Polynomial Bounds on the Slicing Number

Daniel Werner$^1$ and Matthias Lenz$^2$

$^1$ Institut für Informatik, Freie Universität Berlin, Germany
dwerner@mi.fu-berlin.de, http://page.mi.fu-berlin.de/dawerner
$^2$ Institut für Mathematik, Technische Universität Berlin, Germany
lenz@math.tu-berlin.de, http://www.math.tu-berlin.de/~lenz/

Abstract. NOTE: Unfortunately, most of the results mentioned here were already known under the name of "$d$-separated interval piercing". The result that $T_d(m)$ exists was first proved by Gyárfás and Lehel in 1970, see [5]. Later, the result was strengthened by Károlyi and Tardos [9] to match our result. Moreover, their proof (in a different notation, of course) uses ideas very similar to ours and leads to a similar recurrence. Also, our conjecture turns out to be right and was proved for the 2-dimensional case by Tardos and for the general case by Kaiser [8]. An excellent survey article ("Transversals of $d$-intervals") is available on http://www.renyi.hu/~tardos. Still, because of all the work we put into this, we leave the paper available to the public on http://page.mi.fu-berlin.de/dawerner, also because one might find the references useful.

We study the following Gallai-type problem: Assume that we are given a family $X$ of convex objects in $\mathbb{R}^d$ such that among any subset of size $m$, there is an axis-parallel hyperplane intersecting at least two of the objects. What can we say about the number of axis-parallel hyperplanes that sufficient to intersect all sets in the family? In this paper, we show that this number $T_d(m)$ exists, i.e., depends only on $m$ and the dimension $d$, but not on the size of the set $X$. First, we derive a very weak super-exponential bound. Using this result, by a simple proof we are able to show that this number is even polynomially bounded for any fixed $d$. We partly answer open problem 74 on [2], where the planar case is considered, by improving the best known exponential bound to $O(m^2)$.

Keywords: combinatorial geometry, rectangle slicing, independent set, upper bounds, transversal

1 Introduction

Let $X$ and $\mathcal{H}$ be two sets of objects in $\mathbb{R}^d$. An $h \in \mathcal{H}$ is said to be a transversal of $X$, if it intersects each $x \in X$. Investigating the conditions

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under which certain sets have a transversal has been an intensive field of research. An example is the well-known Helly Theorem [7], which states that whenever for a set of convex objects in $\mathbb{R}^d$ every $d+1$ have a nonempty intersection, then they all have a nonempty intersection. Here, $X$ is simply a set of convex objects, and $H$ is the set of all points.

If the given set does not allow a transversal with a single point, one can ask whether there are $k$ points such that their union intersects each set in $X$. If such a set of $k$ points exists, we say that $X$ is $k$-pierceable and call $k$ the piercing number of $X$.

A famous theorem of this type proved by Alon and Kleitman [1] is the following generalization of Helly’s Theorem (for $p = q = d + 1$):

**Theorem (Alon-Kleitman).** Let $X$ be a finite family of convex sets in $\mathbb{R}^d$. Let $p \geq q > d$ be integers such that among any $p$ sets of $X$ there are $q$ sets with a common point. Then $X$ is pierceable with $HD_d(p, q)$ points.³

The crucial part is that this number depends only on $p, q,$ and $d$, but not on the size of $X$. See, e.g., Wenger [12] or Matoušek [10, Ch. 10] for a gentler introduction.

In this paper, we will derive a similar result for a different class $\mathcal{H}$ of objects, namely axis-parallel hyperplanes, i.e., planes of the form $h: x_i = c$, $1 \leq i \leq d, c \in \mathbb{R}$. Analogously to the above notion, we say that a set $X$ of objects is $k$-sliceable, if there are $k$ axis-parallel hyperplanes whose union intersects each $x \in X$. By replacing each object by its bounding box, it suffices to talk about hyperrectangles ("boxes") of course.

**Theorem 1.** Let $X$ be a finite family of convex sets in $\mathbb{R}^d$. Let $m > n$ be integers such that any set of size $m$ can be sliced by $n$ axis-parallel hyperplanes. Then $X$ can be sliced by $T_d(m) < \infty$ axis-parallel hyperplanes.

This can also be stated in a slightly different way. Thereto, let graph $G_X = (V, E)$ be the graph with $V = X$ and $(x, y) \in E$ if and only if there is an axis-parallel hyperplane that intersects both objects.

**Corollary 1.** Let $X$ be a finite family of convex sets in $\mathbb{R}^d$. If $G_X$ does not have an independent set of size $m$, then $X$ can be sliced by $T_d(m) < \infty$ axis-parallel hyperplanes.

Until now, existence of this function was only known for $d = 1, 2$ (see Vatter [11]).

³ The $HD$ stands for Hadwiger and Debrunner, who originally stated this as a conjecture.
The bound we give for \( T_d(m) \) in the proof of existence in Sec. 2 is huge. Using this result though, there is a very simple proof for the following, which we will present in Sec. 3.

**Theorem 2.** \( T_d(m) \in \mathcal{O}(f(d) \cdot m^d) \) for some function \( f \).

That is, for any fixed \( d \) this number is polynomial in the size of the largest independent set, and independent of the total number of objects.

For \( d = 2 \), this also partly settles an open question on [2]: The best known bound [11] was exponential in \( m \), and Theorem 2 gives a quadratic bound.

Because any set of objects that is intersected by a single hyperplane forms a clique in the corresponding graph, we get another nice corollary. For Interval graphs \( G \), which are perfect, it holds that \( \alpha(G) \), the indepence number, is equal to \( \chi(G) \), the clique partition number. The corollary shows that unions of Interval graphs have a similar property.

**Corollary 2.** For any \( d \), there is a constant \( c_d > 0 \) such that the following holds: Let \( I_1, \ldots, I_d \) be interval graphs on the same vertex-set \( V \), and \( \mathcal{I} \) their union. Let \( m \) be the size of the largest independent set in \( \mathcal{I} \). Then \( \alpha(G) \leq \chi(G) \leq c_d \cdot m^d \cdot \alpha(G) \).

And, by using the pigeonhole-principle, we also get a Ramsey-type corollary:

**Corollary 3.** For any \( d \), there is a constant \( c'_d > 0 \) such that the following holds: Let \( I_1, \ldots, I_d \) be interval graphs on the same vertex-set \( V \), and \( \mathcal{I} \) their union. Then \( \mathcal{I} \) either contains a clique or an independent set of size \( c'_d \cdot d^{d+1} \sqrt{n} \) (for \( n \) large enough).

From computational point of view, this problem has also been considered: Dom et al. [3] and Giannopoulos et al. [4] independently of each other showed that the problem of deciding whether a given set of rectangles in \( \mathbb{R}^d \) can be sliced (“stabbed”) by \( k \) hyperplanes is W[1]-hard with respect to \( k \). Also, they both show that the problem for disjoint unit squares in the plane is fixed-parameter tractable, i.e., can be solved in time \( \mathcal{O}((4k + 1)^kn^2) \). Recently it has been shown by Hegernes et al. [6] that the problem is even fixed parameter tractable for disjoint rectangles of arbitrary size.
2 Existence of $T_d(m)$

Let $[d] := \{1, \ldots, d\}$. For $D \subseteq [d]$, we say that $r, r'$ are independent with respect to $D$, if $\text{pr}_D(r) \cap \text{pr}_D(r') = \emptyset$. Two sets are called $j$-disjoint, if they are disjoint with respect to dimension $j$.

We will prove Theorem 1 by induction on the dimension $d$. The problem one faces here is that even if we do not have an independent set of size $m$ in $d$ dimensions, we can not say anything about the size of the largest independent set with respect to lower dimensions: The boxes might all lie on a common hyperplane orthogonal to $e_d$, i.e., form an independent set of size 1, but be pairwise disjoint with respect to $[d - 1]$.

Thus, we need to be a little more careful when doing the dimension reduction. The main observation is that there cannot be too many pairwise $d$-disjoint independent sets of a certain size. This is expressed by the following lemma:

**Lemma 1.** Let $X$ be a set of hyperrectangles in $\mathbb{R}^d$ that does not have an independent set of size $m$. Then we can choose $m - 1$ parallel hyperplanes such that the remaining $4$ boxes are partitioned into $m$ sets with the property that for each of these sets the largest independent set with respect to $[d - 1] = \{1, \ldots, d - 1\}$ is of size less than $dm(m - 1)$.

In order to prove this, we need a simple lemma that states that between two $d$-disjoint independent sets there cannot be too many incidences.

**Lemma 2.** Let $M_1, M_2$ be two independent sets of intervals. Then the total number of incidences is at most $|M_1| + |M_2| - 1$.

**Corollary 4.** If $M_1, M_2$ are two sets of rectangles in $\mathbb{R}^d$ such that

- each $M_i$ is an independent set with respect to $[d - 1]$
- $M_1$ and $M_2$ are disjoint with respect to $d$

then the total number of incidences (dependences) between $M_1$ and $M_2$ is less than $(d - 1) (|M_1| + |M_2|)$.

**Proof.** By assumption, there are no incidences in dimension $d$. For each of the remaining $d - 1$ dimensions the two sets form sets of disjoint intervals, thus by Lemma 2 can have at most $|M_1| + |M_2| - 1$ incidences per dimension. \hfill $\square$

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\(\footnote{\text{i.e., the boxes not yet intersected by any of these } m - 1 \text{ planes.}}\)
Using this, we can prove the next lemma by a counting argument.

**Lemma 3.** Let $X$ be a set of hyperrectangles in $\mathbb{R}^d$ with no independent set of size $m$ and $H$ be a set of $m - 1$ hyperplanes orthogonal to $e_d$, partitioning the rectangles into $m$ sets. Then there is a region whose largest independent set with respect to $\{1, \ldots, d - 1\}$ is of size less than $a := dm(m - 1)$.

**Proof.** Assume we have $m$ independent sets of size $dm(m - 1)$ that are pairwise $d$-disjoint. Choose a subset of exactly this size from each of the sets $M_1, \ldots, M_m$. As any selection of one element from each set $x \in \prod M_i$, of which there are $a^m$, must have at least one dependence with respect to $[d - 1]$ (otherwise we had an independent set of size $m$ in the original instance), and any dependence counts for at most $a^m$ such sets, we need at least $\frac{a^m}{a^m - 2} = a^2$ intersections. But because of Corollary 4, the total number of intersections is at most

$$(d - 1) \cdot \sum_{i \neq j} (|M_i| + |M_j| - 1) < d \cdot \binom{m}{2} \cdot 2a = dm(m - 1) \cdot a = a^2.$$

Thus, at least one of the independent sets must be of smaller size. \(\square\)

Now we come to prove the main lemma. The idea is to sweep about the set with a hyperplane orthogonal to $e_d$ and pick hyperplanes subsequently just before further sweeping would yield a large independent set with respect to $[d - 1]$ on the negative side. See Fig. 2.

This vague argument is formalized in the following Lemma.

**Proof (Lemma 1).** For a set $R$ of (closed) rectangles, let $\beta_R$ be minimal such that the halfspace $x_d \leq \beta_R$ contains a $[d - 1]$ independent set of size $a$. (If such a $\beta_R$ does not exist, we are done by induction.) Let $h: x_d = \beta_R$ be the corresponding hyperplane.

Observe that there cannot be a $[d - 1]$ independent set of size $a$ strictly on the negative side of $h$ then, for otherwise $h$ would not be minimal.

For the set $R$, let $R_h$ denote the set of all rectangles that strictly lie in the halfspace $x_d > \beta_R$, i.e., the ones that have now "seen" the sweeping hyperplane so far.

Now we simply pick hyperplanes $h_i$ as follows: $h_1: x_d = \beta_R$, $h_{i+1}: x_d = \beta_{R_{h_i}}$.

Because of Lemma 3, this process stops (i.e., $\beta$ is undefined) after we have chosen at most $m - 1$ hyperplanes, for otherwise we would $m$
No large \((a = 4)\) independent set with respect to dimension \(x\) on the negative side.

No large \((a = 4)\) independent set with respect to dimension \(x\).

Fig. 1. Schematic drawing of the sweeping procedure.

independent sets with respect to \([d - 1]\) of size \(a\) that are pairwise \(d\)-disjoint.

Thus, we have chosen at most \(m - 1\) hyperplanes, and in each of the at most \(m\) induced regions the largest independent set with respect to \([d - 1]\) is of size less than \(a\).

\[\square\]

**Corollary 5.** \(T_d(m)\) exists.

**Proof.** The existence of \(T_1(m)\) is clear, and Lemma 3 yields

\[T_d(m) < m \cdot T_{d-1}(dm(m - 1)) + m - 1.\]

\[\square\]

### 3 A polynomial bound

Using the existence of \(T_d(m)\) from the previous section, in a straightforward way from this lemma we can derive a much stronger bound for the higher-dimensional case:
Lemma 4. Let \( R \) be a set of boxes in \( \mathbb{R}^d \) that does not have an independent set of size \( m \), then it can be sliced by

\[
T_d(m) \leq (2m - 1)^d \cdot T_d(d) + (m - 1) \cdot 2^d
\]

axis-parallel hyperplanes.

Proof. Given a maximum independent set \( M \) of size \( (m - 1) \), we choose a hyperplane through each of the boundaries. This makes a total of \( (m - 1) \cdot 2^d \) hyperplanes. Any box not intersected yet lies inside one of the at most \((2m - 1)^d\) regions created by these hyperplanes.

The crucial observation now is the following: Any region can have a nonempty projection with at most \( d \) of the boxes in \( M \) (at most one for each direction, as \( M \) is an independent set). Thus, each region can contain an independent of size at most \( d \): Assume we had a box that contained some independent set \( M' \) of size \( d' > d \). Let \( M_d \subset M \) be the set of boxes that have a nonempty intersection with this region. Then \( M - M_d \cup M' \) is an independent set and

\[
|(M - M_d) \cup M'| = |M| - |M_d| + |M'| \geq m - d + d' \geq m,
\]
a contradiction.

Thus, we need at most \( T_d(d) \) additional hyperplanes for each region, making it a total of \((2m - 1)^d \cdot T_d(d) + (m - 1) \cdot 2^d\).

\( \square \)

Corollary 6. For any fixed \( d \) we have \( T_d(m) \in \mathcal{O}(m^d) \).

Observe how this bound is based on the existence of \( T_d(d) \) in the first place!

4 Conclusion and open problems

We have shown that the slicing number for convex objects with bounded independent set exists in arbitrary dimension, and that it is bounded by a polynomial for any fixed \( d \). As during the proof in Sec. 3 we are not very careful with our analysis, we assume that the bound is actually much stronger:

Conjecture 1. For any fixed \( d \), it holds that \( T_d(m) \in \mathcal{O}(m) \).
References