# Polynomial Bounds on the Slicing Number

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**Abstract. NOTE:** Unfortunately, most of the results mentioned here were already known under the name of "$d$-separated interval piercing". The result that $T_d(m)$ exists was first proved by Gyárfás and Lehel in 1970, see [5]. Later, the result was strengthened by Károlyi and Tardos [9] to match our result. Moreover, their proof (in a different notation, of course) uses ideas very similar to ours and leads to a similar recurrence. Also, our conjecture turns out to be right and was proved for the 2-dimensional case by Tardos and for the general case by Kaiser [8]. An excellent survey article ("Transversals of $d$-intervals") is available on `http://www.renyi.hu/~tardos`.
Still, because of all the work we put into this, we leave the paper available to the public on `http://page.mi.fu-berlin.de/dawerner`, also because one might find the references useful.

We study the following Gallai-type problem: Assume that we are given a family $X$ of convex objects in $\mathbb{R}^d$ such that among any subset of size $m$, there is an axis-parallel hyperplane intersecting at least two of the objects. What can we say about the number of axis-parallel hyperplanes that sufficient to intersect all sets in the family?
In this paper, we show that this number $T_d(m)$ exists, i.e., depends only on $m$ and the dimension $d$, but not on the size of the set $X$. First, we derive a very weak super-exponential bound. Using this result, by a simple proof we are able to show that this number is even polynomially bounded for any fixed $d$.
We partly answer open problem 74 on [2], where the planar case is considered, by improving the best known exponential bound to $\mathcal{O}(m^2)$.
*Keywords: combinatorial geometry, rectangle slicing, independent set, upper bounds, transversal*

## 1 Introduction

Let $X$ and $\mathcal{H}$ be two sets of objects in $\mathbb{R}^d$. An $h \in \mathcal{H}$ is said to be a *transversal* of $X$, if it intersects each $x \in X$. Investigating the conditions

⋆ This research was funded by Deutsche Forschungsgemeinschaft within the Research Training Group (Graduiertenkolleg) "Methods for Discrete Structures"

under which certain sets have a transversal has been an intensive field of research. An example is the well-known Helly Theorem [7], which states that whenever for a set of convex objects in $\mathbb{R}^d$ every $d+1$ have a nonempty intersection, then they all have a nonempty intersection. Here, $X$ is simply a set of convex objects, and $\mathcal{H}$ is the set of all points.

If the given set does not allow a transversal with a single point, one can ask whether there are $k$ points such that their union intersects each set in $X$. If such a set of $k$ points exists, we say that $X$ is *$k$-pierceable* and call $k$ the *piercing number* of $X$.

A famous theorem of this type proved by Alon and Kleitman [1] is the following generalization of Helly's Theorem (for $p = q = d + 1$):

**Theorem (Alon-Kleitman).** Let $X$ be a finite family of convex sets in $\mathbb{R}^d$. Let $p \geq q > d$ be integers such that among any $p$ sets of $X$ there are $q$ sets with a common point. Then $X$ is pierceable with $HD_d(p, q)$ points.[3]

The crucial part is that this number depends only on $p, q$, and $d$, but not on the size of $X$. See, e.g., Wenger [12] or Matoušek [10, Ch. 10] for a gentler introduction.

In this paper, we will derive a similar result for a different class $\mathcal{H}$ of objects, namely axis-parallel hyperplanes, i.e., planes of the form $h\colon x_i = c$, $1 \leq i \leq d, c \in \mathbb{R}$. Analogously to the above notion, we say that a set $X$ of objects is *$k$-sliceable*, if there are $k$ axis-parallel hyperplanes whose union intersects each $x \in X$. By replacing each object by its bounding box, it suffices to talk about *hyperrectangles* ("boxes") of course.

**Theorem 1.** *Let $X$ be a finite family of convex sets in $\mathbb{R}^d$. Let $m > n$ be integers such that any set of size $m$ can be sliced by $n$ axis-parallel hyperplanes. Then $X$ can be sliced by $T_d(m) < \infty$ axis-parallel hyperplanes.*

This can also be stated in a slightly different way. Thereto, let graph $G_X = (V, E)$ be the graph with $V = X$ and $(x, y) \in E$ if and only if there is an axis-parallel hyperplane that intersects both objects.

**Corollary 1.** *Let $X$ be a finite family of convex sets in $\mathbb{R}^d$. If $G_X$ does not have an independent set of size $m$, then $X$ can be sliced by $T_d(m) < \infty$ axis-parallel hyperplanes.*

Until now, existence of this function was only known for $d = 1, 2$ (see Vatter [11]).

[3] The $HD$ stands for Hadwiger and Debrunner, who originally stated this as a conjecture.

The bound we give for $T_d(m)$ in the proof of existence in Sec. 2 is huge. Using this result though, there is a very simple proof for the following, which we will present in Sec. 3.

**Theorem 2.** *$T_d(m) \in \mathcal{O}\left(f(d) \cdot m^d\right)$ for some function $f$.*

That is, for any fixed $d$ this number is polynomial in the size of the largest independent set, and independent of the total number of objects.

For $d = 2$, this also partly settles an open question on [2]: The best known bound [11] was exponential in $m$, and Theorem 2 gives a quadratic bound.

Because any set of objects that is intersected by a single hyperplane forms a clique in the corresponding graph, we get another nice corollary. For Interval graphs $G$, which are perfect, it holds that $\alpha(G)$, the indepence number, is equal to $\chi(\overline{G})$, the clique partition number. The corollary shows that unions of Interval graphs have a similar property.

**Corollary 2.** *For any $d$, there is a constant $c_d > 0$ such that the following holds: Let $\mathcal{I}_1, \ldots, \mathcal{I}_d$ be interval graphs on the same vertex-set $V$, and $\mathcal{I}$ their union. Let $m$ be the size of the largest independent set in $\mathcal{I}$. Then $\alpha(G) \leq \chi(\overline{G}) \leq c_d \cdot m^d \cdot \alpha(G)$.*

And, by using the pigeonhole-principle, we also get a Ramsey-type corollary:

**Corollary 3.** *For any $d$, there is a constant $c'_d > 0$ such that the following holds: Let $\mathcal{I}_1, \ldots, \mathcal{I}_d$ be interval graphs on the same vertex-set $V$, and $\mathcal{I}$ their union. Then $\mathcal{I}$ either contains a clique or an independent set of size $c'_d \cdot \sqrt[d+1]{n}$ (for $n$ large enough).*

From computational point of view, this problem has also been considered: Dom et al. [3] and Giannopoulos et al. [4] independently of each other showed that the problem of deciding whether a given set of rectangles in $\mathbb{R}^d$ can be sliced ("stabbed") by $k$ hyperplanes is W[1]-hard with respect to $k$. Also, they both show that the problem for disjoint unit squares in the plane is fixed-parameter tractable, i.e., can be solved in time $\mathcal{O}\left((4k+1)^k n^2\right)$. Recently it has been shown by Heggernes et al. [6] that the problem is even fixed parameter tractable for disjoint rectangles of arbitrary size.

## 2 Existence of $T_d(m)$

Let $[d] := \{1, \ldots, d\}$. For $D \subseteq [d]$, we say that $r, r'$ are *independent with respect to* $D$, if $\mathrm{pr}_D(r) \cap \mathrm{pr}_D(r') = \emptyset$. Two sets are called $j$-disjoint, if they are disjoint with respect to dimension $j$.

We will prove Theorem 1 by induction on the dimension $d$. The problem one faces here is that even if we do not have an independent set of size $m$ in $d$ dimensions, we can not say anything about the size of the largest independent set with respect to lower dimensions: The boxes might all lie on a common hyperplane orthogonal to $e_d$, i.e., form an independent set of size 1, but be pairwise disjoint with respect to $[d-1]$.

Thus, we need to be a little more careful when doing the dimension reduction. The main observation is that there cannot be too many pairwise $d$-disjoint independent sets of a certain size. This is expressed by the following lemma:

**Lemma 1.** *Let $X$ be a set of hyperrectangles in $\mathbb{R}^d$ that does not have an independent set of size $m$. Then we can choose $m-1$ parallel hyperplanes such that the remaining*[4] *boxes are partitioned into $m$ sets with the property that for each of these sets the largest independent set with respect to $[d-1] = \{1, \ldots, d-1\}$ is of size less than $dm(m-1)$.*

In order to prove this, we need a simple lemma that states that between *two* $d$-disjoint independent sets there cannot be too many incidences.

**Lemma 2.** *Let $M_1$, $M_2$ be two independent sets of intervals. Then the total number of incidences is at most $|M_1| + |M_2| - 1$.*

**Corollary 4.** *If $M_1, M_2$ are two sets of rectangles in $\mathbb{R}^d$ such that*

- *each $M_i$ is an independent set with respect to $[d-1]$*
- *$M_1$ and $M_2$ are disjoint with respect to $d$*

*then the total number of incidences (dependences) between $M_1$ and $M_2$ is less than $(d-1)\,(|M_1| + |M_2|)$.*

*Proof.* By assumption, there are no incidences in dimension $d$. For each of the remaining $d-1$ dimensions the two sets form sets of disjoint intervals, thus by Lemma 2 can have at most $|M_1| + |M_2| - 1$ incidences per dimension. □

[4] I.e., the boxes not yet intersected by any of these $m-1$ planes.

Using this, we can prove the next lemma by a counting argument.

**Lemma 3.** *Let $X$ be a set of hyperrectangles in $\mathbb{R}^d$ with no independent set of size $m$ and $H$ be a set of $m-1$ hyperplanes orthogonal to $e_d$, partitioning the rectangles into $m$ sets. Then there is a region whose largest independent set with respect to $\{1,\dots,d-1\}$ is of size less than $a := dm(m-1)$.*

*Proof.* Assume we have $m$ independent sets of size $dm(m-1)$ that are pairwise $d$-disjoint. Choose a subset of exactly this size from each of the sets $M_1,\dots,M_m$. As any selection of one element from each set $x \in \prod M_i$, of which there are $a^m$, must have at least one dependence with respect to $[d-1]$ (otherwise we had an independent set of size $m$ in the original instance), and any dependence counts for at most $a^{m-2}$ such sets, we need at least $\frac{a^m}{a^m-2} = a^2$ intersections. But because of Corollary 4, the total number of intersections is at most

$$(d-1)\cdot\sum_{i\neq j}(|M_i|+|M_j|-1) < d\cdot\binom{m}{2}\cdot 2a = dm(m-1)\cdot a = a^2.$$

Thus, at least one of the independent sets must be of smaller size. □

Now we come to prove the main lemma. The idea is to *sweep* about the set with a hyperplane orthogonal to $e_d$ and pick hyperplanes subsequently just before further sweeping would yield a large independent set with respect to $[d-1]$ on the negative side. See Fig. 2.

This vague argument is formalized in the following Lemma.

*Proof (Lemma 1).* For a set $R$ of (closed) rectangles, let $\beta_R$ be minimal such that the halfspace $x_d \le \beta_R$ contains a $[d-1]$ independent set of size $a$. (If such a $\beta_R$ does not exist, we are done by induction.) Let $h\colon x_d = \beta_R$ be the corresponding hyperplane.

Observe that there cannot be a $[d-1]$ independent set of size $a$ strictly on the negative side of $h$ then, for otherwise $h$ would not be minimal.

For the set $R$, let $R_h$ denote the set of all rectangles that strictly lie in the halfspace $x_d > \beta_R$, i.e., the ones that have now "seen" the sweeping hyperplane so far.

Now we simply pick hyperplanes $h_i$ as follows: $h_1\colon x_d = \beta_R$, $h_{i+1}\colon x_d = \beta_{R_{h_i}}$.

Because of Lemma 3, this process stops (i.e., $\beta$ is undefined) after we have chosen at most $m-1$ hyperplanes, for otherwise we would $m$

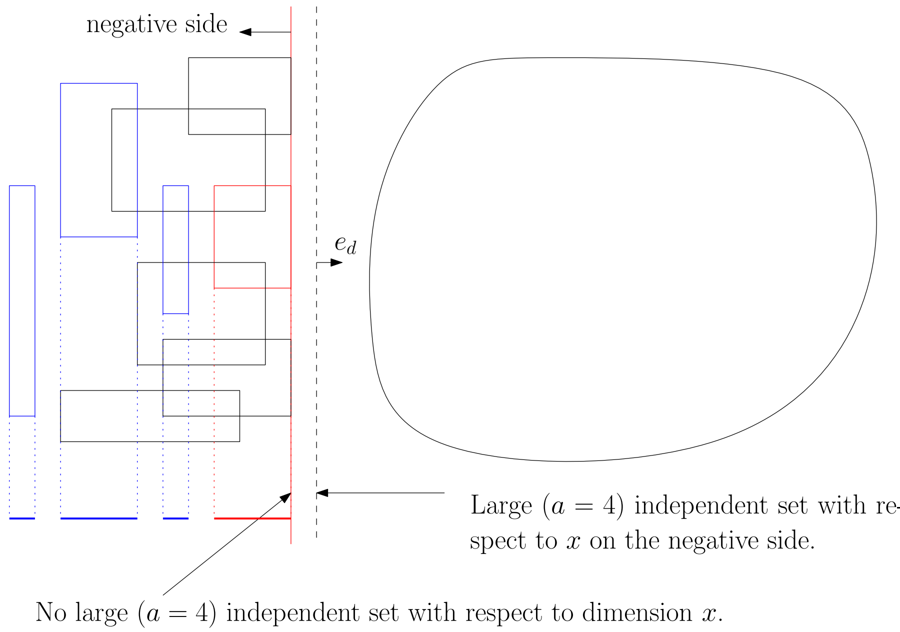


**Fig. 1.** Schematic drawing of the sweeping procedure.

independent sets with respect to $[d-1]$ of size $a$ that are pairwise $d$-disjoint.

Thus, we have chosen at most $m-1$ hyperplanes, and in each of the at most $m$ induced regions the largest independent set with respect to $[d-1]$ is of size less than $a$. $\square$

**Corollary 5.** *$T_d(m)$ exists.*

*Proof.* The existence of $T_1(m)$ is clear, and Lemma 3 yields

$$T_d(m) < m \cdot T_{d-1}\left(dm(m-1)\right) + m - 1.$$

$\square$

## 3 A polynomial bound

Using the existence of $T_d(m)$ from the previous section, in a straightforward way from this lemma we can derive a much stronger bound for the higher-dimensional case:

**Lemma 4.** *Let $R$ be a set of boxes in $\mathbb{R}^d$ that does not have an independent set of size $m$, then it can be sliced by*

$$T_d(m) \leq (2m-1)^d \cdot T_d(d) + (m-1) \cdot 2d$$

*axis-parallel hyperplanes.*

*Proof.* Given a maximum independent set $M$ of size $(m-1)$, we choose a hyperplane through each of the boundaries. This makes a total of $(m-1) \cdot 2d$ hyperplanes. Any box not intersected yet lies inside one of the at most $(2m-1)^d$ regions created by these hyperplanes.

The crucial observation now is the following: Any region can have a nonempty projection with at most $d$ of the boxes in $M$ (at most one for each direction, as $M$ is an independent set). Thus, each region can contain an independent of size at most $d$: Assume we had a box that contained some independent set $M'$ of size $d' > d$. Let $M_d \subset M$ be the set of boxes that have a nonempty intersection with this region. Then $M - M_d \cup M'$ is an independent set and

$$\left|(M - M_d) \uplus M'\right| = |M| - |M_d| + |M'| \geq m - d + d' \geq m,$$

a contradiction.

Thus, we need at most $T_d(d)$ additional hyperplanes for each region, making it a total of $(2m-1)^d \cdot T_d(d) + (m-1) \cdot 2d$. $\square$

**Corollary 6.** *For any fixed $d$ we have $T_d(m) \in \mathcal{O}\left(m^d\right)$.*

Observe how this bound is based on the existence of $T_d(d)$ in the first place!

## 4 Conclusion and open problems

We have shown that the slicing number for convex objects with bounded independent set exists in arbitrary dimension, and that it is bounded by a polynomial for any fixed $d$. As during the proof in Sec. 3 we are not very careful with our analysis, we assume that the bound is actually much stronger:

*Conjecture 1.* For any fixed $d$, it holds that $T_d(m) \in \mathcal{O}(m)$.

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