On the computational complexity of Ham-Sandwich cuts, Helly sets and related problems

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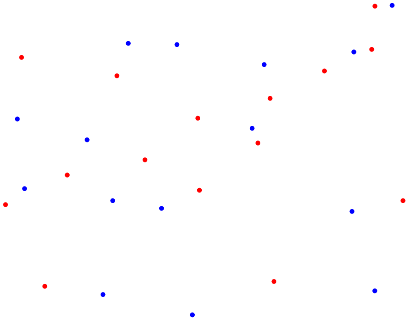
March 12, 2011

The planar case

Let $P = R \cup B$. Then there is a line that *bisects* both sets simultaneously.



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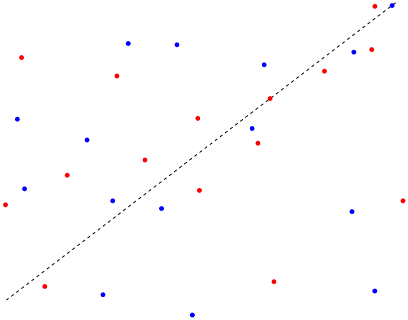

Such a line can be found in linear time!
[Edelsbrunner, Waupotitsch; '86]

General version

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- trivial algorithm: n^{d+1}
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- recently: $O(n \log^d n)$ for well separated point sets [Bárány, Hubard, Jéronimo; '08], [Steiger, Zhao; '09]

The decision problem

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No complexity results known so far.

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- requires $n^{\Omega(d)}$ time, unless 3-SAT can be solved in $2^{o(n)}$

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there exists a *linear* ham-sandwich cut

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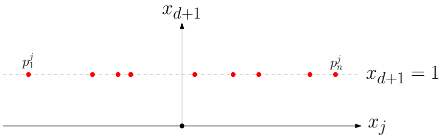

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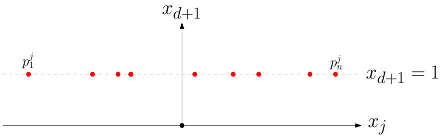

Observe: if $h \cdot p_i^j = 0$ then $h_j = -h_{d+1} s_i$.

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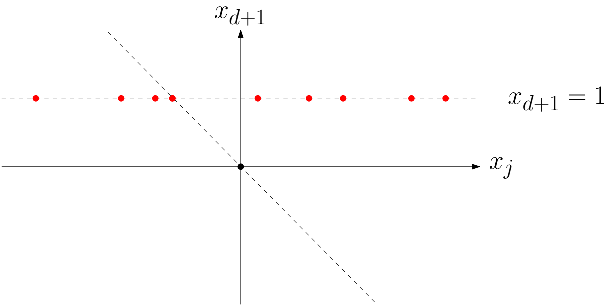

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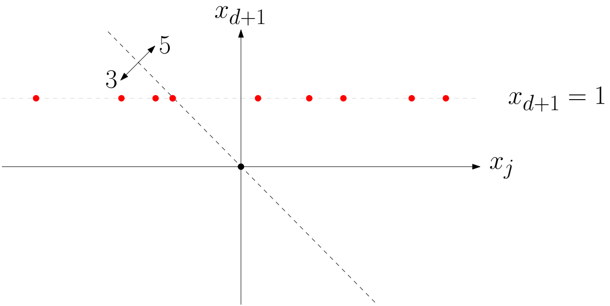

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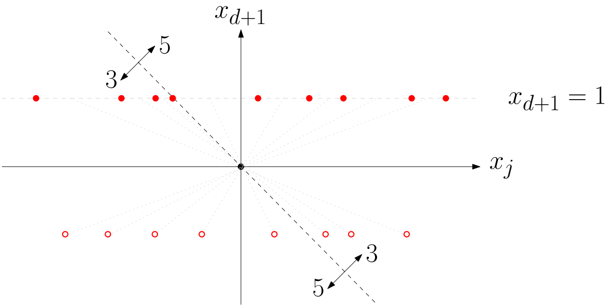

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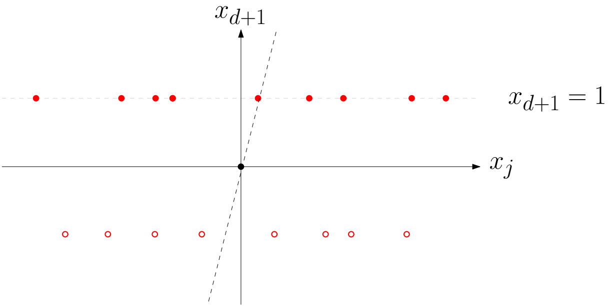

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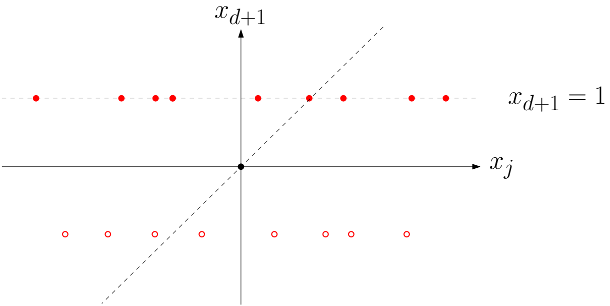

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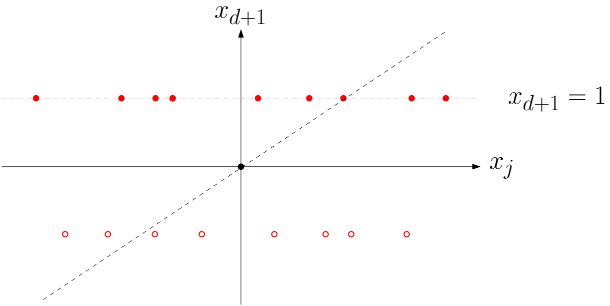

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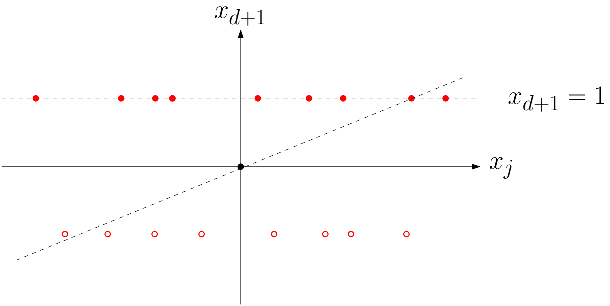

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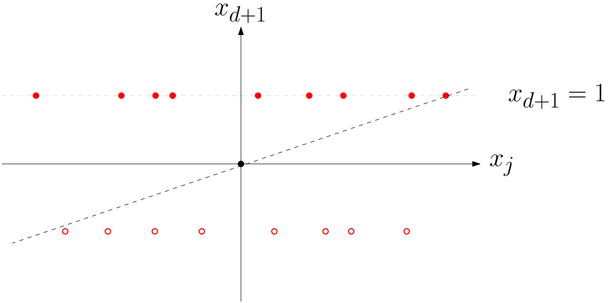

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Set

$$q = -\sum_{i=1}^{d} \mathbf{e}_i$$

and $P_{d+1} = \{q\}$.

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Why it works

Claim:

There are d numbers that sum to 0.

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There is a linear ham-sandwich cut.

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- more specific: Minimum Infeasible Subsystem for LP