathcal{O}_S[t_1, \dots, t_m]/_{<F_S^*(f_1), \dots F_S^*(f_s)>} .$$

For $g = \sum \mu_{\underline{i}} \underline{t}^{\underline{i}} \in \mathcal{O}_{X'}$ one has

$$F^*(g) = \sum \mu_{\underline{i}} t^{p \cdot \underline{i}} \quad \text{where} \quad t^{p \cdot \underline{i}} = t_1^{p \cdot i_1} \cdot \ldots \cdot t_m^{p \cdot i_m}.$$

c) If X is smooth over S, one has locally étale morphisms $\pi : X \to \mathbf{A}_S^n$, hence a diagram, where the right hand squares are by definition cartesian:

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[d, "\pi"'] & X' \arrow[r, "F_S"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\
\mathbf{A}_S^n \arrow[r, "F"] \arrow[dr] & (\mathbf{A}_S^n)' \arrow[r, "F_S"] \arrow[d] & \mathbf{A}_S^n \arrow[d] \\
 & S \arrow[r, "F_S"] & S
\end{tikzcd}$$

For

$$\mathbf{A}_S^n = \mathrm{Spec} \mathcal{O}_S[t_1 \dots t_n]$$

we have

$$(\mathbf{A}_S^n)' = \mathrm{Spec} \mathcal{O}_S[t_1 \dots t_n]$$

and $F^{-1}\mathcal{O}_{(\mathbf{A}_S^n)'}$ is the subsheaf of $\mathcal{O}_{\mathbf{A}_S^n}$ given by $\mathcal{O}_S[t_1^p, \dots, t_n^p]$. Hence $F_* \mathcal{O}_{\mathbf{A}_S^n}$ is freely generated over $\mathcal{O}_{(\mathbf{A}_S^n)'}$ by

$$F_*(t_1^{a_1} \cdot \ldots \cdot t_n^{a_n}) \quad \text{for} \quad 0 \leq a_i < p.$$

We have an isomorphism $f : X \longrightarrow X' \times_{(\mathbf{A}_S^n)'} \mathbf{A}_S^n$ and the left upper square in the diagram is cartesian as well.
In fact, for $x \in X$ we may assume that the maximal ideal $m_{X,x} \subset \mathcal{O}_{X,x}$ is generated by $t_1, \dots t_n$ and, if $\mathcal{O}_x$ is the local ring of x in $X' \times_{(\mathbf{A}_S^n)'} \mathbf{A}_S^n$, then the maximal ideal m_x of $\mathcal{O}_x$ has the same generators. Hence

$$f^* : \mathcal{O}_x \longrightarrow \mathcal{O}_{X,x}$$

is a local homomorphism inducing a surjection

$$m_x \longrightarrow m_{X,x}/m_{X,x}^2.$$

d) The sheaf $F_*\mathcal{O}_X$ is again a locally free $\mathcal{O}_X$-module. Using the notation from part c) it is generated by

$$F_*\pi^*(t_1^{a_1}\dots t_n^{a_n}) \quad \text{for} \quad 0 \le a_i < p.$$

Therefore, for any locally free sheaf $\mathcal{F}$ on X, the sheaf $F_*\mathcal{F}$ is locally free over $\mathcal{O}_{X'}$. For example, if D is a normal crossing divisor on X, then $F_*\Omega^a_{X/S}(\log\ D)$ is locally free.

9.4. Definition. When S has a lifting $\widetilde{S}$ to $\mathbb{Z}/p^2$ (see (8.5)), a *lifting* $\widetilde{F}_{\widetilde{S}}$ of F_S is a finite morphism

$$\widetilde{F}_{\widetilde{S}} : \widetilde{S} \longrightarrow \widetilde{S}$$

whose restriction to S is F_S.

Similarly, if X and X' have liftings $\widetilde{X}$ and $\widetilde{X}'$ to $\widetilde{S}$, a *lifting* $\widetilde{F}$ *of the relative Frobenius* F is a finite morphism

$$\widetilde{F} : \widetilde{X} \longrightarrow \widetilde{X}'$$

which restricts to F.

In particular, (8.13,c) gives rise to an exact sequence of $\mathcal{O}_{\widetilde{X}'}$-modules

$$0 \longrightarrow \widetilde{F}_* p\cdot\mathcal{O}_{\widetilde{X}} \longrightarrow \widetilde{F}_*\mathcal{O}_{\widetilde{X}} \longrightarrow F_*\mathcal{O}_X \longrightarrow 0$$

together with an $\mathcal{O}_{\widetilde{X}'}$-isomorphism

$$p : F_*\mathcal{O}_X \longrightarrow \widetilde{F}_* p\cdot\mathcal{O}_{\widetilde{X}} = p\cdot\widetilde{F}_*\mathcal{O}_{\widetilde{X}}.$$

9.5. Assumptions. For the rest of this lecture we assume S to be a scheme over $\mathbb{F}_p$ with a lifting $\widetilde{S}$ to $\mathbb{Z}/p^2$ and a lifting

$$\widetilde{F}_{\widetilde{S}} : \widetilde{S} \longrightarrow \widetilde{S}$$

of the absolute Frobenius.

Moreover we keep the assumptions made in (8.9). Hence X is supposed to be smooth over S and $D \subset X$ is a normal crossing divisor over S. We write $D' = F_S^*(D)$ for $F_S : X' \to X$.

9.6. Example. If k is a perfect field, $S = \text{Spec}\ k$ and $\widetilde{S} = \text{Spec}\ W_2(k)$, then one takes

$$\widetilde{F}^*_{\widetilde{S}}(x + y\cdot p) = x^p + y^p\cdot p\ .$$

Furthermore, in this case F_S is an isomorphism of fields and X' is isomorphic to X. In particular, X has a lifting to $\widetilde{S} = \text{Spec}\ W_2(k)$ if and only if X' does.

9.7. Proposition. *Let $D \subset X$ be an S-normal crossing divisor of the smooth S-scheme X. Let*

$$\widetilde{D}' \subset \widetilde{X}' \quad \textit{be a lifting of} \quad D' \subset X'$$

to $\widetilde{S}$. Then locally in the Zarisky topology

$$D \subset X \quad \textit{has a lifting} \quad \widetilde{D} \subset \widetilde{X}$$

to $\widetilde{S}$ such that F lifts to

$$\widetilde{F} : \widetilde{X} \longrightarrow \widetilde{X}' \quad \textit{with} \quad \widetilde{F}^* \mathcal{O}_{\tilde{X}'}(-\widetilde{D}') = \mathcal{O}_{\tilde{X}}(-p \cdot \widetilde{D}).$$

Proof: By (8.17,c) we know that a lifting $\widetilde{D} \subset \widetilde{X}$ exists locally. Let

$$\pi : X \longrightarrow \mathbf{A}^n_S = \mathbf{Spec}\ \mathcal{O}_S[t_1, \ldots, t_n]$$

be étale and D_j be the zero set of $\varphi_j = \pi^*(t_j)$. By (8.14,c) we can choose liftings of φ_j to $\widetilde{\varphi}_j \in \mathcal{O}_{\tilde{X}}$ and of

$$\varphi'_j = F^*_S(\varphi_j) = \varphi_j \otimes 1$$

to $\widetilde{\varphi}'_j$ such that $\widetilde{D}_j$ is defined by $\widetilde{\varphi}_j$ and $\widetilde{D}'_j$ by $\widetilde{\varphi}'_j$. We can define

$$\widetilde{F}^* \quad \text{by} \quad \widetilde{F}^*(\widetilde{\varphi'_j}) = \widetilde{\varphi}^p_j.$$

By the explicit description of F in (9.3,c) $\widetilde{F}$ restricts to F and

$$\widetilde{F}^* \mathcal{O}_{\tilde{X}'}(-\widetilde{D}') = \mathcal{O}_{\tilde{X}}(-p \cdot \widetilde{D}).$$

□

9.8. Remark. We have seen in (8.12) that $(\widetilde{X})_{top} = (X)_{top}$ and $(\widetilde{X}')_{top} = (X')_{top}$. By (9.3,a) we have $(X)_{top} \simeq (X')_{top}$ and hence we can regard a lifting $\widetilde{F}$ as a morphism

$$\widetilde{F}^* : \mathcal{O}_{\tilde{X}'} \longrightarrow \mathcal{O}_{\tilde{X}}$$

of sheaves of rings over $(X')_{top}$.

Similarly to (8.22) we have

9.9. Proposition. *Keeping notations and assumptions from (9.7) assume that*

$$D \subset X \quad \textit{has a lifting} \quad \widetilde{D} \subset \widetilde{X}$$

to $\widetilde{S}$ and let $\widetilde{F}_0 : \widetilde{X} \to \widetilde{X}'$ be one lifting of F with

$$\widetilde{F}^*_0 \mathcal{O}_{\tilde{X}'}(-\widetilde{D}') = \mathcal{O}_{\tilde{X}}(-p \cdot \widetilde{D}).$$

Then

$$\{\widetilde{F} : \widetilde{X} \longrightarrow \widetilde{X}'; \quad \widetilde{F}^* \mathcal{O}_{\widetilde{X}'}(-\widetilde{D}') = \mathcal{O}_{\widetilde{X}}(-p \cdot \widetilde{D}), \ \widetilde{F} \ \textit{ lifting of } \ F\}$$

is described by the affine space

$$\widetilde{F}_0^* + \mathrm{Hom}_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log \ D'), F_* \mathcal{O}_X).$$

PROOF: As in (8.22) for isomorphisms of liftings one finds that $\widetilde{F}^* - \widetilde{F}_0^*$ is zero on $p \cdot \mathcal{O}_{\widetilde{X}}$ and induces the zero map from $\mathcal{O}_{X'}$ to $\mathcal{O}_X$. Hence $\widetilde{F}^* - \widetilde{F}_0^*$ induces

$$\widetilde{F}^* - \widetilde{F}_0^* : \mathcal{O}_{X'} \longrightarrow p \cdot \mathcal{O}_X.$$

For $\widetilde{x}', \widetilde{y}' \in \mathcal{O}_{\widetilde{X}'}$ one has

$$(\widetilde{F}^* - \widetilde{F}_0^*)(\widetilde{x}' \cdot \widetilde{y}') = F(x')(\widetilde{F}^* - \widetilde{F}_0^*)(\widetilde{y}') + F(y')(\widetilde{F}^* - \widetilde{F}_0^*)(\widetilde{x}')$$

and

$$p^{-1}(\widetilde{F}^* - \widetilde{F}_0^*) : \mathcal{O}_{X'} \longrightarrow \mathcal{O}_X$$

factorizes through

$$\mathcal{O}_{X'} \xrightarrow{d} \Omega^1_{X'/S} \longrightarrow \mathcal{O}_X$$

where the right hand side morphism is $\mathcal{O}_{X'}$-linear and is again denoted by $(\widetilde{F}^* - \widetilde{F}_0^*)$. For $\widetilde{\varphi}_j \in \mathcal{O}_{\widetilde{X}}$ and $\widetilde{\varphi}'_j \in \mathcal{O}_{\widetilde{X}'}$, local parameters for $\widetilde{D}_j$ and $\widetilde{D}'_j$ respectively, which lift φ_j and $\varphi'_j = \varphi_j \otimes 1$, one has

$$\widetilde{F}^*(\widetilde{\varphi}'_j) = \widetilde{\varphi}_j^p \cdot (1 + p \cdot \widetilde{\lambda})$$

and

$$\widetilde{F}_0^*(\widetilde{\varphi}'_j) = \widetilde{\varphi}_j^p \cdot (1 + p \cdot \widetilde{\lambda}_0)$$

for some $\widetilde{\lambda}, \widetilde{\lambda}_0 \in \mathcal{O}_{\widetilde{X}}$. Therefore

$$(\widetilde{F}^* - \widetilde{F}_0^*)(\widetilde{\varphi}'_j) = \widetilde{\varphi}_j^p(p(\widetilde{\lambda} - \widetilde{\lambda}_0))$$

and

$$(\widetilde{F}^* - \widetilde{F}_0^*)(d\widetilde{\varphi}'_j) = p^{-1} \circ (\widetilde{F}^* - \widetilde{F}_0^*)(\widetilde{\varphi}'_j) \in \mathcal{O}_X(-p \cdot D_j).$$

Hence

$$(\widetilde{F}^* - \widetilde{F}_0^*) : \Omega^1_{X'/S} \longrightarrow \mathcal{O}_X$$

extends to

$$(\widetilde{F}^* - \widetilde{F}_0^*) : \Omega^1_{X'/S}(\log \ D') \longrightarrow \mathcal{O}_X.$$

Conversely, for

$$\varphi \in \mathrm{Hom}_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log \ D'), \mathcal{O}_X)$$

we define
$$\widetilde{F}^* = \widetilde{F}_0^* + \varphi^* \quad \text{by} \quad \widetilde{F}^*(\widetilde{x}) = \widetilde{F}_0^*(\widetilde{x}) - p(\varphi(dx)).$$
We have
$$\mathcal{O}_{X'}(-D') \xrightarrow{d} \Omega^1_{X'/S}(\log\ D')(-D') \xrightarrow{\varphi} \mathcal{O}_X(-p\cdot D)$$
and
$$\widetilde{F}^*(\mathcal{O}_{\widetilde{X}'}(-\widetilde{D}')) = \mathcal{O}_{\widetilde{X}}(-p\cdot \widetilde{D}).$$
□

9.10. Corollary. *Under the assumption of (9.7) assume that X is affine and that*
$$D \subset X \quad \textit{has a lifting} \quad \widetilde{D} \subset \widetilde{X}$$
to $\widetilde{S}$. Then there is a lifting $\widetilde{F} : \widetilde{X} \to \widetilde{X}'$ of F with
$$\widetilde{F}^*\mathcal{O}_{\widetilde{X}'}(-\widetilde{D}') = \mathcal{O}_{\widetilde{X}}(-p\cdot\widetilde{D}).$$

PROOF: One repeats the argument used to prove (8.23), replacing $\widetilde{X}^{(1)}$ by $\widetilde{X}$ and $\widetilde{X}^{(2)}$ by $\widetilde{X}'$ and using (9.7) and (9.9) instead of (8.21) and (8.22).
□

9.11. Remark. Let X be a smooth S-scheme, $\widetilde{X}'$ a lifting of X' and $\widetilde{X}^{(i)}$, for $i = 1, 2$, two liftings of X to $\widetilde{S}$. Assume that we have a lifting
$$\widetilde{F}_2 : \widetilde{X}^{(2)} \longrightarrow \widetilde{X}'$$
and isomorphisms
$$u : \widetilde{X}^{(1)} \longrightarrow \widetilde{X}^{(2)} \quad \text{and} \quad v : \widetilde{X}^{(1)} \longrightarrow \widetilde{X}^{(2)},$$
both lifting the identity. Then, considering again $\mathcal{O}_{\widetilde{X}'}, \mathcal{O}_{\widetilde{X}^{(1)}}$ and $\mathcal{O}_{\widetilde{X}^{(2)}}$ as sheaves of rings on $(X')_{top}$, one has
$$\mathcal{O}_{\widetilde{X}'} \xrightarrow{\widetilde{F}_2^*} \mathcal{O}_{\widetilde{X}^{(2)}} \xrightarrow{(u^*-v^*)} \mathcal{O}_{\widetilde{X}^{(1)}}$$
and
$$(u^* - v^*) \circ \widetilde{F}_2^*(\widetilde{x}') = (u^* - v^*)(d(F^*(x'))) = 0$$
since $F^*(x')$ is a p-th power. We find:
$$(\widetilde{F}_2 \circ u)^* = (\widetilde{F}_2 \circ v)^* : \mathcal{O}_{\widetilde{X}'} \longrightarrow \mathcal{O}_{\widetilde{X}^{(1)}}.$$
In other words, $(\widetilde{F}_2 \circ u)^*$ does not depend on the choice of u.

In (9.10) we used the fact that, for X' affine, the higher cohomology groups of coherent sheaves are zero to obtain the existence of the lifting

$$\widetilde{F} : \widetilde{X} \longrightarrow \widetilde{X}'$$

of F. We have more generally:

9.12. Corollary. *Let X be a smooth scheme and $D \subset X$ be a normal crossing divisor over S. Given liftings*

$$\widetilde{D} \subset \widetilde{X} \quad \textit{and} \quad \widetilde{D}' \subset \widetilde{X}' \quad \textit{of} \quad D \subset X \quad \textit{and} \quad D' \subset X'$$

(respectively) to $\widetilde{S}$, the exact obstruction for lifting F to

$$\widetilde{F} : \widetilde{X} \longrightarrow \widetilde{X}' \quad \textit{with} \quad \widetilde{F}^* \mathcal{O}_{\widetilde{X}'}(-\widetilde{D}') = \mathcal{O}_{\widetilde{X}}(-p \cdot \widetilde{D})$$

is a class

$$[F_{\tilde{X}',\tilde{D}'}] \in H^1(X', \mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log\ D'), F_* \mathcal{O}_X))$$

which does not depend on $(\widetilde{X}, \widetilde{D})$.

Proof: By (9.7) or (9.10) one can cover $\widetilde{X}$ by affine $\widetilde{X}_\alpha$ such that F lifts to $\widetilde{F}_\alpha$ on $\widetilde{X}_\alpha$ with the required property for D. Then by (9.9) $(\widetilde{F}^*_\alpha - \widetilde{F}^*_\beta)$ describes a 1-cocycle with values in

$$\mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log\ D'), F_* \mathcal{O}_X).$$

Changing the $\widetilde{F}_\alpha$ corresponds to changing the cocycle by a coboundary. We define $[F_{\tilde{X}',\tilde{D}'}]$ to be the cohomology class of this cocycle. If $[F_{\tilde{X}',\tilde{D}'}] = 0$ one finds for a possibly finer cover $\{X'_\alpha\}$

$$\varphi_\alpha \in \Gamma(X'_\alpha, \mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log\ D'), F_* \mathcal{O}_X))$$

such that the $\widetilde{F}^*_\alpha + \varphi_\alpha$ glue together to give $\widetilde{F} : \widetilde{X} \to \widetilde{X}'$.

If $\widetilde{X}^{(i)}$ are two liftings, $\widetilde{X}^{(i)}_\alpha$ coverings and $\widetilde{F}^{(i)}_\alpha$ liftings of F, for $i = 1, 2$ we can apply (8.21) or (8.23) to get isomorphisms of liftings

$$u_\alpha : \widetilde{X}^{(1)}_\alpha \longrightarrow \widetilde{X}^{(2)}_\alpha.$$

By (9.11) we have on $\widetilde{X}'_\alpha \cap \widetilde{X}'_\beta$

$$(\widetilde{F}^{(2)}_\alpha \circ u_\alpha)^* - (\widetilde{F}^{(2)}_\beta \circ u_\beta)^* = (\widetilde{F}^{(2)}_\alpha \circ u_\alpha)^* - (\widetilde{F}^{(2)}_\beta \circ u_\alpha)^* =$$

$$u^*_\alpha \circ (\widetilde{F}^{(2)*}_\alpha - \widetilde{F}^{(2)*}_\beta) \in \Gamma(X'_\alpha \cap X'_\beta, \mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log\ D'), F_* \mathcal{O}_X))$$

As u^*_α is the identity on X'_α the cocycle defined by $\widetilde{F}^{(2)}_\alpha$ and $\widetilde{F}^{(2)}_\alpha \circ u_\alpha$ are the same and $\widetilde{F}^{(2)}_\alpha$ and $\widetilde{F}^{(1)}_\alpha$ define the same cohomology class.

□

9.13. The Cartier operator

Let X be a smooth scheme over S and $F : X \to X'$ be the Frobenius relative to S. The key observation is that the differential d in the de Rham complex $\Omega^\bullet_{X/S}$ is $\mathcal{O}_{X'}$-linear as, using the notations of (9.1),

$$dF^*(x \otimes 1) = dx^p = 0.$$

If D is a normal crossing divisor over S, the homology sheaves

$$\mathcal{H}^a = \mathcal{H}^a(F_*\Omega^\bullet_{X/S}(\log\ D))$$

are $\mathcal{O}_{X'}$-modules computed by the following

9.14. Theorem (Cartier, see [9] [34]).
One has an isomorphism of $\mathcal{O}_{X'}$-modules

$$C^{-1} : \Omega^1_{X'/S}(\log\ D') \longrightarrow \mathcal{H}^1(F_*\Omega^\bullet_{X/S}(\log\ D))$$

such that:
a) For $x \in \mathcal{O}_X$ one has

$$C^{-1}(d(x \otimes 1)) = x^{p-1}dx \quad in \quad \mathcal{H}^1.$$

b) If t is a local parameter defining a component of D, then

$$C^{-1}\left(\frac{d(t \otimes 1)}{t \otimes 1}\right) = \frac{dt}{t} \quad in \quad \mathcal{H}^1.$$

c) C^{-1} is uniquely determined by a) and b).

d) For all $a \geq 0$ one has an isomorphism

$$\bigwedge^a C^{-1} : \Omega^a_{X'/S}(\log\ D') \longrightarrow \mathcal{H}^a(F_*\Omega^\bullet_{X/S}(\log\ D))$$

obtained by wedge product from C^{-1}.

Proof: c) is obvious since $\Omega^1_{X'/S}(\log\ D')$ is generated by elements of the form

$$d(x \otimes 1) \ \text{ and } \ \frac{d(t \otimes 1)}{t \otimes 1}.$$

For the existence of C^{-1} let us first assume that $D = \emptyset$. Then

$$(x + y)^{p-1}(dx + dy) - x^{p-1}dx - y^{p-1}dy = df$$

where for

$$\gamma_i \in \mathbb{F}_p \ \text{ with } \ \gamma_i \equiv \frac{1}{p}\binom{p}{i} \ mod\ p$$

we take

$$f = \sum_{i=1}^{p-1} \gamma_i \cdot x^i \cdot y^{p-i} = \frac{1}{p}[(x+y)^p - x^p - y^p].$$

Moreover, one has

$$(y \cdot x)^{p-1} d(y \cdot x) = x^p \cdot y^{p-1} dy + y^p \cdot x^{p-1} dx$$

and $d(x^{p-1}dx) = 0$. Hence, the property a) defines C^{-1}.

For $D \neq \emptyset$, b) is compatible with the definition of C^{-1} on $\Omega^1_{X'/S}$. In fact,

$$C^{-1}\left(t \otimes 1 \cdot \frac{d(t\otimes 1)}{t \otimes 1}\right) = F^*(t\otimes 1) C^{-1}\left(\frac{d(t\otimes 1)}{t\otimes 1}\right) = t^p \frac{dt}{t}.$$

Having defined C^{-1}, we can define $\bigwedge^a C^{-1}$ as well. As in (9.3,c) we have locally a cartesian square

$$\begin{array}{ccc} X & \xrightarrow{F} & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ \mathbf{A}^n_S & \xrightarrow{F} & (\mathbf{A}^n_S)' \end{array}$$

with π and π' étale. Hence to show that $\bigwedge^a C^{-1}$ is an isomorphism it is enough to consider the case

$$X = \mathbf{A}^n_S = \mathbf{Spec}\ \mathcal{O}_S[t_1, \ldots t_n]$$

and D to be the zero set of $t_1 \cdot \ldots \cdot t_r$.

If $\mathcal{B}^a$ is the $\mathbb{F}_p$-vector space freely generated by

$$t_1^{i_1} \cdot \ldots \cdot t_n^{i_n} \cdot \omega_{\alpha_1} \wedge \ldots \wedge \omega_{\alpha_a}$$

for

$$0 \leq i_\nu < p \quad \text{for} \quad \nu = 1, \ldots, n$$
$$1 \leq \alpha_1 < \alpha_2 \ldots < \alpha_a \leq n$$
$$\omega_\nu = \begin{cases} \frac{dt_\nu}{t_\nu} & \nu = 1, \ldots, r \\ dt_\nu & \nu = r+1, \ldots, n \end{cases}$$

then $\mathcal{B}^\bullet$, with the usual differential is a subcomplex of $F_*\Omega^\bullet_{X/S}(\log\ D)$. One has $F_*\Omega^\bullet_{X/S}(\log\ D) = \mathcal{O}_{X'} \otimes_{\mathbb{F}_p} \mathcal{B}^\bullet$ and (9.14) follows from the following claim. $\square$

9.15. Claim. One has
i) $H^0(\mathcal{B}^\bullet) = \mathbb{F}_p$.

ii) $H^1(\mathcal{B}^\bullet)$ has the basis $\{\omega_1, \ldots, \omega_r, t_{r+1}^{p-1}\omega_{r+1}, \ldots, t_n^{p-1}\omega_n\}$.

iii) $H^a(\mathcal{B}^\bullet) = \bigwedge^a H^1(\mathcal{B}^\bullet)$.

Proof: For $n = 1$, this is shown easily:
Obviously ker $(d : \mathcal{B}^0 \longrightarrow \mathcal{B}^1) = \mathbb{F}_p$. For $D = \emptyset$ let us write $\mathcal{K}^\bullet = \mathcal{B}^\bullet$. One has

$$\mathcal{K}^1 = < t^i dt;\ i = 0, \ldots, p-1 >_{\mathbb{F}_p}$$

and

$$d\mathcal{K}^0 = < dt^{i+1} = (i+1) \cdot t^i dt;\ i = 0, \ldots, p-2 >_{\mathbb{F}_p} .$$

For $D \neq \emptyset$ write $\mathcal{L}^\bullet = \mathcal{B}^\bullet$. One has

$$\mathcal{L}^1 = < t^i \frac{dt}{t};\ i = 0, \ldots, p-1 >_{\mathbb{F}_p}$$

and

$$d\mathcal{L}^0 = < dt^i = i \cdot t^i \cdot \frac{dt}{t}; i = 1, \ldots p-1 >_{\mathbb{F}_p} .$$

In both cases (9.15) is obvious. For $n > 1$ one can write

$$\mathcal{B}^\bullet = \underbrace{\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{L}^\bullet \otimes \ldots \otimes_{\mathbb{F}_p} \mathcal{L}^\bullet}_{r \text{ times}} \otimes_{\mathbb{F}_p} \underbrace{\mathcal{K}^\bullet \otimes \ldots \otimes_{\mathbb{F}_p} \mathcal{K}^\bullet}_{n-r \text{ times.}} .$$

By the Künneth formula (A.8)

$$H^a(\mathcal{B}^\bullet) = \sum_{\sum_{i=1}^n \varepsilon_i = a} = H^{\varepsilon_1}(\mathcal{L}^\bullet) \otimes \ldots \otimes H^{\varepsilon_r}(\mathcal{L}^\bullet) \otimes \ldots \otimes H^{\varepsilon_n}(\mathcal{K}^\bullet).$$

which implies a), b) and c).

□

9.16. Notation. Following Deligne-Illusie, we define

$$\Omega^\bullet_{X/S}(A, B) = \Omega^\bullet_{X/S}(\log\ (A + B))(-A)$$

where $A + B$ is a normal crossing divisor over S.

9.17. Corollary. *The Cartier operator induces an isomorphism*

$$\Omega^a_{X'/S}(A', B') \longrightarrow \mathcal{H}^a(F_* \Omega^\bullet_{X/S}(A, B))$$

Proof: By (2.7) the residues of

$$d : \mathcal{O}_X(-A) \longrightarrow \Omega^1_{X/S}(\log\ (A + B))(-A) = \Omega^1_{X/S}(A, B)$$

along the components of A are all 1 and by (2.10)

$$\Omega^\bullet_{X/S}(\log\ (A + B))(-p \cdot A) \longrightarrow \Omega^\bullet_{X/S}(A, B)$$

is a quasi isomorphism. Since

$$F_* \Omega^\bullet_{X/S}(\log\ (A + B))(-p \cdot A) = F_* \Omega^\bullet_{X/S}(\log\ (A + B)) \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(-A')$$

we can apply (9.14).

□

9.18. Duality

Let us keep the notation from (9.16). The wedge product

$$\Omega^{n-i}_{X/S}(\log\ D) \otimes \Omega^i_{X/S}(\log\ D) \xrightarrow{\wedge} \Omega^n_{X/S} \otimes \mathcal{O}_X(D)$$

is a perfect duality of locally free sheaves. Hence one obtains:

9.19. Lemma. $\Omega^{n-i}_{X/S}(A,B) \otimes \Omega^i_{X/S}(B,A) \xrightarrow{\wedge} \Omega^n_{X/S}$ *is a perfect duality.*

9.20. Lemma. *One has a perfect duality*

$$F_*\Omega^{n-i}_{X/S}(A,B) \otimes F_*\Omega^i_{X/S}(B,A) \longrightarrow \Omega^n_{X'/S}$$

given by

$$F_*\Omega^{n-i}_{X/S}(A,B) \otimes F_*\Omega^i_{X/S}(B,A) \xrightarrow[\wedge]{} F_*\Omega^n_{X/S} \longrightarrow \mathcal{H}^n \xrightarrow[C]{} \Omega^n_{X'/S}$$

where C is the Cartier operator.

Proof: In fact, this is nothing but duality for finite flat morphisms ([30], p 239). One has

$$\begin{aligned} F_*\Omega^{n-i}_{X/S}(A,B) &= F_*\mathcal{H}om_{\mathcal{O}_X}(\Omega^i_{X/S}(B,A), \Omega^n_{X/S}) \\ &\simeq \mathcal{H}om_{\mathcal{O}_{X'}}(F_*\Omega^i_{X/S}(B,A), \Omega^n_{X'/S}) \end{aligned}$$

and (9.20) is just saying that $F_*\Omega^n_{X/S} \longrightarrow \Omega^n_{X'/S}$ is given by the Cartier operator. One can do the calculations by hand.

As in the proof of (9.14) it is enough to consider $X = \mathbf{A}^n_S$, A the zero set of $t_1 \cdot \ldots \cdot t_s$ and B the zero set of $t_{s+1} \ldots t_r$. Define, for $a > 0$, $\mathcal{B}^a(A,B)$ to be the $\mathbb{F}_p$-vector space generated by all

$$\varphi = t_1^{i_1} \cdot \ldots \cdot t_n^{i_n} \cdot \omega_{\alpha_1} \wedge \ldots \wedge \omega_{\alpha_a}$$

with

$$\omega_\nu = \begin{cases} \frac{dt_\nu}{t_\nu} & \text{for} \quad \nu = 1, \ldots s, \ldots, r \\ dt_\nu & \text{for} \quad \nu = r+1, \ldots, n \end{cases}$$

where the indices are given by

$$\begin{array}{ll} 0 < i_\nu \leq p & \text{for} \quad \nu = 1, \ldots, s \\ 0 \leq i_\nu < p & \text{for} \quad \nu = s+1, \ldots, r, \ldots, n \\ \text{and by} & 1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_a \leq n. \end{array}$$

Similarly we have $\mathcal{B}^a(B,A)$ by taking as index set

$$\begin{array}{ll} 0 \leq i_\nu < p & \text{for} \quad \nu = 1, \ldots, s \text{ and } \nu = r+1, \ldots, n \\ 0 < i_\nu \leq p & \text{for} \quad \nu = s+1, \ldots, r \\ \text{and} & 1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_a \leq n. \end{array}$$

For $a = n - i$ the only generator δ of $\mathcal{B}^i(B, A)$ with $C(\varphi \wedge \delta) \neq 0$ is

$$\delta = t_1^{j_1} \cdot \ldots \cdot t_n^{j_n} \omega_{\beta_1} \wedge \ldots \wedge \omega_{\beta_i}$$

with

$$\{\beta_1, \ldots, \beta_i\} \cup \{\alpha_1, \ldots, \alpha_{n-i}\} = \{1, \ldots, n\}$$

and

$$i_\nu + j_\nu = \begin{cases} p & \text{for} \quad \nu = 1, \ldots, r \\ p-1 & \text{for} \quad \nu = r+1, \ldots, n. \end{cases}$$

□

9.21. Remark. For $\varphi \in \mathcal{B}^{n-i-1}(A, B)$ and $\delta \in \mathcal{B}^i(B, A)$ the explicit description of the duality in the proof of (9.20) shows that (up to sign)

$$C(d\varphi \wedge \delta) = C(\varphi \wedge d\delta).$$

Hence we obtain as well:

9.22. Corollary. *Under the duality in (9.20) the transposed of the differential d is again d (up to sign).*

§ 10 The proof of Deligne and Illusie [12]

We keep the assumptions from Lectures 8 and 9. Hence X is supposed to be a smooth noetherian S-scheme, $D \subset X$ a S-normal crossing divisor, and S is a noetherian scheme over $\mathbb{Z}/p$ which admits a lifting $\widetilde{S}$ to $\mathbb{Z}/p^2$ as well as a lifting $\widetilde{F}_{\widetilde{S}} : \widetilde{S} \to \widetilde{S}$ of the absolute Frobenius F_S.

10.1. The two term de Rham complex

is defined as

$$\tau_{\leq 1}\ F_*\Omega^\bullet_{X/S}\ (\log\ D).$$

Hence, as explained in (A.26), it is the complex

$$F_*\mathcal{O}_X \longrightarrow Z^1$$

where

$$Z^1 = Ker\ (F_*\Omega^1_{X/S}\ (\log\ D) \longrightarrow F_*\Omega^2_{X/S}\ (\log\ D)).$$

One has a short exact sequence of complexes

$$0 \longrightarrow \mathcal{H}^0 \longrightarrow \tau_{\leq 1}\ F_*\Omega^\bullet_{X/S}\ (\log\ D) \longrightarrow \mathcal{H}^1[-1] \longrightarrow 0$$

given by

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{H}^0 & \longrightarrow & F_*\mathcal{O}_X & & & & \\ & & & & \downarrow & & & & \\ & & & & Z^1 & \longrightarrow & \mathcal{H}^1 & \longrightarrow & 0 \end{array}$$

where $\mathcal{H}^0 = \mathcal{O}_{X'}$ and $\mathcal{H}^1$ is $\mathcal{O}_{X'}$- isomorphic to $\Omega^1_{X'/S}(\log\ D)$ via the Cartier operator (9.14).

10.2. Definition. A *splitting* of $\tau_{\leq 1}F_*\Omega^\bullet_{X/S}(\log\ D)$ is a diagram

$$\begin{array}{ccc} \tau_{\leq 1}F_*\Omega^\bullet_{X/S}(\log\ D) & \xrightarrow{\ \sigma\ } & \mathcal{K}^\bullet \\ & & \uparrow{\scriptstyle\theta} \\ & & \mathcal{H}^0 \oplus \mathcal{H}^1[-1] \end{array}$$

where $\mathcal{K}^\bullet$ is the Čech complex

$$\mathcal{C}^\bullet(\mathcal{U}, \tau_{\leq 1}F_*\Omega^\bullet_{X/S}(\log\ D))$$

associated to some affine open cover $\mathcal{U}$ of X', where σ is the induced morphism, hence a quasi-isomorphism (see (A.6)), and where θ is a quasi-isomorphism. We may assume, of course, that

$$\mathcal{H}^i \xrightarrow{\sigma} \mathcal{H}^i \xrightarrow{\theta^{-1}} \mathcal{H}^i$$

is the identity for $i = 0, 1$.

10.3. Example. Assume that $D \subset X$ and $D' \subset X'$ both lift to

$$\widetilde{D} \subset \widetilde{X} \quad \text{and} \quad \widetilde{D}' \subset \widetilde{X}'$$

on $\widetilde{S}$ and that F lifts to $\widetilde{F} : \widetilde{X} \to \widetilde{X}'$ in such a way that

$$\widetilde{F}^*\mathcal{O}_{\tilde{X}'}(-\widetilde{D}') = \mathcal{O}_{\tilde{X}}(-p \cdot \widetilde{D}).$$

For example, if $S = \operatorname{Spec}\ k$ for a perfect field k and if $D \subset X$ has a lifting $\widetilde{D} \subset \widetilde{X}$ then as we have seen in (9.6) $D' \subset X'$ has a lifting as well. By (9.12) the existence of $\widetilde{F}$ is equivalent to

$$[F_{\tilde{X}',\tilde{D}'}] = 0 \ \text{ in } \ H^1(X', \mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log\ D'), F_*\mathcal{O}_X)).$$

For example it automatically exists if this group vanishes.

Anyway, if the liftings $\widetilde{D}, \widetilde{X}, \widetilde{D}', \widetilde{X}'$ and $\widetilde{F}$ exist, the morphism

$$\widetilde{F}^* : \Omega^1_{\tilde{X}'/\tilde{S}}(\log\ \widetilde{D}') \longrightarrow \Omega^1_{\tilde{X}/\tilde{S}}(\log\ \widetilde{D})$$

verifies
$$\widetilde{F}^*|_{p\cdot\Omega^1_{\tilde{X}'/\tilde{S}}(\log\ \tilde{D}')} = 0.$$
In fact,
$$F^* : \Omega^1_{X'/S}(\log\ D') \longrightarrow \Omega^1_{X/S}(\log\ D)$$
is given by $F^*(d(t \otimes 1)) = d(t^p)$ and hence it is the zero map. We have a commutative diagram

$$\begin{array}{ccc}
\Omega^1_{X'/S}(\log\ D') & \xrightarrow[\simeq]{p} & p\cdot\Omega^1_{\tilde{X}'/\tilde{S}}(\log\ \widetilde{D}') \\
\big\downarrow{\scriptstyle F^*} & & \big\downarrow{\scriptstyle \widetilde{F}^*} \\
\Omega^1_{X/S}(\log\ D) & \xrightarrow[\simeq]{p} & p\cdot\Omega^1_{\tilde{X}/\tilde{S}}(\log\ \widetilde{D})
\end{array}$$

and hence the vertical morphisms are both zero. The same argument shows that the factorization
$$\widetilde{F}^* : \Omega^1_{X'/S}(\log\ D') \longrightarrow \Omega^1_{\tilde{X}/\tilde{S}}(\log\ \widetilde{D})$$
takes values in
$$p\cdot\Omega^1_{\tilde{X}/\tilde{S}}(\log\ \widetilde{D}) = p\cdot\Omega^1_{X/S}(\log\ D).$$
The induced map
$$p^{-1}\circ\widetilde{F}^* : \Omega^1_{X'/S}(\log\ D') \longrightarrow \Omega^1_{X/S}(\log\ D)$$
can be written in coordinates as follows. For $x \in \mathcal{O}_X$ let $\widetilde{x} \in \mathcal{O}_{\tilde{X}}$ be a lifting of x and let $\widetilde{x}' \in \mathcal{O}_{\tilde{X}'}$ be a lifting of $x' = x \otimes 1$. One writes
$$\widetilde{F}^*(\widetilde{x}') = \widetilde{x}^p + p(u(\widetilde{x},\widetilde{x}'))$$
for some $u(\widetilde{x},\widetilde{x}') \in \mathcal{O}_X$. Then
$$p^{-1}\circ\widetilde{F}^*(d\widetilde{x}') = x^{p-1}dx + du(\widetilde{x},\widetilde{x}').$$
In particular the image of $p^{-1}\circ\widetilde{F}^*$ lies in
$$Z^1 \subset F_*\Omega^1_{X/S}(\log\ D)$$
and the composition with $Z^1 \longrightarrow \mathcal{H}^1$ gives back the Cartier operator.

In this example, i.e. if the liftings $\widetilde{D}, \widetilde{X}, \widetilde{D}', \widetilde{X}'$ and $\widetilde{F}$ all exist, we can take $\mathcal{U} = \{X'\}$ and define
$$\theta : \mathcal{O}_{X'} \oplus \Omega^1_{X'/S}(\log\ D')[-1] \longrightarrow \tau_{\leq 1}F_*\Omega^\bullet_{X/S}(\log\ D)$$

by

$$\begin{array}{ccc}
\mathcal{O}_{X'} & \longrightarrow & F_*\mathcal{O}_X \\
\Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle d} \\
\Omega^1_{X'/S} & \xrightarrow[p^{-1}\circ \tilde{F}^*]{} & Z^1
\end{array}$$

and, by (9.14), θ is a quasi-isomorphism.

10.4. Notation. We call a cohomology class

$$\varphi \in \mathbb{H}^1(X', \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, F_*\mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, Z^1))$$

a *splitting cohomology class*, if φ maps to the identity in

$$H^0(X', \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, \mathcal{H}^1)) = \mathbb{H}^1(X', \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, \mathcal{H}^1)[-1]).$$

10.5. Proposition. *The splittings of*

$$\tau_{\leq 1} F_* \Omega^\bullet_{X/S}(\log\ D)$$

are in one to one correspondence with the splitting cohomology classes

$$\varphi \in \mathbb{H}^1(X', \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, F_*\mathcal{O}_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, Z^1)).$$

Proof: Let φ be a splitting cohomology class, realized as cocycle

$$\varphi_{\alpha\beta} \in \Gamma(X'_{\alpha\beta}, \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, F_*\mathcal{O}_X))$$

and

$$\psi_\alpha \in \Gamma(X'_\alpha, \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, Z^1))$$

for some affine open cover $\mathcal{U} = \{X'_\alpha\}$ of X'. Hence, using the notations from (A.6) for the differential in the Čech complex, $\delta\varphi = 0$ and $d\varphi - \delta\psi = 0$. By assumption ψ_α induces the identity in

$$\Gamma(X'_\alpha, \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, \mathcal{H}^1)).$$

Then $\theta = (id, (\varphi_{\alpha\beta}, \psi_\alpha))$ is the map wanted, i.e.

$$\begin{array}{ccc}
\mathcal{O}_{X'} & \xrightarrow{id} & \bigoplus_{\mathcal{U}} \varrho_{\alpha *} F_*\mathcal{O}_X|_{X'_\alpha} \\
{\scriptstyle 0}\Big\downarrow & & \Big\downarrow{\scriptstyle \delta\oplus d} \\
\mathcal{H}^1 & \xrightarrow{(\varphi_{\alpha\beta}, \psi_\alpha)} & \bigoplus_{\mathcal{U}} \varrho_{\alpha\beta *} F_*\mathcal{O}_X|_{X'_{\alpha\beta}} \oplus \bigoplus_{\mathcal{U}} \varrho_{\alpha *}\ Z^1|_{X'_\alpha} \\
\Big\downarrow & & \Big\downarrow{\scriptstyle (-\delta\oplus d,\ 0\oplus -\delta)} \\
0 & \longrightarrow & \bigoplus_{\mathcal{U}} \varrho_{\alpha\beta\gamma *} F_*\mathcal{O}_X|_{X'_{\alpha\beta\gamma}} \oplus \bigoplus_{\mathcal{U}} \varrho_{\alpha\beta *}\ Z^1|_{X'_{\alpha\beta}}.
\end{array}$$

where

$$\varrho_{\alpha_1,\cdots\alpha_r} : X'_{\alpha_1,\cdots\alpha_r} \longrightarrow X'$$

denotes the embedding, and where

$$(-\delta \oplus d,\ 0 \oplus -\delta)(x_{\alpha\beta}, z_\alpha) = (-\delta(x), d(x) - \delta(z)).$$

Conversely, let for some $\mathcal{U}$

$$\tau_{\leq 1}(F_*\Omega^\bullet_{X/S}(\log\ D)) \xrightarrow{\sigma} \mathcal{K}^\bullet \xleftarrow{\theta} \mathcal{H}^0 \oplus \mathcal{H}^1[-1]$$

be a splitting. As $\mathcal{H}^1$ is $\mathcal{O}_{X'}$-locally free, $\sigma \otimes id$ and $\theta \otimes id$ are quasi-isomorphisms of the corresponding complexes tensored with $\mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, \mathcal{O}_{X'})$. We obtain therefore maps

$$\begin{array}{c}
H^0\Big(X', \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, \mathcal{H}^1)\Big) = \mathbb{H}^1\Big(X', \mathcal{H}^1 \otimes \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, \mathcal{O}_{X'})[-1]\Big) \\
\Big\downarrow \\
\mathbb{H}^1\Big(X', \Big[\mathcal{H}^0 \oplus \mathcal{H}^1[-1]\Big] \otimes \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, \mathcal{O}_{X'})\Big) \\
\Big\downarrow \simeq \\
\mathbb{H}^1\Big(X', \tau_{\leq 1}F_*\Omega^\bullet_{X/S}(\log\ D) \otimes \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, \mathcal{O}_{X'})\Big) \\
\Big\downarrow = \\
\mathbb{H}^1(X', \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, F_*\mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, Z^1)) \\
\Big\downarrow \\
H^0\Big(X', \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, \mathcal{H}^1)\Big)
\end{array}$$

where the last map comes from the short exact sequence in (10.1). By definition of a splitting, the composed map is the identity. Therefore, the image of

$$id_{\mathcal{H}^1} \in H^0(X', \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, \mathcal{H}^1))$$

is a splitting cohomology class φ. Obviously, the both constructions are inverse to each other.

□

10.6. Remark. In (10.2) one could have replaced in the definition of a splitting the Čech complex by any complex $\mathcal{K}^\bullet$ bounded below and quasi-isomorphic to $\tau_{\leq 1}F_*\Omega^\bullet_{X/S}(log\ D)$. Then we would have proven as in (10.5), that a splitting cohomology class defines a splitting. However, to get the converse, we need that σ and θ define a map from

$$\mathbb{H}^0(X', \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, \mathcal{O}_{X'}) \otimes \mathcal{K}^\bullet)$$

to
$$\mathbb{H}^1(X', \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^1, \mathcal{O}_{X'}) \otimes \tau_{\leq 1} F_* \Omega^1_{X/S}(\log\ D)).$$
This is of course the case when $\mathcal{K}^\bullet$ is a complex of $\mathcal{O}_{X'}$-modules, but not in general. One needs a bit more knowledge on the derived category; in particular one needs the global Hom (,) in this category.

10.7. Main theorem. *Let X be a smooth scheme over S and $D \subset X$ be a S-normal crossing divisor. Then*

a) *A lifting $\widetilde{D}' \subset \widetilde{X}'$ of $D' \subset X'$ to $\widetilde{S}$ defines a splitting cohomology class*
$$\varphi = \varphi_{(\tilde{X}', \tilde{D}')}.$$
b) *Every splitting cohomology class φ is of the shape $\varphi = \varphi_{(\tilde{X}', \tilde{D}')}$ for some lifting $\widetilde{D}' \subset \widetilde{X}'$ of $D' \subset X'$ to $\widetilde{S}$.*

We will only need part a) in the proof of Theorem (8.3). Even if it might be more elegant to use more formal arguments we will give the necessary calculations in an explicit way for cycles in the Čech-cohomology.

Proof: a) Let $\mathcal{U} = \{X_\alpha\}$ be an affine cover of X, such that the $X_{\alpha\beta}$ are affine, and such that (8.17) and (9.10) give liftings
$$\widetilde{D}_\alpha \subset \widetilde{X}_\alpha \text{ of } D_\alpha \subset X_\alpha$$
to $\widetilde{S}$ and liftings
$$\widetilde{F}_\alpha : \widetilde{X}_\alpha \longrightarrow \widetilde{X}' \text{ of } F : X \longrightarrow X'$$
satisfying
$$\widetilde{F}^*_\alpha \mathcal{O}_{\tilde{X}'}(-\widetilde{D}') = \mathcal{O}_{\tilde{X}_\alpha}(-p \cdot \widetilde{D}_\alpha).$$
For $\widetilde{X}_{\alpha\beta} = \widetilde{X}_\alpha|_{X_{\alpha\beta}}$ (8.23) implies the existence of isomorphisms of liftings
$$u_{\alpha\beta} : \widetilde{X}_{\alpha\beta} \longrightarrow \widetilde{X}_{\beta\alpha}$$
As in the proof of (9.12) one uses (9.9) to define
$$\varphi_{\alpha\beta} = p^{-1} \circ (\widetilde{F}^*_\alpha - (\widetilde{F}_\beta \circ u_{\alpha\beta})^*) \in \Gamma(X'_{\alpha\beta}, \mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log\ D'), F_*\mathcal{O}_X))$$
and by (9.11) $\varphi_{\alpha\beta}$ is a cocycle. On $\widetilde{X}_\alpha$ we obtained in (10.3) a map
$$\psi_\alpha = p^{-1} \circ \widetilde{F}^*_\alpha \in \Gamma(X'_\alpha, \mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log\ D'), Z^1))$$
lifting the Cartier operator C^{-1}.

□

10.8. Claim. One has

$$p^{-1} \circ \widetilde{F}_\beta^*|_{X'_{\beta\alpha}} = p^{-1} \circ (\widetilde{F}_\beta \circ u_{\alpha\beta})^*$$

in

$$\Gamma(X'_{\beta\alpha}, \mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log\ D'), Z')).$$

Proof: For $\widetilde{x}' \in \mathcal{O}_{\tilde{X}'}$ and $\widetilde{x}_\beta \in \mathcal{O}_{\tilde{X}_\beta}$ we can write

$$\widetilde{F}_\beta^*(\widetilde{x}') = \widetilde{x}_\beta^p + p \cdot u(\widetilde{x}_\beta, \widetilde{x}').$$

Since $u^*_{\alpha\beta}|_{p \cdot \mathcal{O}_{X_{\alpha\beta}}}$ is the identity, one obtains

$$u^*_{\alpha\beta}\widetilde{F}_\beta^*(\widetilde{x}') = u^*_{\alpha\beta}(\widetilde{x}_\beta)^p + p \cdot u(\widetilde{x}_\beta, \widetilde{x}').$$

By (10.3) we have

$$p^{-1} \circ \widetilde{F}_\beta^*(d\widetilde{x}') = x^{p-1}dx + du(\widetilde{x}_\beta, \widetilde{x}') = p^{-1} \circ (\widetilde{F}_\beta \circ u_{\alpha\beta})^*.$$

□

Now (10.8) is just saying that $\delta\psi = d\varphi$ and therefore $(\varphi_{\alpha\beta}, \psi_\beta)$ defines a cohomology class $\varphi_{(\tilde{X}', \tilde{D}')}$ in

$$\mathbb{H}^1(X', \mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log\ D'), F_*\mathcal{O}_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log\ D'), Z^1)).$$

By construction its image in

$$H^0(X', \mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log\ D'), \mathcal{H}^1))$$

is given by (ψ_β) and hence it is the Cartier operator.

b) Conversely, let $(\varphi_{\alpha\beta}, \psi_\alpha)$ be the cocycle giving the splitting cohomology class φ for some affine covering $\mathcal{U}' = \{X'_\alpha\}$. First we want to add some coboundary to get a new representative of φ.

By (8.17), (9.10) and (8.21) we can assume that we have:

i) *Liftings to* $\widetilde{S}$:

$$\widetilde{D}_\alpha \subset \widetilde{X}_\alpha \ \text{ of } \ D_\alpha = D|_{X_\alpha} \subset X_\alpha = X|_{X'_\alpha},$$

$$\widetilde{D}'_\alpha \subset \widetilde{X}'_\alpha \ \text{ of } \ D'_\alpha = D'|_{X'_\alpha} \subset X'_\alpha$$

and

$$\widetilde{F}_\alpha : \widetilde{X}_\alpha \longrightarrow \widetilde{X}'_\alpha \ \text{ of } \ F_\alpha = F|_{X_\alpha} : X_\alpha \longrightarrow X'_\alpha$$

with

$$\widetilde{F}_\alpha^* : \mathcal{O}_{\tilde{X}'_\alpha}(-\widetilde{D}'_\alpha) = \mathcal{O}_{\tilde{X}_\alpha}(-p \cdot \widetilde{D}_\alpha).$$

ii) *Isomorphisms of liftings*:

$$u'_{\alpha\beta} : \widetilde{X}'_{\alpha\beta} \longrightarrow \widetilde{X}'_{\beta\alpha} \quad \text{and} \quad u_{\alpha\beta} : \widetilde{X}_{\alpha\beta} \longrightarrow \widetilde{X}_{\beta\alpha}$$

were we keep the notation $\widetilde{X}'_{\alpha\beta} = \widetilde{X}'_{\alpha}|_{X_{\alpha\beta}}$.

For $\widetilde{x}' \in \mathcal{O}_{\widetilde{X}'_{\alpha\beta}}$ we can write $u'^{*}_{\alpha\beta}(\widetilde{x}') = \widetilde{x}' + p \cdot \lambda_{\alpha\beta}(\widetilde{x}')$. Then

$$\widetilde{F}^*_{\alpha} u'^*_{\alpha\beta}(\widetilde{x}') = \widetilde{F}^*_{\alpha}(\widetilde{x}') + p \cdot F^* \lambda_{\alpha\beta}(\widetilde{x}') = \widetilde{F}^*_{\alpha}(\widetilde{x}') + p \cdot \lambda_{\alpha\beta}(\widetilde{x}')^p.$$

Since $d(\lambda_{\alpha\beta}(\widetilde{x}')^p) = 0$ the explicit description of $p^{-1} \circ \widetilde{F}^*_{\alpha}$ in (10.3) gives

10.9. Claim. One has $p^{-1} \circ \widetilde{F}^*_{\alpha}|_{X'_{\alpha\beta}} = p^{-1} \circ (u'_{\alpha\beta} \circ \widetilde{F}_{\alpha})^*$ in

$$\Gamma(X'_{\alpha\beta}, \mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log\ D'),\ Z^1)).$$

Define

$$\theta_{\alpha} = p^{-1} \circ \widetilde{F}^*_{\alpha} \in \Gamma(X'_{\alpha}, \mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log\ D'),\ F_*\mathcal{O}_X)).$$

Replacing $\mathcal{U}'$ by some finer cover if necessary we find

$$f_{\alpha} \in \Gamma(X'_{\alpha}, \mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log\ D'),\ F_*\mathcal{O}_X))$$

such that $df_{\alpha} = \theta_{\alpha} - \psi_{\alpha}$.

10.10. Claim. For $\sigma'_{\alpha\beta} = \varphi_{\alpha\beta} + \delta f_{\alpha}$ the cohomology class φ is represented by the cocycle $(\sigma'_{\alpha\beta}, \theta_{\alpha})$.

Proof: This is obvious since

$$(\sigma'_{\alpha\beta}, \theta_{\alpha}) = (\varphi_{\alpha\beta} + \delta f_{\alpha}, \psi_{\alpha} + df_{\alpha}).$$

□

The main advantage of $\sigma'_{\alpha\beta}$ is that it comes in geometric terms:
Write, using (9.9)

$$\sigma_{\alpha\beta} = p^{-1} \circ \Big((u'_{\alpha\beta} \circ \widetilde{F}_{\alpha})^* - (\widetilde{F}_{\beta} \circ u_{\alpha\beta})^* \Big)$$

in

$$\Gamma\Big(X'_{\alpha\beta}, \mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log\ D')\ , F_*\mathcal{O}_X)\Big).$$

Applying (10.9) to the first and (10.8) to the second summand one gets

$$d\sigma_{\alpha\beta} = \theta_{\alpha} - \theta_{\beta} = df_{\alpha} - df_{\beta} + \psi_{\alpha} - \psi_{\beta} = d(f_{\alpha} - f_{\beta}) + d\varphi_{\alpha\beta}.$$

Hence $g_{\alpha\beta} = \sigma_{\alpha\beta} - \varphi_{\alpha\beta} - (f_\alpha - f_\beta)$ is closed and lives in

$$\Gamma\Big(X'_{\alpha\beta}, \mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log\ D'), \mathcal{O}_{X'})\Big).$$

By (8.22) $u'^*_{\alpha\beta} - g_{\alpha\beta}$ defines a new isomorphism of liftings

$$v'_{\alpha\beta} : \widetilde{X}'_{\alpha\beta} \longrightarrow \widetilde{X}'_{\beta\alpha}.$$

As $\widetilde{F}^*_\alpha = pF^*$ on $p \cdot \mathcal{O}_{X'_\alpha}$, one has $p^{-1} \circ \widetilde{F}^*_\alpha \circ g_{\alpha\beta} = g_{\alpha\beta}$ and

$$\begin{aligned}\sigma'_{\alpha\beta} = \quad & \varphi_{\alpha\beta} + (f_\alpha - f_\beta) = \sigma_{\alpha\beta} - g_{\alpha\beta} = \\ & p^{-1} \circ \Big[(u'_{\alpha\beta} \circ \widetilde{F}_\alpha)^* - (\widetilde{F}_\beta \circ u_{\alpha\beta})^* - \widetilde{F}^*_\alpha \cdot g_{\alpha\beta}\Big] \ .\end{aligned}$$

One obtains

10.11. Claim. $\sigma'_{\alpha\beta} = p^{-1} \circ \Big[(v'_{\alpha\beta} \circ \widetilde{F}_\alpha)^* - (\widetilde{F}_\beta \circ u_{\alpha\beta})^*\Big]$.

The proof of (10.7) ends with

10.12. Claim. The cocycle condition for $\sigma'_{\alpha\beta}$ allows the glueing of $\widetilde{X}'_\alpha$ to $\widetilde{X}'$ using $v'_{\alpha\beta}$.

Proof: One has to show that

$$v'_{\alpha\gamma} = v'_{\beta\gamma} \circ v'_{\alpha\beta}$$

or, by (8.22), that the homomorphism defined there verifies

$$v'^*_{\alpha\beta} \circ v'^*_{\beta\gamma} - v'^*_{\alpha\gamma} = 0.$$

Since $\widetilde{F}^*_\alpha$ is injective, it is enough to show that

$$p^{-1} \circ \widetilde{F}^*_\alpha \circ [v'^*_{\alpha\beta} \circ v'^*_{\beta\gamma} - v'^*_{\alpha\gamma}] = 0$$

as homomorphism in

$$\mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^1_{X'/S}(\log\ D'), F_*\mathcal{O}_X).$$

The cocycle condition for σ' is

$$p^{-1} \circ [\widetilde{F}^*_\alpha \circ v'^*_{\alpha\beta} - u^*_{\alpha\beta} \circ \widetilde{F}^*_\beta - \widetilde{F}^*_\alpha \circ v'^*_{\alpha\gamma} + u^*_{\alpha\gamma} \circ \widetilde{F}^*_\gamma + \widetilde{F}^*_\beta \circ v'^*_{\beta\gamma} - u^*_{\beta\gamma} \circ \widetilde{F}^*_\gamma] = 0.$$

Since

$$\widetilde{F}^*_\alpha \circ v'^*_{\alpha\beta} - u^*_{\alpha\beta} \circ \widetilde{F}^*_\beta$$

is a homomorphism from $\mathcal{O}_{X'}$ to $p \cdot \mathcal{O}_X$, we can replace it by

$$[\widetilde{F}^*_\alpha \circ v'^*_{\alpha\beta} - u^*_{\alpha\beta} \circ \widetilde{F}^*_\beta] \circ v^*_{\beta\gamma}$$

as $v^*_{\beta\gamma}$ is the identity on $\mathcal{O}_{X'_{\beta\gamma}}$. Similarly, we can add some $u^*_{\alpha\beta}$ at the right hand side (see (9.11)) and get

$$0 = p^{-1} \circ [\widetilde{F}^*_\alpha \circ v'^*_{\alpha\beta} \circ v'^*_{\beta\gamma} - u^*_{\alpha\beta} \circ \widetilde{F}^*_\beta \circ v'^*_{\beta\gamma} - \widetilde{F}^*_\alpha \circ v'^*_{\alpha\gamma} +$$

$$+ u^*_{\alpha\gamma} \circ \widetilde{F}^*_\gamma - u^*_{\alpha\beta} \circ \widetilde{F}^*_\beta \circ v'^*_{\beta\gamma} - u^*_{\alpha\beta} u^*_{\beta\gamma} \circ \widetilde{F}^*_\gamma]$$

where all the summands are morphisms from $\mathcal{O}_{\tilde{X}'_\gamma} \longrightarrow \mathcal{O}_{\tilde{X}_\alpha}$. This is the same as

$$0 = p^{-1} \circ \widetilde{F}^*_\alpha \circ (v'^*_{\alpha\beta} \circ v'^*_{\beta\gamma} - v'^*_{\alpha\gamma}) + p^{-1} \circ (u^*_{\alpha\gamma} - u^*_{\alpha\beta} u^*_{\beta\gamma}) \widetilde{F}^*_\gamma.$$

By (9.11) the term on the right is zero and

$$0 = p^{-1} \circ \widetilde{F}^*_\alpha \circ (v'^*_{\alpha\beta} \circ v'^*_{\beta\gamma} - v'^*_{\alpha\gamma}).$$

□

10.13. Splittings of the de Rham complex.

Let us generalize (10.2) to $\tau_{\leq i} F_* \Omega^\bullet_{X/S}(\log\ D)$ for $i > 1$. As remarked in (10.6) one can, using the derived category, replace the complex $\mathcal{K}^\bullet$ in the following definition by any complex $\mathcal{K}^\bullet$ bounded below and quasi-isomorphic to $\tau_{\leq i} F_* \Omega^\bullet_{X/S}(\log\ D)$.

10.14. Definition. A *splitting* of $\tau_{\leq i} F_* \Omega^\bullet_{X/S}(\log\ D)$ is a diagram

$$\begin{tikzcd} \tau_{\leq i} F_* \Omega^\bullet_{X/S}(\log\ D) \arrow[r, "\sigma"] & \mathcal{K}^\bullet \\ & \bigoplus_{j \leq i} \mathcal{H}^j[-j] \arrow[u, "\theta"'] \end{tikzcd}$$

where $\mathcal{K}^\bullet$ is the Čech complex

$$\mathcal{C}^\bullet(\mathcal{U}, \tau_{\leq i} F_* \Omega^\bullet_{X/S}(\log\ D))$$

associated to some affine cover $\mathcal{U}$ of X (and hence σ a quasi-isomorphism) and where θ is a quasi-isormorphism. Here again,

$$\bigoplus_{j \leq i} \mathcal{H}^j[-j]$$

is the complex with zero differential and with $\mathcal{H}^j$ in degree j and $\tau_{\leq i}$ is the filtration explained in (A.26).

10.15. Example. Let us return to the assumptions made in (10.3), i.e. that the liftings $\widetilde{D}, \widetilde{X}, \widetilde{D}', \widetilde{X}'$ and especially $\widetilde{F}$ exist. We had defined there a morphism

$$\psi = p^{-1} \circ \widetilde{F}^* : \Omega^1_{X'/S}(\log\ D') \longrightarrow Z^1$$

which was a lifting of C^{-1}.

We define

$$\psi^j(\omega_1 \wedge \ldots \wedge \omega_j) = \psi(\omega_1) \wedge \psi(\omega_2) \wedge \ldots \wedge \psi(\omega_j)$$

where $\omega_l \in \Omega^1_{X'/S}(\log\ D')$. Since $\psi(\omega_l)$ is closed, the image of ψ^j lies in Z^j and, since the Cartier operator was defined as $\bigwedge^j C^{-1}$ the map ψ^j induces the Cartier operator on in

$$\mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^j_{X'/S}(\log\ D'), \mathcal{H}^j).$$

10.16. Theorem. *Let X be a smooth S-scheme, $D \subset X$ be a normal crossing divisor over S and let*

$$\widetilde{D}' \subset \widetilde{X}' \quad \textit{be a lifting of} \quad D' \subset X'$$

to $\widetilde{S}$. Then the splitting cohomology class $\varphi_{(\tilde{X}', \tilde{D}')}$ of (10.7,a) induces a splitting of

$$\tau_{\leq i} F_* \Omega^\bullet_{X/S}(\log\ D) \quad \textit{for} \quad i < p = \mathrm{char}\ (S).$$

In particular, if $p > \dim_S X$, it induces a splitting of the whole de Rham complex

$$F_* \Omega^\bullet_{X/S}(\log\ D).$$

Proof: Let $(\varphi_{\alpha\beta}, \psi_\alpha)$ be a Čech cocycle for $(\varphi_{\tilde{X}', \tilde{D}'})$ where we regard $(\varphi_{\alpha\beta}, \psi_\alpha)$ as an $\mathcal{O}_{X'}$-homomorphism:

$$(\varphi, \psi) : \Omega^1_{X'/S}(\log\ D') \longrightarrow \mathcal{C}^1(F_*\mathcal{O}_X) \oplus \mathcal{C}^0(Z^1).$$

We define

$$(\varphi, \psi)^{\otimes j}(\omega_1, \otimes \cdots \otimes \omega_j) \in \mathcal{C}^j(\tau_{\leq i} F_* \Omega^\bullet_{X/S}(\log\ D))$$

for all $0 < j \leq i$ and all

$$\omega_1 \otimes \cdots \otimes \omega_j \in \bigotimes_1^j \Omega^1_{X'/S}(\log\ D')$$

by the following inductive formula:
For any cocycle

$$\underline{b} := (b_j, \ldots, b_0),\ b_l \in \mathcal{C}^l(F_* \Omega^{j-l}_{X/S}(\log\ D)),$$

with

$$db_{j-s} + (-1)^{j-s} \delta b_{j-s-1} = 0 \quad \text{for all} \quad 0 \leq s \leq j,$$

we define

$$\underline{b} \otimes (\varphi, \psi)(\omega_{j+1}) = (a_{j+1}, \ldots, a_0) =: \underline{a},$$

with
$$a_l \in \mathcal{C}^l(F_*\Omega^{j+1-l}_{X/s}(\log\ D)),$$
by the rule
$$a_{j+1-s} := (-1)^s b_{j-s} \cup \varphi + b_{j+1-s} \cup \psi$$
where
$$(b_{j-s} \cup \varphi)_{\alpha_0 \ldots \alpha_{j+1-s}} := (b_{j-s})_{\alpha_0 \ldots \alpha_{j-s}} \cdot \varphi_{\alpha_{j-s}, \alpha_{j+1-s}}(\omega_{j+1})$$
and
$$(b_{j+1-s} \cup \psi)_{\alpha_0 \ldots \alpha_{j+1-s}} := (b_{j+1-s})_{\alpha_0 \ldots \alpha_{j+1-s}} \cdot \psi_{\alpha_{j+1-s}}(\omega_{j+1}).$$
One has
$$da_{j+1-s} + (-1)^{j+1-s}\delta a_{j-s} = 0$$
and therefore $\underline{a}$ is a Čech cocycle.

We have, for $j \leq i$, a diagram

$$\begin{array}{ccc}
(\Omega^1_{X'/S}(\log\ D'))^{\otimes\ j} & \xrightarrow{(\varphi,\psi)^{\otimes\ j}} & \mathcal{C}^j(\tau_{\leq i} F_* \Omega^1_{X/S}(\log\ D))_{\text{closed}} \\
\downarrow{\scriptstyle \pi^j} & & \downarrow \\
\Omega^j_{X'/S}(\log\ D') & \xrightarrow{\bigwedge^j C^{-1}} & \mathcal{H}^j
\end{array}$$

and any section δ^j of π^j allows to define
$$\theta_j = (\bigwedge^j C^{-1})^{-1} \circ \delta^j \circ (\varphi, \psi)^{\otimes j}.$$
The splitting
$$\theta : \bigoplus_{j \leq i} \mathcal{H}^j[-j] \longrightarrow \mathcal{K}^\bullet \quad \text{is} \quad \theta = \bigoplus_{j \leq i} \theta_j[-j].$$
Such sections δ^j exist for $j \leq i < \text{char}\ (S)$:
$$\delta^j(\omega_1 \otimes \ldots \otimes \omega_j) = \frac{1}{j!} \sum_{s \in \Sigma_j} sign\ (s) \cdot \omega_{s(1)} \wedge \ldots \wedge \omega_{s(j)},$$
where Σ_j denotes the symmetric group.

□

10.17. Corollary. *Let $D = A + B$ in (10.16). Then for $i < p$ the splitting cohomology class $\varphi_{(\tilde{X}, \tilde{D})}$ induces a splitting of*
$$\tau_{\leq i} F_* \Omega^\bullet_{X/S}(A, B)$$

as well, i.e. a quasi-isomorphism

$$\theta : \bigoplus_{j \leq i} \Omega^j_{X'/S}(A', B')[-j] \longrightarrow \mathcal{K}^\bullet(A, B)$$

where $\mathcal{K}^\bullet(A, B)$ *is the Čech complex of*

$$\tau_{\leq i} F_* \Omega^\bullet_{X/S}(A, B).$$

Proof: As in (9.17) one obtains from (2.7) and (2.10) a quasi-isomorphism

$$\begin{array}{c} F_*\Omega^1_{X/S}(\log\ (A+B)) \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(-A') \\ \| \\ F_*(\Omega^\bullet_{X/S}(\log\ (A+B))(-p \cdot A)) \\ \Big\downarrow \\ F_*\Omega^\bullet_{X/S}(A, B)\ . \end{array}$$

For $\mathcal{K}^\bullet$, the Čech complex of $\tau_{\leq i} F_* \Omega^\bullet_{X/S}(\log\ D)$, we have a quasi-isomorphism

$$\mathcal{K}^\bullet \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(-A) \longrightarrow \mathcal{K}^\bullet(A, B)$$

and the existence of θ follows from (10.16).

□

10.18. Remark. In (10.3) and (10.15) we have seen that if both $D' \subset X'$ and $D \subset X$ lift to $\widetilde{D}' \subset \widetilde{X}'$ and $\widetilde{D} \subset \widetilde{X}$, and if F lifts to: $\widetilde{F} : \widetilde{X} \to \widetilde{X}'$ with

$$\widetilde{F}^* \mathcal{O}_{\widetilde{X}'}(-\widetilde{D}') = \mathcal{O}_{\widetilde{X}}(-p \cdot \widetilde{D}),$$

then

$$\psi = p^{-1} \circ \widetilde{F}^* : \Omega^1_{X'/S}(\log\ D') \longrightarrow Z^1 \subset \Omega^1_{X/S}(\log\ D)$$

lifts the Cartier operator, and gives an especially nice splitting of

$$\tau_{\leq 1} F_* \Omega^\bullet_{X/S}(\log\ D).$$

This defines

$$\bigwedge^l \psi : \Omega^l_{X'/S}(\log\ D') \longrightarrow Z^l \subset \Omega^l_{X/S}(\log\ D)$$

lifting the Cartier operator, and therefore one obtains a quasi-isomorphism:

$$\bigoplus_j \Omega^j_{X'/S}(\log\ D')[-j] \xrightarrow{\bigwedge^\bullet \psi} F_* \Omega^\bullet_{X/S}(\log\ D),$$

which gives via (10.17) a quasi-isomorphism

$$\bigoplus_j \Omega^j_{X'/S}(A',B')[-j] \xrightarrow{\bigwedge^\bullet \psi} F_*\Omega^\bullet_{X/S}(A,B)$$

if $D = A+B$. In particular, there is here no restriction on $\dim_S X$ in this case.

In general one has

10.19. Proposition. *Let X, A and B be as in (10.17). Then the splitting cohomology class $\varphi_{(\tilde{X}',\tilde{D}')}$ induces a splitting of*

$$F_*\Omega^\bullet_{X/S}(A,B)$$

when $\dim_S X \leq p$ *and S is affine.*

Proof: Of course this is nothing but (10.17) if $\dim_S X < p$.
For $\dim_S X \geq p$, we observe first that whenever $j : U \to X$ is the embedding of an open set such that $(X - U)$ is a divisor, then for coherent sheaves $\mathcal{F}$ on U and $\mathcal{G}$ on X one has

$$\mathcal{H}om_{\mathcal{O}_X}(j_*\mathcal{F},\mathcal{G})_x = \begin{cases} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}|_U)_x & \text{for } x \in U \\ 0 & \text{for } x \in (X-U), \end{cases}$$

that is

$$\mathcal{H}om_{\mathcal{O}_X}(j_*\mathcal{F},\mathcal{G}) = j_!\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}|_U).$$

Let us consider the $\mathcal{O}_{X'}$-maps defined in (10.17) for $0 \leq l \leq p-1$:

$$\Omega^l_{X'/S}(B',A') \longrightarrow \mathcal{C}^l(F_*\mathcal{O}_X) + \cdots + \mathcal{C}^0(Z^l)$$
$$(\varphi_{\alpha_0\ldots\alpha_l}, \ \ldots \ , \varphi_{\alpha_0})$$

with the cocycle condition

$$d\varphi_{\alpha_0\ldots\alpha_k} + (-1)^l\delta\varphi_{\alpha_0\ldots\alpha_{k-1}} = 0.$$

The composite map

$$\Omega^l_{X'/S}(B',A') \longrightarrow \mathcal{C}^0(Z^l) \longrightarrow \mathcal{C}^0(\mathcal{H}^l)$$

is just $\bigwedge^l C^{-1}$.
Applying for $1 \leq k \leq l \leq p-1$ the functor $\mathcal{H}om_{\mathcal{O}_{X'}}(-,\Omega^n_{X'/S})$, where $n = \dim_S X$, one obtains $\mathcal{O}_{X'}$-linear maps (see (9.19) and (9.20))

$$(j_{\alpha_0\ldots\alpha_k})_!F_*\Omega^{n-(l-k)}_{X/S}(A,B) \xrightarrow{\varphi^\vee_{\alpha_0\ldots\alpha_k}} \Omega^{n-l}_{X'/S}(A',B')$$

and therefore $\mathcal{O}_{X'}$-maps

$$F_*\Omega^{n-(l-k)}_{X/S}(A,B) \xrightarrow{\varphi^\vee_{\alpha_0\ldots\alpha_k}} \mathcal{C}^k(\Omega^{n-l}_{X'/S}(A',B')).$$

For $k=0$, one has an exact sequence

$$0 \longrightarrow Z^l \longrightarrow F_*\Omega^l_{X/S}(B,A) \longrightarrow F_*\Omega^{l+1}_{X/S}(B,A),$$

and applying (9.20) again, one obtains that

$$\mathcal{H}om_{\mathcal{O}_{X'}}(Z^l,\Omega^n_{X'/S}) = \frac{F_*\Omega^{n-l}_{X/S}(A,B)}{dF_*\Omega^{n-l-1}_{X/S}(A,B)}$$

which gives similarly a $\mathcal{O}_{X'}$-linear map

$$\frac{F_*\Omega^{n-l}_{X/S}(A,B)}{dF_*\Omega^{n-l-1}_{X/S}(A,B)} \xrightarrow{\varphi^\vee_{\alpha_0}} \mathcal{C}^0(\Omega^{n-l}_{X'/S}(A',B')).$$

The cocycle condition tells us that

$$\varphi^\vee_{\alpha_0\ldots\alpha_k}\circ d + (-1)^l\delta\varphi^\vee_{\alpha_0\ldots\alpha_{k-1}} = 0.$$

This means that $\varphi^\vee$ defines a map of complexes

$$\tau_{\geq n-l}F_*\Omega^\bullet_{X/S}(A,B)[(n-l)] \longrightarrow \tau_{\leq l}\mathcal{C}^\bullet(\Omega^{n-l}_{X'/S}(A',B')),$$

where

$$\tau_{\geq n-l}F_*\Omega^\bullet_{X/S}(A,B)[(n-l)] :=$$

$$\frac{F_*\Omega^{n-l}_{X/S}(A,B)}{dF_*\Omega^{n-l}_{X/S}(A,B)} \longrightarrow F_*\Omega^{n-l+1}_{X/S}(A,B) \longrightarrow \cdots \longrightarrow F_*\Omega^n_{X/S}(A,B).$$

The composite map

$$\mathcal{H}^{n-l} \longrightarrow \tau_{\geq n-l}F_*\Omega^\bullet_{X/S}(A,B)[(n-l)] \longrightarrow \mathcal{C}^\bullet(\Omega^{n-l}_{X'/S}(A',B'))$$

is given by $\bigwedge^{n-l} C$. As $\tau_{\geq n-l}$ maps to $\tau_{\geq n-l+1}$, we find in this way a $\mathcal{O}_{X'}$-map

$$\tau_{\geq n-p+1}F_*\Omega^\bullet_{X/S}(A,B) \xrightarrow{\varphi^\vee} \bigoplus_{i=n-p+1}^{n} \mathcal{C}^\bullet(\Omega^i_{X'/S}(A',B'))[-i]$$

which is a quasi-isomorphism. In particular, for any open set $U'\subset X'$ and any $\mathcal{O}_{X'}$-sheaf $\mathcal{F}'$ one has

$$\mathbb{H}^l(U',\tau_{\geq n-p+1}F_*\Omega^\bullet_{X/S}(A,B)\otimes\mathcal{F}') = \bigoplus_{i=n-p+1}^{n} H^{l-i}(U',\Omega^i_{X'/S}(A',B')\otimes\mathcal{F}').$$

Consider now $n = p$ as needed to finish the proof of (10.19). From the exact sequence

$$0 \longrightarrow \mathcal{H}^0 \longrightarrow F_*\Omega^\bullet_{X/S}(A,B) \longrightarrow \tau_{\geq 1} F_*\Omega^\bullet_{X/S}(A,B) \longrightarrow 0$$

we obtain an exact sequence of hypercohomology groups

$$\mathbb{H}^p(F_*\Omega^\bullet_{X/S}(A,B) \otimes \mathcal{H}^{p^\vee}) \longrightarrow \bigoplus_{a=1}^{p} H^{p-a}(\Omega^a_{X'/S}(A',B') \otimes \mathcal{H}^{p^\vee}) \longrightarrow$$
$$\longrightarrow H^{p+1}(\mathcal{H}^0 \otimes \mathcal{H}^{p^\vee})$$

where $\mathcal{H}^{p^\vee} := \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{H}^p, \mathcal{O}_{X'})$.

As $\dim_S X = p$ and S is affine, one has $H^{p+1}(\mathcal{H}^0 \otimes \mathcal{H}^{p^\vee}) = 0$, and therefore

$$\bigwedge^p C \in H^0(\Omega^p_{X'/S}(A',B') \otimes \mathcal{H}^{p^\vee})$$

lifts to some

$$\Gamma \in \mathbb{H}^p(F_*\Omega^\bullet_{X/S}(A,B) \otimes \mathcal{H}^{p^\vee}).$$

Representing Γ by a Čech cocycle

$$\mathcal{H}^p \xrightarrow{[\Gamma]} \mathcal{C}^p(F_*\mathcal{O}_X) + \cdots + \mathcal{C}^0(F_*\Omega^p_{X/S}(A,B)),$$

and taking a common refinement $\mathcal{U}$ of the Čech covers defining φ and Γ, one obtains altogether a quasi-isomorphism

$$(\varphi, [\Gamma] \circ \bigwedge^p C^{-1}) : \bigoplus_j \Omega^j_{X'/S}(A',B')[-j] \longrightarrow \mathcal{C}^\bullet(F_*\Omega^\bullet_{X/S}(A,B)).$$

□

10.20. Remark. If $\dim_S X = n > p$, and S is affine, one considers the exact sequence

$$0 \longrightarrow \tau_{\leq n-p} F_*\Omega^\bullet_{X/S}(A,B) \longrightarrow F_*\Omega^\bullet_{X/S}(A,B) \longrightarrow \tau_{\geq n-p+1}\Omega^\bullet_{X/S}(A,B) \longrightarrow 0$$

giving the short exact sequences

$$\mathbb{H}^q(F_*\Omega^\bullet_{X/S}(A,B) \otimes \mathcal{H}^{q^\vee}) \longrightarrow \bigoplus_{a=n-p+1}^{n} H^{q-a}(\Omega^a_{X'/S}(A',B') \otimes \mathcal{H}^{q^\vee}) \longrightarrow$$
$$\longrightarrow H^{q+1}(\tau_{\leq n-p} F_*\Omega^\bullet_{X/S}(A,B) \otimes \mathcal{H}^{q^\vee})$$

for all $p \leq q \leq n$. If $n - p \leq p - 1$, then

$$H^{q+1}(\tau_{\leq n-p} F_*\Omega^\bullet_{X/S}(A,B) \otimes \mathcal{H}^{q^\vee}) = \bigoplus_{a=0}^{n-p} H^{q+1-a}(\Omega^a_{X'/S}(A',B') \otimes \mathcal{H}^{q^\vee}).$$

If those groups are vanishing, then the same argument as above shows that one obtains a splitting of $F_*\Omega^\bullet_{X/S}(A,B)$.

For example, take $D=\emptyset$. For $n=p+1$, one requires the vanishing of

$$H^{p+1}(\mathcal{H}^{p^\vee}),\ H^p(\Omega^1_{X'/S}\otimes\mathcal{H}^{p^\vee}) \text{ and of } H^{p+1}(\Omega^1_{X'/S}\otimes\mathcal{H}^{p+1^\vee}),$$

that is, via duality, the vanishing of

$$H^0(\Omega^p_{X'/S}\otimes\Omega^{p+1}_{X'/S}) \quad\text{and}\quad H^1(\Omega^{p\otimes 2}_{X'/S}).$$

Using (10.19) it is now quite easy to prove theorem (8.3) and some generalizations.

10.21. Theorem. *Let $f: X\to S$ be a smooth proper S-scheme, $\dim_S X\le p$, and let $D\subset X$ be a S-normal crossing divisor. Assume that there exists a lifting*

$$\widetilde{D}'\subset\widetilde{X}' \quad\textit{of}\quad D'\subset X'$$

to $\widetilde{S}$. Let $D=A+B$. Then one has:

a) *The $\mathcal{O}_S$-sheaves*

$$E_1^{ab}=R^bf_*\Omega^a_{X/S}(A,B)$$

are locally free and compatible with arbitrary base change.

b) *The Hodge to de Rham spectral sequence*

$$E_1^{ab}\Longrightarrow \mathbb{R}^{a+b}f_*\Omega^\bullet_{X/S}(A,B)$$

degenerates in E_1.

Proof: Assuming a), part b) follows if one knows that $\mathbb{R}^lf_*\Omega^\bullet_{X/S}(A,B)$ is a locally free $\mathcal{O}_S$-module of rank

$$\sum_{a+b=l}\operatorname{rank}_{\mathcal{O}_S}(E_1^{ab}).$$

Hence for a) and b) we can assume S to be affine.

(10.17), for $i=\dim_S X<p$, or (10.19) for $\dim_S X=p$ imply that

$$\mathbb{R}^lf_*\Omega^\bullet_{X/S}(A,B)=\mathbb{R}^lf'_*(F_*\Omega^\bullet_{X/S}(A,B))=\bigoplus_a R^{l-a}f'_*\Omega^a_{X'/S}(A',B').$$

Hence, if a) holds true for $f: X\to S$, then

$$\mathbb{R}^lf_*\Omega^\bullet_{X/S}(A,B)=\bigoplus_a F_S^*R^{l-a}f_*\Omega^a_{X/S}(A,B)$$

is locally free of the right rank and one obtains b).

By "cohomology and base change" ([50], II §5, for example), there exist for $a = 0, \ldots, l$ bounded complexes $\mathcal{E}_a^\bullet$ of vector bundles on S, such that

$$\mathcal{H}^l(\mathcal{E}_a^\bullet) = R^{l-a} f_* \Omega^a_{X/S}(A,B)$$

and, for any affine map $\varphi : T \longrightarrow S$,

$$\mathcal{H}^l(\varphi^* \mathcal{E}_a^\bullet) = R^{l-a} f_{T*} \Omega^a_{X_T/T}(A_T, B_T)$$

where $X_T,\ A_T,\ B_T,\ X'_T,\ f_T : X_T \to T$ and $f'_T : X'_T \to T$ are obtained by pullback from $X,\ A,\ B,\ X',\ f$ and $f' : X' \to S$.

For example, $R^{l-a} f'_* \Omega^a_{X'/S}(A', B')$ is given by $\mathcal{H}^l(F_S^* \mathcal{E}_a^\bullet)$ and hence $\mathbb{R}^l f_* \Omega^\bullet_{X/S}(A,B)$ by the l-th homology of the complex

$$F_S^* \mathcal{E}^\bullet \ \text{ for } \ \mathcal{E}^\bullet = \bigoplus_a \mathcal{E}_a^\bullet .$$

To prove a), we have to show that for all l, the sheaf

$$\mathcal{H}^l(\mathcal{E}^\bullet) = \bigoplus_a \ \mathcal{H}^l(\mathcal{E}_a^\bullet)$$

is locally free. If this is wrong, then we take l_0 to be the maximal l with $\mathcal{H}^l(\mathcal{E}^\bullet)$ not locally free. Hence, if $\partial_\bullet$ denotes the differential in $\mathcal{E}^\bullet$, $\ker\ \partial_{l_0}$ is a vector bundle, let us say of rank r, but the image of

$$\partial_{l_0 - 1} : \mathcal{E}^{l_0 - 1} \longrightarrow \ker\ \partial_{l_0}$$

is not a subbundle.

For some closed point $s \in S$ one finds an infinitesimal neighbourhood $\hat{S}$, for example one of the form

$$\hat{S} = \operatorname{Spec}(\mathcal{O}_{S,s}/m^\mu_{S,s}) \ \text{ for } \ \mu >> 0,$$

such that $\operatorname{im}(\partial_{l_0-1}|_{\hat{S}})$ is not a subbundle of $\ker(\partial_{l_0}|_{\hat{S}})$. Let us write $\hat{S} = \operatorname{Spec} R$ where R is an Artin ring. For any R-module M let $\lg(M)$ denote the lenght.

∂_{l_0-1} is represented by a matrix Δ_{l_0-1}. For the point $s \in \hat{S}$ one has

$$h = \operatorname{rank}_{k(s)} \mathcal{H}^{l_0}(\mathcal{E}^\bullet \otimes k(s)) > \frac{\lg(\mathcal{H}^{l_0}(\mathcal{E}^\bullet \otimes R))}{\lg(R)} .$$

Let $n_0 \subset R$ be the ideal generated by the $(r-h+1)$ minors of Δ_{l_0-1}. Then

$$\mathcal{H}^{l_0}(\mathcal{E}^\bullet \otimes R/n_0)$$

is free of rank h as an R/n_0-module. If $n'_0 \subset n_0 \subset R$ is another ideal with $n'_0 \neq n_0$, then

$$\lg(\mathcal{H}^{l_0}(\mathcal{E}^\bullet \otimes R/n'_0)) < h \cdot \lg(R/n'_0).$$

Repeating this construction, starting with R/n_0 instead of R we find after finitly many steps some ideal n such that the R/n modules

$$\mathcal{H}^l(\mathcal{E}^\bullet \otimes R/n)$$

are free of rank $h(l)$ over R/n for all l, but for some l' and for all ideals $n' \subset n$ with $n' \neq n$ one has

$$\lg(\mathcal{H}^{l'}(\mathcal{E}^\bullet \otimes R/n')) < h(l') \cdot \lg(R/n') \ .$$

In particular, this holds true for the ideal n' of R generated by $F_{\hat{S}}^* n$. Let us write $T' = \operatorname{Spec}(R/n')$ and $T = \operatorname{Spec}(R/n)$. We have affine morphisms

$$\begin{array}{ccccc} T' & \xrightarrow{\delta} & T & \xrightarrow{j} & T' \\ \downarrow & & \downarrow & & \downarrow \\ S & \xrightarrow{F_S} & S & \xrightarrow{=} & S \ . \end{array}$$

$\mathcal{H}^l(\mathcal{E}^\bullet|_T)$ is free of rank $h(l)$ over R/n for all l and hence

$$\mathcal{H}^l(\delta^* j^* \mathcal{E}^\bullet|_{T'}) = \mathcal{H}^l((F_S^*\mathcal{E}^\bullet)|_{T'})$$

is free of rank $h(l)$ over R/n'. The Hodge to de Rham spectral sequence implies that

$$\begin{aligned} \lg(\mathcal{H}^l(\mathcal{E}^\bullet|_{T'})) &= \sum_a \lg(R^{l-a} f_{T'*}\Omega^a_{X_{T'}/T'}(A_{T'}, B_{T'})) \geq \\ &\geq \lg(\mathbb{R}^l f_{T'*}\Omega^\bullet_{X_{T'}/T'}(A_{T'}, B_{T'})). \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathbb{R}^l f_{T'*}\Omega^\bullet_{X_{T'}/T'}(A_{T'}, B_{T'}) &= \mathbb{R}^l f'_*(F_*\Omega^\bullet_{X/S}(A,B) \otimes f'^*(R/n')) = \\ &= \bigoplus_a R^{l-a} f'_*(\Omega^a_{X'/S}(A', B') \otimes f'^*(R/n')). \end{aligned}$$

We can apply base change and find the latter to be

$$\bigoplus_a \mathcal{H}^l(\delta^*\mathcal{E}^\bullet_a|_T) = \mathcal{H}^l(\delta^*\mathcal{E}^\bullet|_T) \ .$$

Since the sheaves $\mathcal{H}^l(\mathcal{E}^\bullet|_T)$ are locally free, we have altogether

$$\begin{aligned} \lg(\mathcal{H}^l(\mathcal{E}^\bullet|_{T'})) &\geq \lg(\mathcal{H}^l(\delta^*\mathcal{E}^\bullet|_T)) = \\ &= \lg(\delta^*\mathcal{H}^l(\mathcal{E}^\bullet|_T)) = h(l) \cdot \lg(R/n') \ . \end{aligned}$$

For $l = l'$ this contradicts the choice of n and n'. Hence $\mathcal{H}^l(\mathcal{E}^\bullet)$ is locally free. $\square$

10.22. Remark. As shown in [12], 4.1.2. it is enough to assume that for all a and b the dimension of $H^b(X_s, \Omega^a_{X_s}(A_s, B_s))$ is finite for closed points $s \in S$ and that the conjugate spectral sequence

$$ {}_cE_2^{ij} = R^i f'_{T*}\mathcal{H}^j(F_*\Omega^\bullet_{X_T/T}(A_T, B_T)) \Longrightarrow \mathbb{R}^{i+j} f_{T*}\Omega^\bullet_{X_T/T}(A_T, B_T) $$

satisfies

$$ {}_cE_2^{ij} = {}_cE_\infty^{ij} \quad \text{for} \quad i + j = l $$

and for all T, in order to obtain (10.21).

10.23. Corollary. *Let K be a field of characteristic zero, X be a smooth proper scheme over K and $D = A + B$ be a reduced normal crossing divisor defined over K. Then the Hodge to de Rham spectral sequence*

$$ E_1^{ab} = H^b(X, \Omega^a_X(A, B)) \Longrightarrow \mathbb{H}^{a+b}(X, \Omega^\bullet_X(A, B)) $$

degenerates in E_1.

Proof: By flat base change we may assume that K is of finite type over $\mathbb{Q}$. Hence we find a ring R, of finite type over $\mathbb{Z}$, such that K is the quotient field over R. Let $f : \mathcal{X} \to \mathrm{Spec} R$ be a proper morphism with $X = \mathcal{X} \times_R K$, and $\mathcal{A}$ and $\mathcal{B}$ divisors on $\mathcal{X}$ with $A = \mathcal{A}|_X$ and $B = \mathcal{B}|_X$.
Replacing $\mathrm{Spec} R$ by some open affine subscheme, we may assume that f is smooth, that $\mathcal{D} = \mathcal{A} + \mathcal{B}$ is a normal crossing divisor over $\mathrm{Spec} R$ and that

$$ \mathbb{R}^l f_*\Omega^\bullet_{\mathcal{X}/\mathrm{Spec} R}(\mathcal{A}, \mathcal{B}) \quad \text{and} \quad R^b f_*\Omega^a_{\mathcal{X}/\mathrm{Spec}\ R}(\mathcal{A}, \mathcal{B}) $$

are locally free for all l, a and b. Take a closed point $s \in \mathrm{Spec} R$ with

$$ \mathrm{char}\ k(s) = p > \dim_S X. $$

Hence

$$ X_s = \mathcal{X} \times_R k(s) \longrightarrow S = k(s)\ , \quad A_s = \mathcal{A}|_{X_s} \quad \text{and} \quad B_s = \mathcal{B}|_{X_s} $$

satisfy the assumptions made in (10.21) and

$$ \sum_{a+b=l} \mathrm{rank}\ R^b f_*\Omega^a_{\mathcal{X}/\mathrm{Spec}\ R}(\mathcal{A}, \mathcal{B}) = \mathrm{rank}\ \mathbb{R}^l f_*\Omega^\bullet_{\mathcal{X}/\mathrm{Spec}\ R}(\mathcal{A}, \mathcal{B}). $$

□

As mentioned in (3.18) the corollary (10.23) finally ends the proof of (3.2) in characteristic zero, and hence of the different vanishing theorems and applications discussed in Lectures 5 - 7. A slightly different argument, avoiding the use of (3.19) or (3.22) can be found at the end of this lecture.

As promised we are now able to prove (3.2,b and c) for fields of characteristic $p \neq 0$ as well.

PROOF OF (3.2,B) AND C) IN CHARACTERISTIC $p \neq 0$: Recall, that on the projective manifold X we considered the invertible sheaves

$$\mathcal{L}^{(i)} = \mathcal{L}^{(i,D)} = \mathcal{L}^{i}(-[\frac{i \cdot D}{N}]) \quad \text{where}$$

$$D = \sum_{j=1}^{r} \alpha_i D_j$$

is a normal crossing divisor and $\mathcal{L}$ an invertible sheaf with $\mathcal{L}^N = \mathcal{O}_X(D)$. For N prime to char k, we constructed in §3, for $i = 0, \ldots, N-1$, integrable logarithmic connections

$$\nabla_{(i)} : \mathcal{L}^{(i)^{-1}} \longrightarrow \Omega^1_X(\log\ D^{(i)}) \otimes \mathcal{L}^{(i)^{-1}}$$

with poles along

$$D^{(i)} = \sum_{\substack{j=1 \\ \frac{i \cdot \alpha_j}{N} \notin \mathbb{Z}}}^{r} D_j.$$

The residue of $\nabla_{(i)}$ along $D_j \subset D^{(i)}$ is given by multiplication with

$$(i \cdot \alpha_j - N \cdot [\frac{i \cdot \alpha_j}{N}]) \cdot N^{-1}.$$

If A and B are reduced divisors such that A, B and $D^{(i)}$ have pairwise no common component then we want to prove:

□

10.24. Claim. Let k be a perfect field, N prime to char $k = p$ and assume that $p \geq \dim\ X$. If X, D, A and B admit a lifting to $W_2(k)$, then the spectral sequence

$$E_1^{ab} = H^b(X, \Omega^a_X(\log\ (A + B + D^{(i)}))(-B) \otimes \mathcal{L}^{(i)^{-1}}) \Longrightarrow$$
$$\mathbb{H}^{a+b}(X, \Omega^\bullet_X(\log\ (A + B + D^{(i)}))(-B) \otimes \mathcal{L}^{(i)^{-1}})$$

degenerates in E_1.

PROOF: Let $F : X \to X'$ be the relative Frobenius morphism and $\mathcal{L}', D', A'$ and B' be the sheaf and the divisors on X' obtained by field extensions. Then

$$F^* \mathcal{L}'^{(i,D')} = F^* \mathcal{L}'^{(i)} = \mathcal{L}^{p \cdot i}(-p \cdot [\frac{i \cdot D}{N}])$$

contains
$$\mathcal{L}^{(p\cdot i)} = \mathcal{L}^{p\cdot i}(-[\frac{p\cdot i\cdot D}{N}]).$$
Since p is prime to N one has $D^{(i)} = D^{(p\cdot i)}$. The connection
$$\nabla_{(p\cdot i)^{-1}} : \mathcal{O}_X(-B)\otimes\mathcal{L}^{(p\cdot i)^{-1}} \longrightarrow \Omega^1_X(\log\ (A+B+D^{(i)}))(-B)\otimes\mathcal{L}^{(p\cdot i)^{-1}}$$
induces a connection with the same poles on
$$\mathcal{O}_X(-B)\otimes F^*\mathcal{L}'^{(i)^{-1}}$$
whose residues along $D_j \subset D^{(i)}$ are given by multiplication with
$$N^{-1}\cdot p\cdot i\cdot\alpha_j - [\frac{p\cdot i\cdot\alpha_j}{N}] + ([\frac{p\cdot i\cdot\alpha_j}{N}] - p[\frac{i\cdot\alpha_j}{N}])\ .$$
Obviously this number is zero modulo p. Since
$$[\frac{p\cdot i\cdot\alpha_j}{N}] \leq \frac{p\cdot i\cdot\alpha_j}{N} < p\cdot([\frac{i\cdot\alpha_j}{N}]+1) = p\cdot[\frac{i\cdot\alpha_j}{N}]+p$$
(2.10) implies that the complexes
$$\Omega^\bullet_X\ (\log\ (A+B+D^{(i)}))(-B)\otimes F^*\mathcal{L}'^{(i)^{-1}}$$
and
$$\Omega^\bullet_X(\log\ (A+B+D^{(i)}))(-B)\otimes\mathcal{L}^{(p\cdot i)^{-1}}$$
are quasi-isomorphic.

10.25. Claim. *The complex*
$$F_*(\Omega^\bullet_X(\log\ (A+B+D^{(i)}))(-B)\otimes F^*\mathcal{L}'^{(i)^{-1}})$$
is isomorphic to the complex
$$F_*(\Omega^\bullet_X(\log\ (A+B+D^{(i)}))(-B))$$
tensorized with $\mathcal{L}'^{(i)^{-1}}$.

PROOF: Let $\pi : Y \to X$ be the cyclic cover obtained by taking the N-th root out of D, let $F : Y \to Y'$ be the relative Frobenius of Y. We have the induced diagram
$$\begin{array}{ccc} Y & \xrightarrow{F} & Y' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ X & \xrightarrow{F} & X' \end{array}$$

and, on $X' - \mathrm{Sing}(D_{red})$,

$$\pi'_* \mathcal{O}_{Y'} = Ker(d : \pi'_* F_* \mathcal{O}_Y \longrightarrow \pi'_* F_* \Omega^1_Y) =$$

$$Ker(d : F_*(\bigoplus_{j=0}^{N-1} \mathcal{L}^{(j)^{-1}}) \longrightarrow F_*(\bigoplus_{j=0}^{N-1} \Omega^1_X(\log\ D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}})).$$

Since F^* of the i-th eigenspace $\mathcal{L}'^{(i)}$ of $\pi'_* \mathcal{O}_{Y'}$ lies in the $p \cdot i$-th eigenspace $\mathcal{L}^{(i \cdot p)}$ of $\pi_* \mathcal{O}_Y$ one has

$$\begin{aligned} \mathcal{L}'^{(i)^{-1}} &= Ker(\nabla_{(p\cdot i)} : F_* \mathcal{L}^{(p\cdot i)^{-1}} \longrightarrow F_*(\Omega^1_X(\log\ D^{(i)}) \otimes \mathcal{L}^{(p\cdot i)^{-1}})) \\ &= Ker(\nabla_{(p\cdot i)} : (F_* \mathcal{O}_X) \otimes \mathcal{L}'^{(i)^{-1}} \longrightarrow (F_* \Omega^1_X(\log\ D^{(i)})) \otimes \mathcal{L}'^{(i)^{-1}}) \end{aligned}$$

By the Leibniz rule $\nabla_{(p\cdot i)}$ restricted to $(F_* \mathcal{O}_X) \otimes \mathcal{L}'^{(i)^{-1}}$ is nothing but $d \otimes id$ as claimed.

□

From (10.19) and by base change

$$\dim \mathbb{H}^l(X, \Omega^\bullet_X(\log\ (A + B + D^{(i)}))(-B) \otimes \mathcal{L}^{(p\cdot i)^{-1}}) =$$

$$\dim \mathbb{H}^l(X', F_*(\Omega^\bullet_X(\log\ (A + B + D^{(i)}))(-B) \otimes \mathcal{L}^{(p\cdot i)^{-1}})) =$$

$$\sum_{a+b=l} \dim H^b(X', \omega^a_{X'}(\log\ (A' + B' + D'^{(i)}))(-B') \otimes \mathcal{L}'^{(i)^{-1}} =$$

$$\sum_{a+b=l} \dim H^b(X, \omega^a_X(\log\ (A + B + D^{(i)}))(-B) \otimes \mathcal{L}^{(i)^{-1}} \geq$$

$$\dim \mathbb{H}^l(X, \Omega^\bullet_X(\log\ (A + B + D^{(i)}))(-B) \otimes \mathcal{L}^{(i)^{-1}}).$$

For some $\nu > 0$ one has $p^\nu \equiv 1\ mod\ N$. Repeating the argument $\nu - 1$ times one finds

$$\dim \mathbb{H}^l(X, \Omega^\bullet_X(\log\ (A + B + D^{(i)}))(-B) \otimes \mathcal{L}^{(i)^{-1}}) \geq$$

$$\dim \mathbb{H}^l(X, \Omega^\bullet_X(\log\ (A + B + D^{(i)}))(-B) \otimes \mathcal{L}^{(p^{\nu-1}\cdot i)^{-1}}) \geq \cdots$$

$$\cdots \geq \sum_{a+b=l} \dim H^b(X, \Omega^a_X(\log\ (A + B + D^{(i)}))(-B) \otimes \mathcal{L}^{(i)^{-1}}) \geq$$

$$\dim \mathbb{H}^l(X, \Omega^\bullet_X(\log\ (A + B + D^{(i)}))(-B) \otimes \mathcal{L}^{(i)^{-1}})\ .$$

Hence all the inequalities must be equalities and one obtains (10.24).

□

2. Proof of (3.2,b) and c) in characteristic 0:
The arguments used in (10.23) to reduce (10.23) to (10.21) show as well that 3.2,b and c in characteristic 0 follow from (10.24).

□

§ 11 Vanishing theorems in characteristic p.

In this lecture we start with the elegant proof of the Akizuki-Kodaira-Nakano vanishing theorem, due to Deligne, Illusie and Raynaud [12]. Then we will discuss some generalizations. However they only seem to be of interest if one assumes that one has embedded resolutions of singularities in characteristic p.

11.1. Lemma. *Let k be a perfect field, let X be a proper smooth k-scheme and D a normal crossing divisor, both admitting a lifting $\widetilde{D} \subset \widetilde{X}$ to $W_2(k)$. Let $\mathcal{M}$ be a locally free $\mathcal{O}_X$-module. Then, for $l <$ char(k) one has*

$$\sum_{a+b=l} \dim\ H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{M}) \leq \sum_{a+b=l} \dim\ H^b(X, \Omega^a_X(\log\ D) \otimes F^*_X\mathcal{M}).$$

Proof: By (10.16) we have

$$\dim\ \mathbb{H}^l(X, \Omega^\bullet_X(\log\ D) \otimes F^*_X\mathcal{M}) = \sum_{a+b=l} \dim\ H^b(X', \Omega^a_{X'}(\log\ D') \otimes \mathcal{M}')$$

for the sheaf $\mathcal{M}' = {pr_1}^*\mathcal{M}$ on $X' = X \times_{F_S} S$ and $D' = D \times_{F_S} S$. By base change the right hand side is

$$\sum_{a+b=l} \dim\ H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{M}).$$

The Hodge to de Rham spectral sequence implies that the left hand side is smaller than or equal to

$$\sum_{a+b=l} \dim\ H^b(X, \Omega^a_X(\log\ D) \otimes F^*_X\mathcal{M}).$$

□

11.2. Corollary. *Under the assumptions of (11.1) assume that $\mathcal{M}$ is invertible. Then*

$$\sum_{a+b=l} \dim\ H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{M}) \leq \sum_{a+b=l} \dim\ H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{M}^p).$$

11.3. Corollary (Deligne, Illusie, Raynaud, see [12]).
For $a+b < \text{Min}\ \{\text{char}(k),\ \dim\ X\}$ and $\mathcal{L}$ ample and invertible, one has under the assumptions of (11.1.)

$$H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{L}^{-1}) = 0.$$

Proof: For ν large enough, and $\mathcal{L}^{-1} = \mathcal{M}$ one has

$$H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{M}^{p^\nu}) = 0$$

for $b < \dim\ X$. By (11.2) one has

$$H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{M}^{p^{\nu-1}}) = 0$$

and after finitely many steps one obtains (11.3)

□

The following corollary is, as well known in characteristic zero, a direct application of (11.3) for $a = 0$. It will be needed in our discussion of possible generalizations of (11.3).

11.4. Corollary. *Let k be a perfect field, let X be a proper smooth k-scheme with $\dim X \leq \text{char}\ k$ and let $\mathcal{L}$ be a numerically effective sheaf (see (5.5)). Assume that $\mathcal{A}$ is a very ample sheaf, such that X and $\mathcal{A}$ lift to $\widetilde{X}$ and $\widetilde{\mathcal{A}}$ over $W_2(k)$, with $\widetilde{\mathcal{A}}$ very ample over $\widetilde{S}$. Then one has:*

a) $\mathcal{A}^{\dim X+1} \otimes \mathcal{L} \otimes \omega_X$ *is generated by global sections.*

b) $\mathcal{A}^{\dim X+2} \otimes \mathcal{L} \otimes \omega_X$ *is very ample.*

Proof: By (11.3) and Serre duality $H^1(X, \mathcal{A}^{\dim X} \otimes \mathcal{L} \otimes \omega_X) = 0$. Hence one has a surjection

$$H^0(X, \mathcal{A}^{\dim X+1} \otimes \mathcal{L} \otimes \omega_X) \longrightarrow H^0(H, \mathcal{A}^{\dim X} \otimes \mathcal{L} \otimes \omega_H)$$

where H is a smooth zero divisor of a general section of $\mathcal{A}$. Since $\mathcal{A}$ lifts to $W_2(k)$, we can choose H such that it lifts to $W_2(k)$ as well. By induction on $\dim\ X$ we can assume that

$$\mathcal{A}^{\dim X} \otimes \mathcal{L} \otimes \omega_H$$

is generated by global sections and, moving H, we obtain a).
Part b) follows directly from a) (see [30], II. Ex.7.5).

□

11.5. Proposition. *Let k be a perfect field of characteristic $p > 0$, let X be a proper smooth k-variety, let D be an effective normal crossing divisor and $\mathcal{L}$ be an invertible sheaf on X. Assume that (X, D) and $\mathcal{L}$ admit liftings to $W_2(k)$*

and that one has:

$(*)$ *For some* $\nu_0 \in \mathbb{N}$ *and all* $\nu \geq 0$ *the sheaf* $\mathcal{L}^{\nu_0+\nu} \otimes \mathcal{O}_X(-D)$ *is ample.*

Then, for $a+b < \dim\ X \leq \mathrm{char}\ k$ *one has*

$$H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{L}^{-1}) = 0.$$

PROOF: By (5.7), the assumption $(*)$ implies that $\mathcal{L}$ is numerically effective. Let us choose μ_0 such that $\mathcal{L}^{\mu_0\cdot\nu_0}(-\mu_0 \cdot D)$ is very ample and

$$\mathcal{L}^{\mu_0\cdot\nu_0}(-\mu_0 \cdot D) \otimes \omega_X^{-1} \quad \text{and} \quad \mathcal{L}^{\mu_0\cdot\nu_0}(-\mu_0 \cdot D + D_{red}) \otimes \omega_X^{-1}$$

are both ample. From (11.4) we find for $n = \dim X$ that both,

$$\mathcal{L}^{\mu_0\cdot\nu_0(n+3)+\nu}(-\mu_0(n+3)\cdot D) \quad \text{and} \quad \mathcal{L}^{\mu_0\cdot\nu_0(n+3)+\nu}(-\mu_0(n+3)\cdot D + D_{red})$$

are very ample for all $\nu \geq 0$. Hence the assumption $(*)$ in (11.5) can be replaced by

$(**)$ *For some* $\nu_0 \in \mathbb{N}$ *and all* $\nu \geq 0$ *the sheaves* $\mathcal{L}^{\nu_0+\nu} \otimes \mathcal{O}_X(-D)$ *and* $\mathcal{L}^{\nu_0+\nu} \otimes \mathcal{O}_X(-D + D_{red})$ *are very ample.*

Choose $\eta \in \mathbb{N} - \{0\}$ such that $N = p^\eta + 1 > \nu_0$ and $[\frac{D}{N}] = 0$. Let H be the zero set of a general section of $\mathcal{L}^N \otimes \mathcal{O}_X(-D)$. In §3 we constructed an integrable logarithmic connection $\nabla_{(i)}$ on the sheaf

$$\mathcal{L}^{(i)^{-1}} = \mathcal{L}^{-i}([\frac{i\cdot(D+H)}{N}])\ .$$

Let $F : X \to X'$ be the relative Frobenius morphism and $\mathcal{L}', D', H'$ the sheaf and the divisors on X', obtained by field extension via $F_{\mathrm{Spec}\ k} : k \to k$ from $\mathcal{L}, D$ and H.

As we have seen in the proof of (10.24) and in (10.25) one has an inclusion of complexes

$$(F_*\Omega^\bullet_X(\log\ (D+H))) \otimes \mathcal{L}'^{(i)^{-1}} \longrightarrow F_*(\Omega^\bullet_X(\log\ (D+H)) \otimes \mathcal{L}^{(p\cdot i)^{-1}}).$$

This inclusion is a quasi-isomorphism and as in (10.24) one obtains from (10.19) and base change the inequalities

$$\dim \mathbb{H}^l(X, \Omega^\bullet_X(\log\ (D+H)) \otimes \mathcal{L}^{(1)^{-1}}) \leq$$

$$\sum_{a+b=l} \dim H^b(X, \Omega^a_X(\log\ (D+H)) \otimes \mathcal{L}^{(1)^{-1}}) =$$

$$\sum_{a+b=l} \dim H^b(X', \Omega^a_{X'}(\log\ (D'+H')) \otimes \mathcal{L}'^{(1)^{-1}}) =$$

$$\dim \mathbb{H}^l(X, \Omega^\bullet_X(\log\ (D+H)) \otimes \mathcal{L}^{(p)^{-1}}) \leq \cdots$$

$$\dim \mathbb{H}^l(X, \Omega^\bullet_X(\log\ (D+H)) \otimes \mathcal{L}^{(p^\gamma)^{-1}}) \leq$$

$$\sum_{a+b=l} H^b(X, \Omega^a_X(\log\ (D+H)) \otimes \mathcal{L}^{(p^\gamma)^{-1}})$$

for all $\gamma > 0$. For $\gamma = \eta$ we have $p^\eta = N-1$ and

$$\mathcal{L}^{(p^\eta)} = \mathcal{L}^{(N-1)} = \mathcal{L}^{N-1}(-[\tfrac{(N-1)\cdot(D+H)}{N}]) =$$

$$= \mathcal{L}^{N-1}(-D - [\tfrac{-D}{N} + \tfrac{(N-1)\cdot H}{N}]) = \mathcal{L}^{N-1}(-D + D_{red}).$$

Hence $\mathcal{L}^{(p^\eta)}$ is ample and from (11.3) we obtain, for

$$l < \dim\ X \leq \text{char}\ k,$$

that

$$\sum_{a+b=l} H^b(X, \Omega^a_X(\log\ (D+H)) \otimes \mathcal{L}^{(p^\eta)^{-1}}) = 0.$$

Since $\mathcal{L}^{(1)} = \mathcal{L}$ we obtain for $a+b < \dim\ X \leq \text{char}\ k$

$$H^b(X, \Omega^a_X(\log\ (D+H)) \otimes \mathcal{L}^{-1}) = 0.$$

Finally, since $\mathcal{L}^N \otimes \mathcal{O}_X(-D)$ lifts to $W_2(k)$, we can choose H such that H and $D|_H$ both lift to $W_2(k)$ and the exact sequence

$$H^{b-1}(H, \Omega^{a-1}_H(\log\ D)|_H) \otimes \mathcal{L}^{-1}) \longrightarrow H^b(X, \Omega^a_X(\log\ D) \otimes \mathcal{L}^{-1}) \longrightarrow$$

$$\longrightarrow H^b(X, \Omega^a_X(\log\ (D+H)) \otimes \mathcal{L}^{-1}).$$

allows to prove (11.5) by induction.

□

11.6. Remarks..

a) By (5.7) and (5.4,d) the assumption (*) in (11.5) implies that $\mathcal{L}$ is numerically effective and of maximal Iitaka dimension.

b) If, on the other hand, $\mathcal{L}$ is numerically effective and of maximal Iitaka dimension, then there exists some effective divisor D such that the sheaf

$$\mathcal{L}^{\nu_0+\nu} \otimes \mathcal{O}_X(-D)$$

is ample for some $\nu_0 \in \mathbb{N}$ and all $\nu \geq 0$. However, in general, this divisor is not a normal crossing divisor and henceforth (11.5) is of no use.

If one assumes that the embedded resolution of singularities holds true over k and even over $W_2(k)$, (11.5) would give an affirmative answer to

11.7. Problem. *Let k be a perfect field of characteristic $p > 0$, let X be a proper smooth k-variety and $\mathcal{L}$ an invertible sheaf. Assume that X and $\mathcal{L}$ admit liftings to $W_2(k)$, that $\mathcal{L}$ is numerically effective and that $\kappa(\mathcal{L}) = \dim X$.*

Does this imply that

$$H^b(X, \mathcal{L}^{-1}) = 0 \quad \textit{for} \quad b < \dim X \leq \operatorname{char} k \ ?$$

11.8. Remark.
a) If $\dim X = 2$ then (11.5) gives an affirmative answer to the problem (11.7), since we have imbedded resolution of singularities for curves on surfaces. In the surface case however, [12], Cor. 2.8, gives the vanishing of $H^b(X, \mathcal{L}^{-1})$, for $b < 2$, without assuming that $\mathcal{L}$ lifts to $W_2(k)$.
b) As mentioned in Lecture 1 and 8, even if we restrict ourselves to the case where $\mathcal{L}$ is semi-ample and of maximal Iitaka dimension, we do not know the answer to problem (11.7) for higherdimensional X.

§ 12 Deformation theory for cohomology groups

In this lecture we will recall D. Mumford's description of higher direct image sheaves, already used in (10.21), and their base change properties and, following Green and Lazarsfeld [26], deduce the deformation theory for cohomology groups.

12.1. Theorem (Mumford). *Let $g : Z \longrightarrow Y$ be a projective flat morphism of noetherian schemes, let $Y_0 \subset Y$ be an affine open subscheme,*

$$Z_0 = g^{-1}(Y_0) \quad , \quad g_0 = g|_{Z_0}$$

and let $\mathcal{B}$ be a locally free sheaf on Z. Then there exists a bounded complex $(\mathcal{E}^\bullet, \delta_\bullet)$ of locally free $\mathcal{O}_{Y_0}$ modules of finite rank such that

$$\mathcal{H}^b(\mathcal{E}^\bullet \otimes \mathcal{F}) = R^b g_{0_*}(\mathcal{B}|_{Z_0} \otimes g_0^* \mathcal{F})$$

for all coherent sheaves $\mathcal{F}$ on Y_0.

In order to construct $\mathcal{E}^\bullet$, D. Mumford uses in [50], II, §5, the description of higher direct images by Čech complexes. The " Coherence Theorem" of Grauert-Grothendieck allows to realize $(\mathcal{E}^\bullet)$ as a complex of locally free sheaves of finite rank. The proof of (12.1) can be found as well in [30], III, §12.

From (12.1) one obtains easily the base change theorems of Grauert and Grothendieck, as well as the ones used at the end of Lecture 10.

12.2. Example. Let $y \in Y_0$ be a point and $\mathcal{F} = k(y)$. For $Z_y = g^{-1}(y)$ one obtains

$$\begin{array}{ccc} \mathcal{H}^b(\mathcal{E}^\bullet \otimes k(y)) & \xrightarrow{=} & H^b(Z_y, B|_{Z_y}) \\ {\scriptstyle\tau}\big\uparrow & & {\scriptstyle\eta}\big\uparrow \\ \mathcal{H}^b(\mathcal{E}^\bullet) \otimes k(y) & \xrightarrow{=} & R^b g_*(\mathcal{B}) \otimes k(y), \end{array}$$

where η is the base change morphism ([30], III, 9.3.1). In general, due to the fact that the images of

$$\delta_{b-1} : \mathcal{E}^{b-1} \longrightarrow \mathcal{E}^b \quad \text{and} \quad \delta_b : \mathcal{E}^b \longrightarrow \mathcal{E}^{b+1}$$

are not subbundles of $\mathcal{E}^b$ and $\mathcal{E}^{b+1}, \tau$ and hence η will be neither injective nor surjective.

12.3. Example. Let X be a projective manifold, defined over an algebraically closed field k and let $Y \subset \mathrm{Pic}^0(X)$ be a closed subscheme,

$$Z = X \times Y \quad \text{and} \quad g = pr_2 : Z \longrightarrow Y.$$

Recall that on Z we have a Poincaré bundle $\mathcal{P}$ (see for example [50]), i.e. an invertible sheaf $\mathcal{P}$ such that

$$\mathcal{P}|_{g^{-1}(y)} \simeq \mathcal{N}_y,$$

if $\mathcal{N}_y$ is the linebundle on X corresponding to

$$y \in Y \subset \mathrm{Pic}^0(X).$$

In more fancy terms, the functor

$$T \longmapsto \mathcal{P}ic(X \times T/T)$$

is represented by a locally noetherian group-scheme Pic(X), whose connected component containing zero is $\mathrm{Pic}^0(X)$. The invertible sheaf $\mathcal{P}$ is the restriction of the universal bundle on $X \times \mathrm{Pic}(X)$ to

$$X \times Y \subset X \times \mathrm{Pic}^0(X)$$

(see [28]). For $y \in Y$ let $T_{y,Y} = (m_{y,Y}/m_{y,Y}^2)^*$ be the Zariski tangent space. We have an exact sequence

$$0 \longrightarrow T_{y,Y}^* \longrightarrow \mathcal{O}_{y,Y}/m_{y,Y}^2 \longrightarrow k(y) \longrightarrow 0.$$

Since

$$g^*(T_{y,Y}^*) \otimes_{\mathcal{O}_Z} \mathcal{P} = g^*(T_{y,Y}^*) \otimes \mathcal{N}_y$$

one obtains the exact sequence

$$0 \longrightarrow g^*(T^*_{y,Y}) \otimes \mathcal{N}_y \longrightarrow g^*(\mathcal{O}_{y,Y}/m^2_{y,Y}) \otimes \mathcal{P} \longrightarrow \mathcal{N}_y \longrightarrow 0$$

on $X \simeq g^{-1}(y)$. If, identifying X with $g^{-1}(y)$,

$$\zeta \in H^1(X, g^*(T^1_{y,Y})) = H^1(X, \mathcal{O}_X) \otimes T^*_{y,Y}$$

is the extension class of this sequence, the induced edge morphism

$$H^b(X, \mathcal{N}_y) \longrightarrow H^{b+1}(X, g^*(T^*_{y,Y}) \otimes \mathcal{N}_y) = H^{b+1}(X, \mathcal{N}_y) \otimes_{k(y)} T^*_{y,Y}$$

is the cup-product with ζ.

12.4.

Keeping the notations from (12.3), let $\mathcal{M}$ be a locally free sheaf on X and $\mathcal{B} = \mathcal{P} \otimes pr_1^* \mathcal{M}$. Since the exact sequence

$$0 \longrightarrow g^*(T^*_{y,Y}) \otimes \mathcal{N}_y \otimes \mathcal{M} \longrightarrow g^*(\mathcal{O}_{y,Y}/m^2_{y,Y}) \otimes \mathcal{B} \longrightarrow \mathcal{N}_y \otimes \mathcal{M} \longrightarrow 0$$

is obtained from

$$0 \longrightarrow g^*(T^*_{y,Y}) \otimes \mathcal{N}_y \longrightarrow g^*(\mathcal{O}_{y,Y}/m^2_{y,Y}) \otimes \mathcal{P} \longrightarrow \mathcal{N}_y \longrightarrow 0$$

by tensorproduct with $\mathcal{M}$, the induced edge morphism

$$H^b(X, \mathcal{M} \otimes \mathcal{N}_y) \longrightarrow H^{b+1}(X, \mathcal{M} \otimes \mathcal{N}_y) \otimes_{k(y)} T^*_{y,Y}$$

is again the cup-product with ζ. Let us finally remark that ζ induces a morphism

$$T_{y,Y} \xrightarrow{\ \zeta\ } H^1(X, \mathcal{O}_X)$$

which, due to the universal property of $\mathcal{P}$ is injective. In fact, if we represent $\tau \in T_{y,Y}$ by a morphism

$$\zeta' : D = \text{Spec } \ k[\epsilon] \longrightarrow Y \quad \text{with} \quad \zeta'(< \epsilon >) = y \ ,$$

where $k[\epsilon] = k[t]/t^2$ is the ring of dual numbers, then for $\tau \neq 0$ the pullback of $\mathcal{P}$ to $X \times D$ is non trivial and hence the extension class $\zeta(\tau)$ of

$$0 \longrightarrow \mathcal{N}_y \xrightarrow{\ \cdot\epsilon\ } \mathcal{P}|X \times D \longrightarrow \mathcal{N}_y \longrightarrow 0$$

is non zero. If $Y = \text{Pic}^0(X)$ one has $\dim T_{y,Y} = \dim H^1(X, \mathcal{O}_X)$ and ζ is surjective as well.

12.5. Notations. For $X, \mathcal{M}$ as above let us write

$$S^b(X, \mathcal{M}) = \{y \in \text{Pic}^0(X); H^b(X, \mathcal{N}_y \otimes \mathcal{M}) \neq 0\}.$$

The first part of the following lemma is well known and an easy consequence of the semicontinuity of the dimensions of cohomology groups. To fix notations we will prove it nevertheless.

12.6. Lemma.
a) $S^b(X, \mathcal{M})$ is a closed subvariety of $\mathrm{Pic}^0(X)$.

b) If $Y \subset S^b(X, \mathcal{M})$ is an ireducible component and

$$m = \mathrm{Min}\{\dim H^b(X, \mathcal{N}_y \otimes \mathcal{M});\ y \in Y\},$$

then the set

$$U = \{y \in Y;\ \dim H^b(X, \mathcal{N}_y \otimes \mathcal{M}) = m\}$$

is open and dense in Y.

c) For $y \in U$ and $\zeta : T_{y,Y} \hookrightarrow H^1(X, \mathcal{O}_X)$ as in (12.4) the cup-products

$$\zeta(T_{y,Y}) \otimes H^{b-1}(X, \mathcal{N}_y \otimes \mathcal{M}) \longrightarrow H^b(X, \mathcal{N}_y \otimes \mathcal{M})$$

and

$$\zeta(T_{y,Y}) \otimes H^b(X, \mathcal{N}_y \otimes \mathcal{M}) \longrightarrow H^{b+1}(X, \mathcal{N}_y \otimes \mathcal{M})$$

are both zero.

Proof: For any open affine $P_0 \subset \mathrm{Pic}^0(X)$ let $\mathcal{E}^\bullet$ be the complex from (12.1) describing the higher direct images of

$$\mathcal{B}|_{X \times P_o} = pr_1^* \mathcal{M} \otimes \mathcal{P}|_{X \times P_o}$$

and there base change. If we write

$$\mathcal{W}_b = \mathrm{Coker}(\delta_{b-1} : \mathcal{E}^{b-1} \longrightarrow \mathcal{E}^b),$$

then for $\mathcal{F}$ coherent on P_0 we have

$$\mathcal{W}_b \otimes \mathcal{F} = \mathrm{Coker}(\delta_{b-1} : \mathcal{E}^{b-1} \otimes \mathcal{F} \longrightarrow \mathcal{E}^b \otimes \mathcal{F})$$

and an exact sequence

$$0 \longrightarrow \mathcal{H}^b(\mathcal{E}^\bullet \otimes \mathcal{F}) \longrightarrow \mathcal{W}_b \otimes \mathcal{F} \longrightarrow \mathcal{E}^{b+1} \otimes \mathcal{F} \longrightarrow \mathcal{W}_{b+1} \otimes \mathcal{F} \longrightarrow 0.$$

If $S^b(X, \mathcal{M}) \neq \mathrm{Pic}^0(X)$, then $S^b(X, \mathcal{M})$ is just the locus where $\mathcal{W}_b \to \mathcal{E}^{b+1}$ is not a subbundle. Obviously this condition defines a closed subscheme of P_0. For $y \in Y \cap P_0 = Y_0$ we have

$$\dim \mathcal{H}^b(\mathcal{E}^\bullet \otimes k(y)) = -\mathrm{rank}(\mathcal{E}^{b+1}) + \dim(\mathcal{W}_b \otimes k(y)) + \dim(\mathcal{W}_{b+1} \otimes k(y))$$

and the open set U in b) is nothing but the locus where both,

$$\mathcal{W}_b \otimes \mathcal{O}_{Y_0} \quad \text{and} \quad \mathcal{W}_{b+1} \otimes \mathcal{O}_{Y_0}$$

are locally free $\mathcal{O}_{Y_0}$ modules. On U the sequence

$$0 \longrightarrow \mathcal{H}^b(\mathcal{E}^\bullet|_U) \longrightarrow \mathcal{W}_b|_U \longrightarrow \mathcal{E}^{b+1}|_U \longrightarrow \mathcal{W}_{b+1}|_U \longrightarrow 0$$

is an exact sequence of vector bundles and

$$\mathcal{H}^b(\mathcal{E}^\bullet \otimes \mathcal{F}) = \mathcal{H}^b(\mathcal{E}|_U) \otimes \mathcal{F}$$

for all coherent $\mathcal{O}_U$ modules $\mathcal{F}$. In particular for $y \in U$ the exact sequence

$$0 \longrightarrow \mathcal{E}^\bullet \otimes T^*_{y,Y} \longrightarrow \mathcal{E}^\bullet \otimes \mathcal{O}_{y,Y}/m^2_{y,Y} \longrightarrow \mathcal{E}^\bullet \otimes k(y) \longrightarrow 0$$

induces

$$\begin{array}{ccccc}
\mathcal{H}^b(\mathcal{E}^\bullet \otimes T^*_{y,Y}) & \longrightarrow & \mathcal{H}^b(\mathcal{E}^\bullet \otimes \mathcal{O}_{y,Y}/m^2_{y,Y}) & \longrightarrow & \mathcal{H}^b(\mathcal{E}^\bullet \otimes k(y)) \\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow \\
\mathcal{H}^b(\mathcal{E}^\bullet) \otimes T^*_{y,Y} & \longrightarrow & \mathcal{H}^b(\mathcal{E}^\bullet) \otimes \mathcal{O}_{y,Y}/m^2_{y,Y} & \longrightarrow & \mathcal{H}^b(\mathcal{E}^\bullet) \otimes k(y)
\end{array}$$

and the edge morphisms

$$\mathcal{H}^i(\mathcal{E}^\bullet \otimes k(y)) \longrightarrow \mathcal{H}^{i+1}(\mathcal{E}^\bullet \otimes T^*_{y,Y})$$

are zero for $i = b$ and $b-1$. Using (12.1) we have identified in (12.4) this edge morphism with the cup-product

$$H^i(X, \mathcal{M} \otimes \mathcal{N}_y) \longrightarrow H^{i+1}(X, \mathcal{M} \otimes \mathcal{N}_y) \otimes_{k(y)} T^*_{y,Y}$$

with the extension class $\zeta \in H^1(X, \mathcal{O}_X) \otimes T^*_{y,Y}$.

$\square$

12.7. Corollary(Green, Lazarsfeld [26]). *If $Y \subset S^b(X, \mathcal{M})$ is an irreducible component and $y \in Y$ is a point in general position then*

$$\operatorname{codim}_{\mathrm{Pic}^0(X)}(Y) \geq \operatorname{codim}(\Gamma \subset H^1(X, \mathcal{O}_X))$$

where

$$\Gamma = \{\varphi \in H^1(X, \mathcal{O}_X);\ \alpha \cup \varphi = 0 \ \ \textit{and} \ \ \beta \cup \varphi = 0 \ \ \textit{for all}$$

$$\alpha \in H^{b-1}(X, \mathcal{N}_y \otimes \mathcal{M}) \ \ \textit{and} \ \ \beta \in H^b(X, \mathcal{N}_y \otimes \mathcal{M})\}.$$

12.8. Remark. Even if one seems to loose some information, in the applications of (12.7) in Lecture 13 we will replace Γ by the larger space

$$\{\varphi \in H^1(X, \mathcal{O}_X);\ \beta \cup \varphi = 0 \ \text{ for all } \ \beta \in H^b(X, \mathcal{N}_y \otimes \mathcal{M})\}$$

in order to obtain lower bounds for $\operatorname{codim}_{\mathrm{Pic}^0(X)}(S^b(X, \mathcal{M}))$ for certain invertible sheaves $\mathcal{M}$.

§ 13 Generic vanishing theorems [26], [14]

In this section we want to use (12.7) and Hodge-duality to prove some bounds for

$$\operatorname{codim}_{\mathrm{Pic}^0(X)}(S^b(X,\mathcal{M}))$$

for the subschemes $S^b(X,\mathcal{M})$ introduced in §12. In particular, we lose a little bit the spirit of the previous lectures, where we tried to underline as much as possible the *algebraic* aspects of vanishing theorems. Everything contained in this lecture is either due Green-Lazarsfeld [26] or to H. Dunio [14]. The use of Hodge duality will force us to assume that X is a complex manifold. Without mentioning it we will switch from the algebraic to the analytic language and use the comparison theorem of [56] whenever needed.

13.1. Notations. Let X be a projective complex manifold. The Picard group $\mathrm{Pic}(X)$ is $H^1(X,\mathcal{O}_X^*)$ and, using the exponential sequence, $\mathrm{Pic}^0(X)$ is identified with

$$H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z}).$$

Let $\mathcal{P}$ be the Poincaré bundle on $X \times \mathrm{Pic}^0(X)$, and $g = pr_2$. If

$$\zeta : T_{y,\mathrm{Pic}^0(X)} \longrightarrow H^1(X,\mathcal{O}_X)$$

is the extension class of

$$0 \longrightarrow g^*T_{y,\mathrm{Pic}^0(X)} \otimes \mathcal{N}_y \longrightarrow g^*(\mathcal{O}_{y,\mathrm{Pic}^0}/m^2_{y,\mathrm{Pic}^0}) \otimes \mathcal{P} \longrightarrow \mathcal{N}_y \longrightarrow 0$$

then ζ is the identity. Let

$$\mathrm{Alb}(X) = H^0(X,\Omega^1_X)^*/H_1(X,\mathbb{Z})$$

be the Albanese variety of X and

$$\alpha : X \longrightarrow \mathrm{Alb}(X)$$

be the Albanese map. The morphism α induces an isomorphism

$$\alpha^* : H^0(\mathrm{Alb}(X),\Omega^1_{\mathrm{Alb}(X)}) \longrightarrow H^0(X,\Omega^1_X).$$

In particular,

$$\dim \alpha(X) = \mathrm{rank}_{\mathcal{O}_X}(\mathrm{im}(H^0(X,\Omega^1_X) \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow \Omega^1_X)) \ .$$

13.2. Theorem (Green-Lazarsfeld [26]). *Let X be a complex projective manifold and*

$$S^b(X) = \{y \in \mathrm{Pic}^0(X);\ H^b(X,\mathcal{N}_y) \neq 0\}.$$

Then

$$\operatorname{codim}_{\mathrm{Pic}^0(X)}(S^b(X)) \geq \dim(\alpha(X)) - b.$$

In particular, if $\mathcal{N} \in \mathrm{Pic}^0(X)$ is a generic line bundle, then $H^b(X,\mathcal{N}) = 0$ for $b < \dim \alpha(X)$.

PROOF: Using (12.7) or (12.8) one obtains (13.2) from

□

13.3. Claim. *Assume that $H^b(X,\mathcal{N}_y) \neq 0$ then for*

$$\Gamma = \{\varphi \in H^1(X,\mathcal{O}_X);\ \beta \cup \varphi = 0 \ \textit{ for all } \ \beta \in H^b(X,\mathcal{N}_y)\}$$

one has

$$\operatorname{codim}(\Gamma \subset H^1(X,\mathcal{O}_X)) \geq \dim(\alpha(X)) - b.$$

PROOF:
Step 1: $\mathcal{N}_y$ is a flat unitary bundle on X, obtained from a unitary representation of the fundamental group. In particular the conjugation of harmonic forms with values in $\mathcal{N}_y$ gives a complex antilinear isomorphism, the so called Hodge duality,

$$\iota : H^b(X,\Omega^a_X \otimes \mathcal{N}_y) \longrightarrow H^a(X,\Omega^b_X \otimes \mathcal{N}_y^{-1})$$

(see (13.5) and (13.6) for generalizations). Moreover, if $\varphi \in H^1(X,\mathcal{O}_X)$ and if $\omega = \bar{\varphi} \in H^0(X,\Omega^1_X)$ is the Hodge-dual of φ, then for $\beta \in H^b(X,\Omega^a_X \otimes \mathcal{N}_y)$ one has

$$\iota(\beta \cup \varphi) = \iota(\beta) \wedge \omega \in H^a(X,\Omega^{b+1}_X \otimes \mathcal{N}_y^{-1}).$$

Hence

$$\iota(\Gamma) = \bar{\Gamma} \subset H^0(X,\Omega^1_X)$$

is the subspace of forms $\omega \in H^0(X,\Omega^1_X)$ such that

$$\beta \wedge \omega = 0 \ \text{ for all } \ \beta \in H^0(X,\Omega^b_X \otimes \mathcal{N}_y^{-1}).$$

Step 2: Consider the natural map

$$\gamma : H^0(X,\Omega^1_X) \otimes \mathcal{O}_X \longrightarrow \Omega^1_X.$$

Since all one-forms are pullback of one-forms on $\alpha(X) \subset \mathrm{Alb}(X)$, the subsheaf $\mathrm{im}(\gamma)$ of Ω^1_X is of rank $\dim \alpha(X)$ and

$$r = \operatorname{rank}\gamma(\overline{\Gamma} \otimes \mathcal{O}_X) = \dim \overline{\Gamma} - \operatorname{rank}(\ker(\gamma) \cap \overline{\Gamma} \otimes \mathcal{O}_X) \geq$$

$$\dim \overline{\Gamma} - \dim \ker(\gamma) = \dim \overline{\Gamma} - (\dim H^0(X,\Omega^1_X) - \dim \alpha(X))$$

$$= \dim \alpha(X) - \operatorname{codim}(\Gamma \subset H^1(X,\mathcal{O}_X)) \ .$$

We assumed that $H^0(X,\Omega^b_X \otimes \mathcal{N}_y^{-1}) \neq 0$. Hence we have at least one element

$$\beta \in H^0(X,\Omega^b_X \otimes \mathcal{N}_y^{-1}) \ \text{ and } \ \beta \wedge \gamma(\overline{\Gamma} \otimes \mathcal{O}_X) = 0.$$

Since

$$(\wedge^b \Omega^1_X) \otimes (\wedge^{n-b}\Omega^1_X) \longrightarrow \Omega^n_X$$

is a nondegenerate pairing, for $n = \dim X$, we find some meromorphic differential form

$$\delta \in \Omega_X^{n-b} \otimes \mathbb{C}(X) \quad \text{with} \quad \delta \wedge \beta \neq 0.$$

Hence δ lies in

$$\Omega_X^{n-b} \otimes \mathbb{C}(X) - \{\gamma(\overline{\Gamma} \otimes \mathcal{O}_X) \wedge \Omega_X^{n-b-1} \otimes \mathbb{C}(X)\}.$$

This however is only possible if $n - b \leq n - r$ or $b \geq r$. Altogether we find

$$b \geq \dim \alpha(X) - \operatorname{codim}(\Gamma \subset H^1(X, \mathcal{O}_X)).$$

□

13.4.

If one tries to use the same methods for $S^b(X, \mathcal{M})$ one has to make sure that $H^b(X, \mathcal{M} \otimes \mathcal{N}_y)$ is in Hodge duality with $H^0(X, \Omega_X^b \otimes \mathcal{M}' \otimes \mathcal{N}_y^*)$ for some sheaf $\mathcal{M}'$. As shown in (3.23) this holds true for the sheaves $\mathcal{L}^{(i)}$ arising from cyclic coverings, at least if one considers $\Omega_X^b(\log\ D)$ instead of Ω_X^b. More generally one has:

13.5. Theorem (K. Timmerscheidt [59]). *Let D be a normal crossing divisor on X, let $\mathcal{V}$ be a locally free sheaf and*

$$\nabla : \mathcal{V} \longrightarrow \Omega_X^1(\log\ D) \otimes \mathcal{V}$$

an integrable logarithmic connection. Assume that for all components D_i of D, the real part of all eigenvalues of $\operatorname{res}_{D_i}(\nabla)$ lies in $(0,1)$
(which implies that conditions (*) and (!) of (2.8) are satisfied, and that $\mathcal{V}$ is the canonical extension defined by Deligne [10]).
Assume moreover that the local constant system

$$V = \ker(\nabla : \mathcal{V}|_U \longrightarrow \Omega_U^1 \otimes \mathcal{V}|_U)$$

is unitary for $U = X - D$. Then one has:
a) The Hodge to de Rham spectral sequence

$$E_1^{ab} = H^b(X, \Omega_X^a(\log\ D) \otimes \mathcal{V}) \longrightarrow \mathbb{H}^{a+b}(X, \Omega_X^\bullet(\log\ D) \otimes \mathcal{V})$$

degenerates at E_1.
b) There exists a $\mathbb{C}$ - antilinear isomorphism

$$\iota : H^b(X, \Omega_X^a(\log\ D) \otimes \mathcal{V}) \longrightarrow H^a(X, \Omega_X^b(\log\ D) \otimes \mathcal{V}^*(-D_{\mathrm{red}}))$$

such that for $\varphi \in H^1(X, \mathcal{O}_X)$ and $\omega = \overline{\varphi} \in H^0(X, \Omega_X^1)$ the diagram

$$\begin{array}{ccc} H^b(X, \Omega_X^a(\log\ D) \otimes \mathcal{V}) & \xrightarrow{\ \iota\ } & H^a(X, \Omega^b(\log\ D) \otimes \mathcal{V}^*(-D_{\mathrm{red}})) \\ {\scriptstyle \cup\varphi}\big\downarrow & & {\scriptstyle \wedge\omega}\big\downarrow \\ H^{b+1}(X, \Omega_X^a(\log\ D) \otimes \mathcal{V}) & \xrightarrow{\ \iota\ } & H^a(X, \Omega_X^{b+1}(\log\ D) \otimes \mathcal{V}^*(-D_{\mathrm{red}})) \end{array}$$

commutes.

13.6. Examples.
a) If $D = 0$ and if $\mathcal{N}_y$ is the invertible sheaf corresponding to $y \in \mathrm{Pic}^0(X)$, then $\mathcal{N}_y$ has an integrable connection ∇, whose kernel is a unitary rank one local constant system. In this case (13.5) is wellknown and proven by the usual arguments from classical Hodge-theory, applied to $\mathcal{N}_y \otimes \mathcal{N}_y^*$ see [11].
b) If the sheaf $\mathcal{V}$ in (13.5) is of the form $\mathcal{V} = \mathcal{L}^{(i)^{-1}}$ for

$$D = \sum_{j=1}^{r} \alpha_j D_j \quad \text{and} \quad \mathcal{L}^N = \mathcal{O}_X(D),$$

then the assumptions made in (13.5) are satisfied whenever

$$\frac{i \cdot \alpha_j}{N} \notin \mathbb{Z} \ \text{ for } \ j = 1, ..., r.$$

Hence, using the notations from (3.2), one has $D^{(i)} = D^{(N-i)} = D_{red}$ and

$$\mathcal{L}^{(i)}(-D_{red}) = \mathcal{L}^i(-[\frac{i \cdot D}{N}] - D_{red}) = \mathcal{L}^i(-\{\frac{i \cdot D}{N}\}) =$$

$$\mathcal{L}^i([\frac{-i \cdot D}{N}]) = \mathcal{L}^{i-N}([\frac{(N-i) \cdot D}{N}]) = \mathcal{L}^{(N-i)^{-1}}$$

Hence (3.2) and (3.23) imply (13.5) for $\mathcal{M} = \mathcal{L}^{(i)^{-1}}$.
c) Finally, for $\mathcal{M} = \mathcal{L}^{(i)^{-1}} \otimes \mathcal{N}_y$ one can use (13.6,a) on the finite covering Y of X obtained by taking the N−th root out of D and the arguments used to prove (3.23) imply (13.5)in that case.

13.7. Corollary (H. Dunio [14]). *Keeping the assumptions made in (13.5) and the notations introduced in (13.1) and (12.5) one has*

$$\mathrm{codim}_{\mathrm{Pic}_0(X)}(S^b(X, \mathcal{V})) \geq \dim(\alpha(X)) - b.$$

PROOF: Again, it is sufficient to give a lower bound for

$$\mathrm{codim}(\Gamma \subset H^1(X, \mathcal{O}_X))$$

where

$$\Gamma = \{\varphi \in H^1(X, \mathcal{O}_X);\ \beta \cup \varphi = 0 \ \text{ for all } \ \beta \in H^b(X, \mathcal{N}_y \otimes \mathcal{V})\},$$

or using (13.5,b), for

$$\mathrm{codim}(\bar{\Gamma} \subset H^0(X, \Omega^1_X))$$

where

$$\bar{\Gamma} = \{\omega \in H^0(X, \Omega^1_X); \beta \wedge \omega = 0 \ \text{ for all}$$
$$\beta \in H^0(X, \Omega^b_X(\log\ D) \otimes \mathcal{N}_y^* \otimes \mathcal{V}^*(-D_{red}))\}.$$

As in (13.3), if

$$\gamma : H^0(X, \Omega^1_X) \otimes \mathcal{O}_X \longrightarrow \Omega^1_X$$

is the natural map, one has

$$r = \operatorname{rank}(\gamma(\bar{\Gamma} \otimes \mathcal{O}_X)) \geq \dim \alpha(X) - \operatorname{codim}(\bar{\Gamma} \subset H^0(X, \Omega^1_X)).$$

Assume that one has some

$$0 \neq \beta \in H^0(X, \Omega^b_X(\log\ D) \otimes \mathcal{N}^*_y \otimes \mathcal{V}^*(-D_{red})).$$

Let $v_1, ..., v_s$ be a basis of

$$\mathcal{N}^*_y \otimes \mathcal{V}^*(-D_{red}) \otimes \mathbb{C}(X),$$

then one has $\beta = \sum_{i=1}^s \beta_i v_i$ for some $\beta_i \in \Omega^b_X \otimes \mathbb{C}(X)$ and $\beta_i \wedge \omega = 0$ for all i and all

$$\omega \in \gamma(\bar{\Gamma} \otimes \mathcal{O}_X) \otimes \mathbb{C}(X).$$

As in (13.3) this is only possible if $b \geq r$.

$\square$

13.8. Example. If $\mathcal{L}$ is an invertible sheaf on X and if D is a normal crossing divisor let $N \in \mathbb{N}$ be larger than the multiplicities of the components of D. If $\mathcal{L}^N = \mathcal{O}_X(D)$ then $\mathcal{V} = \mathcal{L}^{-1}$ satisfies the assumptions made in (13.7) and

$$H^b(X, \mathcal{L}^{-1} \otimes \mathcal{N}_y) = 0$$

for $y \in \operatorname{Pic}^0(X)$ in general position and $b < \dim \alpha(X)$.
On the other hand, (5.12,e) tells us, that

$$H^b(X, \mathcal{L}^{-1} \otimes \mathcal{N}_y) = 0$$

for all $y \in \operatorname{Pic}^0(X)$ and $b < \kappa(\mathcal{L})$. Hence (13.7) is only of interest if

$$\dim \alpha(X) - \kappa(\mathcal{L}) > 0.$$

In this situation the bounds given in (13.7) can be improved. The generic vanishing theorem remains true for

$$b < \dim \alpha(X) + \kappa(\mathcal{L}) - \dim \alpha'(Z)$$

where Z is a desingularization of the image of the rational map

$$\Phi_\nu : X \longrightarrow \mathbb{P}(H^0(X, \mathcal{L}^\nu))$$

for ν sufficiently large (see (5.3)) and where

$$\alpha' : Z \longrightarrow \operatorname{Alb}(Z)$$

is the Albanese map of Z. To be more precise:

13.9. Assumptions and Notations. Let X be a complex projective manifold, let $\mathcal{L}$ be an invertible sheaf on X, let

$$D = \sum_{j=1}^{r} \alpha_j D_j$$

be a normal crossing divisor and let N be a natural number with

$$0 < \alpha_j < N \quad \text{for} \quad j = 1, ..., r.$$

Assume that either $\mathcal{L}^N(-D)$ is semi-ample or that (more generally) $\mathcal{L}^N(-D)$ is numerically effective and

$$\kappa(\mathcal{L}^N(-D)) = \nu(\mathcal{L}^N(-D))$$

(see (5.9) and (5.11)). For some $\mu > 0$ the rational map (see (5.3))

$$\Phi_\mu : X \longrightarrow \Phi_\mu(X) \subset \mathbb{P}(H^0(X, \mathcal{L}^\mu))$$

has an irreducible general fibre and $\dim(\Phi_\mu(X)) = \kappa(\mathcal{L})$. For such a μ let Z be a desingularization of $\Phi_\mu(X)$ and X' a blowing up of X such that the induced rational map

$$\Phi' : X' \longrightarrow Z$$

is a morphism of manifolds. Φ'^* defines a morphism

$$\Phi^* : \mathrm{Pic}^0(Z) \longrightarrow \mathrm{Pic}^0(X') = \mathrm{Pic}^0(X)$$

and $\Phi^*(\mathrm{Pic}^0(Z))$ is an abelian subvariety of $\mathrm{Pic}^0(X)$ independent of the desingularization choosen. Let

$$\alpha : X \longrightarrow \mathrm{Alb}(X) \quad \text{and} \quad \alpha' : Z \longrightarrow \mathrm{Alb}(Z)$$

be the Albanese maps.

13.10. Theorem (H. Dunio [14]). *Under the assumptions made in (13.9) one has:*

a) $S^b(X, \mathcal{L}^{-1}) = 0$ *for* $b < \kappa(\mathcal{L})$.

b) $S^b(X, \mathcal{L}^{-1})$ *lies in the subgroup of* $\mathrm{Pic}^0(X)$, *which is generated by torsion elements and by* $\Phi^*(\mathrm{Pic}^0(Z))$, *for* $b = \kappa(\mathcal{L})$.

c) $\mathrm{codim}_{\mathrm{Pic}^0(X)}(S^b(X, \mathcal{L}^{-1})) \geq \dim\ \alpha(X) - \dim\ \alpha'(Z) + \kappa(\mathcal{L}) - b$.

PROOF: a) is nothing but (5.12,e) and it has already be shown twice in these notes. Nevertheless, when we prove (13.10,c) it will come out again.

First of all, since $S^b(X, \mathcal{L}^{-1})$ is compatible with blowing ups of X, we may assume that the rational map $\Phi : X \to Z$ is a morphism. Moreover, as in the proof of (5.12), we can assume that $\mathcal{L}^N(-D)$ is semi-ample or even, replacing N and D by some common multiple, that

$$\mathcal{L}^N(-D) = \mathcal{O}_X(H)$$

where H is a non singular divisor and $D + H$ a normal crossing divisor. Since

$$\mathcal{L} = \mathcal{L}^{(1,D)} = \mathcal{L}^{(1,D+H)}$$

we can as well assume that $\mathcal{L}^N = \mathcal{O}_X(D)$. By (13.5,b) or (13.6,c) the space

$$H^b(X, \mathcal{L}^{-1} \otimes \mathcal{N}_y)$$

is Hodge dual to

$$H^0(X, \Omega^b_X(\log\ D) \otimes \mathcal{L}^{(N-1)^{-1}} \otimes \mathcal{N}_y^{-1}).$$

Let $\mathcal{G}^b_\Phi \hookrightarrow \Omega^b_X(\log\ D)$ be the largest subsheaf which over some open non empty subvariety of X coincides with

$$\Phi^*\Omega^\kappa_Z \wedge \Omega^{b-\kappa}_X(\log\ D).$$

Of course $\mathcal{G}^b_\Phi = 0$ for $b < \kappa = \kappa(\mathcal{L})$ and $\delta \in \mathcal{G}^b_\Phi \otimes \mathbb{C}(X)$ if and only if δ is a meromorphic b-form with $\delta \wedge \Phi^*\Omega^1_Z = 0$.

Since $\mathcal{L}^{(N-1)} \subseteq \mathcal{L}^{N-1}$ and since

$$(\mathcal{L}^{(N-1)})^N = \mathcal{O}_X(N \cdot D_{red} - D),$$

we have $\kappa(\mathcal{L}^{(N-1)}) = \kappa(\mathcal{L})$ and $(\mathcal{L}^{(N-1)})^\mu$ contains $\mathcal{L}^N$ for some $\mu > 0$.

13.11. Claim. If $\mathcal{M}$ is an invertible sheaf such that $\mathcal{M}^\mu$ contains $\mathcal{L}^N$ for some $\mu > 0$, then

$$H^0(X, \mathcal{G}^b_\Phi \otimes \mathcal{M}^{-1} \otimes \mathcal{N}_y^{-1}) = H^0(X, \Omega^b_X(\log\ D) \otimes \mathcal{M}^{-1} \otimes \mathcal{N}_y^{-1}).$$

PROOF: The methods used to prove (13.11) are due to F. Bogomolov [6].
A section $\beta \in H^0(X, \Omega^b_X(\log\ D) \otimes \mathcal{M}^{-1} \otimes \mathcal{N}_y^{-1})$ gives an inclusion

$$\beta : \mathcal{M} \longrightarrow \Omega^b_X(\log\ D) \otimes \mathcal{N}_y^{-1}$$

and we have to show that $\Phi^*\Omega^1_Z \wedge \beta(\mathcal{M}) = 0$.

If $\tau : X' \longrightarrow X$ is generically finite and $D' = \tau^* D$ a normal crossing divisor, then $\tau^*\Omega^b_X(\log\ D)$ is a subsheaf of $\Omega^b_{X'}(\log\ D')$. In fact, if (locally) D' is the zero set of $x'_1, ..., x'_r$ and if x is a local parameter on X defining one component of D, then

$$\tau^* x = \prod_{j=1}^{r} {x'}_j^{\nu_j} \quad \text{and} \quad \tau^* \frac{dx}{x} = \sum_{j=1}^{r} \nu_j \frac{dx'_j}{x'_j} \in \Omega^b_{X'}(\log\ D').$$

Hence β induces

$$\beta' : \tau^*\mathcal{M} \longrightarrow \Omega^b_{X'}(\log\ D') \otimes \tau^*(\mathcal{N}_y^{-1}).$$

Since

$$\beta'(\tau^*\mathcal{M}) \wedge \tau^*\Phi^*\Omega^1_Z = 0$$

implies that

$$\beta(\mathcal{M}) \wedge \Phi^*\Omega^1_Z = 0,$$

we can replace X by X' whenever we like. For example, if

$$s_0, ..., s_\kappa \in H^0(X, \mathcal{L}^\nu) \subset H^0(X, \mathcal{M}^\mu)$$

are choosen such that the functions

$$\frac{s_1}{s_0}, ..., \frac{s_\kappa}{s_0}$$

are algebraic independent, we can take X' as a desingularization of the covering obtained by taking the μ-th root out of $s_0, s_1, ..., s_\kappa$. Hence to prove (13.11) we may assume that $\mathcal{M}$ itself has sections

$$s_0, ..., s_\kappa \quad \text{with} \quad f_1 = \frac{s_1}{s_0}, ..., f_\kappa = \frac{s_\kappa}{s_0}$$

algebraic independent. From (13.5) we know that $d(\beta(s_i)) = 0$, which by the Leibniz rule implies

$$0 = d(\beta(s_i)) = d(f_i \cdot \beta(s_0)) = d(f_i) \wedge \beta(s_0).$$

However, $d(f_1), ..., d(f_\kappa)$ are generators of $\Phi^*\Omega^1_Z$ over some non empty open subset.

□

Part a) of (13.10) follows from (13.11) since for $b < \kappa$ the sheaf $\mathcal{G}^b_\Phi = 0$.

If $b = \kappa$ then

$$\mathcal{G}^b_\Phi = \Phi^*\omega_Z \otimes \mathcal{O}_X(\Delta)$$

for some effective divisor Δ on X, not meeting the general fibre F of Φ. Hence, for $y \in S^\kappa(X, \mathcal{L}^{-1})$ (13.11) implies that $\mathcal{L}^{(N-1)^{-1}} \otimes \mathcal{N}_y^{-1}|_F$ has a non trivial section and therefore $\mathcal{N}_y^{-1}|_F = \mathcal{L}^{(N-1)}|_F$. The divisor $D + H$ does not meet the general fibre F and, as we claimed in (13.10,b), $N \cdot y \in \Phi^*(\mathrm{Pic}^0(Z))$.

13.12. Remark. If $\mathcal{L}$ is semi-ample and $b > \kappa$, then a similar argument shows that $\mathcal{L}^{(N-1)^{-1}} \otimes \mathcal{N}_y^{-1}|_F \otimes \Omega_F^{b-\kappa}$ has a non trivial section. This implies, as we have seen in the proof of (13.2), that those $\mathcal{N}_y|_F$ are corresponding to points y in a subvariety of $\mathrm{Pic}^0(F)$ of codimension larger than or equal to

$$\dim(\alpha(F)) - b + \kappa = \dim(\alpha(X)) - \dim(\alpha(Z)) - b + \kappa,$$

which gives (13.10,c).

We instead generalise the argument used in step 2 of the proof of (13.3):
If

$$\Gamma = \{\varphi \in H^1(X, \mathcal{O}_X);\ \beta \cup \varphi = 0 \ \text{ for all } \ \beta \in H^b(X, \mathcal{L}^{-1} \otimes \mathcal{N}_y)\}$$

then the Hodge dual of Γ is

$$\bar{\Gamma} = \{\omega \in H^0(X, \Omega_X^1);\ \beta \cup \omega = 0 \ \text{ for all } \ \beta \in H^0(X, \mathcal{G}_\Phi^b \otimes \mathcal{L}^{(N-1)^{-1}} \otimes \mathcal{N}_y^{-1})\}.$$

If γ is the composed map

$$H^0(X, \Omega_X^1) \otimes \mathcal{O}_X \longrightarrow \Omega_X^1 \longrightarrow \Omega_{X/Z}^1$$

then

$$H^0(X, \mathcal{G}_\Phi^b \otimes \mathcal{L}^{(N-1)^{-1}} \otimes \mathcal{N}_y^{-1}) \neq 0$$

implies that

$$\gamma(\bar{\Gamma} \otimes \mathcal{O}_X) \wedge \bar{\beta} = 0 \ \text{ for some } \ \bar{\beta} \in \Omega_{X/Z}^{b-\kappa} \otimes \mathbb{C}(X)$$

or, in other terms, that

$$\Omega_{X/Z}^{n-b} \otimes \mathbb{C}(X) \neq \gamma(\bar{\Gamma} \otimes \mathcal{O}_X) \wedge \Omega_{X/Z}^{n-b-1} \otimes \mathbb{C}(X).$$

Again this is only possible for

$$n - b \leq n - \kappa - \mathrm{rank}(\gamma(\bar{\Gamma} \otimes \mathcal{O}_X))$$

or

$$b - \kappa \geq \mathrm{rank}(\gamma(\bar{\Gamma} \otimes \mathcal{O}_X)).$$

However,

$$\mathrm{rank}(\gamma(\bar{\Gamma} \otimes \mathcal{O}_X)) \geq \dim(\alpha(X)) - \dim(\alpha'(Z)) - \mathrm{codim}(\bar{\Gamma} \subset H^0(X, \Omega_X^1))$$

and (13.10,c) follows from (12.7)

$\square$

13.13. Remarks.
a) If $Z(\varphi)$ denotes the zero locus of a global one-form φ and

$$w(X) = \mathrm{Max}\{\mathrm{codim}_X Z(\varphi); \varphi \in H^0(X, \Omega^1_X)\},$$

then a second result of Green and Lazarsfeld [26] says that, for a generic line bundle

$$\mathcal{N} \in \mathrm{Pic}^0(X) \quad \text{and} \quad a + b < w(X),$$

one has

$$H^b(X, \Omega^a_X \otimes \mathcal{N}) = 0.$$

b) In [27] Green and Lazarsfeld obtain moreover a more explicit description of the subvarieties $S^b(X)$ of $\mathrm{Pic}^0(X)$. They show that the irreducible components of $S^b(X)$ are translates of subtori of $\mathrm{Pic}^0(X)$. This description generalizes results due to A. Beauville [5], who studied $S^1(X)$ and showed the same result in this case.

c) Finally, C. Simpson recently gave in [58] a complete description of the $S^b(X)$ and similar "degeneration loci". In particular he showed that the components of $S^b(X)$ are even translates of subtori of $\mathrm{Pic}^0(X)$ by points of finite order, a result conjectured and proved for $b = 1$ by A. Beauville.

d) Writing these notes we would have liked to prove the generic vanishing theorems for invertible sheaves in the algebraic language used in the first part. However, we were not able to replace the use of Hodge duality by some algebraic argument.

APPENDIX: Hypercohomology and spectral sequences

1.

In this appendix, we list some formal properties of cohomology of complexes that we are using throughout these notes. However we do not pretend making a complete account on this topic. In particular, we avoid the use of the derived category, which is treated to a broad extend in the literature (see [60], [29], [7], [8], [33], [31]).

2.

Through this section X is a variety over a commutative ring k.

3.

We consider *complexes* $\mathcal{F}^\bullet$ *of sheaves* of $\mathcal{O}$-modules, where $\mathcal{O}$ is a sheaf of commutative rings. For example

$$\mathcal{O} = \mathbb{Z}\ ,\ \mathcal{O} = k \ \text{ or } \ \mathcal{O} = \ \text{structure sheaf of } X.$$

Any map of $\mathcal{O}$-modules

$$\sigma : \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet$$

between two such complexes induces a map of *cohomology sheaves*:

$$\mathcal{H}^i(\sigma) : \mathcal{H}^i(\mathcal{F}^\bullet) \longrightarrow \mathcal{H}^i(\mathcal{G}^\bullet)$$

where $\mathcal{H}^i(\mathcal{F}^\bullet)$ is the sheaf associated to the presheaf

$$U \mapsto \frac{\ker\ \Gamma(U,\mathcal{F}^i) \to \Gamma(U,\mathcal{F}^{i+1})}{\operatorname{im}\ \Gamma(U,\mathcal{F}^{i-1}) \to \Gamma(U,\mathcal{F}^i)}$$

in the given topology.
One says that σ is a *quasi-isomorphism* if $\mathcal{H}^i(\sigma)$ is an isomorphism for all i.

4.

We will only consider complexes $\mathcal{F}^\bullet$ which are *bounded below*, that means $\mathcal{F}^i = 0$ for i sufficiently negative.

5. Example: the analytic de Rham complex.

X is a complex manifold. Then the standard map

$$\mathbb{C} \longrightarrow \ \left(\ \mathcal{O}_X \to \Omega^1_X \to \Omega^2_X \to \Omega^3_X \to \cdots\ \right)$$

from the constant sheaf $\mathbb{C}$ to the analytic de Rham complex $\Omega^\bullet_X$ is a quasi-isomorphism as by the so called "Poincaré lemma"

$$\mathcal{H}^i(\Omega^\bullet_X) = 0 \quad \text{for} \quad i > 0, \text{ and } \mathcal{H}^0(\Omega^\bullet_X) = \mathbb{C}.$$

6. Example: the Čech complex.

Let $\mathcal{U} = \{U_\alpha; \alpha \in A\}$, for $A \subset \mathbb{N}$, be some open covering of the variety X defined over k. To a bounded below complex $\mathcal{F}^\bullet$ one associates its Čech complex $\mathcal{G}^\bullet$ defined as follows.

$$\mathcal{G}^i := \bigoplus_{a \geq 0} \mathcal{C}^a(\mathcal{U}, \mathcal{F}^{i-a})$$

where

$$\mathcal{C}^a(\mathcal{U}, \mathcal{F}^{i-a}) = \prod_{\alpha_0 < \alpha_1 < \ldots < \alpha_a} \varrho_* \mathcal{F}^{i-a}|_{U_{\alpha_0 \cdots \alpha_a}}.$$

Here, for any

$$U_{\alpha_0 \ldots \alpha_a} := U_{\alpha_0} \cap \ldots \cap U_{\alpha_a}$$

ϱ denotes the open embedding

$$\varrho = \varrho_{\alpha_0 \ldots \alpha_a} : U_{\alpha_0 \ldots \alpha_a} \longrightarrow X,$$

and for any sheaf $\mathcal{F}$, one writes $\varrho_* \mathcal{F}$ for the sheaf associated to the presheaf

$$U \mapsto \Gamma(U_{\alpha_0 \ldots \alpha_a} \cap U, \mathcal{F}).$$

As $\mathcal{F}^i = 0$ for $i << 0$, the direct sum in the definition of $\mathcal{G}$ has finitely many summands.

The differential Δ of $\mathcal{G}^\bullet$ is defined by

$$\Delta(s) = (-1)^i \delta s + d_{\mathcal{F}^\bullet} s \quad \text{for} \quad s \in \mathcal{C}^a(\mathcal{U}, \mathcal{F}^{i-a}),$$

where δ is the Čech differential defined by

$$(\delta s)_{\alpha_0 \ldots \alpha_{a+1}} = \sum_0^{a+1} (-1)^l s_{\alpha_0 \ldots \hat{\alpha}_l \ldots \alpha_{a+1}}|_{U_{\alpha_0 \cdots \alpha_{a+1}}}$$

and $d_{\mathcal{F}^\bullet}$ is the differential of $\mathcal{F}^\bullet$. Then the natural map

$$\sigma : \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet$$

defined by

$$\mathcal{F}^i \xrightarrow{\varrho} \prod_{\alpha \in A} \varrho_* \mathcal{F}^i|_{U_\alpha} = \mathcal{C}^0(\mathcal{U}, \mathcal{F}^i)$$

is a quasi-isomorphism.

To show this one considers first a single sheaf $\mathcal{F}$ (see [30], III 4.2) and then one computes that, whenever one has a double complex

$$\begin{tikzcd}
 & & & & & \\
{} \arrow[r] & \mathcal{K}^{1,i-1} \arrow[r] \arrow[u] & \mathcal{K}^{1,i} \arrow[r] \arrow[u] & \mathcal{K}^{1,i+1} \arrow[r] \arrow[u] & {} & \\
{} \arrow[r] & \mathcal{K}^{0,i-1} \arrow[r] \arrow[u] & \mathcal{K}^{0,i} \arrow[r] \arrow[u] & \mathcal{K}^{0,i+1} \arrow[r] \arrow[u] & {} \arrow[r, phantom] & {} \arrow[u, "d_{vert}"'] \\
{} \arrow[r] & \mathcal{F}^{i-1} \arrow[r] \arrow[u] & \mathcal{F}^{i} \arrow[r] \arrow[u] & \mathcal{F}^{i+1} \arrow[r] \arrow[u] & \ldots & {} \arrow[r, "d_{hor}"'] & {}
\end{tikzcd}$$

such that $\mathcal{F}^i \longrightarrow \mathcal{K}^{\bullet,i}$ is a quasi-isomorphism for all i, then $\mathcal{F}^\bullet \longrightarrow \mathcal{D}^\bullet$ is a quasi-isomorphism as well, where $\mathcal{D}^\bullet$ is the associated *double complex*:

$$\mathcal{D}^i := \bigoplus_a \mathcal{K}^{a,i-a} \longrightarrow \mathcal{D}^{i+1} := \bigoplus_a \mathcal{K}^{a,i+1-a}$$

with differential $(-1)^i d_{vert} + d_{hor}$.

7.

If $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are two complexes of $\mathcal{O}$-modules bounded below one defines the *tensor product* $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$ by

$$(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)^i := \bigoplus_a \mathcal{F}^a \otimes \mathcal{G}^{i-a}$$

with differential from

$$\mathcal{F}^a \otimes \mathcal{G}^{i-a} \quad \text{to} \quad \mathcal{F}^{a+1} \otimes \mathcal{G}^{i-a} \oplus \mathcal{F}^a \otimes \mathcal{G}^{i+1-a}. \ \text{ given by}$$

$$d(f_a \otimes g_{i-a}) = df_a \otimes g_{i-a} + (-1)^a f_a \otimes dg_{i-a}.$$

As both $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are bounded below, a takes finitely many values, and $(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)$ is a complex of $\mathcal{O}$-modules bounded below.

8.

If in 7 we assume moreover that *locally* the $\mathcal{O}$-modules $\mathcal{F}^a$ and $\mathcal{G}^b$ are *free* and $d\mathcal{F}^{a-1}$ as well as $d\mathcal{G}^{b-1}$ are subbundles for all a and b, then locally one has some decomposition

$$\begin{array}{ccccccc}
\mathcal{F}^a & = & d\mathcal{F}^{a-1} & \oplus & \mathcal{H}^a(\mathcal{F}^\bullet) & \oplus & \mathcal{F}'^a \\
\mathcal{G}^b & = & d\mathcal{G}^{b-1} & \oplus & \mathcal{H}^b(\mathcal{G}^\bullet) & \oplus & \mathcal{G}'^b
\end{array}$$

where

$$\begin{array}{cccc}
d: & \mathcal{F}'^a & \longrightarrow & d\mathcal{F}^a \\
d: & \mathcal{G}'^b & \longrightarrow & d\mathcal{G}^b
\end{array}$$

are isomorphisms. In particular,

$$\begin{array}{rcl}\mathcal{F}^a \otimes \mathcal{G}^b & = & d(\mathcal{F}^{a-1} \otimes d\mathcal{G}^{b-1} + \mathcal{F}^{a-1} \otimes \mathcal{H}^b + \mathcal{H}^a \otimes \mathcal{G}^{b-1}) \\ & \oplus & \mathcal{H}^a(\mathcal{F}^\bullet) \otimes \mathcal{H}^b(\mathcal{G}^\bullet) \\ & \oplus & (d\mathcal{F}^{a-1} + \mathcal{H}^a(\mathcal{F}^\bullet)) \otimes \mathcal{G}'^b \\ & \oplus & \mathcal{F}'^a \otimes (d\mathcal{G}^{b-1} + \mathcal{H}^b(\mathcal{G}^\bullet)) \\ & \oplus & \mathcal{F}'^a \otimes \mathcal{G}'^b\end{array}$$

and therefore one has the Künneth decomposition

$$\mathcal{H}^i(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet) = \bigoplus_a \mathcal{H}^a(\mathcal{F}^\bullet) \otimes \mathcal{H}^{i-a}(\mathcal{G}^\bullet).$$

9.

The map $\sigma : \mathcal{F}^\bullet \longrightarrow \mathcal{I}^\bullet$ is called an *injective resolution of* $\mathcal{F}^\bullet$ if $\mathcal{I}^\bullet$ is a complex of $\mathcal{O}$-modules bounded below, σ is a quasi-isomorphism, and the sheaves $\mathcal{I}^i$ are injective for all i, i.e.:

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{B}, \mathcal{I}^i) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{A}, \mathcal{I}^i)$$

is surjective for any injective map $\mathcal{A} \longrightarrow \mathcal{B}$ of sheaves of $\mathcal{O}$-modules. It is an easy fact that if $\mathcal{O}$ is a constant commutative ring, for example $\mathcal{O} = \mathbb{Z}$, $\mathcal{O} = k$, then every complex of $\mathcal{O}$-modules which is bounded below admits an injective resolution (see [33], (6.1)).

10.

From now on we assume that $\mathcal{O}$ is a constant commutative ring. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}$-modules, bounded below. One defines the *hypercohomology group* $\mathbb{H}^a(X, \mathcal{F}^\bullet)$, to be the $\mathcal{O}$-module

$$\mathbb{H}^a(X, \mathcal{F}^\bullet) := \frac{\ker\ \Gamma(X, \mathcal{I}^a) \to \Gamma(X, \mathcal{I}^{a+1})}{\mathrm{im}\ \Gamma(X, \mathcal{I}^{a-1}) \to \Gamma(X, \mathcal{I}^a)}.$$

One verifies that this definition does not depend on the injective resolution choosen (see [30] III, 1.0.8).

In particular, if $\sigma : \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet$ is a *quasi-isomorphism*, then σ induces an *isomorphism of the hypercohomology groups*:

$$\mathbb{H}^a(X, \mathcal{F}^\bullet) \xrightarrow{\sim} \mathbb{H}^a(X, \mathcal{G}^\bullet)$$

(by taking an injective resolution $\mathcal{I}^\bullet$ of $\mathcal{G}^\bullet$ which is also an injective resolution of $\mathcal{F}^\bullet$).

11.

By definition, $H^a(X, \mathcal{I}) = 0$ for $a > 0$ if $\mathcal{I}$ is an injective sheaf. We will verify in (A.28) that if $\sigma : \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet$ is a quasi-isomorphism and if $H^a(X, \mathcal{G}^i) = 0$

for all $a > 0$ and all i, then

$$\mathbb{H}^a(X, \mathcal{F}^\bullet) = \frac{\ker\ \Gamma(X, \mathcal{G}^a) \to \Gamma(X, \mathcal{G}^{a+1})}{\mathrm{im}\ \Gamma(X, \mathcal{G}^{a-1}) \to \Gamma(X, \mathcal{G}^a)}.$$

We call $(\mathcal{G}^\bullet, \sigma)$ an *acyclic resolution* of $\mathcal{F}^\bullet$ in this case.

12.

$\mathbb{H}$ transforms *short exact sequences*

$$0 \longrightarrow \mathcal{A}^\bullet \longrightarrow \mathcal{B}^\bullet \longrightarrow \mathcal{C}^\bullet \longrightarrow 0$$

of complexes of $\mathcal{O}$-modules which are bounded below into *long exact sequences*

$$\cdots \longrightarrow \mathbb{H}^i(\mathcal{A}^\bullet) \longrightarrow \mathbb{H}^i(\mathcal{B}^\bullet) \longrightarrow \mathbb{H}^i(\mathcal{C}^\bullet) \longrightarrow \mathbb{H}^{i+1}(\mathcal{A}^\bullet) \longrightarrow \dots$$

of $\mathcal{O}$-modules.

13.

We assume now that the complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules, bounded below, has subcomplexes

$$\ldots \subset Filt_{i-1} \ \subset Filt_i \subset \ldots \subset \mathcal{F}^\bullet$$

$$(\text{or} \quad \ldots \subset Filt^i \subset Filt^{i-1} \subset \ldots \subset \mathcal{F}^\bullet)$$

such that

$$\textstyle\bigcup_i \ Filt_i = \mathcal{F}^\bullet$$

$$(\text{or} \quad \textstyle\bigcup_i \ Filt^i = \mathcal{F}^\bullet)$$

and such that

$$Filt_i = 0 \ \text{ for } \ i << 0$$

$$(\text{or} \quad Filt^i = 0 \ \text{ for } \ i \gg 0).$$

One says that $\mathcal{F}^\bullet$ is *filtered* by the subcomplexes $Filt_i$ (or $Filt^i$). Via (A.12), this filtration defines a *filtration on the hypercohomology groups*:

$$Filt_i\mathbb{H}^a(X, \mathcal{F}^\bullet) := \ \mathrm{im}(\mathbb{H}^a(X, Filt_i) \longrightarrow \mathbb{H}^a(X, \mathcal{F}^\bullet))$$

$$(\text{or} \ \ Filt^i\mathbb{H}^a(X, \mathcal{F}^\bullet) := \ \mathrm{im}(\mathbb{H}^a(X, Filt^i) \longrightarrow \mathbb{H}^a(X, \mathcal{F}^\bullet))).$$

This just means that the group $\mathbb{H}^a(X, \mathcal{F}^\bullet)$ has subgroups

$$\ldots \subset Filt_{i-1}\mathbb{H}^a(X, \mathcal{F}^\bullet) \subset Filt_i\mathbb{H}^a(X, \mathcal{F}^\bullet) \subset \ldots \subset \mathbb{H}^a(X, \mathcal{F}^\bullet)$$

$$(\text{or} \quad \ldots \subset Filt^i\mathbb{H}^a(X, \mathcal{F}^\bullet) \subset Filt^{i-1}\mathbb{H}^a(X, \mathcal{F}^\bullet) \subset \ldots \subset \mathbb{H}^a(X, \mathcal{F}^\bullet)).$$

We pass from *increasing* to *decreasing* filtrations by setting

$$Filt_i := Filt^{-i}.$$

14.

We define

$$Gr_i := Filt_i / Filt_{i-1}.$$

One has a diagram of exact sequences

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & Filt_{i-1} \arrow[r] \arrow[d] & Filt_i \arrow[r] \arrow[d] & Gr_i \arrow[r] & 0 \\
 & & \mathcal{F}^\bullet \arrow[r, "\text{identity}"] \arrow[d] & \mathcal{F}^\bullet \arrow[d] & & \\
0 \arrow[r] & Gr_i \arrow[r] & \mathcal{F}^\bullet / Filt_{i-1} \arrow[r] \arrow[d] & \mathcal{F}^\bullet / Filt_i \arrow[r] \arrow[d] & 0 & \\
 & & 0 & 0 & &
\end{tikzcd}$$

which gives via (A.12):

$$\begin{aligned}
Gr_i(Filt_\bullet \mathbb{H}^a(\mathcal{F}^\bullet)) &:= Filt_i \mathbb{H}^a(\mathcal{F}^\bullet)/Filt_{i-1}\mathbb{H}^a(\mathcal{F}^\bullet) \\
&= \frac{\mathbb{H}^a(Filt_i)}{\mathbb{H}^a(Filt_{i-1}) + \mathbb{H}^{a-1}(\mathcal{F}^\bullet / Filt_i)} \\
&= \frac{\ker(\mathbb{H}^a(Gr_i) \to \mathbb{H}^{a+1}(Filt_{i-1}))}{\ker(\mathbb{H}^a(Gr_i) \to \mathbb{H}^a(\mathcal{F}^\bullet / Filt_{i-1}))}.
\end{aligned}$$

We define

$$E_\infty^{a-i,i} := Gr_i(Filt_\bullet \mathbb{H}^a(\mathcal{F}^\bullet)) \quad \text{and} \quad E_2^{a-i,i} := \mathbb{H}^a(Gr_i).$$

Obviously $E_\infty^{a-i,i}$ is a *subquotient* of $E_2^{a-i,i}$.

15.

The formations of *spectral sequences* has the aim to compute $E_\infty^{a-i,i}$ only in terms of $E_2^{s,t}$, by filtering the terms $\mathbb{H}^{a+1}(Filt_{i-1})$ and $\mathbb{H}^a(\mathcal{F}^\bullet / Filt_{i-1})$ appearing in the description (A.14) by the induced filtrations:

$$Filt_l \ \mathbb{H}^{a+1}(Filt_{i-1}) := \operatorname{im}(\mathbb{H}^{a+1}(Filt_l) \longrightarrow \mathbb{H}^{a+1}(Filt_{i-1}))$$

for $l \leq i-1$ and

$$Filt_l\ \mathbb{H}^a(\mathcal{F}^\bullet/Filt_{i-1}) := \mathrm{im}(\mathbb{H}^a(Filt_l/Filt_{i-1}) \longrightarrow \mathbb{H}^a(\mathcal{F}^\bullet/Filt_{i-1}))$$

for $l \geq i-1$, and by computing the corresponding graded quotients.

16.

If we assume that for some a, the filtration $Filt_i\mathbb{H}^a(\mathcal{F}^\bullet)$ is *exhausting*, that is:

$$\bigcup_i Filt_i\mathbb{H}^a(\mathcal{F}^\bullet) = \mathbb{H}^a(\mathcal{F}^\bullet),$$

then

$$Gr\ \mathbb{H}^a(\mathcal{F}^\bullet) = \bigoplus_i E_\infty^{a-i,i}$$

is the corresponding graded group. In particular, assume that $\mathbb{H}^a(\mathcal{F}^\bullet)$ and $E_\infty^{a-i,i}$ are *free $\mathcal{O}$-modules* (where $\mathcal{O}$ is $\mathbb{Z}$ or k), and that $\mathbb{H}^a(\mathcal{F}^\bullet)$ is of finite rank. Then

$$\mathrm{rank}_{\mathcal{O}}\mathbb{H}^a(\mathcal{F}^\bullet) = \sum_i \mathrm{rank}_{\mathcal{O}}E_\infty^{a-i,i}.$$

If $E_2^{a-i,i}$ is also free, then $\mathrm{rank}_{\mathcal{O}}\mathbb{H}^a(\mathcal{F}^\bullet) \leq \sum_i \mathrm{rank}_{\mathcal{O}}E_2^{a-i,i}$ and one has equality if and only if

$$E_\infty^{a-i,i} = E_2^{a-i,i} \quad \text{for all} \quad i.$$

17. Example: One step filtration:

$$0 = Filt_{s-1} \subset Filt_s = \mathcal{F}^\bullet.$$

Then $\mathbb{H}^a(\mathcal{F}^\bullet) = \mathbb{H}^a(Gr_s) = E_\infty^{a-s,s} = E_2^{a-s,s}$.

18. Example: Two steps filtration:

$$0 = Filt_{s-2} \subset Filt_{s-1} \subset Filt_s = \mathcal{F}^\bullet$$

Then one has the exact sequence

$$0 \longrightarrow Gr_{s-1} \longrightarrow \mathcal{F}^\bullet \longrightarrow Gr_s \longrightarrow 0$$

and

$$0 \longrightarrow E_\infty^{a-(s-1),(s-1)} \longrightarrow \mathbb{H}^a(\mathcal{F}^\bullet) \longrightarrow E_\infty^{a-s,s} \longrightarrow 0$$

with

$$\begin{aligned} E_\infty^{a-(s-1),(s-1)} &= \mathbb{H}^a(Gr_{s-1})/\mathbb{H}^{a-1}(Gr_s) \\ E_\infty^{a-s,s} &= \ker\ \mathbb{H}^a(Gr_s) \longrightarrow \mathbb{H}^{a+1}(Gr_{s-1}) \end{aligned}$$

19.

Define the differential d_2 as the connecting morphism of the vertical exact sequence on the right hand side (or of the above horizontal exact sequence) of the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Gr_{i-1} \arrow[r] \arrow[d, equal] & Filt_i/Filt_{i-2} \arrow[r] \arrow[d] & Gr_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Gr_{i-1} \arrow[r] & Filt_{i+1}/Filt_{i-2} \arrow[r] \arrow[d] & Filt_{i+1}/Filt_{i-1} \arrow[r] \arrow[d] & 0 \\
 & & Gr_{i+1} \arrow[r, "\text{identity}"] \arrow[d] & Gr_{i+1} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

and the $\mathcal{O}$-module $E_3^{a-i,i}$ by

$$E_3^{a-i,i} = \frac{\ker \ E_2^{a-i,i} \xrightarrow{d_2} E_2^{a-i+2,i-1}}{\mathrm{im}\ E_2^{a-i-2,i+1} \xrightarrow{d_2} E_2^{a-i,i}}.$$

For $\varepsilon = 1, 2$ and the ε-steps filtration one has

$$E_\infty^{a-i,i} = E_{\varepsilon+1}^{a-i,i},$$

as we saw in (A.17) and (A.18), and the filtration on $\mathbb{H}^a(\mathcal{F}^\bullet)$ is exhausting.

20.

The right vertical exact sequence of the diagram (A.19) gives a surjection

$$\mathbb{H}^a(Filt_{i+1}/Filt_{i-1}) \xrightarrow{\gamma} \ker(\mathbb{H}^a(Gr_{i+1}) \to \mathbb{H}^{a+1}(Gr_i))$$

and the middle horizontal sequence induces a morphism

$$\ker(\mathbb{H}^a(Gr_{i+1}) \to \mathbb{H}^{a+1}(Gr_i)) \xrightarrow{d_3'} \mathbb{H}^{a+1}(Gr_{i-1})/\mathrm{im}\ \mathbb{H}^a(Gr_i)$$

Replacing i by $i-1$, one obtains maps

$$\mathbb{H}^a(Filt_i/Filt_{i-2}) \xrightarrow{\gamma} \ker(\mathbb{H}^a(Gr_i) \to \mathbb{H}^{a+1}(Gr_{i-1})) \xrightarrow{d_3'} \frac{\mathbb{H}^{a+1}(Gr_{i-2})}{\mathrm{im}\ \mathbb{H}^a(Gr_{i-1})}$$

where γ is surjective. As the extension

$$0 \longrightarrow Gr_{i-2} \longrightarrow Filt_i/Filt_{i-3} \longrightarrow Filt_i/Filt_{i-2} \longrightarrow 0$$

lifts to an extension

$$0 \longrightarrow Filt_{i-2}/Filt_{i-4} \longrightarrow Filt_i/Filt_{i-4} \longrightarrow Filt_i/Filt_{i-2} \longrightarrow 0$$

the image of $d_3' \circ \gamma$ lies in fact in

$$\frac{\ker\ \mathbb{H}^{a+1}(Gr_{i-2}) \to \mathbb{H}^{a+2}(Gr_{i-3})}{\mathrm{im}\ \mathbb{H}^{a}(Gr_{i-1}) \to \mathbb{H}^{a+1}(Gr_{i-2})} = E_3^{a-i+3,i-2}\ ,$$

as well as the image of d_3'. The map d_3' factorizes through $E_3^{a-i,i}$. The resulting map is written as

$$d_3 : E_3^{a-i,i} \longrightarrow E_3^{a-i+3,i-2}.$$

One defines

$$E_4^{a-i,i} = \frac{\ker\ E_3^{a-i,i} \xrightarrow{d_3} E_3^{a-i+3,i-2}}{\mathrm{im}\ E_3^{a-i-3,i+2} \xrightarrow{d_3} E_3^{a-i,i}}\ .$$

By construction a class in $E_3^{a-i,i}$ may be represented by a class in

$$\mathbb{H}^a(Filt_i/Filt_{i-2})$$

and similarly a class in $E_4^{a-i,i}$ may be represented by a class in

$$\mathbb{H}^a(Filt_i/Filt_{i-3}).$$

More generally, one defines inductively in the same vein differentials

$$E_r^{a-i,i} \xrightarrow{d_r} E_r^{a-i+r,i-r+1},$$

and $\mathcal{O}$-modules:

$$E_{r+1}^{a-i,i} := \frac{\ker\ E_r^{a-i,i} \xrightarrow{d_r} E_r^{a-i+r,i-r+1}}{\mathrm{im}\ E_r^{a-i-r,i+r-1} \xrightarrow{d_r} E_r^{a-i,i}}$$

which are subquotients of E_r. A class in $E_{r+1}^{a-i,i}$ is represented by a lifting to $\mathbb{H}^a(Filt_i/Filt_{i-r})$.

When $Filt_{\sigma-1} = 0$, and $Filt_\sigma \neq 0$, one defines the induced filtration on $Filt_{\sigma+\rho}$ by

$$Filt_l\ Filt_{\sigma+\rho} = \begin{cases} Filt_l & \text{if } \ l \leq \sigma+\rho \\ Filt_{\sigma+\rho} & \text{if } \ l \geq \sigma+\rho\ . \end{cases}$$

Then one has:

$$E_\infty^{a-i,i}(\mathbb{H}^a(Filt_{\sigma+\varrho})) = E_{\varrho+2}^{a-i,i}$$

and the filtration on $\mathbb{H}^a(Filt_{\sigma+\varrho})$ is exhausting.

21.

This shows that in general, for going from $\mathbb{H}^a(Filt_{\sigma+\varrho})$ to $\mathbb{H}^a(\mathcal{F}^\bullet)$ one has to introduce infinitely many r and groups $E_r^{a-i,i}$, a reason for the notation $E_\infty^{a-i,i}$.

One says that the *spectral sequence* with E_2 term $E_2^{a-i,i}$ and d_2 differential d_2 (simply noted $(E_2^{a-i,i}, d_2)$) *degenerates* in E_r if:

The filtration on $\mathbb{H}^a(\mathcal{F}^\bullet)$ is exhausting and $E_\infty^{a-i,i} = E_r^{a-i,i}$ for all i, or equivalently $d_{r+l} = 0$ for all $l \geq 0$.

With this terminology, we have seen in (A.20) that a $(\varrho+1)$-steps filtration defines an E_2 spectral sequence which degenerates in $E_{\varrho+2}$.

22.

Under the assumptions of (A.16), assume moreover that $E_r^{a-i,i}$ is also a free $\mathcal{O}$-module (for example if $\mathcal{O}$ is a field). Then one has:

$$\operatorname{rank}_{\mathcal{O}} \mathbb{H}^a(\mathcal{F}^\bullet) \leq \sum_i \operatorname{rank}_{\mathcal{O}} E_r^{a-i,i}$$

for all r, and the spectral sequence degenerates in E_r if and only if this is an equality.

23.

By (A.19), to say that the spectral *sequence degenerates* in E_2 means that for all i, one has an exact sequence

$$0 \longrightarrow \mathbb{H}^a(Filt_{i-1}) \longrightarrow \mathbb{H}^a(Filt_i) \longrightarrow \mathbb{H}^a(Gr_i) \longrightarrow 0,$$

and of course that the filtration is exhausting.

24.

One says that the spectral sequence (E_2, d_2) *converges* to $\mathbb{H}^a(\mathcal{F}^\bullet)$ if it degenerates in E_r for some r. We sometimes write

$$(E_2^{a-i,i}, d_2) \Longrightarrow \mathbb{H}^a(\mathcal{F}^\bullet)$$

instead of "(E_2, d_2) converges to $\mathbb{H}^a(\mathcal{F}^\bullet)$".

25. The Hodge to de Rham spectral sequence.

On $\mathcal{F}^\bullet$, a complex of $\mathcal{O}$-modules, bounded below, we define the *Hodge filtration* (often called the stupid filtration) by:

$$Filt^i\mathcal{F}^\bullet = \mathcal{F}^{\geq i} = Filt_{-i}\mathcal{F}^\bullet$$

where

$$(\mathcal{F}^{\geq i})^l = \begin{cases} 0 & \text{if } l < i \\ \mathcal{F}^l & \text{if } l \geq i. \end{cases}$$

Then $Gr_{-i} = Filt_{-i}/Filt_{-i-1} = \mathcal{F}^i[-i]$ where $[\alpha]$ means:

$$(\mathcal{F}^\bullet[\alpha])^l = \mathcal{F}^{l+a}.$$

This is the so called *shift* by α to the right. The E_2 spectral sequence reads:

$$E_2^{a-i,i} = \mathbb{H}^a(Gr_i) = \mathbb{H}^a(\mathcal{F}^{-i}[i]) = H^{a+i}(\mathcal{F}^{-i})$$

where the differential d_2 goes to

$$\mathbb{H}^{a+1}(Gr_{i-1}) = \mathbb{H}^{a+1}(\mathcal{F}^{-(i-1)}[i-1]) = H^{a+i}(\mathcal{F}^{-i+1})$$

and is just induced by the differential in the complex.

This spectral sequence is usually rewritten as an E_1 spectral sequence by setting

$$E_1^{-i,a+i} = E_2^{a-i,i} \quad \text{or} \quad E_1^{\alpha,\beta} = E_2^{\beta+2\alpha,-\alpha}$$

with differentials:

$$\begin{array}{ccc} E_2^{\beta+2\alpha,-\alpha} & \xrightarrow{d_2} & E_2^{\beta+2\alpha+2,-\alpha-1} \\ \| & & \| \\ E_1^{\alpha,\beta} & \xrightarrow{d_1} & E_1^{\alpha+1,\beta} \end{array}$$

The E_1 spectral sequence obtained is called the Hodge to de Rham spectral sequence, at least when $\mathcal{F}^\bullet$ is some de Rham complex on X, possibly with some poles, possibly with non-trivial coefficients . . .

For a given a, one has

$$\mathbb{H}^a(\mathcal{F}^\bullet) = \mathbb{H}^a(\mathcal{F}^{\leq a+1})$$

where

$$(\mathcal{F}^{\leq i})^l = \begin{cases} \mathcal{F}^l & \text{if } l \leq i \\ 0 & \text{if } l > i. \end{cases}$$

In particular this is a finite complex, on which the Hodge filtration induces a finite step filtration. Therefore the E_1 Hodge to de Rham spectral sequence always converges.

To say that it degenerates in E_1 means that for any i one has exact sequences

$$0 \longrightarrow \mathbb{H}^a(\mathcal{F}^{\geq i+1}) \longrightarrow \mathbb{H}^a(\mathcal{F}^{\geq i}) \longrightarrow H^{a-i}(\mathcal{F}^i) \longrightarrow 0.$$

Putting those sequences together, one obtains exact sequences

$$0 \longrightarrow \mathbb{H}^a(\mathcal{F}^{\geq i+j}) \longrightarrow \mathbb{H}^a(\mathcal{F}^{\geq i}) \longrightarrow \mathbb{H}^a(\mathcal{F}^{[i,i+j)}) \longrightarrow 0$$

where

$$(\mathcal{F}^{[i,i+j)})^l = \begin{cases} \mathcal{F}^l & \text{if } l \in [i, i+j) \\ 0 & \text{if not.} \end{cases}$$

If moreover, one knows that $\mathbb{H}^a(\mathcal{F}^\bullet)$ is a free $\mathcal{O}$-module of finite rank, as well as $H^{a-i}(\mathcal{F}^i)$, then the E_1 Hodge to de Rham spectral sequence degenerates in E_1 if and only if

$$\operatorname{rank}_{\mathcal{O}} \mathbb{H}^a(\mathcal{F}^\bullet) = \sum_i \operatorname{rank}_{\mathcal{O}} H^{a-i}(\mathcal{F}^i).$$

26. The conjugate spectral sequence.

On $\mathcal{F}^\bullet$, a complex of $\mathcal{O}$-modules bounded below, we defined the τ-filtration:

$$(\tau_{\leq i}\mathcal{F}^\bullet)^l = \begin{cases} \mathcal{F}^l & \text{for } l < i \\ \ker\, d & \text{for } l = i \\ 0 & \text{otherwise.} \end{cases}$$

Then $Gr_i = \mathcal{H}^i[-i]$ is the cohomology sheaf in degree i. The E_2 spectral sequence reads

$$E_2^{a-i,i} = \mathbb{H}^a(Gr_i) = H^{a-i}(\mathcal{H}^i)$$

where the d_2 differential goes to

$$\mathbb{H}^{a+1}(Gr_{i-1}) = H^{a-i+2}(\mathcal{H}^{i-1}).$$

It is called the conjugate spectral sequence. For a given a, one has

$$\mathbb{H}^a(\mathcal{F}^\bullet) = \mathbb{H}^a(\tau_{\leq a+1}\mathcal{F}^\bullet).$$

However $\tau_{\leq a+1}\mathcal{F}^\bullet$ is a finite complex on which the τ-filtration induces a finite step filtration. Therefore the E_2-conjugate spectral sequence always converges. Furthermore, if $\sigma : \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet$ is a *quasi-isomorphism* then σ induces quasi-isomorphisms

$$\tau_{\leq i}\mathcal{F}^\bullet \longrightarrow \tau_{\leq i}\mathcal{G}^\bullet$$

for all i, and therefore σ induces an isomorphism of the conjugate spectral sequences.

To say that the conjugate spectral sequence degenerates in E_2 means that one has exact sequences

$$0 \longrightarrow \mathbb{H}^a(\tau_{\leq i-1}) \longrightarrow \mathbb{H}^a(\tau_{\leq i}) \longrightarrow H^{a-i}(\mathcal{H}^i) \longrightarrow 0$$

for all i. If $\mathbb{H}^a(\mathcal{F}^\bullet)$ is a free $\mathcal{O}$-module of finite rank, as well as $\mathbb{H}^{a-i}(\mathcal{H}^i)$, then the degeneration in E_2 is equivalent to

$$\operatorname{rank}_{\mathcal{O}}\mathbb{H}^a(\mathcal{F}^\bullet) = \sum_i \operatorname{rank}_{\mathcal{O}} H^{a-i}(\mathcal{H}^i).$$

27. The Leray spectral sequence.

Let $f : X \longrightarrow Y$ be a morphism between two k-varieties, and let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}$-modules on X, bounded below.

Let $\mathcal{F}^\bullet \longrightarrow \mathcal{I}^\bullet$ be an injective resolution. We consider the *direct image functor* (already used and defined but not named in 6):

$$(f_*\mathcal{K})_x = \varinjlim_{x \in U} H^0(f^{-1}(U), \mathcal{K})$$

for any $\mathcal{O}$-sheaf $\mathcal{K}$ on X and any open set U in Y. In particular, by definition $H^0(X, \mathcal{K}) = H^0(Y, f_*\mathcal{K})$. Therefore one has

$$\mathbb{H}^a(X, \mathcal{F}^\bullet) = \frac{\ker \quad H^0(Y, f_*\mathcal{I}^a) \longrightarrow H^0(Y, f_*\mathcal{I}^{a+1})}{\mathrm{im}\ H^0(Y, f_*\mathcal{I}^{a-1}) \longrightarrow H^0(Y, f_*\mathcal{I}^a)}.$$

One verifies immediately that, by definition, $f_*\mathcal{I}^i$ is an injective sheaf as well, which allows to write (A.10):

$$\mathbb{H}^a(X, \mathcal{F}^\bullet) = \mathbb{H}^a(Y, (f_*\mathcal{I}^\bullet)),$$

where $f_*\mathcal{I}^\bullet$ is the complex

$$(f_*\mathcal{I}^\bullet)^l = f_*\mathcal{I}^l.$$

One considers the conjugate spectral sequence for $(f_*\mathcal{I}^\bullet)$:

$$E_2^{a-i,i} = H^{a-i}(Y, \mathcal{H}^i(f_*\mathcal{I}^\bullet)).$$

One defines

$$R^i f_*\mathcal{F}^\bullet := \mathcal{H}^i(f_*\mathcal{I}^\bullet).$$

By definition

$$\begin{aligned}(R^i f_*\mathcal{F}^\bullet)_x : &= \varinjlim_{x \in U} \frac{\ker \ H^0(f^{-1}(U), \mathcal{I}^i) \to H^0(f^{-1}(U), \mathcal{I}^{i+1})}{\mathrm{im}\ H^0(f^{-1}(U), \mathcal{I}^{i-1}) \to H^0(f^{-1}(U), \mathcal{I}^i)} \\ &= \varinjlim_{x \in U} \mathbb{H}^i(f^{-1}(U), \mathcal{F}^\bullet)\end{aligned}$$

In particular, $R^i f_*\mathcal{F}^\bullet$ does not depend on the injective resolution chosen.

The E_2 spectral sequence reads

$$E_2^{a-i,i} = H^{a-i}(Y, R^i f_*\mathcal{F}^\bullet),$$

with d_2 differential to $H^{a-i+2}(Y, R^{i-1} f_*\mathcal{F}^\bullet)$, and is called the *Leray spectral sequence for f*.

As the conjugate spectral sequence for $(f_*\mathcal{I}^\bullet)$ converges to $\mathbb{H}^a(f_*\mathcal{I}^\bullet)$ (A.26), the *Leray spectral sequence for f always converges to* $\mathbb{H}^a(\mathcal{F}^\bullet)$.
In particular, if $\mathcal{F}$ is just a sheaf for which $R^if_*\mathcal{F} = 0$ for $i > 0$, one has:

$$H^a(X,\mathcal{F}) = H^a(Y, f_*\mathcal{F}) \text{ for all } a.$$

28.

Let $\mathcal{G}^\bullet$ be a complex of $\mathcal{O}$-modules, bounded below, such that $H^i(\mathcal{G}^j) = 0$ for $i > 0$ and all j. Then the E_1 Hodge to de Rham spectral sequence $E_1^{ij} = H^j(\mathcal{G}^i)$ degenerates in E_2 and one has

$$\begin{aligned} E_\infty^{i,0} = E_2^{i,0} \quad &= \tfrac{\ker \ H^0(\mathcal{G}^i)\to H^0(\mathcal{G}^{i+1})}{\operatorname{im}\ H^0(\mathcal{G}^{i-1})\to H^0(\mathcal{G}^i)} \\ &= \mathbb{H}^i(\mathcal{G}^\bullet) \\ &= \mathbb{H}^i(\mathcal{F}^\bullet) \text{ for any quasi-isomorphism } \quad \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet. \end{aligned}$$

29.

Take for $\mathcal{F}^\bullet$ a complex of quasi-coherent sheaves (for example some de Rham complex). We consider a collection of very ample Cartier divisors D_α with empty intersection, such that the open covering of X defined by $U_\alpha := X - D_\alpha$ consists of affine varieties. Then one has:

$$H^a(X, \varrho_*\mathcal{F}^j|_{U_{\alpha_0\ldots\alpha_i}}) = H^a(U_{\alpha_0\ldots\alpha_i}, \mathcal{F}^j|_{U_{\alpha_0\cdots\alpha_i}})$$

for all a where $\varrho : U_{\alpha_0\ldots\alpha_i} \longrightarrow X$ is the natural embedding of the affine set $U_{\alpha_0\ldots\alpha_i}$. In fact, one has

$$\begin{aligned} (R^i\varrho_*\mathcal{F}^j)_x \ &= \ \varinjlim_{x\in V} \ H^i(V\cap U_{\alpha_0\cdots\alpha_i}, \mathcal{F}^j) \\ &= \ 0 \ \text{ for } \ i>0 \end{aligned}$$

and one applies (A.27). By (A.28) one obtains:

$$\mathbb{H}^a(X,\mathcal{F}^\bullet) = \frac{\ker \ \oplus C^i(\mathcal{U},\mathcal{F}^{a-i}) \to \oplus\ C^{i'}(\mathcal{U},\mathcal{F}^{a+1-i'})}{\operatorname{im}\ \oplus C^{i'}(\mathcal{U},\mathcal{F}^{a-1-i'}) \to \oplus\ C^i(\mathcal{U},\mathcal{F}^{a-i})}$$

where

$$\begin{aligned} C^i(\mathcal{U},\mathcal{F}^j) \ &= H^0(X, \mathcal{C}^i(\mathcal{U},\mathcal{F}^j)) \\ &= \textstyle\bigoplus_{\alpha_0<\ldots<\alpha_i} H^0(U_{\alpha_0\ldots\alpha_i},\mathcal{F}^j). \end{aligned}$$

References

[1] Y. Akizuki, S. Nakano: Note on Kodaira-Spencer's proof of Lefschetz's theorem. Proc. Jap. Acad., Ser A **30** (1954), 266 - 272

[2] D. Arapuro: A note on Kollár's theorem. Duke Math. Journ. **53** (1986), 1125 - 1130

[3] D. Arapuro and D.B. Jalte: On Kodaira vanishing for singular varieties. Proc. Amer. Math. Soc. **105** (1989), 911 - 916

[4] I. Bauer, S. Kosarew: On the Hodge spectral sequences for some classes of non-complete algebraic manifolds. Math. Ann. **284** (1989), 577 - 593

[5] A. Beauville: Annulation du H^1 et systèmes paracanoniques sur les surfaces. Journ. reine angew. Math. **388** (1988), 149 - 157 & 219 - 220

[6] F. Bogomolov: Unstable vector bundles and curves on surfaces. Proc. Intern. Congress of Math., Helsinki 1978, 517 - 524

[7] A. Borel et al.: Intersection Cohomology. Progress in Math. **50** (1984) Birkhäuser

[8] A. Borel et al.: Algebraic D-Modules. Perspectives in Math. **2** (1987) Acadamic Press

[9] P. Cartier: Une nouvelle opération sur les formes différentielles. C. R. Acad. Sci., Paris, **244** (1957), 426 - 428

[10] P. Deligne: Equations différentielles à points singuliers réguliers. Springer Lect. Notes Math. **163** (1970)

[11] P. Deligne: Théorie de Hodge II. Publ. Math., Inst. Hautes Etud. Sci. **40** (1972), 5 - 57

[12] P. Deligne, L. Illusie: Relèvements modulo p^2 et décomposition du complexe de de Rham. Inventiones math. **89** (1987), 247 - 270

[13] J.-P. Demailly: Une généralisation du théorème d'annulation de Kawamata - Viehweg. C. R. Acad. Sci. Paris. **309** (1989), 123 - 126

[14] H. Dunio: Über generische Verschwindungssätze. Diplomarbeit, Essen 1991

[15] L. Ein, R. Lazarsfeld: Global generation of pluricanonical and adjoint linear series on smooth projective threefolds. manuscript

[16] H. Esnault: Fibre de Milnor d'un cône sur une courbe plane singulière. Inventiones math. **68** (1982), 477 - 496

[17] H. Esnault, E. Viehweg: Revêtement cycliques. Algebraic Threefolds, Proc. Varenna 1981. Springer Lect. Notes in Math. **947** (1982), 241 - 250

[18] H. Esnault, E. Viehweg: Sur une minoration du degré d'hypersurfaces s'annulant en certains points. Math. Ann. **263** (1983), 75 - 86

[19] H. Esnault, E. Viehweg: Two-dimensional quotient singularities deform to quotient singularities. Math. Ann. **271** (1985), 439 - 449

[20] H. Esnault, E. Viehweg: Logarithmic De Rham complexes and vanishing theorems. Inventiones math. **86** (1986), 161 - 194

[21] H. Esnault, E. Viehweg: Revêtements cycliques II, Proc. Algebraic Geometry, La Rabida 1984, Hermann, Travaux en cours **23** (1987), 81 - 96

[22] H. Esnault, E. Viehweg: Vanishing and non-vanishing theorems. Proc. Hodge Theory, Luminy 1987 (Ed.: Barlet, Elzein, Esnault, Verdier, Viehweg). Astérisque **179 - 180** (1989), 97 - 112

[23] H. Esnault, E. Viehweg: Effective bounds for semipositive sheaves and the height of points on curves over complex function fields. Compositio math. **76** (1990), 69 - 85

[24] T. Fujita: On Kähler fibre spaces over curves. J. Math. Soc. Japan **30** (1978), 779 - 794

[25] H. Grauert, O. Riemenschneider: Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen. Inventiones math. **11** (1970), 263 - 292

[26] M. Green, R. Lazarsfeld: Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville. Inventiones math. **90** (1987), 389 - 407

[27] M. Green, R. Lazarsfeld: Higher obstructions to deforming cohomology groups of line bundles. Journal of the AMS. **4** (1991), 87 - 103

[28] A. Grothendieck: Fondements de la Géométrie Algébrique. Séminaire Bourbaki 1957 - 62

[29] R. Hartshorne: Residues and Duality. Springer Lect. Notes in Math. **20** (1966)

[30] R. Hartshorne: Algebraic Geometry. Graduate Texts in Mathematics 52. Springer Verlag, New York, 1977

[31] L. Illusie: Catégories dérivées et dualité. Travaux de J.-L. Verdier. L'Enseignement Math. **36** (1990), 369 - 391

[32] L. Illusie: Réduction semi-stable et décomposition de complexes de de Rham à coefficients. Duke Math. Journ. **60** (1990), 139 - 185

[33] B. Iversen: Cohomology of Sheaves. Universitext, Springer-Verlag (1986)

[34] N. Katz: Nilpotent Connections and the Monodromy Theorem. Publ. Math., Inst. Hautes Etud. Sci. **39** (1970), 175 - 232

[35] Y. Kawamata: Characterization of abelian varieties. Compositio math. **43** (1981), 253 - 276

[36] Y. Kawamata: A generalization of Kodaira-Ramanujam's vanishing theorem. Math. Ann. **261** (1982), 43 - 46

[37] Y. Kawamata: Pluricanonical systems on minimal algebraic varieties. Inventiones math. **79** (1985), 567 - 588

[38] Y. Kawamata, K. Matsuda, M. Matsuki: Introduction to the mimimal model problem. Algebraic Geometry, Sendai 1985. Adv. Stud. Pure Math. **10** (1987), 283 - 360

[39] K. Kodaira: On a differential-geometric method in the theory of analytic stacks. Proc. Natl. Acad. Sci. USA **39** (1953), 1268 - 1273

[40] J. Kollár: Higher direct images of dualizing sheaves I. Ann. Math. **123** (1986), 11 - 42

[41] J. Kollár: Higher direct images of dualizing sheaves II. Ann. Math. **124** (1986), 171 - 202

[42] I. Kosarew, S. Kosarew: Kodaira vanishing theorems on non-complete algebraic manifolds. Math. Z. **205** (1990), 223 - 231

[43] K. Maehara: Vanishing theorems. manuscript

[44] K. Maehara: Kawamata covering and logarithmic de Rham complexes. manuscript

[45] Y. Miyaoka: On the Mumford-Ramanujam vanishing theorem on a surface. Proc. of the conference "Géométrie alg. d'Angers 1979", Oslo 1979, 239 - 248

[46] S. Mori: Classification of higher-dimensional varieties. Algebraic Geometry. Bowdoin 1985. Proc. of Symp. in Pure Math. **46** (1987), 269 - 331

[47] D. Mumford: Pathologies of modular surfaces. Am. J. Math. **83** (1961), 339 - 342

[48] D. Mumford: The topology of normal singularities of an algebraic surface and a criterion for simplicity. Publ. Math. IHES, **9** (1961), 229 - 246

[49] D. Mumford: Pathologies, III. Amer. Journ. Math. **89** (1967), 94 - 104

[50] D. Mumford: Abelian Varieties. Oxford University Press (1970)

[51] C.P. Ramanujam: Remarks on the Kodaira vanishing theorem. Journ. Indian Math. Soc. **36** (1972), 41 - 51

[52] M. Raynaud: Contre-exemple au "Vanishing Theorem" en caractéristique $p > 0$. C. P. Ramanujam - A tribute. Studies in Mathematics 8, Tata Institute of Fundamental Research, Bombay (1978), 273 - 278

[53] I. Reider: Vector bundles of rank 2 and linear systems on algebraic surfaces. Ann. of Math. **127** (1988), 309 - 316

[54] M. Saito: Introduction to mixed Hodge Modules. Proc. Hodge Theory, Luminy 1987 (Ed.: Barlet, Elzein, Esnault, Verdier, Viehweg). Astérisque **179 - 180** (1989), 145 - 162

[55] J.-P. Serre: Faisceaux algébriques cohérents. Ann. of Math. **61** (1955), 197 - 278

[56] J.-P. Serre: Géométrie algébrique et géométrie analytique. Ann. Inst. Fourier **6** (1956), 1 - 42

[57] B. Shiffman, A.J. Sommese: Vanishing Theorems on Complex Manifolds. Progress in Math. 56, Birkhäuser (1985)

[58] C. Simpson: Subspaces of moduli spaces of rank one local systems. manuscript

[59] K. Timmerscheidt: Mixed Hodge theory for unitary local systems. Journ. reine angew. Math. **379** (1987), 152 - 171

[60] J.L. Verdier: Catégories dérivées, état 0. SGA 4 1/2 (Ed.: Deligne) Springer Lect. Notes in Math. **569** (1977), 262 - 311

[61] J.L. Verdier: Classe d'homologie associée à un cycle. Séminaire de Géometrie Analytique, Astérisque **36/37** (1976), 101-151

[62] E. Viehweg: Rational singularities of higher dimensional schemes. Proc. AMS **63** (1977), 6 - 8

[63] E. Viehweg: Vanishing theorems. Journ. reine angew. Math. **335** (1982), 1 - 8

[64] E. Viehweg: Vanishing theorems and positivity in algebraic fibre spaces. Proc. ICM, Berkeley 1986, 682 - 688

[65] O. Zariski: The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface. Ann. of Math. **76** (1962), 560 - 615

Hélène Esnault and Eckart Viehweg
Fachbereich 6, Mathematik
Universität GH Essen
Universitätsstr. 3
D-W-4300 Essen1
Germany

Index