Hardness of discrepancy and related problems

parameterized by the dimension

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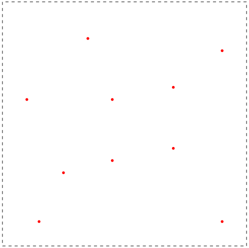
How equally distributed is a point set?

Let P be a finite set of points in the d-dimensional unit cube and O be a subset of $\mathbb{R}^d$. We set

$$D_O(P) := \left| \frac{|P \cap O|}{|P|} - vol(O) \right|.$$



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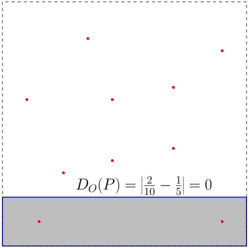

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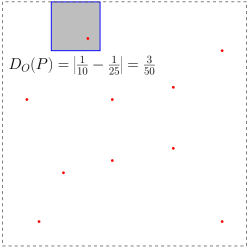

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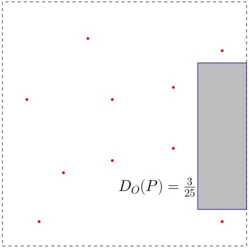

The *Star Discrepancy* is defined as

$$D^*(P) = \max_{I \in \mathcal{I}^*} D_I(P)$$

where $\mathcal{I}^*$ is the set of all boxes inside the unit cube that contain the origin.

The *Box Discrepancy* is defined as

$$D(P) = \max_{I \in \mathcal{I}} D_I(P)$$

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Fixed-parameter tractability and parameterized hardness

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- A problem is $W[1]$–hard if the k–$\textsc{Clique}$ problem can be reduced to it by a parameterized reduction.

The MAXIMUM-EMPTY-SUBINTERVAL problem

Given: A finite point set P inside the d-dimensional unit cube, a number V.
Question: Is there a box inside $[0,1]^d$ containing the origin and none of the points that has volume at least V?

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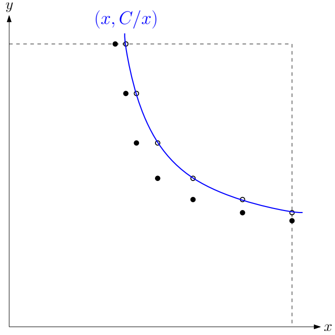
- We will place points into k orthogonal planes to encode the vertices
- and additionally into their pairwise product to encode the edges.
- Observe: As the origin must be contained, the planes can be considered separately.

Let $\mu > 1$ and $C := 1/\mu^{n-1} < 1$. In each of the k planes, we place $n+1$ points (n large rectangles) as follows.



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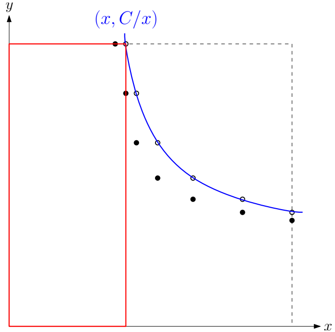

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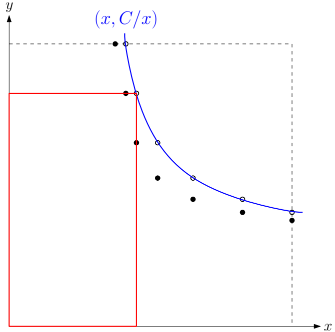

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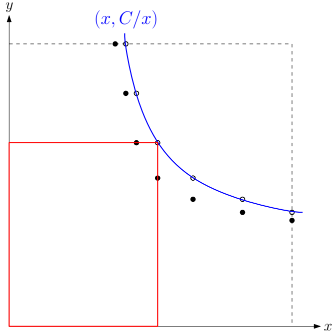

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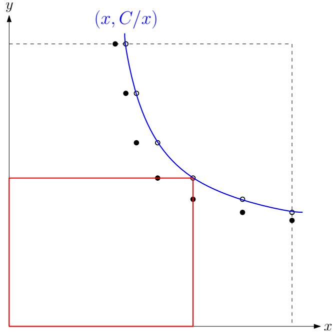

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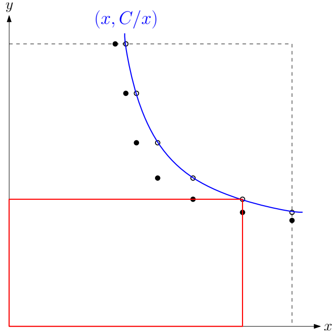

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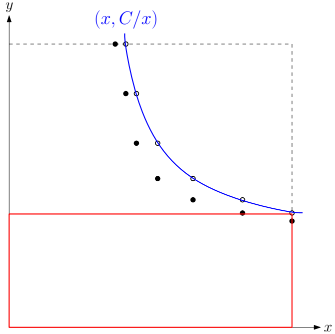

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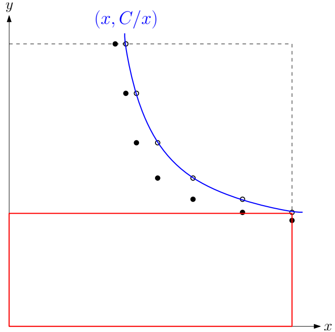

- A selection of k such rectangles corresponds to a subset of vertices of G.

How to forbid certain large rectangles?

We want to forbid boxes corresponding to u in the i–th $\mathbb{R}^2$ and to v in the j–th $\mathbb{R}^2$ for $uv \notin E$.

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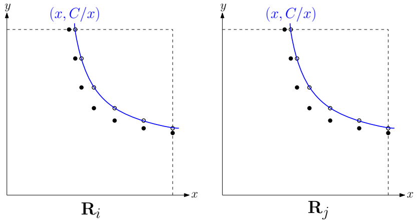

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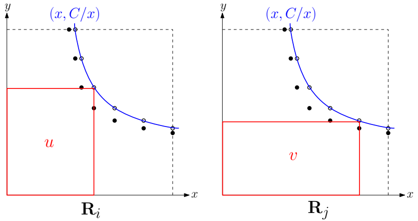

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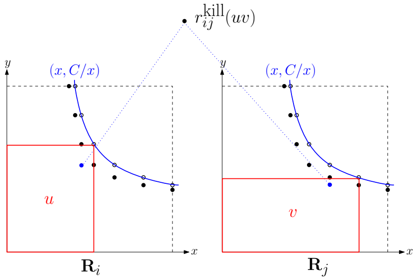

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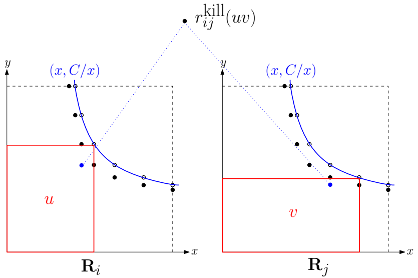

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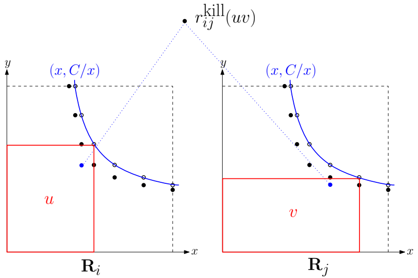

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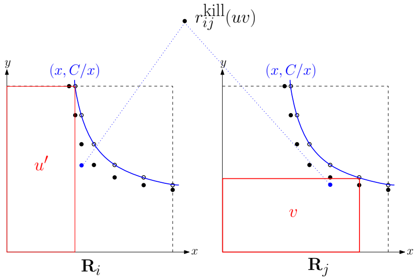

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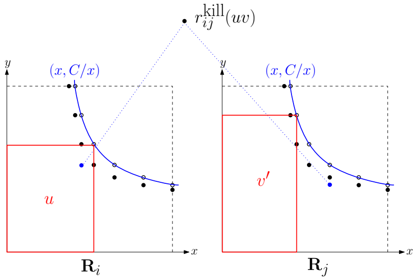

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Correctness

Lemma

G has a k–clique iff there is an empty box of size C^k.

Theorem

MAXIMUM-EMPTY-BOX *is W[1]–hard with respect to the dimension.*

Corollary

Unless W[1] $=$ FPT, there is no algorithm running in time $\mathcal{O}\left(f(d) \cdot |P|^c\right)$ for this problem.

An even stronger result

- Observe: If there is no k–clique, we need to choose at least one rectangle of size at most C/μ in one plane.

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- An empty box can have a total volume of at most C^k/μ.
- Choosing μ large creates a large gap between positive and negative instances.
- Approximating the problem by, e. g., a factor of $1/2^{|P|}$ is NP–hard!

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The proof can be modified to show the W[1]–hardness of

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Thank You.