

# WKB Approximation of the Nonlinear Schrödinger-Newton Equations

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**Abstract.** In this paper we present a WKB approximation for spherically-symmetric solutions of the Schrödinger-Newton equations. These are nonlinear modifications of the ordinary Schrödinger equation involving gravitational self-interaction of the wavefunction. Applying the WKB procedure leads to two different nonlinear differential equations for the gravitational potential  $U$  for positive and negative values of  $U$ . Both equations can be solved analytically. The corresponding wavefunctions that are regular within the neighbourhood of the turning point are calculated and compared to the numerical solutions. In the last section the asymptotic behaviour of the eigenvalues is derived by aid of a modified Bohr-Sommerfeld quantization rule.

## 1. Introduction

The Schrödinger-Newton equations form a nonlinear system of differential equations obtained by coupling together the ordinary Schrödinger equation and Newton's field equation for the gravitational potential. The equations describe a particle moving in its own gravitational field, where the field is generated by the particle's probability density via classical Newton's field equation. The name *Schrödinger-Newton equations* was originally introduced by Penrose in [1]. He also gave a heuristic motivation for the model provided by the equations. In the following, Schmidt [2] extended the Schrödinger-Newton equations to many particles and provided a model of the measurement process. Moroz et al. performed numerical studies in [3]. In a second work [4] they proved that a class of bounded solutions for the Schrödinger-Newton equations exists. The equations are also known as Choquard's equation from plasma physics. As far back as in 1977 Lieb proved the existence and the uniqueness of the ground state for Choquard's equation in [5] using functional analytical methods.

Since we are dealing with classical gravitational interaction, it seems reasonable to derive a quasi-classical approximation for the problem. We adopt the ordinary WKB-method for the time-independent one-dimensional Schrödinger equation and for a given potential  $U$ , in order to reduce the Schrödinger-Newton equations approximately to a nonlinear ODE which can be solved analytically.

The paper is structured as now to be told: In section 2, we introduce the Schrödinger-Newton equations with their general properties. A numerical analysis, slightly different from the one in [3] is presented in section 3. To come to the main

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aspect in section 4, we develop a WKB-approximation for the Schrödinger-Newton equations that reproduces most of the numerical results. Further, an asymptotic formula for the eigenvalues is derived in the last section.

## 2. The Schrödinger-Newton equations

Considering a single particle of mass  $m$ , the time-independent Schrödinger-Newton equations take the form

$$\Delta\psi = -\frac{2m}{\hbar^2}(E - \phi)\psi, \quad (1)$$

$$\Delta\phi = -4\pi Gm^2|\psi|^2, \quad (2)$$

where  $\psi$  is the wavefunction,  $\phi$  denotes the gravitational potential,  $E$  is the energy eigenvalue and  $G$  is the gravitational constant. In this context, the probability density  $|\psi|^2$  multiplied by  $m$  is interpreted as the particle's matter density. Following [3], we redefine the variables

$$U := \frac{2m}{\hbar^2}(E - \phi) \quad \text{and} \quad S := \frac{\hbar\psi}{\sqrt{8\pi Gm^3}}, \quad (3)$$

and receive a simplified pair of equations of real functions  $S$  and  $U$

$$\Delta S = -SU, \quad (4)$$

$$\Delta U = -S^2 \quad (5)$$

that we may call the modified Schrödinger-Newton equations. Here, as in the previous papers [4] and [3], we shall restrict our attention to the spherically-symmetric solutions. It can be easily verified that the system (4) and (5) has a scaling invariance with respect to a one-parameter group of scaling transformations  $\Gamma$  that is given by

$$\Gamma(\mu^2, \mu) : \{S, U, r\} \mapsto \{\mu^2 S, \mu^2 U, \mu^{-1} r\}. \quad (6)$$

We are only interested in solutions  $S$  and  $U$  that are normalizable. So we can assume the normalization integral (total probability)

$$I = \int_0^\infty r^2 S(r)^2 dr \quad (7)$$

to be finite. It transforms under rescaling  $\Gamma$  according to  $I \mapsto \mu I$ . Consequently, choosing  $\mu = I^{-1}$  will set the total probability equal to one. We require the solutions  $\{S(r), U(r)\}$  to be smooth at the origin. This implies that their derivatives vanish at  $r = 0$ . Writing  $S(0) = S_0$  and  $U(0) = U_0$  and integrating (4) as (5) twice with respect to  $r$ , we obtain integral equations for  $S$  and  $U$ :

$$S(r) = S_0 - \int_0^r x \left(1 - \frac{x}{r}\right) S(x)U(x) dx, \quad (8)$$

$$U(r) = U_0 - \int_0^r x \left(1 - \frac{x}{r}\right) S(x)^2 dx. \quad (9)$$

We may use the integral solutions to derive asymptotic forms for  $S$  and  $U$  (cf. [3, 4]). Near  $r = 0$ , Taylor series expansion yields

$$S(r) = S_0 - \frac{S_0 U_0}{6} r^2 + \mathcal{O}(r^4) \quad (10)$$

$$U(r) = U_0 - \frac{S_0^2}{6} r^2 + \mathcal{O}(r^4). \quad (11)$$

Equation (9) can be written in a different form in order to indicate the asymptotic behaviour for large  $r$

$$U(r) = U_0 - \int_0^r xS(x)^2 dx + \frac{1}{r} \int_0^r x^2 S(x)^2 dx \xrightarrow{r \rightarrow \infty} U_\infty + \frac{I}{r}, \quad (12)$$

where we only assumed the solutions to be normalizable. In this case, both integrals necessarily converge. The normalized energy eigenvalue is then written as

$$E = \frac{\mu^2}{2} \lim_{r \rightarrow \infty} U(r) = \frac{U_\infty}{2I^2}. \quad (13)$$

The wavefunction  $S$  exhibits the usual exponential behaviour for large values of  $r$ . Setting  $U$  equal to  $U_\infty$ , we find from (4)

$$S(r) \sim \frac{k}{r} e^{-\sqrt{|U_\infty|}r}. \quad (14)$$

### 3. Numerical Analysis

In principle, the Schrödinger-Newton equations can be integrated by standard numerical routines. However, the  $1/r$ -singularity of the Laplacian operator requires extremely small step widths tending to zero with  $r$  decreasing. In order to circumvent the  $1/r$ -term, we transform (4) and (5) into an autonomous system of differential equations. We suppose to achieve a considerable improvement of the numerical accuracy by avoiding the singularity. First of all the modified Schrödinger-Newton equations are written as one fourth order equation

$$\frac{dS}{dr} = F\left(r, S, \frac{d^2S}{dr^2}, \frac{d^3S}{dr^3}, \frac{d^4S}{dr^4}\right), \quad \text{where } F: \mathbf{R}^5 \rightarrow \mathbf{R}. \quad (15)$$

Taking into account the one-parameter group of scaling transforms that is generated by  $\Gamma(\mu^2, \mu)$ , we can, according to [6] reduce the order of (15) by one. To this end, consider the transformation

$$S(r) \mapsto r^2 S(r) = x(t), \quad (16)$$

$$U(r) \mapsto r^2 U(r) = y(t) \quad \text{with } r = e^t. \quad (17)$$

Substitution into (4) and (5) yields an autonomous system for  $\{x(t), y(t)\}$

$$\ddot{x}(t) - 3\dot{x}(t) + (2 + y(t))x(t) = 0, \quad (18)$$

$$\ddot{y}(t) - 3\dot{y}(t) + 2y(t) + x(t)^2 = 0. \quad (19)$$

Here, the dot denotes differentiation with respect to  $t$ . We found that integrating the autonomous equations (18) and (19) instead of the original system increases the numerical accuracy essentially. Nevertheless, for the sake of clarity, we shall refer to the original solutions  $S$  and  $U$  in the following paragraph.

We shall now turn to the bound states of (4) and (5). The solutions can be enumerated in the way that the  $n$ -th solution has  $n - 1$  zeros. Further, using the scaling freedom (6) we can set  $U_0 = 1$  without loss of generality. The solutions therefore only depend on the single free parameter  $S_0 > 0$ . Let  $S_0$  be an element of  $M_0 = ]a_0, b_0]$ , where  $M_0$  has to be a sufficiently large interval. We choose the positive initial value  $\xi_1 = (a_0 + b_0)/2$  and start the integration for the  $n$ -th solution. One can show that if the calculated function  $S_1$  has more than  $n - 1$  zeros, then  $\xi_1 < S_0$  and else  $\xi_1 > S_0$ . Thus we can define a recursion sequence  $\{M_i\}$  as

$$M_{i+1} = ]a_{i+1}, b_{i+1}] = \begin{cases} ]\xi_i, b_i] & \text{if } N_i > n - 1 \\ ]a_i, \xi_i] & \text{else} \end{cases} \quad (20)$$

with  $N_i$  counting the zeros of the calculated function  $S_i$  and  $\xi_i = (a_{i-1} + b_{i-1})/2$ . It follows by induction that  $M_0 \supset M_1 \supset \dots \supset M_n \supset \dots \supset \{S_0\}$ . Simultaneously the sequence of functions  $\{S_i\}$  converges to the wanted solution  $S$  as we get the potential  $U$ . The corresponding eigenvalues can be easily calculated using equation (13). Here, in contrast to the procedure suggested in [3], it is not necessary to make a difference  $n$  even or odd.

As figure 2 shows,  $E(n)$  exhibits power law behaviour. We have plotted the first twenty eigenvalues and conjecture the asymptotic behaviour for large  $n$  to be

$$E_n \propto n^{-2} \quad \text{with } n \rightarrow \infty. \quad (21)$$

Moroz et al. found a deviating law in [3]. However, they calculated only the first ten eigenvalues, which might solve this contradiction. To verify (21), we use Levin's u-sequence acceleration method (cf. [7]). We may define  $U_n = \ln(E_n/E_{n+1})/\ln((n+1)/n)$ . If the u-sequence elements are assigned to  $u_1 = U_1$ ,  $u_2 = U_2 - U_1$ ,  $u_3 = U_3 - U_2$  and so forth, then the Levin formula for the limit of  $\{U_n\}$  is given by

$$U^{(N)} = \frac{\sum_{n=1}^N (-1)^{n-1} \binom{N}{n} n^{N-2} \frac{U_n}{u_n}}{\sum_{n=1}^N (-1)^{n-1} \binom{N}{n} n^{N-2} \frac{1}{u_n}}. \quad (22)$$

We find  $U^{(20)} = -2.03435\dots$  in good agreement with the results of the WKB approximation in section 5. In order to estimate the numerical error of our eigenvalues, we multiply (4) by  $r^2$  and integrate the obtained equation from 0 to infinity. It follows

$$0 = \int_0^\infty r^2 S U dr = \frac{2m}{\hbar^2} \int_0^\infty r^2 S (E - \phi) dr \simeq \frac{2m}{\hbar^2} \int_0^\infty r^2 S \varepsilon dr, \quad (23)$$

where  $\varepsilon$  denotes the deviation of the numerical eigenvalue from the exact one. By calculating the corresponding error for  $S$ , it can be shown that  $S$  is numerically sufficiently exact determined compared to  $U$ . Consequently

$$\varepsilon = \left| \int_0^\infty r^2 S U dr \right| \left| \frac{2m}{\hbar^2} \int_0^\infty r^2 S dr \right|^{-1} \quad (24)$$

provides a crude estimate for the absolute error of the normalized eigenvalue. It indicates when the discretization error becomes relevant and renders the numerical solutions to be unstable. We observed that  $\varepsilon$  blows up, when  $n$  gets larger than 20.

#### 4. WKB-Approximation

We are aiming at developing an approximate solution for the Schrödinger-Newton equations that is asymptotically valid. So it seems reasonable to adopt quasi-classical methods as the WKB approximation. Usually, this method is used to construct approximate bound solutions of the ordinary one-dimensional Schrödinger equation with a given potential  $U$  for large quantum numbers  $n$  (cf. [8]). If we define  $X(r) = rS(r)$ , then we can write equation (4) as a one-dimensional Schrödinger equation

$$X''(r) = -U(r)X(r). \quad (25)$$

As we know from the general WKB procedure,  $X$  can be approximated as follows

$$X(r) = C U(r)^{-\frac{1}{4}} \operatorname{Re} \left\{ \exp \left( i \int_{r_0}^r U(x)^{\frac{1}{2}} dx \right) \right\}. \quad (26)$$

Here, we restrict the WKB solution to its real part. The integration constant  $r_0$  is determined by the condition  $U(r_0) = 0$  (turning point). Without loss of generality we may assume that  $U$  is positive for  $r < r_0$  (classically allowed region) and negative, if  $r > r_0$  (classically forbidden region). We shall regard these two domains separately.

Our first step will be to calculate  $U$  from the field equation (5) for  $U > 0$ . Since we focus our attention on the large quantum numbers limit, we expect the function  $X$  to be rapidly oscillating. Thus let

$$\bar{X}(r) = \frac{C}{\pi} U(r)^{-\frac{1}{4}} \int_0^{2\pi} \cos^2 \Theta d\Theta = C U(r)^{-\frac{1}{4}} \quad (27)$$

be the averaged value of  $X$ , where  $\Theta$  denotes the wavefunction's phase contribution. Then,  $\bar{X}$  approximately satisfies the equation  $\bar{X}'' = -U\bar{X}$  and we may therefore use the ansatz  $S(r) = r^{-1}\bar{X}(r)$ . Substituting this expression into (5) we obtain the WKB equation for the classically allowed domain:

$$U''(r) + \frac{2}{r}U'(r) + \frac{C^2}{r^2}U(r)^{-\frac{1}{2}} = 0. \quad (28)$$

The correct choice of  $C$  will guarantee  $I = 1$  in (7). If we integrate (28) from 0 to  $r < r_0$ , we get

$$r^2 U'(r) = -C^2 \int_0^r U(x)^{-\frac{1}{2}} dx + \beta. \quad (29)$$

We will show later that  $r^2 U'(r)$  converges to  $\beta < 0$  at the origin. The solution then necessarily diverges. Since  $\beta$  is negative, there is an open neighbourhood of  $r = 0$ , within that  $U'(r)$  is negative. Consequently,  $U$  is monotonically decreasing on the interval  $]0, r_0]$ .

Although being a modification of the well-known Emden-Fowler equation, the WKB equation represents a special case, in which no solution is to be found in literature. § Note that (28) has a scaling invariance with respect to dilatations of the argument  $U(r) \mapsto U(\mu r)$ , where  $\mu$  is a positive, real parameter. Similar to the proceedings in the second section, we can apply the same procedure as above, producing an autonomous WKB equation

$$\ddot{x}(t) + \dot{x}(t) + C^2 x(t)^{-\frac{1}{2}} = 0, \quad (30)$$

that results from the transformation  $x(t) = U(e^t)$ . As  $U$ , the function  $x$  ought to be monotonically decreasing. Thus on the parameter set  $M = ]-\infty, \ln r_0]$  the inverse function  $t(x)$  exists and we may define  $p(x) = \dot{x}(t(x))$ . Differentiation yields

$$\ddot{x}(t) = \frac{dp}{dt} = \frac{dp}{dx} \frac{dx}{dt} = p'(x)p(x). \quad (31)$$

If we now label the independent variable by  $x$ , then we can rewrite (30) into

$$p(x)p'(x) + p(x) + C^2 x^{-\frac{1}{2}} = 0. \quad (32)$$

Further, replacing the square root term by  $s = -2C\sqrt{x}$  provides an Abelian differential equation of the second kind for  $P(s) = p(-2C\sqrt{x})$

$$P(s)P'(s) + \frac{s}{2C^2}P(s) = 1. \quad (33)$$

§ We found a solution for the case that  $C^2$  is negative in [9]. However we have considerable doubts concerning the correctness of the solution presented there.

For the last equation a parametric solution can be found in literature (cf. [9]). Introducing the parameter  $\alpha = P(s) + s^2/(4C^2)$  and differentiating implicitly with respect to  $\alpha$  gives

$$\frac{d\alpha}{d\alpha} = 1 = \frac{dP}{ds} \frac{ds}{d\alpha} + \frac{s}{2C^2} \frac{ds}{d\alpha}. \quad (34)$$

A comparison to (33) reveals that  $s'(\alpha) = P(s(\alpha))$ , which leads to the special Riccati equation

$$s'(\alpha) + \frac{s(\alpha)^2}{4C^2} - \alpha = 0. \quad (35)$$

If we substitute  $s(\alpha) = 4C^2 u'(\alpha) u(\alpha)^{-1}$ , with  $u(\alpha)$  denoting the Airy functions

$$u(\alpha) = k_1 \text{Ai}((2C)^{\frac{2}{3}} \alpha) + k_2 \text{Bi}((2C)^{\frac{2}{3}} \alpha), \quad (36)$$

we find our parametric solution

$$U(\alpha) = x(\alpha) = \frac{s(\alpha)^2}{4C^2} = 4C^2 \left( \frac{u'(\alpha)}{u(\alpha)} \right)^2, \quad (37)$$

where we used the relation  $s = -2C\sqrt{x}$ . The parametrization of  $r = e^t$  is easy to find. From

$$t - t_0 = \int dt = \int \frac{dt}{dx} dx = 2 \int \frac{1}{u(\alpha)} \frac{du}{d\alpha} d\alpha = 2 \ln \frac{u}{u_0} \quad (38)$$

it follows that

$$r(\alpha) = e^{t(\alpha)} = u(\alpha)^2. \quad (39)$$

Note that the initial conditions  $U_0 = 1$  and  $U'_0 = 0$ , we chose for the Schrödinger-Newton equations, are not consistent with the WKB potential  $U$ . This becomes clear if we expand (37) into a power series with respect to  $\alpha$ . Let  $\alpha_1$  be a zero of  $r(\alpha)$ . Then, within a neighbourhood of  $\alpha_1$  we can derive

$$U(\alpha) = \left( \frac{u'(\alpha)}{u(\alpha)} \right)^2 = \frac{1}{(\alpha - \alpha_1)^2} + \frac{2\alpha_1}{3} + \mathcal{O}(|\alpha - \alpha_1|). \quad (40)$$

According to (39), we can expand  $r(\alpha)$  into

$$r(\alpha - \alpha_1) = u'(\alpha_1)^2 (\alpha - \alpha_1)^2 + \mathcal{O}(|\alpha - \alpha_1|^6), \quad (41)$$

which yields

$$U(r) = \frac{u'(\alpha)^2}{r} + \frac{2\alpha_1}{3} + \mathcal{O}(r^{\frac{1}{2}}). \quad (42)$$

This verifies the assumption  $\beta < 0$ , we used in (29). We shall now have a look at the classically forbidden region: Due to the fact that the general WKB solution (26) has a singularity at  $r_0$ , it provides an acceptable approximation only far away from the turning point. As we know from (12),  $U$  is approximately constant for large  $r$ . Assuming that  $U \approx U_\infty$  for all  $r \geq r_1 > r_0$  we may write

$$X(r) \simeq C \exp \left( - \int_{r_0}^{r_1} \sqrt{|U(x)|} dx - \int_{r_1}^r \sqrt{|U_\infty|} dx \right) \quad \forall r > r_1, \quad (43)$$

If we summarize all constant terms to the normalization constant  $C$ , then  $X$  takes the following simple form

$$X(r) \simeq C \exp \left( -\sqrt{|U_\infty|} r \right), \quad (44)$$

in accordance to the asymptotic formula (14) derived previously. Substitution into the field equation yields the WKB equation for the classically forbidden region:

$$U''(r) + \frac{2}{r}U'(r) + \frac{C^2}{r^2}e^{-2\sqrt{|U_\infty|r}} = 0. \quad (45)$$

We will see later that the solution of (45) can be extended to the whole region  $r > r_0$ , though we restricted the domain of definition of  $X$  to  $r > r_1 > r_0$ . Defining  $Y(r) = rU(r)$ ,  $Y(r)$  obeys the separable equation

$$Y''(r) + \frac{C^2}{r}e^{-2\sqrt{|U_\infty|r}} = 0. \quad (46)$$

Integrating once from  $r$  to  $\infty$  and then, a second time from  $r_0$  to  $r$  provides the solution

$$Y(r) = K + Y'_\infty r - \frac{C^2 e^{-2\sqrt{|U_\infty|r}}}{2\sqrt{|U_\infty|}} - C^2 r \operatorname{Ei}(-2\sqrt{|U_\infty|r}), \quad (47)$$

where all integration constants are again summarized to one constant that is labeled by  $K$ . The last term denotes the well-known exponential integral function (cf. [10])

$$\operatorname{Ei}(r) = \int_{-\infty}^r \frac{e^x}{x} dx. \quad (48)$$

As a numerical comparison of the last two terms to the others in (47) shows, those can be neglected. If we further take into account that  $Y'_\infty = U_\infty$ , we may approximate  $U(r)$  by

$$U(r) \simeq U_\infty + \frac{K_R}{r}, \quad (49)$$

which is consistent with the results from the first section. The only thing to do now is to fit both solution's branches to each other at  $r_0$ . The turning point is determined by  $U(r_0) = 0$ , which yields the condition

$$U(\alpha_0) = 0 \quad \iff \quad u'(\alpha_0) = 0, \quad (50)$$

$$r(\alpha_0) = r_0 \quad \iff \quad u(\alpha_0) = \sqrt{r_0}. \quad (51)$$

The parameter  $\alpha$  is restricted to the interval  $[\alpha_0, \alpha_1]$ , where  $\alpha_1$  determines the origin  $r = 0$ . The choice of  $\alpha_0$  fixes the slope  $U'_0$  at the turning point. It can be verified that the left-handed limit of  $U'(r)$  exists and is simply given by

$$\left. \frac{dU}{dr} \right|_{r \rightarrow r_0} = \frac{\alpha_0}{r_0}, \quad \text{where } r < r_0. \quad (52)$$

The corresponding WKB wavefunction is given by (26). However, there is a procedure which yields an approximation for the WKB wavefunction that is regular within the vicinity of the turning point (cf. [11]). It can be proved that

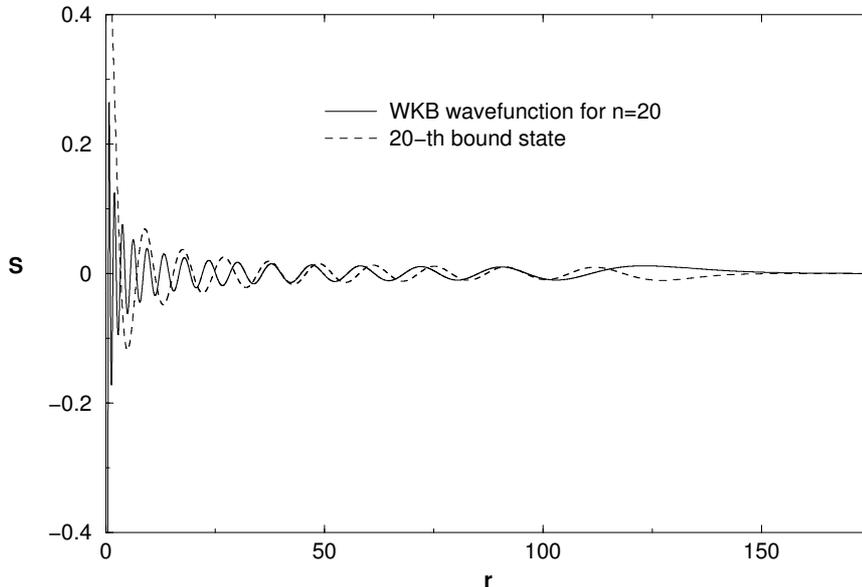
$$\chi(r) = C \left| \frac{\zeta(r)}{U(r)} \right|^{\frac{1}{4}} \operatorname{Ai}(\zeta(r)) \quad (53)$$

converges uniformly to the WKB wavefunction (26) far away from the turning point and is an exact solution of the Schrödinger equation (25) at the turning point up to first order. The argument  $\zeta(r)$  is defined by

$$\frac{2}{3}\zeta(r)^{\frac{3}{2}} = \int_{r_0}^r U(r')^{\frac{1}{2}} dr' \quad \text{for } U > 0, \quad (54)$$

$$\frac{2}{3}\zeta(r)^{\frac{3}{2}} = e^{\pm \frac{3\pi i}{2}} \int_{r_0}^r |U(r')|^{\frac{1}{2}} dr' \quad \text{for } U < 0. \quad (55)$$

With some effort, both integrals can be solved analytically. Of course, within the classically allowed domain the wavefunction's representation is parameterized by  $\alpha$  and we can compare the numerical to the WKB wavefunctions on the whole domain  $r > 0$ . Figure 1 shows such an example for  $n = 20$ . Due to the singularity of the WKB



**Figure 1.** Numerical and WKB wavefunction for  $n = 20$ .

potential at  $r = 0$ , the WKB wavefunctions diverge there, too. As a consequence, each wavefunction has infinitely many zeros, which accumulate within a neighbourhood of the origin.

Despite of qualitative confirmation of the functions in figure 1, one can observe that their phases deviate from each other. Nevertheless the WKB procedure provides generally reliable results for the eigenvalues. Since the procedure is valid especially for the semi-classical limit of large quantum numbers, we suppose the phase behaviour of the WKB wavefunction to improve only for much larger  $n$ , i.e. for higher bound states. However, we cannot examine our conjecture here.

## 5. Asymptotic behaviour of the eigenvalues

The numerical results suggest a power law for the  $n$ -dependence of the eigenvalues. We expect the asymptotic  $n$ -dependence to be  $E_n \propto n^{-2}$ . Usually, the Bohr-Sommerfeld quantization rule allows to calculate the eigenvalues for a fixed  $n$ , if the potential  $U$  is given. It is possible to derive a modified quantization rule for the Schrödinger-Newton equations.

We can calculate the number of zeros by means of Cauchy's residue theorem from the potential. Consider an analytical continuation of the solutions  $\{S, U\}$  of the Schrödinger-Newton equations to complex values  $z$ . Then, the number  $n$  of zeros is

given by

$$2\pi \left( n + \frac{1}{2} \right) = \oint_{\mathcal{C}} |U(z)|^{\frac{1}{2}} dz , \quad (56)$$

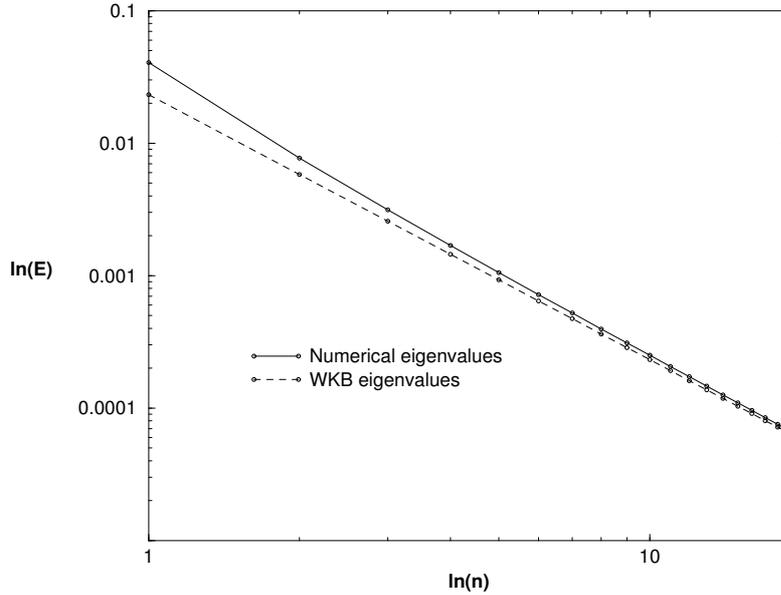
where the curve  $\mathcal{C}$  encloses the turning point  $z_0$  as all complex zeros of the wavefunction  $S$ . If we bear in mind the WKB potential's singularity at the origin, then the expression (56) transforms into

$$2\pi \left( n + \frac{1}{8} \right) = \int_0^{r_0} U(r)^{\frac{1}{2}} dr . \quad (57)$$

Since the WKB approximation is only valid for large  $n$ , we might neglect the term  $\pi/4$  on the right-hand side of the last equation. As already mentioned, the WKB potential has a scaling invariance with respect to dilatations  $\Gamma : r \mapsto \mu^{-1}r$ . Under rescaling the phase integral transforms according to

$$\begin{aligned} \tilde{P}(\tilde{r}_0) &:= (\Gamma \circ P)(\tilde{r}_0) = \int_0^{\tilde{r}_0} \tilde{U}(r)^{\frac{1}{2}} dr = \int_0^{\tilde{r}_0} U(\mu r)^{\frac{1}{2}} dr \\ &= \mu^{-1} \int_0^{\mu \tilde{r}_0} U(\rho)^{\frac{1}{2}} d\rho = \mu^{-1} P(r_0) . \end{aligned} \quad (58)$$

The tilde labels all rescaled quantities. Hence, the number of zeros formally transforms as  $n \mapsto \mu^{-1}n$ . We might understand this scaling property as follows: If we calculate a wavefunction  $S$  from the Schrödinger-Newton equation with a given WKB potential  $U$ , then under rescaling of  $U$  the number  $n$  of zeros changes to  $\tilde{n}$ , assuming  $n$  and  $\tilde{n}$  are integer.



**Figure 2.** The first 19 numerical and WKB eigenvalues.

We shall now work out the scaling behaviour of the eigenvalues. As we know from previous considerations, the eigenvalue can be expressed by terms of the potential's

asymptotic formula (12). We assume that the asymptotic expression already provides a valid approximation in the vicinity of the turning point. Thus using  $U(r_0) = 0$ , we can rewrite equation (13) into

$$E = \frac{1}{2} \frac{1}{r_0^3 U'(r_0)}, \quad (59)$$

which transforms under rescaling  $\Gamma$  according to  $E \mapsto \mu^2 E$ . The transformed eigenvalue  $\tilde{E}$  represents the eigenvalue corresponding to the quantum number  $\tilde{n}$ . A comparison to the scaling properties of  $n$  yields immediately

$$\frac{\tilde{E}}{E} = \left(\frac{n}{\tilde{n}}\right)^2 \implies \tilde{E} \propto \tilde{n}^{-2}. \quad (60)$$

This result is to be read on condition that  $\tilde{n}$  is large. It underpins our suspicion concerning the asymptotical  $n$ -dependence of the eigenvalues as figure 2 shows. It further gives an analytical justification for (21).

## 6. Summary

In this paper we have presented a semi-classical approximation of the spherically-symmetric Schrödinger-Newton equations, which provides support for the numerical results. We have found explicit solutions for the potential as for the wavefunction in the limit of high quantum numbers  $n$ . We further derived an asymptotically valid power law for the  $n$ -dependence of the eigenvalues.

Extensions of the present study containing generalizations to the non-stationary case and to non-spherically-symmetric solutions would be desirable.

## References

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