Hardness of discrepancy and related problems
parameterized by the dimension

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Basics
- Geometric Discrepancy
- Parameterized Complexity
- Maximum-Empty-Subinterval

Our Results
- Overview

Hardness of Maximum-Empty-Subinterval
- The construction
- Encoding vertices
- Encoding edges
- Correctness
- Approximation

Conclusion
- Adaption to the other problems
How equally distributed is a point set?

Let $P$ be a finite set of points in the $d$-dimensional unit cube and $O$ be a subset of $\mathbb{R}^d$. We set

$$D_O(P) := \left| \frac{|P \cap O|}{|P|} - \text{vol}(O) \right|.$$
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The *Star Discrepancy* is defined as

\[ D^*(P) = \max_{I \in \mathcal{I}^*} D_I(P) \]

where $\mathcal{I}^*$ is the set of all boxes inside the unit cube that contain the origin.

The *Box Discrepancy* is defined as

\[ D(P) = \max_{I \in \mathcal{I}} D_I(P) \]

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$L$ is **fixed-parameter tractable**, if it can be decided in $O(f(k) \cdot |x|^c)$ time whether $(x, k) \in L$.

A problem is $W[1]$–hard if the $k$–CLIQUE problem can be reduced to it by a parameterized reduction.
The Maximum-Empty-Subinterval problem

**Given:** A finite point set $P$ inside the $d$-dimensional unit cube, a number $V$.

**Question:** Is there a box inside $[0, 1]^d$ containing the origin and none of the points that has volume at least $V$?
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- Observe: As the origin must be contained, the planes can be considered separately.
Let $\mu > 1$ and $C := 1/\mu^{n-1} < 1$. In each of the $k$ planes, we place $n + 1$ points ($n$ large rectangles) as follows.
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![Diagram showing the placement of points and rectangles.](attachment:image.png)
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A selection of $k$ such rectangles corresponds to a subset of vertices of $G$. 
How to forbid certain large rectangles?

We want to forbid boxes corresponding to $u$ in the $i$–th $\mathbb{R}^2$ and to $v$ in the $j$–th $\mathbb{R}^2$ for $uv \notin E$. 
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- Add a point in the product of the two planes ($\mathbb{R}^4$).

\[ (x, C/x) \]
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- Do this for all $1 \leq i \neq j \leq k$ and all $uv \notin E$. 
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Lemma

$G$ has a $k$–clique iff there is an empty box of size $C^k$.

Theorem

**Maximum-Empty-Box** is $W[1]$–hard with respect to the dimension.

Corollary

Unless $W[1] = FPT$, there is no algorithm running in time $O(f(d) \cdot |P|^c)$ for this problem.
An even stronger result

- Observe: If there is no $k$–clique, we need to choose at least one rectangle of size at most $C/\mu$ in one plane.
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- An empty box can have a total volume of at most $C^k/\mu$.
- Choosing $\mu$ large creates a large gap between positive and negative instances.
- Approximating the problem by, e. g., a factor of $1/2^{|P|}$ is \text{NP–hard}!
Shrink and lift

The proof can be modified to show the $W[1]$–hardness of

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Thank You.