Semantics of Higher-Order Logic
(ESSLLI 2006 Course Notes)

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1 Types and Frames

We consider (classical) higher-order logic formulated as Church's simple type theory [Chu40] based on the simply typed λ-calculus. We first define the set of (simple) types $\mathcal{T}$.

Definition 1 (Types) *The set of types $\mathcal{T}$ is the least set such that*

- $o \in \mathcal{T}$,
- $\iota \in \mathcal{T}$, *and*
- $(\alpha \to \beta) \in \mathcal{T}$ *whenever* $\alpha, \beta \in \mathcal{T}$.

Furthermore, we assume types are generated freely. That is, $o \neq \iota$, $o \neq (\alpha \to \beta)$ *for any* $\alpha, \beta \in \mathcal{T}$, $\iota \neq (\alpha \to \beta)$ *for any* $\alpha, \beta \in \mathcal{T}$, *and if* $(\alpha \to \beta) = (\gamma \to \delta)$, *then* $\alpha = \gamma$ *and* $\beta = \delta$.

Since types are generated freely, we can prove properties by induction on types:
Induction on Types: We can prove a property $\varphi(\alpha)$ holds for all types α by proving

- $\varphi(o)$
- $\varphi(\iota)$
- If $\varphi(\alpha)$ and $\varphi(\beta)$, then $\varphi(\alpha \to \beta)$.

Definition 2 *A* typed family of sets $\mathcal{D}$ *is a collection of sets* $\mathcal{D}_\alpha$ *indexed by types* $\alpha \in \mathcal{T}$. *A* typed family of nonempty sets $\mathcal{D}$ *is a typed family of sets such that* $\mathcal{D}_\alpha$ *is nonempty for each* $\alpha \in \mathcal{T}$. *Given two typed families of sets* $\mathcal{D}$ *and* $\mathcal{D}'$, *a* typed function $f : \mathcal{D} \to \mathcal{D}'$ *is a collection of functions* $f_\alpha : \mathcal{D}_\alpha \to \mathcal{D}'_\alpha$ *indexed by types* $\alpha \in \mathcal{T}$. *We use* $\mathfrak{F}_\mathcal{T}(\mathcal{D}, \mathcal{D}')$ *to denote the set of typed functions from a typed family of sets* $\mathcal{D}$ *to a typed family of sets* $\mathcal{D}'$.

Recursion on Types: We can define typed families $\mathcal{D}_\alpha$ indexed by types $\alpha \in \mathcal{T}$ by specifying:

- $\mathcal{D}_o$
- $\mathcal{D}_\iota$
- A rule for forming $\mathcal{D}_{\alpha\to\beta}$ given $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$.

Fix two distinct values T and F.
Example: A Standard Frame:

- Let $\mathcal{D}_o$ be $\{\mathtt{T}, \mathtt{F}\}$.
- Let $\mathcal{D}_\iota$ be $\mathbb{N}$ (the set of natural numbers $\{0, 1, 2, \ldots\}$).
- Let $\mathcal{D}_{\alpha\to\beta}$ be $\mathcal{D}_\beta^{\mathcal{D}_\alpha}$ (all functions from $\mathcal{D}_\alpha$ to $\mathcal{D}_\beta$).

Note that $\mathcal{D}_{\iota\to o}$ is the set of characteristic functions for sets in $\mathcal{P}(\mathbb{N})$. That is, $X \subseteq \mathbb{N}$ corresponds to $\chi_X \in \mathcal{D}_{\iota\to o}$ where $\chi_X(n) = \mathtt{T}$ iff $n \in X$.

Definition 3 (Frame) *A typed family* $\mathcal{D}$ *of nonempty sets is called a* frame *if* $\mathcal{D}_{\alpha\to\beta} \subseteq \mathcal{D}_\beta^{\mathcal{D}_\alpha}$ *for all types* α *and* β. *The frame is* standard *if* $\mathcal{D}_{\alpha\to\beta} = \mathcal{D}_\beta^{\mathcal{D}_\alpha}$ *for all types* α *and* β.

Standard frames $\mathcal{D}$ are determined by $\mathcal{D}_o$ and $\mathcal{D}_\iota$. That is, if $\mathcal{D}$ and $\mathcal{E}$ are standard frames such that $\mathcal{D}_o = \mathcal{E}_o$ and $\mathcal{D}_\iota = \mathcal{E}_\iota$, then $\mathcal{D} = \mathcal{E}$. **Proof:** A trivial induction on types.

In general, frames $\mathcal{D}$ are not determined by $\mathcal{D}_o$ and $\mathcal{D}_\iota$.

WARNING: Suppose $\mathcal{D}$ is a frame, $\mathcal{E}$ is a standard frame, $\mathcal{D}_\iota = \mathcal{E}_\iota$, and $\mathcal{D}_o = \mathcal{E}_o$. Then, we **do not necessarily have** $\mathcal{D}_\alpha \subseteq \mathcal{E}_\alpha$ for all types α.

One could try to prove $\mathcal{D}_\alpha \subseteq \mathcal{E}_\alpha$ by induction on types. At the function type, one must prove $\mathcal{D}_{\alpha\to\beta}$ is a subset of $\mathcal{E}_{\alpha\to\beta}$ while inductively assuming $\mathcal{D}_\alpha \subseteq \mathcal{E}_\alpha$ and $\mathcal{D}_\beta \subseteq \mathcal{E}_\beta$. This reduces to proving $\mathcal{D}_\beta^{\mathcal{D}_\alpha} \subseteq \mathcal{E}_\beta^{\mathcal{E}_\alpha}$.

A priori, a result stating $A \subseteq C$ and $B \subseteq D$ imply $B^A \subseteq D^C$ may seem plausable. However, such a result does not hold in general. In particular,

$$\{1\}^{\{1\}} = \{\{\langle 1, 1\rangle\}\} \not\subseteq \{\{\langle 1, 1\rangle, \langle 2, 1\rangle\}\} = \{1\}^{\{1,2\}}$$

The relationship between frames and standard frames agreeing on base types is not so simple.

2 Cantor's Theorem

Cantor's Theorem states that the powerset of A is always bigger than A. Two versions of Cantor's Theorem are the surjective version and the injective version. The **surjective Cantor Theorem** states that there does not exist a surjection from A onto $\mathcal{P}(A)$. The **injective Cantor Theorem** states that there does not exist an injection from $\mathcal{P}(A)$ into A. In the simply typed setting, the type ι will correspond to the set A and the type $\iota \to o$ will correspond to the set $\mathcal{P}(A)$.

Proposition 4 (Surjective Cantor in Standard Frames) *Suppose* $\mathcal{D}$ *is a standard frame with* $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$. *There is no surjection in* $\mathcal{D}_{\iota\to(\iota\to o)}$.

Proof. Suppose $G \in \mathcal{D}_{\iota\to\iota\to o}$ is a surjection. That is, $G : \mathcal{D}_\iota \to \mathcal{D}_{\iota\to o}$ is a surjection. Let D be the diagonal set $\{x \in \mathcal{D}_\iota | G(x)(x) = \mathtt{F}\}$. Intuitively, $G(x)(x) = \mathtt{F}$ means "$x \notin G(x)$." Consider $\chi_D : \mathcal{D}_\iota \to \mathcal{D}_o$ defined by

$$\chi_D(x) := \begin{cases} \mathtt{T} & \text{if } x \in D \\ \mathtt{F} & \text{otherwise.} \end{cases}$$

Since $\mathcal{D}$ is standard, $\chi_D \in \mathcal{D}_{\iota\to o}$. Since G is surjective, $G(X) = \chi_D$ for some $X \in \mathcal{D}_\iota$. Note that $G(X)(X) = \chi_D(X)$ and so $G(X)(X) = \mathtt{T}$ iff $\chi_D(X) = \mathtt{T}$ iff $G(X)(X) = \mathtt{F}$. This is a contradiction.

Does the surjective Cantor Theorem also hold in more general frames?

Example 5 *Let* $\mathcal{D}$ *be a standard frame in which* $\mathcal{D}_o = \{\mathtt{T}\}$. *Note that* $\mathcal{D}_{\iota\to o}$ *has exactly one element, so clearly there is a surjection from* $\mathcal{D}_\iota$ *onto* $\mathcal{D}_{\iota\to o}$. *Since* $\mathcal{D}$ *is standard, the surjection is in* $\mathcal{D}_{\iota\to\iota\to o}$. **Surjective Cantor Fails in this frame. For Cantor, it is vital $\mathcal{D}_o$ has at least two elements.**

Example 6 *Consider a frame* $\mathcal{D}$ *defined by* $\mathcal{D}_o := \{\mathtt{T}, \mathtt{F}\}$, $\mathcal{D}_\iota := \mathbb{N}$ *and taking* $\mathcal{D}_{\alpha\to\beta}$ *to be the set of* constant *functions from* $\mathcal{D}_\alpha$ *to* $\mathcal{D}_\beta$. *In this case* $\mathcal{D}_{\iota\to o}$ *contains two elements (the constant* T *function and the constant* F *function). Clearly, there is a surjection from* $\mathbb{N}$ *to the two element set* $\mathcal{D}_{\iota\to o}$. However, *no such surjection is a constant function. So, no such surjections are in* $\mathcal{D}_{\iota\to\iota\to o}$. *In fact,* $\mathcal{D}_{\iota\to\iota\to o}$ *is also a two element set and neither of the two elements are surjective functions.* **Surjective Cantor Holds in this frame.**

Example 7 *Consider a frame* $\mathcal{D}$ *defined by* $\mathcal{D}_o := \{\mathtt{T}, \mathtt{F}\}$, $\mathcal{D}_\iota := \mathbb{N}$, *taking* $\mathcal{D}_{\alpha\to o}$ *to be the set of* constant *functions from* $\mathcal{D}_\alpha$ *to* $\mathcal{D}_o$, *and taking* $\mathcal{D}_{\alpha\to\beta}$ *to be the set of* all *functions from* $\mathcal{D}_\alpha$ *to* $\mathcal{D}_\beta$ *if* $\beta \neq o$. *Again,* $\mathcal{D}_{\iota\to o}$ *contains two elements, so there is a surjection from* $\mathbb{N}$ *onto* $\mathcal{D}_{\iota\to o}$. *In this case,* $\mathcal{D}_{\iota\to\iota\to o}$ *is the set of all functions, so any such surjection is in* $\mathcal{D}_{\iota\to\iota\to o}$. **Surjective Cantor Fails in this frame.**

In Example 5, surjective Cantor fails because $\mathcal{D}$ cannot appropriately interpret the logic involved in defining the diagonal set. In Example 7, surjective Cantor fails because $\mathcal{D}$ cannot appropriately interpret the λ-calculus involved in defining the diagonal set.

3 Lambda Terms

Remark 8 (Signatures, Variables, Parameters) *To define terms, we start with a signature $\mathcal{S}$ of typed (logical) constants, a collection of variables $\mathcal{V}$, and a collection of parameters $\mathcal{P}$. For each type α, let $\mathcal{S}_\alpha \subseteq \mathcal{S}$ [$\mathcal{V}_\alpha \subseteq \mathcal{V}$, $\mathcal{P}_\alpha \subseteq \mathcal{P}$] be the set of constants [variables, parameters] of type α. Let $\aleph_s$ be a fixed infinite cardinal. We assume the set of parameters $\mathcal{P}_\alpha$ has cardinality $\aleph_s$ for each type α. Similarly, we assume $\mathcal{V}_\alpha$ is countably infinite for each type α.*

To formulate higher-order logic, Church (cf. [Chu40]) assumed the signature included logical constants $\neg_{o\to o}$, $\vee_{o\to o\to o}$ and $\Pi^\alpha_{(\alpha\to o)\to o)}$ for each type α. From these, one can define the other logical operators as in [Chu40]. Equality at type α can be defined by Leibniz equality.

The alternative pursued in [And02a] is to have primitive equality $=^\alpha_{\alpha\to\alpha\to o}$ in the signature for each type α. The logical connectives and quantifiers can be defined from primitive equality (assuming full extensionality).

In the general case, we will not assume any logical constants to be in the signature $\mathcal{S}$. Without (enough) logical constants we should not consider the type theory to be "higher-order logic" since one cannot define the same sets using the restricted language as one can define using higher-order logic. Each collection of logical constants yields a fragment of higher-order logic. To avoid confusion with other formulations of higher-order logic, we will call these *fragments of elementary type theory* (*with η*) or *fragments of extensional type theory* depending on what forms extensionality are assumed. These fragments allow us to study what logical constants must be available for instantiations to prove particular theorems. In other words, these fragments allow us to measure how much (set) comprehension is necessary to prove particular theorems (in the presence or absence of extensionality).

Definition 9 (Wffs) *The set of* well-formed formulas *(or* terms*) of type α over a signature $\mathcal{S}$ is denoted by $wff_\alpha(\mathcal{S})$ (or wff_α when the signature $\mathcal{S}$ is clear in context). These sets are defined inductively as follows.*

- $x_\alpha \in wff_\alpha(\mathcal{S})$ *for each variable* $x_\alpha \in \mathcal{V}_\alpha$.
- $W_\alpha \in wff_\alpha(\mathcal{S})$ *for each parameter* $W_\alpha \in \mathcal{P}_\alpha$.
- $c_\alpha \in wff_\alpha(\mathcal{S})$ *for each constant* $c_\alpha \in \mathcal{S}_\alpha$.
- $[\mathbf{F}_{\alpha\to\beta}\,\mathbf{A}_\alpha] \in wff_\beta(\mathcal{S})$ *for each* $\mathbf{F} \in wff_{\alpha\to\beta}(\mathcal{S})$ *and* $\mathbf{A} \in wff_\alpha(\mathcal{S})$.
- $[\lambda x_\alpha \mathbf{B}_\beta] \in wff_{\alpha\to\beta}(\mathcal{S})$ *for each variable* $x_\alpha \in \mathcal{V}_\alpha$ *and* $\mathbf{B} \in wff_\beta(\mathcal{S})$.

We define $\mathbf{Free}(\mathbf{A}_\alpha) \subset \mathcal{V}$ to be the set of free variables in $\mathbf{A}$ in the usual way. We also define $\mathbf{Consts}(\mathbf{A}_\alpha) \subset \mathcal{S}$ [$\mathbf{Params}(\mathbf{A}_\alpha) \subset \mathcal{P}$] to be the set of constants [parameters] that occur in $\mathbf{A}$. We call a term $\mathbf{A}_\alpha$ closed *if there are no free variables in $\mathbf{A}$. Let $cwff_\alpha(\mathcal{S})$ (or $cwff_\alpha$) be the set of all closed terms of type α. For any set of formulas $\Phi \subseteq wff_\alpha(\mathcal{S})$, we define $\mathbf{Free}(\Phi) := \bigcup_{\mathbf{A}\in\Phi} \mathbf{Free}(\mathbf{A})$ and $\mathbf{Params}(\Phi) := \bigcup_{\mathbf{A}\in\Phi} \mathbf{Params}(\mathbf{A})$.*

We next make precise which logical constants we will consider.

Definition 10 (Logical Constants) *We define the set of* logical constants *to be*

$$\{\neg_{o\to o}, \wedge_{o\to o\to o}, \vee_{o\to o\to o}\}$$
$$\cup\, \{\Pi^\alpha_{(\alpha\to o)\to o} \mid \alpha \in \mathcal{T}\} \cup \{\Sigma^\alpha_{(\alpha\to o)\to o} \mid \alpha \in \mathcal{T}\} \cup \{=^\alpha_{\alpha\to\alpha\to o} \mid \alpha \in \mathcal{T}\}.$$

4 Interpreting Lambda Terms in Frames

Let $\mathcal{D}$ be a standard frame. An *assignment* φ into $\mathcal{D}$ is a typed function $\varphi : \mathcal{V} \to \mathcal{D}$ from variables to domains (so $\varphi(x_\alpha) \in \mathcal{D}_\alpha$). We use $\varphi, [\mathsf{a}/x]$ to denote the assignment which agrees with φ except that $(\varphi, [\mathsf{a}/x])(x_\alpha) := \mathsf{a}$.

Let $\mathcal{I} : (\mathcal{P} \cup \mathcal{S}) \to \mathcal{D}$ be a typed function mapping parameters and logical constants into domains. Given $\mathcal{I}$, we can inductively interpret terms $\mathbf{M}_\beta$ as elements $\mathcal{E}_\varphi(\mathbf{M})$ of $\mathcal{D}_\beta$ (depending on an assignment φ) as follows:

Definition 11 *Given $\mathcal{I} : (\mathcal{P} \cup \mathcal{S}) \to \mathcal{D}$, we define $\mathcal{E}_\varphi(\mathbf{M})$ by induction on $\mathbf{M}$:*

1. $\mathcal{E}_\varphi(x) := \varphi(x)$ *for variables* $x \in \mathcal{V}$.
2. $\mathcal{E}_\varphi(c) := \mathcal{I}(c)$ *for parameters or logical constants* c.
3. $\mathcal{E}_\varphi(\mathbf{F}_{\alpha\to\beta}\mathbf{A}_\alpha) := \mathcal{E}_\varphi(\mathbf{F})(\mathcal{E}_\varphi(\mathbf{A}))$ – *using the fact that* $\mathcal{E}_\varphi(\mathbf{F}_{\alpha\to\beta}) \in \mathcal{D}_{\alpha\to\beta}$ *is a function from* $\mathcal{D}_\alpha$ *to* $\mathcal{D}_\beta$.
4. *Let* $\mathcal{E}_\varphi([\lambda x_\alpha \mathbf{B}_\beta])$ *be the function from* $\mathcal{D}_\alpha$ *to* $\mathcal{D}_\beta$ *taking* $\mathsf{a} \in \mathcal{D}_\alpha$ *to* $\mathcal{E}_{\varphi,[\mathsf{a}/x]}(\mathbf{B})$. *This function is in* $\mathcal{D}_{\alpha\to\beta}$ *since* $\mathcal{D}$ *is standard.*

We only used the fact that the frame was standard to interpret the λ-abstraction. In general, we need only assume $\mathcal{D}$ is a frame in which $\mathcal{D}_{\alpha\to\beta}$ contains any function we need to interpret λ-abstractions. Such frames are called *combinatory*.

Every λ-term can be converted to a λ-term which only have very special λ-abstractions. Restricting our attention to such terms provides a way to define an evaluation function by defining the function only on these special terms.

Definition 12 (SK-Combinatory Formulas) *For all types α, β, and γ, we define two families of closed formulas we call* combinators*:*

- $\mathbf{K}_{\alpha\to\beta\to\alpha} := \lambda x_\alpha \lambda y_\beta\, x$
- $\mathbf{S}_{(\alpha\to\beta\to\gamma)\to(\alpha\to\beta)\to\alpha\to\gamma} := \lambda u_{\alpha\to\beta\to\gamma} \lambda v_{\alpha\to\beta} \lambda w_\alpha\, [u\, w\, [v\, w]]$

We define the set of **SK**-combinatory formulas *to be the least subset of $\bigcup_\alpha wff_\alpha(\mathcal{S})$ containing every $\mathbf{K}$ and $\mathbf{S}$, every constant, every parameter, every variable, and closed under application.*

Definition 13 (Combinatory) *Let $\mathcal{D}$ be a frame. We say $\mathcal{D}$ is* combinatory *if for all types α, β and γ there exist elements $\mathsf{k} \in \mathcal{D}_{\alpha\to\beta\to\alpha}$ and $\mathsf{s} \in \mathcal{D}_{(\alpha\to\beta\to\gamma)\to(\alpha\to\beta)\to\alpha\to\gamma}$ such that*

1. $\mathsf{k}(\mathsf{a})(\mathsf{b}) = \mathsf{a}$ *for every* $\mathsf{a} \in \mathcal{D}_\alpha$ *and* $\mathsf{b} \in \mathcal{D}_\beta$, *and*
2. $\mathsf{s}(\mathsf{g})(\mathsf{f})(\mathsf{a}) = \mathsf{g}(\mathsf{a})(\mathsf{f}(\mathsf{a}))$ *for every* $\mathsf{a} \in \mathcal{D}_\alpha$, $\mathsf{f} \in \mathcal{D}_{\alpha\to\beta}$ *and* $\mathsf{g} \in \mathcal{D}_{\alpha\to\beta\to\gamma}$.

Proposition 14 *Let $\mathcal{D}$ be a combinatory frame. Let $\mathcal{I} : (\mathcal{P} \cup \mathcal{S}) \to \mathcal{D}$ be a typed function mapping parameters and logical constants into domains. There is a function $\mathcal{E}$ such that for every assignment φ and term $\mathbf{M}_\beta$, $\mathcal{E}_\varphi(\mathbf{M}_\beta) \in \mathcal{D}_\beta$ and the following hold:*

1. $\mathcal{E}_\varphi(x) = \varphi(x)$ *for variables* $x \in \mathcal{V}$.
2. $\mathcal{E}_\varphi(c) = \mathcal{I}(c)$ *for parameters or logical constants* c.
3. $\mathcal{E}_\varphi(\mathbf{F}_{\alpha\to\beta}\mathbf{A}_\alpha) = \mathcal{E}_\varphi(\mathbf{F})(\mathcal{E}_\varphi(\mathbf{A}))$ – *using the fact that* $\mathcal{E}_\varphi(\mathbf{F}_{\alpha\to\beta}) \in \mathcal{D}_{\alpha\to\beta}$ *is a function from* $\mathcal{D}_\alpha$ *to* $\mathcal{D}_\beta$.
4. $\mathcal{E}_\varphi([\lambda x_\alpha \mathbf{B}_\beta])$ *is the function from* $\mathcal{D}_\alpha$ *to* $\mathcal{D}_\beta$ *taking* $\mathsf{a} \in \mathcal{D}_\alpha$ *to* $\mathcal{E}_{\varphi,[\mathsf{a}/x]}(\mathbf{B})$.

Proof. Essentially one reduces λ-calculus to combinatory logic and then inductively defines $\mathcal{E}$ on combinatory formulas where all the λ-abstractions are hidden in $\mathbf{K}$ or $\mathbf{S}$.

One can compare Proposition 14 above to Theorem 3.6.12 in [Bro04] or Theorem 1 in [And72b].

In the interpretation $\mathcal{I}$ above, we did not assume anything special about the domain $\mathcal{D}_o$ or the interpretation of logical constants, e.g., $\mathcal{I}(\neg)$. Usually, we want to make some assumptions about the interpretation of logical constants.

For now, assume $\mathcal{D}_o$ is $\{\mathtt{T}, \mathtt{F}\}$. Clearly, there is a function $n : \mathcal{D}_o \to \mathcal{D}_o$ which corresponds to negation (reversing $\mathtt{T}$ and $\mathtt{F}$). This function satisfies the basic logical property of negation. For each logical constant c_α we can define an analogous property $\mathfrak{L}_c(\mathsf{a})$ for $\mathsf{a} \in \mathcal{D}_\alpha$.

Definition 15 *Let $\mathcal{D}$ be a frame with $\mathcal{D} = \{\mathtt{T}, \mathtt{F}\}$. For each logical constant c_α and element $\mathsf{a} \in \mathcal{D}_\alpha$, we define the properties $\mathfrak{L}_c(\mathsf{a})$ in Table 1. We say the frame $\mathcal{D}$* realizes *a logical constant c_α if there is some $\mathsf{a} \in \mathcal{D}_\alpha$ such that $\mathfrak{L}_c(\mathsf{a})$ holds.*

prop.	where	holds when	for all
$\mathfrak{L}_{\neg}(\mathsf{n})$	$\mathsf{n} \in \mathcal{D}_{o\to o}$	$\mathsf{n}(\mathsf{a}) = \mathtt{T}$ iff $\mathsf{a} = \mathtt{F}$	$\mathsf{a} \in \mathcal{D}_o$
$\mathfrak{L}_{\vee}(\mathsf{d})$	$\mathsf{d} \in \mathcal{D}_{o\to o\to o}$	$\mathsf{d}(\mathsf{a})(\mathsf{b}) = \mathtt{T}$ iff $\mathsf{a} = \mathtt{T}$ or $\mathsf{b} = \mathtt{T}$	$\mathsf{a}, \mathsf{b} \in \mathcal{D}_o$
$\mathfrak{L}_{\wedge}(\mathsf{c})$	$\mathsf{c} \in \mathcal{D}_{o\to o\to o}$	$\mathsf{c}(\mathsf{a})(\mathsf{b}) = \mathtt{T}$ iff $\mathsf{a} = \mathtt{T}$ and $\mathsf{b} = \mathtt{T}$	$\mathsf{a}, \mathsf{b} \in \mathcal{D}_o$
$\mathfrak{L}_{\Pi^\alpha}(\pi)$	$\pi \in \mathcal{D}_{(\alpha\to o)\to o}$	$\pi(\mathsf{f}) = \mathtt{T}$ iff $\forall \mathsf{a} \in \mathcal{D}_\alpha\ \mathsf{f}(\mathsf{a}) = \mathtt{T}$	$\mathsf{f} \in \mathcal{D}_{\alpha\to o}$
$\mathfrak{L}_{\Sigma^\alpha}(\sigma)$	$\sigma \in \mathcal{D}_{(\alpha\to o)\to o}$	$\sigma(\mathsf{f}) = \mathtt{T}$ iff $\exists \mathsf{a} \in \mathcal{D}_\alpha\ \mathsf{f}(\mathsf{a}) = \mathtt{T}$	$\mathsf{f} \in \mathcal{D}_{\alpha\to o}$
$\mathfrak{L}_{=^\alpha}(\mathsf{q})$	$\mathsf{q} \in \mathcal{D}_{\alpha\to\alpha\to o}$	$\mathsf{q}(\mathsf{a})(\mathsf{b}) = \mathtt{T}$ iff $\mathsf{a} = \mathsf{b}$	$\mathsf{a}, \mathsf{b} \in \mathcal{D}_\alpha$

Table 1. Logical Properties

Later we generalize these properties to applicative structures with a function $\upsilon : \mathcal{D}_o \to \{\mathtt{T}, \mathtt{F}\}$ (see Definition 73).

We say the frame *realizes* $\neg$ if there is a function n in $\mathcal{D}_{o\to o}$ such that $\mathfrak{L}_\neg(\mathsf{n})$. For any logical constant c_α, we say $\mathcal{D}$ realizes c if there is an element $\mathsf{a} \in \mathcal{D}_\alpha$ such that $\mathfrak{L}_c(\mathsf{a})$.

Proposition 16 (Surjective Cantor in Combinatory Frames) *Suppose $\mathcal{D}$ is a combinatory frame with $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$ and $\mathcal{D}$ realizes $\neg$. There is no surjection in $\mathcal{D}_{\iota\to(\iota\to o)}$.*

Proof. Let $\mathcal{S} = \{\neg\}$ and $\mathcal{I}$ be any interpretation of constants and parameters such that $\mathcal{I}(\neg) \in \mathcal{D}_{o\to o}$ is the function interchanging $\mathtt{T}$ and $\mathtt{F}$. Suppose $\mathsf{g} \in \mathcal{D}_{\iota\to\iota\to o}$ is a surjection. Let $g_{\iota\to\iota\to o}$ be a variable and φ be an assignment such that $\varphi(g) = \mathsf{g}$. Using Proposition 14 we can interpret the term

$$\mathbf{D} := [\lambda x_\iota [\neg [g\, x\, x]]$$

as $\mathcal{E}_\varphi(\mathbf{D})$ in $\mathcal{D}_{\iota\to o}$ such that $\mathcal{E}_\varphi(\mathbf{D})(\mathsf{a}) = \mathcal{I}(\neg)(\mathsf{g}(\mathsf{a})(\mathsf{a}))$ holds. Since g is surjective, $\mathsf{g}(X) = \mathcal{E}_\varphi(\mathbf{D})$ for some $X \in \mathcal{D}_\iota$. Note that $\mathsf{g}(X)(X) = \mathcal{E}_\varphi(\mathbf{D})(X)$ and so $\mathsf{g}(X)(X) = \mathtt{T}$ iff $\mathcal{E}_\varphi(\mathbf{D})(X) = \mathtt{T}$ iff $\mathsf{g}(X)(X) = \mathtt{F}$. This is a contradiction.

Note: Suppose $\mathsf{g} : \mathcal{D}_\iota \to \mathcal{D}_{\iota\to o}$ is a surjection, but $\mathsf{g} \notin \mathcal{D}_{\iota\to\iota\to o}$. We cannot repeat the argument above since we cannot find an assignment φ such that $\varphi(g) = \mathsf{g}$.

To prove the injective Cantor Theorem, one needs more than just negation.

Proposition 17 (Injective Cantor in Combinatory Frames) *Suppose $\mathcal{D}$ is a combinatory frame with $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$ and $\mathcal{D}$ realizes $\neg$, $\wedge$, $=^\iota$ and $\Sigma^{\iota\to o}$. There is no injection in $\mathcal{D}_{(\iota\to o)\to\iota}$.*

Proof. Let $\mathcal{S}$ be $\{\neg, \wedge, =^\iota, \Sigma^{\iota\to o}\}$ and $\mathcal{I}$ be any interpretation of constants and parameters such that the following conditions hold:

- $\mathcal{I}(\neg)(\mathsf{a}) = \mathtt{T}$ iff $\mathsf{a} = \mathtt{F}$ (where $\mathsf{a} \in \mathcal{D}_o$).
- $\mathcal{I}(\wedge)(\mathsf{a})(\mathsf{b}) = \mathtt{T}$ iff $\mathsf{a} = \mathtt{T}$ and $\mathsf{b} = \mathtt{T}$ (where $\mathsf{a}, \mathsf{b} \in \mathcal{D}_o$).
- $\mathcal{I}(=^\iota)(\mathsf{a})(\mathsf{b}) = \mathtt{T}$ iff $\mathsf{a} = \mathsf{b}$ (where $\mathsf{a}, \mathsf{b} \in \mathcal{D}_\iota$).
- $\mathcal{I}(\Sigma^{\iota\to o})(\mathsf{f}) = \mathtt{T}$ iff there is some $\mathsf{a} \in \mathcal{D}_\iota$ such that $\mathsf{f}(\mathsf{a}) = \mathtt{T}$ (where $\mathsf{f} \in \mathcal{D}_{\iota\to o}$).

This is possible since we have assumed $\mathcal{D}$ realizes these logical constants.

Suppose $\mathsf{h} \in \mathcal{D}_{(\iota\to o)\to\iota}$ is an injection. Let $h \in \mathcal{V}_{(\iota\to o)\to\iota}$ be a variable and φ be an assignment with $\varphi(h) = \mathsf{h}$. Let $\mathcal{E}$ be the function given by Proposition 14. We define a "diagonal set"

$$\mathsf{d} := \mathcal{E}_\varphi([\lambda y_\iota\, [\exists w_{\iota\to o}\, [h\, w \;=^\iota\; y \,\wedge\, \neg[w\, y]]]]) \in \mathcal{D}_{\iota\to o}$$

where $\exists w_{\iota\to o}\mathbf{M}$ is shorthand for $[\Sigma^{\iota\to o}[\lambda w_{\iota\to o}\mathbf{M}]]$. Note that

$$\mathsf{d}(\mathsf{h}(\mathsf{d})) = \mathcal{E}_{\varphi,[\mathsf{h}(\mathsf{d})/y]}(\exists w_{\iota\to o} {\scriptstyle\blacksquare}\, h\, w \;=^\iota\; y \,\wedge\, \neg[w\, y]).$$

Assume $\mathsf{d}(\mathsf{h}(\mathsf{d})) = \mathtt{F}$. Then

$$\mathcal{E}_{\varphi,[\mathsf{h}(\mathsf{d})/y],[\mathsf{d}/w]}(\neg[w\, y]) = (\mathcal{E}(\neg)(\mathsf{d}(\mathsf{h}(\mathsf{d})))) = \mathtt{T}.$$

By the property of $\mathcal{I}(=^\iota)$, we have $\mathcal{E}_{\varphi,[\mathsf{h}(\mathsf{d})/y],[\mathsf{d}/w]}(h\, w \;=^\iota\; y) = \mathtt{T}$. By the property of $\mathcal{I}(\wedge)$ we know

$$\mathcal{E}_{\varphi,[\mathsf{h}(\mathsf{d})/y],[\mathsf{d}/w]}([h\, w \;=^\iota\; y \,\wedge\, \neg[w\, y]]) = \mathtt{T}.$$

Finally, by the property of $\mathcal{I}(\Sigma^{\iota\to o})$ we conclude

$$\mathsf{d}(\mathsf{h}(\mathsf{d})) = \mathcal{E}_{\varphi,[\mathsf{h}(\mathsf{d})/y]}([\exists w_{\iota\to o} {\scriptstyle\blacksquare}\, h\, w \;=^\iota\; y \,\wedge\, \neg[w\, y]]) = \mathtt{T}$$

which contradicts our assumption.

Hence we must have $\mathsf{d}(\mathsf{h}(\mathsf{d})) = \mathtt{T}$. By the property of $\mathcal{I}(\Sigma^{\iota\to o})$ there must exist some $\mathsf{w} \in \mathcal{D}_{\iota\to o}$ such that

$$\mathcal{E}_{\varphi,[\mathsf{h}(\mathsf{d})/y],[\mathsf{w}/w]}([h\, w \;=^\iota\; y \,\wedge\, \neg[w\, y]]) = \mathtt{T}.$$

This implies $\mathcal{E}_{\varphi,[\mathsf{h}(\mathsf{d})/y],[\mathsf{w}/w]}(\neg[w\, y]) = \mathtt{T}$ and

$$\mathsf{h}(\mathsf{w}) = \mathcal{E}_{\varphi,[\mathsf{h}(\mathsf{d})/y],[\mathsf{w}/w]}(h\, w) = \mathcal{E}_{\varphi,[\mathsf{h}(\mathsf{d})/y],[\mathsf{w}/w]}(y) = \mathsf{h}(\mathsf{d}).$$

Since h is an injective function, $\mathsf{w} = \mathsf{d}$. Thus $\mathsf{w}(\mathsf{h}(\mathsf{d})) = \mathsf{d}(\mathsf{h}(\mathsf{d})) = \mathtt{T}$. This contradicts $\mathcal{E}_{\varphi,[\mathsf{h}(\mathsf{d})/y],[\mathsf{w}/w]}(\neg[w\, y]) = \mathtt{T}$.

5 A Frame without Negation

We can inductively define $\mathcal{D}_\alpha$ along with a relation $\leq^\alpha$ on $\mathcal{D}_\alpha$:

- $\mathcal{D}_o := \{\mathtt{T}, \mathtt{F}\}$. Let $\leq^o$ be the reflexive ordering with $\mathtt{F} \leq^o \mathtt{T}$ and $\mathtt{T} \not\leq^o \mathtt{F}$.
- $\mathcal{D}_\iota := \mathbb{N}$(any nonempty set would work). Let $\leq^\iota$ be the usual ordering on the natural numbers.
- Let $\mathcal{D}_{\alpha\to\beta}$ be the set of all monotone functions $f : \mathcal{D}_\alpha \to \mathcal{D}_\beta$. That is, f satisfying

$$\forall x, y \in \mathcal{D}_\alpha . x \leq^\alpha y \Rightarrow f(x) \leq^\beta f(y)$$

Let $\leq^{\alpha\to\beta}$ be the relation on $\mathcal{D}_{\alpha\to\beta}$ defined by $f \leq^{\alpha\to\beta} g$ iff

$$\forall x, y \in \mathcal{D}_\alpha . x \leq^\alpha y \Rightarrow f(x) \leq^\beta g(y)$$

One can show $\leq$ is reflexive on each $\mathcal{D}_\alpha$ by an easy induction on types.

$\mathcal{D}$ is clearly a frame. The fact that each $\mathcal{D}_\alpha$ is nonempty follows by induction on types, using reflexivity of $\leq^\beta$ to show a constant function is in $\mathcal{D}_{\alpha\to\beta}$.

$\mathcal{D}$ is also combinatory. One can either directly check that the combinator functions are in $\mathcal{D}$, or one can inductively define $\mathcal{E}$ (checking directly that $\mathcal{E}$ is well-defined on λ-abstractions) and then apply $\mathcal{E}$ to $\mathbf{K}$ and $\mathbf{S}$ in order to prove $\mathcal{D}$ is combinatory. We sketch the second possibility:

As we inductively define $\mathcal{E}$ as in Definition 11, we can check that $\mathcal{E}$ is monotone in the assignment. That is, we write $\varphi \leq \psi$ when $\varphi(x) \leq \psi(x)$ for all x. We check $\mathcal{E}_\varphi(\mathbf{M}) \leq \mathcal{E}_\psi(\mathbf{M})$ whenever $\varphi \leq \psi$ as we define $\mathcal{E}$ by induction on $\mathbf{M}$. To know $\mathcal{E}_\varphi(c) \leq \mathcal{E}_\psi(c)$ for constants and parameters c, we use the fact that $\leq$ is reflexive. When we interpret λ-abstractions, we then know $\mathcal{E}_{\varphi,[\mathsf{a}/x]}(\mathbf{B}) \leq \mathcal{E}_{\varphi,[\mathsf{b}/x]}(\mathbf{B})$ whenever $\mathsf{a} \leq \mathsf{b}$. Hence the function $\mathcal{E}_\varphi([\lambda x \mathbf{B}])$ is in $\mathcal{D}_{\alpha\to\beta}$. (We also need to know $\varphi \leq \varphi$ for any assignment φ which follows from the fact that $\leq^\alpha$ is reflexive on each $\mathcal{D}_\alpha$.)

Note that the negation function interchanging $\mathtt{T}$ and $\mathtt{F}$ is not monotone since $\mathtt{F} \leq^o \mathtt{T}$, but $\mathtt{T} \not\leq^o \mathtt{F}$. Hence $\mathcal{D}_{o\to o}$ contains three functions: the constant false function, the constant true function, and the identity function.

Note: Even though the frame does not realize $\neg$, the surjective Cantor theorem does hold: That is, there is no surjection in $\mathcal{D}_{\iota\to\iota\to o}$.

Proof Outline:

1. For each $n \in \mathbb{N}$, let $F_n : \mathcal{D}_\iota \to \mathcal{D}_o$ be defined by $F_n(m) := \mathtt{T}$ iff $n \leq m$. $F_n \in \mathcal{D}_{\iota\to o}$. (We need to know $F_n(k) \leq^o F_n(l)$ whenever $k \leq^\iota l$. This reduces to showing $n \leq k \leq l$ implies $n \leq l$.)
2. $\mathcal{D}_{\iota\to o}$ is infinite.
3. Suppose $\mathsf{g} \in \mathcal{D}_{\iota\to\iota\to o}$ is surjective.

4. There must be some $N \in \mathbb{N}$ such that $\mathsf{g}(N) = F_0$ (the constant $\mathtt{T}$) function.
5. For any $n \geq N$ and $m \in \mathbb{N}$, we have $\mathsf{g}(N)(0) \leq \mathsf{g}(n)(m)$ and hence $\mathsf{g}(n)(m) = \mathtt{T}$.
6. Thus $\mathsf{g}(n) = F_0$ for all $n \geq N$.
7. g has a finite range while $\mathcal{D}_{\iota \to o}$ is infinite; contradiction.

Likewise, there is no injection h in $\mathcal{D}_{(\iota \to o) \to \iota}$ since the range of h must be both infinite and contained in the finite set of natural numbers between $\mathsf{h}(K_{\mathtt{F}})$ and $\mathsf{h}(K_{\mathtt{T}})$ (where $K_{\mathtt{F}}$ and $K_{\mathtt{T}}$ are the constant $\mathtt{F}$ and constant $\mathtt{T}$ functions).

6 Logical Relation Frames

We will use the idea of logical relations to define domains of function types given domains of base types and relations on these domains. The construction will also generalize from n-ary relations to A-ary relations where A is an arbitrary nonempty index set. The logical relations construction will give examples of combinatory frames in which versions of Cantor's theorem fail. The frames will realize some (but not necessarily all) logical constants.

Let A be any nonempty set. For any nonempty set $\mathcal{D}$, we can consider a function $f : A \to \mathcal{D}$ to be an A-tuple of elements of $\mathcal{D}$. Similarly, we can consider a set $\mathcal{R} \subseteq \mathcal{D}^A$ to be an A-ary relation (i.e., a collection of A-tuples).

To define a frame, we can start with any nonempty sets $\mathcal{D}_\iota$ and $\mathcal{D}_o$. We also start with any $\mathcal{R}_\iota \subseteq \mathcal{D}_\iota^A$ and $\mathcal{R}_o \subseteq \mathcal{D}_o^A$ which contain all the constant functions. Then we can define the remaining domains by induction on types. Furthermore, when we define $\mathcal{D}_\alpha$, we will also define $\mathcal{R}_\alpha \subseteq \mathcal{D}_\alpha^A$.

It is useful to define polymorphic operations S and K on functions into a frame. These operations behave similarly to the **S** and **K** combinators. The operations will be used to define logical relation frames and to verify that these frames are combinatory.

Definition 18 *Let $\mathcal{D}$ be a frame and A be a nonempty set. For each type α and $\mathsf{a} \in \mathcal{D}_\alpha$, let $K(\mathsf{a}) : A \to \mathcal{D}_\alpha$ be the constant function defined by $K(\mathsf{a})(x) := \mathsf{a}$ for each $x \in A$. For all types α and β and functions $p : A \to \mathcal{D}_{\alpha\to\beta}$ and $f : A \to \mathcal{D}_\alpha$, let $S(p, f) : A \to \mathcal{D}_\beta$ be defined by $S(p, f)(x) := p(x)(f(x))$ for each $x \in A$.*

Definition 19 (Logical Relation Frame) *Let $\mathcal{D}$ be a frame and A be a nonempty set. We say $\mathcal{D}$ is an* A-ary logical relation frame *with relation $\mathcal{R}$ if*

- *$\mathcal{R}_\alpha \subseteq \mathcal{D}_\alpha^A$ for all types α,*
- *$K(\mathsf{a}) \in \mathcal{R}_\alpha$ for all $\mathsf{a} \in \mathcal{D}_\alpha$ and all types α, and*
- *for all types α and β*

$$\mathcal{D}_{\alpha\to\beta} = \{h : \mathcal{D}_\alpha \to \mathcal{D}_\beta \mid (h \circ f) \in \mathcal{R}_\beta \text{ for every } f \in \mathcal{R}_\alpha\}$$

and

$$\mathcal{R}_{\alpha\to\beta} = \{p : A \to \mathcal{D}_{\alpha\to\beta} \mid S(p, f) \in \mathcal{R}_\beta \text{ for every } f \in \mathcal{R}_\alpha\}.$$

If we start with appropriate values of $\mathcal{D}$ and $\mathcal{R}$ at base types, we can extend this by induction on types to obtain a logical relation frame.

Definition 20 (Logical Relation Extension) *Let A, $\mathcal{D}_\iota$, $\mathcal{D}_o$, $\mathcal{R}_\iota \subseteq \mathcal{D}_\iota^A$ and $\mathcal{R}_o \subseteq \mathcal{D}_o^A$ be nonempty sets. By induction on types, we extend $\mathcal{D}$ and $\mathcal{R}$ to be defined on all types by*

$$\mathcal{D}_{\alpha\to\beta} := \{h : \mathcal{D}_\alpha \to \mathcal{D}_\beta \mid (h \circ f) \in \mathcal{R}_\beta \text{ for every } f \in \mathcal{R}_\alpha\}$$

and

$$\mathcal{R}_{\alpha\to\beta} := \{p : A \to \mathcal{D}_{\alpha\to\beta} \mid S(p, f) \in \mathcal{R}_\beta \text{ for every } f \in \mathcal{R}_\alpha\}$$

for all types α and β. We say $(\mathcal{D}, \mathcal{R})$ is the A-ary logical relation extension *of $(\mathcal{D}_\iota, \mathcal{D}_o, \mathcal{R}_\iota, \mathcal{R}_o)$.*

We now give conditions under which logical relation extensions define logical relation frames. First, we need every element to be related to itself by $\mathcal{R}$. That is, we need to know the operator K defined above maps into $\mathcal{R}_\alpha$ for each type α.

Lemma 21 *Let A, $\mathcal{D}_\iota$, $\mathcal{D}_o$, $\mathcal{R}_\iota \subseteq \mathcal{D}_\iota^A$ and $\mathcal{R}_o \subseteq \mathcal{D}_o^A$ be nonempty sets. Assume*

- *$K(\mathsf{b}) \in \mathcal{R}_o$ for every $\mathsf{b} \in \mathcal{D}_o$ and*
- *$K(\mathsf{c}) \in \mathcal{R}_\iota$ for every $\mathsf{c} \in \mathcal{D}_\iota$.*

Let $(\mathcal{D}, \mathcal{R})$ be the A-ary logical relation extension of $(\mathcal{D}_\iota, \mathcal{D}_o, \mathcal{R}_\iota, \mathcal{R}_o)$. Then $K(\mathsf{a}) \in \mathcal{R}_\alpha$ for each type α and $\mathsf{a} \in \mathcal{D}_\alpha$.

Proof. We know $K(\mathsf{a}) \in \mathcal{D}_\alpha$ for each $\mathsf{a} \in \mathcal{D}_\alpha$ by assumption for base types $\alpha \in \{\iota, o\}$. Assume α is a function type $(\gamma \to \beta)$. Let $h \in \mathcal{D}_{\gamma\to\beta}$ be given. We must verify $K(h) : A \to \mathcal{D}_{\gamma\to\beta}$ is in $\mathcal{R}_{\gamma\to\beta}$. Let $f \in \mathcal{R}_\gamma$ be given. We must check $S(K(h), f) \in \mathcal{R}_\beta$. For each $x \in A$, we have

$$S(K(h), f)(x) = K(h)(x)(f(x)) = h(f(x)).$$

Thus $S(K(h), f) = (h \circ f) \in \mathcal{R}_\beta$ since $h \in \mathcal{D}_{\gamma\to\beta}$.

We can now use constant functions to demonstrate that every domain of such a logical relation extension is nonempty.

Lemma 22 *Let A, $\mathcal{D}_\iota$, $\mathcal{D}_o$, $\mathcal{R}_\iota \subseteq \mathcal{D}_\iota^A$ and $\mathcal{R}_o \subseteq \mathcal{D}_o^A$ be nonempty sets. Assume*

- *$K(\mathsf{b}) \in \mathcal{R}_o$ for every $\mathsf{b} \in \mathcal{D}_o$ and*
- *$K(\mathsf{c}) \in \mathcal{R}_\iota$ for every $\mathsf{c} \in \mathcal{D}_\iota$.*

Let $(\mathcal{D}, \mathcal{R})$ be the A-ary logical relation extension of $(\mathcal{D}_\iota, \mathcal{D}_o, \mathcal{R}_\iota, \mathcal{R}_o)$. Then $\mathcal{D}_\alpha$ is nonempty for each type α.

Proof. We prove $\mathcal{D}_\alpha$ is nonempty by induction on the type α. By assumption, $\mathcal{D}_o$ and $\mathcal{D}_\iota$ are nonempty. Assume $\mathcal{D}_\beta$ is nonempty and choose some $\mathsf{b} \in \mathcal{D}_\beta$. Let $k_{\mathsf{b}}^\gamma : \mathcal{D}_\gamma \to \mathcal{D}_\beta$ be the constant function $k_{\mathsf{b}}^\gamma(\mathsf{c}) := \mathsf{b}$ for all $\mathsf{c} \in \mathcal{D}_\gamma$. For any $f \in \mathcal{R}_\beta$, $(k_{\mathsf{b}}^\gamma \circ f) = K(\mathsf{b}) \in \mathcal{R}_\beta$ by Lemma 21. Hence $k_{\mathsf{b}}^\gamma \in \mathcal{D}_{\gamma\to\beta}$ and so $\mathcal{D}_{\gamma\to\beta}$ is nonempty.

Since the domains are nonempty, the logical relation extension is a frame.

Lemma 23 *Let A, $\mathcal{D}_\iota$, $\mathcal{D}_o$, $\mathcal{R}_\iota \subseteq \mathcal{D}_\iota^A$ and $\mathcal{R}_o \subseteq \mathcal{D}_o^A$ be nonempty sets. Assume*

- *$K(\mathsf{b}) \in \mathcal{R}_o$ for every $\mathsf{b} \in \mathcal{D}_o$ and*
- *$K(\mathsf{c}) \in \mathcal{R}_\iota$ for every $\mathsf{c} \in \mathcal{D}_\iota$.*

Let $(\mathcal{D}, \mathcal{R})$ be the A-ary logical relation extension of $(\mathcal{D}_\iota, \mathcal{D}_o, \mathcal{R}_\iota, \mathcal{R}_o)$. Then $\mathcal{D}$ is an A-ary logical relation frame with relation $\mathcal{R}$.

Proof. By Lemma 22, we know each $\mathcal{D}_\alpha$ is nonempty. By the definition of $\mathcal{D}$ at function types, we know each $\mathcal{D}_{\alpha\to\beta} \subseteq \mathcal{D}_\beta^{\mathcal{D}_\alpha}$. Hence $\mathcal{D}$ is a frame.

We know $\mathcal{R}_\alpha \subseteq \mathcal{D}_\alpha^A$ on base types by assumption and every function type by definition. By Lemma 21, for any $\mathsf{a} \in \mathcal{D}_\alpha$ we know $K(\mathsf{a}) \in \mathcal{R}_\alpha$. Finally, we know

$$\mathcal{D}_{\alpha\to\beta} = \{h : \mathcal{D}_\alpha \to \mathcal{D}_\beta \mid (h \circ f) \in \mathcal{R}_\beta \text{ for every } f \in \mathcal{R}_\alpha\}$$

and

$$\mathcal{R}_{\alpha\to\beta} = \{p : A \to \mathcal{D}_{\alpha\to\beta} \mid S(p, f) \in \mathcal{R}_\beta \text{ for every } f \in \mathcal{R}_\alpha\}.$$

for function types by definition.

We now prove that logical relation frames are combinatory (cf. Definition 13).

Theorem 24 *Let A be a nonempty set and $\mathcal{D}$ be an A-ary logical relation frame with relation $\mathcal{R}$. Then $\mathcal{D}$ is combinatory.*

Proof. Let α and β be types. We must verify there is a $\mathsf{k} \in \mathcal{D}_{\alpha\to\beta\to\alpha}$ such that $\mathsf{k}(\mathsf{a})(\mathsf{b}) = \mathsf{a}$ for every $\mathsf{a} \in \mathcal{D}_\alpha$ and $\mathsf{b} \in \mathcal{D}_\beta$. This, of course, supplies the only possible definition of k as the function $\mathsf{k} : \mathcal{D}_\alpha \to \mathcal{D}_\alpha^{\mathcal{D}_\beta}$ *defined* by $\mathsf{k}(\mathsf{a})(\mathsf{b}) := \mathsf{a}$. To check $\mathsf{k} \in \mathcal{D}_{\alpha\to\beta\to\alpha}$, we first check that $\mathsf{k} : \mathcal{D}_\alpha \to \mathcal{D}_{\beta\to\alpha}$. For each $\mathsf{a} \in \mathcal{D}_\alpha$, we must verify $\mathsf{k}(\mathsf{a}) \in \mathcal{D}_{\beta\to\alpha}$. Let $f \in \mathcal{R}_\beta$ be given. Since $(\mathsf{k}(\mathsf{a}) \circ f) : A \to \mathcal{D}_\alpha$ is the constant a function, we know $(\mathsf{k}(\mathsf{a}) \circ f) \in \mathcal{R}_\alpha$ by Lemma 21. Hence $\mathsf{k}(\mathsf{a}) \in \mathcal{D}_{\beta\to\alpha}$. To check $\mathsf{k} \in \mathcal{D}_{\alpha\to\beta\to\alpha}$, let $g \in \mathcal{R}_\alpha$ be given. In order to check $(\mathsf{k} \circ g) \in \mathcal{R}_{\beta\to\alpha}$, let $f \in \mathcal{R}_\beta$ be given. Now, we must check that $S(\mathsf{k} \circ g, f) \in \mathcal{R}_\alpha$. For each $x \in A$, we compute

$$S(\mathsf{k} \circ g, f)(x) = (\mathsf{k} \circ g)(x)(f(x)) = \mathsf{k}(g(x))(f(x)) = g(x).$$

Thus $S(\mathsf{k} \circ g, f) = g \in \mathcal{R}_\alpha$.

Checking that the **S** combinators have interpretations is similar.

Theorem 24 ensures logical relation frames have domains sufficient to interpret λ-terms (cf. Proposition 14)

7 Models with Specified Sets

Let A be any nonempty set and $\mathcal{B} \subseteq \mathcal{P}(A)$ be a collection of sets such that $\emptyset, A \in \mathcal{B}$. We will now demonstrate how to construct a combinatory frame (cf. Definitions 3 and 13) where $\mathcal{D}_\iota = A$, $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$ and the sets in $\mathcal{B}$ correspond to the sets represented in $\mathcal{D}_{\iota\to o}$. In other words, we can specify precisely what sets should be in $\mathcal{D}_{\iota\to o}$ by choosing $\mathcal{B}$. The resulting evaluation induces a model for different signatures depending on properties of $\mathcal{B}$. We can use the fact that some logical constants cannot be interpreted in the corresponding model to determine that these logical constants must be used in (instantiations involved in) any proof of a theorem. We will define the frame as a logical relation frame (cf. Definition 19).

Let A_1 and A_2 be sets and $f : A_1 \to A_2$ be a function. This function f induces an *inverse image* function $f^{-1} : \mathcal{P}(A_2) \to \mathcal{P}(A_1)$ defined by

$$f^{-1}(Y) := \{x \in A_1 \mid f(x) \in Y\}.$$

Also, if $y \in A_2$, then we write $f^{-1}(y)$ to denote $f^{-1}(\{y\})$. If $B_1 \subseteq \mathcal{P}(A_1)$ and $B_2 \subseteq \mathcal{P}(A_2)$, then we write $f^{-1} : B_2 \to B_1$ for the restriction of f^{-1} to B_2 if $f^{-1}(Y) \in B_1$ whenever $Y \in B_2$. In particular, given a nonempty set A, $\mathcal{B} \subseteq \mathcal{P}(A)$, and $f : A \to A$ as above, we write $f^{-1} : \mathcal{B} \to \mathcal{B}$ to denote this restriction if for every $X \in \mathcal{B}$, $\{x \in A \mid f(x) \in X\} \in \mathcal{B}$.

We define a logical relation frame by defining $\mathcal{D}$ and $\mathcal{R}$ on base types.

Definition 25 *Let A be a nonempty set and $\mathcal{B} \subseteq \mathcal{P}(A)$ be such that $\emptyset, A \in \mathcal{B}$. Let $\mathcal{D}_\iota := A$ and $\mathcal{D}_o := \{\mathtt{T}, \mathtt{F}\}$. Let*

$$\mathcal{R}_\iota := \{f : A \to A \mid f^{-1}(X) \in \mathcal{B} \text{ for every } X \in \mathcal{B}\}$$

and

$$\mathcal{R}_o := \{f : A \to \mathcal{D}_o \mid f^{-1}(\mathtt{T}) \in \mathcal{B}\}.$$

Let $(\mathcal{D}, \mathcal{R})$ be the A-ary logical relation extension $(\mathcal{D}_\iota, \mathcal{D}_o, \mathcal{R}_\iota, \mathcal{R}_o)$. We call $\mathcal{D}$ the $\mathcal{B}$-specified frame with relation $\mathcal{R}$.

Note that in Definition 25 for any $f \in \mathcal{R}_\iota$ we know $f^{-1} : \mathcal{P}(A) \to \mathcal{P}(A)$ restricts to a function $f^{-1} : \mathcal{B} \to \mathcal{B}$.

We use Lemma 23 to determine $\mathcal{D}$ is a frame.

Lemma 26 *Let A be a nonempty set, $\mathcal{B} \subseteq \mathcal{P}(A)$ be such that $\emptyset, A \in \mathcal{B}$ and $\mathcal{D}$ be the $\mathcal{B}$-specified frame with relation $\mathcal{R}$. Then $\mathcal{D}$ is an A-ary logical relation frame with relation $\mathcal{R}$.*

Proof. Using Lemma 23, we are only required to verify $K(\mathsf{b}) \in \mathcal{R}_o$ for every $\mathsf{b} \in \mathcal{D}_o$ and $K(\mathsf{c}) \in \mathcal{R}_\iota$ for every $\mathsf{c} \in \mathcal{D}_\iota$. Note that

$$(K(\mathtt{T}))^{-1}(\mathtt{T}) = \{x \in A \mid K(\mathtt{T})(x) = \mathtt{T}\} = A \in \mathcal{B}$$

and

$$(K(\mathtt{F}))^{-1}(\mathtt{T}) = \{x \in A \mid K(\mathtt{F})(x) = \mathtt{T}\} = \emptyset \in \mathcal{B}.$$

Hence $K(\mathsf{b}) \in \mathcal{R}_o$ for every $\mathsf{b} \in \mathcal{D}_o$. Let $\mathsf{c} \in \mathcal{D}_\iota$ be given. For any set $X \in \mathcal{B}$, $(K(\mathsf{c}))^{-1}(X)$ is either A (if $\mathsf{c} \in X$) or $\emptyset$ (if $\mathsf{c} \notin X$). In either case, $K(\mathsf{c})^{-1}(X) \in \mathcal{B}$ for every $X \in \mathcal{B}$. Hence $K(\mathsf{c}) \in \mathcal{R}_\iota$. Thus $\mathcal{D}$ is an A-ary logical relation frame with relation $\mathcal{R}$. In particular, $\mathcal{D}$ is a frame.

Since $\mathcal{B}$-specified frames are logical relation frames, we know they are combinatory. We now obtain further results about $\mathcal{B}$-specified frames.

Lemma 27 *Let A be a nonempty set, $\mathcal{B} \subseteq \mathcal{P}(A)$ be such that $\emptyset, A \in \mathcal{B}$ and $\mathcal{D}$ be the $\mathcal{B}$-specified frame with relation $\mathcal{R}$. The identity function on A is in $\mathcal{R}_\iota$.*

Proof. The identity function $id : A \to A$ clearly satisfies the property that $id^{-1}(X) = X \in \mathcal{B}$ whenever $X \in \mathcal{B}$. Hence $id \in \mathcal{R}_\iota$.

We will now record the correspondence between $\mathcal{B}$ and the domain $\mathcal{D}_{\iota\to o}$.

Lemma 28 *Let A be a nonempty set, $\mathcal{B} \subseteq \mathcal{P}(A)$ be such that $\emptyset, A \in \mathcal{B}$ and $\mathcal{D}$ be the $\mathcal{B}$-specified frame with relation $\mathcal{R}$. Let $\chi_X : \mathcal{D}_\iota \to \mathcal{D}_o$ be*

$$\chi_X(a) := \begin{cases} \mathtt{T} & \text{if } a \in X \\ \mathtt{F} & \text{if } a \notin X \end{cases}$$

for each $X \in \mathcal{B}$ and $a \in A$. That is, χ_X is the characteristic function of X. Then $\chi_X \in \mathcal{D}_{\iota\to o}$ for each $X \in \mathcal{B}$ and the map $X \mapsto \chi_X$ is a bijection from $\mathcal{B}$ to $\mathcal{D}_{\iota\to o}$.

Proof. Let $X \in \mathcal{B}$ be given. To determine $\chi_X \in \mathcal{D}_{\iota\to o}$, let $f \in \mathcal{R}_\iota$ be given. To check $(\chi_X \circ f) \in \mathcal{R}_o$, we must verify $(\chi_X \circ f)^{-1}(\mathtt{T}) \in \mathcal{B}$. We compute

$$(\chi_X \circ f)^{-1}(\mathtt{T}) = f^{-1}(\chi_X{}^{-1}(\mathtt{T})) = f^{-1}(X) \in \mathcal{B}$$

since $f \in \mathcal{R}_\iota$ and $X \in \mathcal{B}$.

The fact that $X \mapsto \chi_X$ is injective is trivial, since any $X \in \mathcal{B}$ is determined by its characteristic function. To verify surjectivity, let $h \in \mathcal{D}_{\iota\to o}$ be given. Hence $h : A \to \{\mathtt{T}, \mathtt{F}\}$ and for every $f \in \mathcal{R}_\iota$ we know $(h \circ f) \in \mathcal{R}_o$. By Lemma 27, $id \in \mathcal{R}_\iota$ and so $h = (h \circ id) \in \mathcal{R}_o$. By definition of $\mathcal{R}_o$, this means $h^{-1}(\mathtt{T}) \in \mathcal{B}$. Let $X := h^{-1}(\mathtt{T}) \in \mathcal{B}$. Note that $\chi_X(x) = \mathtt{T}$ iff $x \in X$ iff $h(x) = \mathtt{T}$. Thus $h = \chi_X$ and we are done.

Lemma 29 *Let A be a nonempty set, $\mathcal{B} \subseteq \mathcal{P}(A)$ be such that $\emptyset, A \in \mathcal{B}$ and $\mathcal{D}$ be the $\mathcal{B}$-specified frame with relation $\mathcal{R}$. For each type α, $\mathcal{D}_{\iota\to\alpha} \subseteq \mathcal{R}_\alpha$. Furthermore, $\mathcal{D}_{\iota\to o} = \mathcal{R}_o$ and $\mathcal{D}_{\iota\to\iota} = \mathcal{R}_\iota$.*

Proof. By Lemma 27, $id \in \mathcal{R}_\iota$. Consequently, if $h \in \mathcal{D}_{\iota\to\alpha}$, then $h = (h \circ id)$ is in $\mathcal{R}_\alpha$. We conclude the inclusion $\mathcal{D}_{\iota\to\alpha} \subseteq \mathcal{R}_\alpha$.

At base types, we verify the reverse inclusions.

- o Suppose $f \in \mathcal{R}_o$. Let $X := f^{-1}(\mathtt{T}) \in \mathcal{B}$. Since $f = \chi_X$, we know $f \in \mathcal{D}_{\iota\to o}$ by Lemma 28.
- ι Suppose $f \in \mathcal{R}_\iota$. That is, $f : A \to A$ and $f^{-1}(X) \in \mathcal{B}$ for any $X \in \mathcal{B}$. Since $\mathcal{D}_\iota = A$, $f : \mathcal{D}_\iota \to \mathcal{D}_\iota$. To check $f \in \mathcal{D}_{\iota\to\iota}$, let $g \in \mathcal{R}_\iota$ be given. For any $X \in \mathcal{B}$, $f^{-1}(X) \in \mathcal{B}$ and so $(f \circ g)^{-1}(X) = g^{-1}(f^{-1}(X)) \in \mathcal{B}$. This establishes $(f \circ g) \in \mathcal{R}_\iota$ and so $f \in \mathcal{D}_{\iota\to\iota}$.

We turn now to the question of which logical constants are realized by such frames. Note that we have assumed $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$ so that Definition 15 makes sense for $\mathcal{D}$. To interpret the logical constants $\neg$, $\vee$, or $\wedge$ we must assume closure conditions on $\mathcal{B}$.

Lemma 30 *Let A be a nonempty set, $\mathcal{B} \subseteq \mathcal{P}(A)$ be such that $\emptyset, A \in \mathcal{B}$ and $\mathcal{D}$ be the $\mathcal{B}$-specified frame with relation $\mathcal{R}$. We have the following:*

$\neg$ *$\mathcal{D}$ realizes $\neg$ iff $\mathcal{B}$ is closed under complements relative to A (i.e., $A \setminus X \in \mathcal{B}$ whenever $X \in \mathcal{B}$).*

$\vee$ *$\mathcal{D}$ realizes $\vee$ iff $\mathcal{B}$ is closed under binary unions.*

$\wedge$ *$\mathcal{D}$ realizes $\wedge$ iff $\mathcal{B}$ is closed under binary intersections.*

Proof. For each logical constant $c_\alpha \in \{\neg, \vee, \wedge\}$, there is only one possible function a which will satisfy the property $\mathfrak{L}_c(\mathsf{a})$ with respect to id. That is, we are forced to interpret each of these logical constant as the function determined by truth tables. We must check in each case that this function is actually in the appropriate domain.

$\neg$ Let $n : \mathcal{D}_o \to \mathcal{D}_o$ be the function defined by $n(\mathtt{T}) := \mathtt{F}$ and $n(\mathtt{F}) := \mathtt{T}$. Suppose $\mathcal{B}$ is closed under complements relative to A. Let $f \in \mathcal{R}_o$ be given. Then $f : A \to \{\mathtt{T}, \mathtt{F}\}$ and $f^{-1}(\mathtt{T}) \in \mathcal{B}$. Hence

$$(n \circ f)^{-1}(\mathtt{T}) = f^{-1}(n^{-1}(\mathtt{T})) = f^{-1}(\mathtt{F}) = (A \setminus f^{-1}(\mathtt{T})) \in \mathcal{B}$$

since $\mathcal{B}$ is closed under complements relative to A. Thus $n \in \mathcal{D}_{o\to o}$ witnesses that $\mathcal{D}$ realizes $\neg$.

On the other hand, if we know $n \in \mathcal{D}_{o\to o}$ and $X \in \mathcal{B}$, then $\chi_X \in \mathcal{D}_{\iota\to o} \subseteq \mathcal{R}_o$ by Lemmas 28 and 29. Hence $n \circ \chi_X \in \mathcal{R}_o$ and so $(A \setminus X) = \{a \in A \mid n(\chi_X(a)) = \mathtt{T}\} \in \mathcal{B}$.

$\vee$ Let $d : \mathcal{D}_o \to \mathcal{D}_o^{\mathcal{D}_o}$ be the function defined by

$$d(\mathsf{a})(\mathsf{b}) := \begin{cases} \mathtt{T} & \text{if } \mathsf{a} = \mathtt{T} \text{ or } \mathsf{b} = \mathtt{T} \\ \mathtt{F} & \text{otherwise.} \end{cases}$$

First, note that $d : \mathcal{D}_o \to \mathcal{D}_{o\to o}$ since $d(\mathtt{T})$ is the constant $\mathtt{T}$ function and $d(\mathtt{F})$ is the identity function (both of which are in $\mathcal{D}_{o\to o}$ since the frame $\mathcal{D}$ is combinatory).

Suppose $\mathcal{B}$ is closed under binary unions. To check $d \in \mathcal{D}_{o\to o\to o}$, we let $f, g \in \mathcal{R}_o$ and verify $S(d \circ f, g) \in \mathcal{R}_o$. For each $x \in A$, we compute

$$S(d \circ f, g)(x) = d(f(x))(g(x)).$$

Hence

$$\begin{aligned} S(d \circ f, g)(x) = \mathtt{T} \ &\text{iff} \ d(f(x))(g(x)) = \mathtt{T} \\ &\text{iff} \ f(x) = \mathtt{T} \text{ or } g(x) = \mathtt{T}. \end{aligned}$$

Thus $(S(d \circ f, g))^{-1}(\mathtt{T}) = f^{-1}(\mathtt{T}) \cup g^{-1}(\mathtt{T}) \in \mathcal{B}$ since $\mathcal{B}$ is closed under binary unions. Therefore, $d \in \mathcal{D}_{o\to o\to o}$ and $\mathcal{D}$ realizes $\vee$.

Suppose $d \in \mathcal{D}_{o\to o\to o}$ and $X, Y \in \mathcal{B}$. Then $S(d \circ \chi_X, \chi_Y) \in \mathcal{R}_o$. Thus $X \cup Y = \{a \in A \mid d(\chi_X(a))(\chi_Y(a)) = \mathtt{T}\} \in \mathcal{B}$.

$\wedge$ Analogous to the $\vee$ case.

Definition 31 *Let A be a nonempty set. We call $\mathcal{B} \subseteq \mathcal{P}(A)$ a* field of sets *if $\emptyset, A \in \mathcal{B}$ and $\mathcal{B}$ is closed under complements, finite unions and finite intersections.*

Theorem 32 *Let A be a nonempty set, $\mathcal{B} \subseteq \mathcal{P}(A)$ be a field of sets and $\mathcal{D}$ be the $\mathcal{B}$-specified frame with relation $\mathcal{R}$. Then $\mathcal{D}$ realizes $\neg$, $\vee$ and $\wedge$.*

Proof. This follows immediately from Lemma 30.

8 A Combinatory Frame in which Surjective Cantor Fails

We will use a $\mathcal{B}$-specified frame to construct an extensional model (without negation) in which both forms of Cantor's Theorem Fail.

Let A be the real interval $(-1, 1)$ and

$$\mathcal{B} := \{(a, 1) \mid -1 \le a \le 1\} \subseteq \mathcal{P}(A).$$

The emptyset is in $\mathcal{B}$ as the (trivial) interval $(1, 1)$ and A is in $\mathcal{B}$ as the interval $(-1, 1)$. The set $\mathcal{B}$ is closed under unions and intersections, but is not closed under complements. Let $\mathcal{D}$ be the $\mathcal{B}$-specified frame with relation $\mathcal{R}$ (cf. Definition 25). By Lemma 30, we know $\mathcal{D}$ realizes $\vee$ and $\wedge$, but does not realize $\neg$.

Definition 33 *Let C and D be sets of real numbers. A function $f : C \to D$ is* left continuous *if for any $a \in C$,*

$$\lim_{x \to a^-} f(x) = f(a).$$

Lemma 34 *Let $f : (-1, 1) \to (-1, 1)$ be any function. $f \in \mathcal{R}_\iota$ iff f is nondecreasing and left continuous.*

Proof. Suppose $f \in \mathcal{R}_\iota$. Assume there exists x and y with $-1 < x < y < 1$ and $f(y) < f(x)$. Hence $x \in f^{-1}((f(y), 1)) \in \mathcal{B}$. Since every interval in $\mathcal{B}$ is upward closed, we must have $y \in f^{-1}((f(y), 1))$, a contradiction. Thus f is nondecreasing. Next, if f is not left continuous, then there is some $a \in (-1, 1)$ and increasing sequence $-1 < x_1 < x_2 < \cdots < a$ such that $x_n \to a$ but $f(x_n) \not\to f(a)$ as $n \to \infty$. Since f is nondecreasing, $\{f(x_n)\}_n$ is a bounded, monotone sequence and hence convergent. Let L be its limit. Since $L \neq f(a)$ and each $-1 < f(x_n) \le f(a)$, we must have $-1 < L < f(a)$. Hence we know $a \in f^{-1}((L, 1)) \in \mathcal{B}$. There is some $b \in [-1, 1]$ such that $f^{-1}((L, 1)) = (b, 1)$. Since $a \in (b, 1)$, we know $a > b$. Choose some $c \in (b, a) \subseteq f^{-1}((L, 1))$. Since $f(x_n) \le L < f(c)$ and f is nondecreasing, $x_n < c$ for every n. Since $x_n \to a$ as $n \to \infty$, we conclude $a \le c$, contradicting our choice of c. Thus f is left continuous.

Conversely, suppose f is nondecreasing and left continuous. For any $b \in [-1, 1]$, we prove $f^{-1}((b, 1)) \in \mathcal{B}$. If $f^{-1}((b, 1))$ is empty, then we are done. Hence we can assume $f^{-1}((b, 1))$ is nonempty. Let

$$a := \inf(\{y \mid f(y) > b\}) \in [-1, 1).$$

Then for every $x \in (a, 1)$, there exists some $y \in (a, x)$ such that $f(y) > b$. Since f is nondecreasing, $f(x) \ge f(y) > b$ and so $x \in f^{-1}((b, 1))$. Thus $(a, 1) \subseteq f^{-1}((b, 1)) \subseteq (-1, 1)$. If $a = -1$, then we are done. Assume $a > -1$. To verify $f^{-1}((b, 1)) \subseteq (a, 1)$, assume there is some $x \in (-1, a]$ with $f(x) > b$. Since f is left continuous, there is some $c \in (-1, x)$ such that $f(c) > b$. This $c < a$ contradicts the definition of a as the infimum of such values.

Note that $\mathcal{B}$ is a topology on $(-1, 1)$. By the definition of $\mathcal{R}_\iota$, we know $f \in \mathcal{R}_\iota$ iff f is continuous with respect to this topology. Lemma 34 establishes continuity of $f : (-1, 1) \to (-1, 1)$ with respect to the topology $\mathcal{B}$ corresponds to f being nondecreasing and left continuous with respect to the usual topology on $(-1, 1)$.

Recall (cf. Definition 18) for any $\mathsf{a} \in \mathcal{D}_\alpha$ we use the notation $K(\mathsf{a})$ to denote the constant function $K(\mathsf{a}) : (-1, 1) \to \mathcal{D}_\alpha$ defined by $K(\mathsf{a})(x) := \mathsf{a}$. Since $\mathcal{D}$ is an A-ary logical relation frame with relation $\mathcal{R}$ (cf. Definition 19), we know $K(\mathsf{a}) \in \mathcal{R}_\alpha$. Also, for any $p : (-1, 1) \to \mathcal{D}_{\alpha\to\beta}$ and $f : (-1, 1) \to \mathcal{D}_\alpha$ we use the notation $S(p, f) : (-1, 1) \to \mathcal{D}_\beta$ to denote the function $S(p, f)(x) := p(x)(f(x))$. Since $\mathcal{D}$ is an A-ary logical relation frame with relation $\mathcal{R}$, we know $S(p, f) \in \mathcal{R}_\beta$ whenever $p \in \mathcal{R}_{\alpha\to\beta}$ and $f \in \mathcal{R}_\alpha$.

Lemma 35 *Suppose $g \in \mathcal{R}_{\iota\to o}$, $x, y \in (-1, 1)$ and $g(x)(y) = \mathtt{T}$. For any $y' \in (y, 1)$, $g(x)(y') = \mathtt{T}$. Also, for any $x' \in (x, 1)$, $g(x')(y) = \mathtt{T}$.*

Proof. Since $g \in \mathcal{R}_{\iota\to o}$, $g(x) \in \mathcal{D}_{\iota\to o}$. By Lemma 28, there is some $a \in [-1,1]$ such that $g(x) = \chi_{(a,1)}$. Since

$$\chi_{(a,1)}(y) = g(x)(y) = \mathtt{T},$$

we know $y > a$. Hence for any $y' \in (y,1)$ we have

$$g(x)(y') = \chi_{(a,1)}(y') = \mathtt{T}.$$

By Lemma 21, we know $K(y) \in \mathcal{R}_\iota$. Since $g \in \mathcal{R}_{\iota\to o}$ we must have $S(g, K(y)) \in \mathcal{R}_o$. That is, there is some $b \in [-1,1]$ such that

$$\{z \in (-1,1) \mid g(z)(y) = \mathtt{T}\} = \{z \in (-1,1) \mid S(g,K(y))(z) = \mathtt{T}\} = (b,1).$$

Since $g(x)(y) = \mathtt{T}$, we have $x > b$. Hence $g(x')(y) = \mathtt{T}$ for any $x' \in (x,1)$.

We can now characterize the members of $\mathcal{D}_{\iota\to\iota\to o}$ and $\mathcal{R}_{\iota\to o}$.

Lemma 36 *Let $g : (-1,1) \to \{\mathtt{T},\mathtt{F}\}^{(-1,1)}$ be any function. The following are equivalent:*

1. $g \in \mathcal{D}_{\iota\to\iota\to o}$.
2. $g \in \mathcal{R}_{\iota\to o}$.
3. *There exists a nonincreasing left continuous $l : (-1,1) \to \Re$ (the reals) such that $g(x)(y) = \mathtt{T}$ iff $l(x) < y$ for every $x, y \in (-1,1)$.*

Proof. **(1)⇒(2)** This follows from $\mathcal{D}_{\iota\to\iota\to o} \subseteq \mathcal{R}_{\iota\to o}$ (cf. Lemma 29).

(2)⇒(3) Suppose $g \in \mathcal{R}_{\iota\to o}$. Then $g(x) \in \mathcal{D}_{\iota\to o}$ for each $x \in (-1,1)$. By Lemma 28, for every x there is some $l(x) \in [-1,1]$ such that

$$g(x) = \chi_{(l(x),1)}.$$

Clearly, $g(x)(y) = \mathtt{T}$ iff $l(x) < y$ for every $x, y \in (-1,1)$.

First we check l is nonincreasing. Suppose $-1 < a < b < 1$ and $-1 \le l(a) < l(b) \le 1$. Choose some $c \in (l(a), l(b))$. Since $l(a) < c$ and $l(b) \not< c$, we know $g(a)(c) = \mathtt{T}$ and $g(b)(c) = \mathtt{F}$, contradicting Lemma 35.

To prove l is left continuous, suppose there is some $a \in (-1,1)$ and increasing sequence $-1 < x_1 < x_2 < \cdots < a$ where $x_n \to a$ and $l(x_n) \not\to l(a)$ as $n \to \infty$. Since l is nonincreasing, $l(x_1) \ge l(x_2) \ge \cdots$ is a monotone sequence and has a limit $L \in [-1,1]$. Also, $l(x_n) \ge l(a)$ for every n and $L \neq l(a)$, so we must have $L > l(a)$. Choose some $c \in (l(a), L)$. Since $g \in \mathcal{R}_{\iota\to o}$ and $K(c) \in \mathcal{R}_\iota$, we know $S(g, K(c)) \in \mathcal{R}_o$. Hence there is some $d \in [-1,1]$ such that $S(g, K(c)) = \chi_{(d,1)}$. Note that for any $x \in (-1,1)$, we have

$$\begin{aligned} x > d \ &\text{ iff } \ S(g,K(c))(x) = \mathtt{T} \\ &\text{ iff } \ g(x)(c) = \mathtt{T} \\ &\text{ iff } \ l(x) < c. \end{aligned}$$

By the choice of c, we have $l(a) < c$ and so (by the equivalence) $a > d$. Since $x_n \to a$, there must be some N with $x_N > d$. Consequently, $l(x_N) < c$ by the equivalence. This contradicts $l(x_N) \ge L > c$. Thus l is left continuous.

(3)⇒(1) Suppose $l : (-1,1) \to \Re$ is a nonincreasing left continuous function such that $g(x)(y) = \mathtt{T}$ iff $l(x) > y$. We first must check $g : \mathcal{D}_\iota \to \mathcal{D}_{\iota\to o}$. This follows from Lemma 28 since

$$g(x) = \chi_{(l(x),1)} \in \mathcal{D}_{\iota\to o}$$

for each $x \in \mathcal{D}_\iota$. To check $g \in \mathcal{D}_{\iota\to\iota\to o}$, let $f_1 \in \mathcal{R}_\iota$ be given. To check $g \circ f_1 \in \mathcal{R}_{\iota\to o}$, let $f_2 \in \mathcal{R}_\iota$ be given. By Lemma 34, we know f_1 and f_2 are nondecreasing and left continuous. We must check $S(g \circ f_1, f_2) \in \mathcal{R}_o$. That is,

$$\{x \in (-1,1) \mid g(f_1(x))(f_2(x)) = \mathtt{T}\} \in \mathcal{B}.$$

By our assumption about l, we know

$$\begin{aligned} &\{x \in (-1,1) \mid g(f_1(x))(f_2(x)) = \mathtt{T}\} \\ &= \{x \in (-1,1) \mid l(f_1(x)) < f_2(x)\}. \end{aligned}$$

If $\{x \in (-1,1) \mid l(f_1(x)) < f_2(x)\}$ is empty, then we are done. Assume $\{x \in (-1,1) \mid l(f_1(x)) < f_2(x)\}$ is nonempty and let

$$a := \inf\{y \in (-1,1) \mid l(f_1(y)) < f_2(y)\}.$$

We will determine $\{x \in (-1,1) \mid l(f_1(x)) < f_2(x)\} = (a,1) \in \mathcal{B}$.

Suppose $x \in (a,1)$. By our choice of a, there is some $y \in (a,x)$ with $l(f_1(y)) < f_2(y)$. Since $y < x$, f_1 and f_2 are nondecreasing and l is nonincreasing, $f_1(y) \le f_1(x)$, $l(f_1(x)) \le l(f_1(y))$ and $f_2(y) \le f_2(x)$. Hence

$$l(f_1(x)) \le l(f_1(y)) < f_2(y) \le f_2(x).$$

Thus

$$(a,1) \subseteq \{x \in (-1,1) \mid l(f_1(x)) < f_2(x)\} \subseteq (-1,1).$$

To verify $\{x \in (-1,1) \mid l(f_1(x)) < f_2(x)\} \subseteq (a,1)$ assume there is some $x \in (-1,a]$ with $l(f_1(x)) < f_2(x)$. That is, $l(f_1(x)) - f_2(x) < 0$. Since l, f_1 and f_2 are left continuous, there is some $y \in (-1,x)$ such that $l(f_1(y)) - f_2(y) < 0$. This $y < x \le a$ contradicts the choice of a as the infimum of such values.

Lemma 37 *There is a surjection $\mathsf{g} \in \mathcal{D}_{\iota\to\iota\to o}$ and an injection $\mathsf{h} \in \mathcal{D}_{(\iota\to o)\to\iota}$.*

Proof. Let $\mathsf{g} : (-1,1) \to \{\mathtt{T},\mathtt{F}\}^{(-1,1)}$ be the function

$$\mathsf{g}(x)(y) := \begin{cases} \mathtt{T} & \text{if } -2x < y \\ \mathtt{F} & \text{otherwise} \end{cases}$$

Lemma 36 proves $\mathsf{g} \in \mathcal{D}_{\iota\to\iota\to o}$ using the function $l : (-1,1) \to \Re$ defined by $l(x) := -2x$. By definition, $\mathsf{g}(x)(y) = \mathtt{T}$ iff $l(x) < y$. Clearly l is nonincreasing and left continuous. The graph of this function g is shown as a binary relation in Figure 1.

Define $\mathsf{h} : \mathcal{D}_{\iota\to o} \to \mathcal{D}_\iota$ by

$$\mathsf{h}(\chi_{(a,1)}) := \frac{-a}{2}$$

(using Lemma 28). To check $\mathsf{h} \in \mathcal{D}_{(\iota\to o)\to\iota}$, let $p \in \mathcal{R}_{\iota\to o}$ be given. By Lemma 36, there is a nonincreasing left continuous $q : (-1,1) \to \Re$ such that $p(x)(y) = \mathtt{T}$ iff $q(x) < y$ for every $x, y \in (-1,1)$. That is, $p(x) = \chi_{(q(x),1)}$ for every $x \in (-1,1)$. We must check $(\mathsf{h} \circ p) \in \mathcal{R}_\iota$. Let $(b,1) \in \mathcal{B}$ be given. We must ensure

$$\{x \in (-1,1) \mid \mathsf{h}(p(x)) > b\} \in \mathcal{B}.$$

We compute

$$\begin{aligned} \{x \in (-1,1) \mid \mathsf{h}(p(x)) > b\} &= \{x \in (-1,1) \mid \mathsf{h}(\chi_{(q(x),1)}) > b\} \\ &= \{x \in (-1,1) \mid \tfrac{-q(x)}{2} > b\} = \{x \in (-1,1) \mid q(x) < -2b\}. \end{aligned}$$

Let $c := \inf\{y \in (-1,1) \mid q(y) < -2b\}$. We will determine

$$\{x \in (-1,1) \mid q(x) < -2b\} = (c,1) \in \mathcal{B}.$$

First, suppose $x \in (c,1)$. By the choice of c, there is a $y \in (c,x)$ such that $q(y) < -2b$. Since q is nonincreasing and $x > y$, we know $q(x) \le q(y) < -2b$. Hence

$$(c,1) \subseteq \{x \in (-1,1) \mid q(x) < -2b\} \subseteq (-1,1).$$

Next, suppose $x \in (-1, c]$ and $q(x) < -2b$. Since q is left continuous, there is some $y \in (-1, x)$ such that $q(y) < -2b$. This $y < x \leq c$ contradicts the choice of c as the infimum of such values. Thus

$$\{x \in (-1,1) \mid q(x) < -2b\} = (c, 1) \in \mathcal{B}.$$

Therefore, $(\mathsf{h} \circ p) \in \mathcal{R}_\iota$ and so $\mathsf{h} \in \mathcal{D}_{(\iota\to o)\to\iota}$.

Finally, we check that g is surjective and h is injective by showing $\mathsf{g} \circ \mathsf{h}$ is the identity. For any $\chi_{(a,1)} \in \mathcal{D}_{\iota\to o}$, we compute

$$\mathsf{g}(\mathsf{h}(\chi_{(a,1)}))(y) = \mathsf{g}(\frac{-a}{2})(y) = \mathtt{T}$$

iff $-2(\frac{-a}{2}) < y$ iff $a < y$. Thus $\mathsf{g}(\mathsf{h}(\chi_{(a,1)})) = \chi_{(a,1)}$. In particular, g is surjective and h is injective.

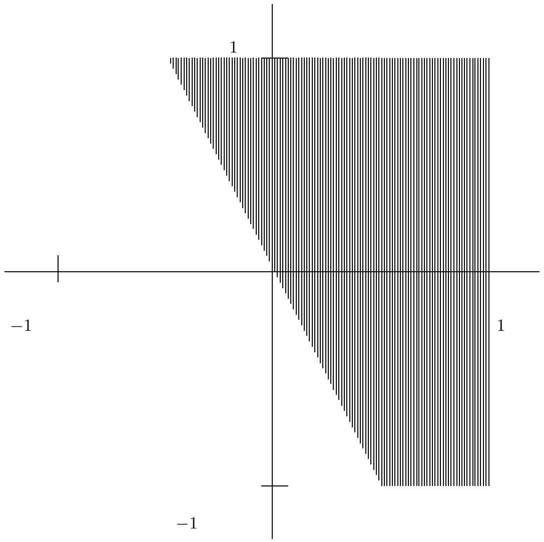

Fig. 1. The Surjection g as a Binary Relation.

9 A Combinatory Frame in which Injective Cantor Fails

We have constructed a model in which the injective Cantor theorem fails due to the model lacking negation. A more interesting problem is to find an extensional model which realizes $\neg$, $\wedge$, and $\vee$ (and more) but fails to satisfy injective Cantor. This must rely on the model failing to realize $\Sigma^{\iota\to o}$ or $=^\iota$. We construct such a model here which realizes neither $\Sigma^{\iota\to o}$ nor $=^\iota$ and fails to satisfy injective Cantor.

Let $\mathcal{B} \subseteq \mathcal{P}(\mathbb{N})$ be the collection of sets X of natural numbers where X is finite or cofinite. $\mathcal{B}$ is clearly a field of sets. Let $\mathcal{D}$ be the $\mathcal{B}$-specified frame with relation $\mathcal{R}$ (cf. Definition 25). We will use this frame to construct a model in which the injective Cantor theorem fails to hold.

By Definition 25, we know $\mathcal{D}_\iota := \mathbb{N}$, so $\mathcal{D}_\iota$ is countably infinite. Since there are countably many finite and cofinite subsets of $\mathbb{N}$, $\mathcal{B}$ is also countably infinite. By Lemma 28, the function taking each $X \in \mathcal{B}$ to a characteristic function χ_X in $\mathcal{D}_{\iota\to o}$ is a bijection. Hence $\mathcal{D}_{\iota\to o}$ is also countably infinite.

Since $\mathcal{D}_{\iota\to o}$ and $\mathcal{D}_\iota$ are countably infinite, there is an injection $h : \mathcal{D}_{\iota\to o} \to \mathcal{D}_\iota$. The existence of such an h certainly does *not* contradict the injective Cantor theorem. The existence of such an h is often referred to as *Skolem's Paradox*. As is well-known, Skolem's Paradox is not really a paradox, since the injection h exists at the meta-level. This is far different from saying there is such an $h \in \mathcal{D}_{(\iota\to o)\to\iota}$. That is, we know the injection h exists, but we do not know it exists *in the model*. We will prove that for the $\mathcal{B}$-specified model, any such injection h *is* in $\mathcal{D}_{(\iota\to o)\to\iota}$. This means that with respect to this model, Skolem's Paradox really does provide a counterexample to this version of Cantor's "theorem". This is still not a paradox. It is merely a model demonstrating that the injective Cantor "theorem" is not a theorem if the signature does not include enough quantifiers or equalities.

Definition 38 *Let C be a set and $(a_n)_{n\in\mathbb{N}}$ be a sequence with $a_n \in C$. We say the sequence $(a_n)_{n\in\mathbb{N}}$* converges to *$a \in C$ (written $a_n \to_n a$) if there exists an $N \in \mathbb{N}$ such that $a_n = a$ for every $n > N$. We say a function $g : \mathbb{N} \to C$ is* eventually constant *if there exists some $a \in C$ such that $g(n) \to_n a$. For each type α, let*

$$\mathcal{K}_\alpha := \{g : \mathbb{N} \to \mathcal{D}_\alpha \mid g \textit{ is eventually constant}\}.$$

Let C and D be sets and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n : D \to C$. We say the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise *to a function $f : D \to C$ (written $f_n \to_n^p f$) if $f_n(d) \to_n f(d)$ for every $d \in D$. For all types α and β, let*

$$\mathcal{K}^p_{\alpha\to\beta} := \{g : \mathbb{N} \to \mathcal{D}_{\alpha\to\beta} \mid g \textit{ converges pointwise to some } f : \mathcal{D}_\alpha \to \mathcal{D}_\beta\}.$$

We start by examining the domains of base type.

Lemma 39 $\mathcal{D}_{\iota\to o} = \mathcal{R}_o = \mathcal{K}_o$.

Proof. By Lemma 29, we know $\mathcal{R}_o = \mathcal{D}_{\iota\to o}$.

To verify $\mathcal{D}_{\iota\to o} \subseteq \mathcal{K}_o$, let $f \in \mathcal{D}_{\iota\to o}$ be given. By Lemma 28, $f = \chi_X$ for some $X \in \mathcal{B}$. If X is finite, then $f(n) = \mathtt{F}$ for every $n > N$ where $N \in \mathbb{N}$ is such that $X \subseteq \{0, \ldots, N\}$. If X is cofinite, then $f(n) = \mathtt{T}$ for every $n > N$ where $N \in \mathbb{N}$ is such that $(\mathbb{N} \setminus X) \subseteq \{0, \ldots, N\}$.

Next suppose $f \in \mathcal{K}_o$. That is, there is some $N \in \mathbb{N}$ and $\mathsf{b} \in \{\mathtt{T}, \mathtt{F}\}$ such that $f(n) = \mathsf{b}$ for every $n > N$. If $\mathsf{b} = \mathtt{F}$, then $f^{-1}(\mathtt{T})$ is finite since

$$f^{-1}(X) \subseteq \{0, \ldots, N\}.$$

If $\mathsf{b} = \mathtt{T}$, then $f^{-1}(\mathtt{T})$ is cofinite since

$$(\mathbb{N} \setminus f^{-1}(\mathtt{T})) \subseteq \{0, \ldots, N\}.$$

In either case $f \in \mathcal{R}_o$.

Definition 40 *We say a function $f : \mathbb{N} \to \mathbb{N}$ is* uniformly unbounded *if for every $j \in \mathbb{N}$ there is some $I \in \mathbb{N}$ such that $f(i) > j$ for every $i > I$.*

Lemma 41 *Let $f : \mathbb{N} \to \mathbb{N}$ be any function. The function $f \in \mathcal{R}_\iota$ iff either f is eventually constant or f is uniformly unbounded.*

Proof. Suppose $f \in \mathcal{R}_\iota$ and f is not eventually constant. We will prove f is uniformly unbounded. Given $j \in \mathbb{N}$, since $\{0, \ldots, j\} \in \mathcal{B}$ we must have $f^{-1}(\{0, \ldots, j\}) \in \mathcal{B}$. Assume $f^{-1}(\{0, \ldots, j\})$ is cofinite. Then there must be some $L \in \{0,\ldots,j\}$) such that $f^{-1}(L)$ is cofinite. That is, $(\mathbb{N}\setminus f^{-1}(L))$ is finite. There must be some $N \in \mathbb{N}$ such that

$$(\mathbb{N}\setminus f^{-1}(L)) \subseteq \{0,\ldots,N\}.$$

However, this implies $f(n) = L$ for every $n > N$, contradicting our assumption that f is not eventually constant. Hence $f^{-1}(\{0,\ldots,j\}) \in \mathcal{B}$ must be finite and there must be some $I \in \mathbb{N}$ such that $f^{-1}(\{0,\ldots,j\}) \subseteq \{0,\ldots,I\}$. This means for any $i > I$ we have $f(i) > j$. Thus f is uniformly unbounded.

Next suppose f is eventually constant. There exist $N, L \in \mathbb{N}$ such that $f(n) = L$ for all $n > N$. In this case, for any $X \subseteq \mathbb{N}$, $f^{-1}(X)$ is finite if $L \notin X$ and $f^{-1}(X)$ is cofinite if $L \in X$. In either case $f^{-1}(X) \in \mathcal{B}$. In particular, this holds for $X \in \mathcal{B}$ and so $f \in \mathcal{R}_\iota$.

Finally, suppose f is uniformly unbounded. For every $j \in \mathbb{N}$ there is some $I \in \mathbb{N}$ such that $f(i) > j$ for every $i > I$. To verify $f \in \mathcal{R}_\iota$, let $X \in \mathcal{B}$ be given. Suppose X is finite and let $j_1 \in \mathbb{N}$ be such that $X \subseteq \{0,\ldots,j_1\}$. Let $I_1 \in \mathbb{N}$ be such that $f(i) > j_1$ for every $i > I_1$. This implies $f^{-1}(X) \subseteq \{0,\cdots,I_1\}$. Hence $f^{-1}(X)$ is finite and $f^{-1}(X) \in \mathcal{B}$. Suppose X is cofinite and let $j_2 \in \mathbb{N}$ be such that $(\mathbb{N}\setminus X) \subseteq \{0,\ldots,j_2\}$. Let $I_2 \in \mathbb{N}$ be such that $f(i) > j_2$ for every $i > I_2$. This implies

$$(\mathbb{N}\setminus f^{-1}(X)) = f^{-1}(\mathbb{N}\setminus X) \subseteq \{0,\cdots,I_2\}.$$

Hence $f^{-1}(X)$ is cofinite and $f^{-1}(X) \in \mathcal{B}$. Thus $f \in \mathcal{R}_\iota$.

We can use this to prove that $\mathcal{D}$ does not realize $=^\iota$, in spite of the fact that $\mathcal{B}$ contains every singleton $\{n\}$ where $n \in \mathbb{N}$.

Lemma 42 *The frame $\mathcal{D}$ does not realize $=^\iota$.*

Proof. Assume $\mathsf{q} \in \mathcal{D}_{\iota\to\iota\to o}$ satisfies $\mathfrak{L}_{=^\iota}(\mathsf{q})$. That is, $\mathsf{q}(m)(n) = \mathtt{T}$ iff $n = m$ for every $m, n \in \mathcal{D}_\iota$. Let $id : \mathbb{N} \to \mathcal{D}_\iota$ be the identity function and $f : \mathbb{N} \to \mathcal{D}_\iota$ be defined by $f(2n) := 2n$ and $f(2n+1) := 2n$. Clearly, both f and id are uniformly unbounded. Hence $f, id \in \mathcal{R}_\iota$ by Lemma 41. Thus $\mathsf{q} \circ f \in \mathcal{R}_{\iota\to o}$ and $S(\mathsf{q} \circ f, id) \in \mathcal{R}_o$. However,

$$S(\mathsf{q} \circ f, id)^{-1}(\mathtt{T}) = \{n \in \mathbb{N} \mid \mathsf{q}(f(n))(n) = \mathtt{T}\} = \{n \in \mathbb{N} \mid n \text{ is even}\} \notin \mathcal{B}$$

contradicting $S(\mathsf{q} \circ f, id) \in \mathcal{R}_o$.

We next prove $\mathcal{R}_\alpha$ is closed under the operation of making finitely many changes to functions.

Definition 43 *Let C be a set and $f, g : \mathbb{N} \to C$ be functions. We say f and g are* eventually equal *if there exists some $N \in \mathbb{N}$ such that $f(n) = g(n)$ for every $n > N$.*

Lemma 44 *Suppose $f \in \mathcal{R}_\alpha$ and $g : \mathbb{N} \to \mathcal{D}_\alpha$ are eventually equal. Then $g \in \mathcal{R}_\alpha$.*

Proof. We prove this by induction on the type α.

o By Lemma 39, the function f is eventually constant. Hence g is also eventually constant and $g \in \mathcal{R}_o$.

ι By Lemma 41, the function f is eventually constant or uniformly unbounded. If f is eventually constant, then so is g and hence $g \in \mathcal{R}_\iota$. Similarly, if f is uniformly unbounded, then so is g and hence $g \in \mathcal{R}_\iota$.

$\beta \to \gamma$ Let $h \in \mathcal{R}_\beta$ be given. We must check $S(g, h) \in \mathcal{R}_\gamma$. Since $f \in \mathcal{R}_{\beta\to\gamma}$, we know $S(f, h) \in \mathcal{R}_\beta$. Since f and g are eventually equal, there is some N such that $f(n) = g(n)$ for every $n > N$. Thus

$$S(f, h)(n) = f(n)(h(n)) = g(n)(h(n)) = S(g, h)(n).$$

That is, $S(f, h)$ and $S(g, h)$ are also eventually equal. By the inductive hypothesis, $S(g, h) \in \mathcal{R}_\gamma$ and hence $g \in \mathcal{R}_{\beta\to\gamma}$.

We can conclude the sets $\mathcal{K}_\alpha$ are lower bounds for the sets $\mathcal{R}_\alpha$.

Lemma 45 *For every type α, $\mathcal{K}_\alpha \subseteq \mathcal{R}_\alpha$.*

Proof. Let $f \in \mathcal{K}_\alpha$ be given. By definition, there is some $\mathsf{a} \in \mathcal{D}_\alpha$ such that $f(n) \to_n \mathsf{a}$. By Lemma 21, we know $K(\mathsf{a}) \in \mathcal{R}_\alpha$. Since $f(n) \to_n \mathsf{a}$, f and $K(\mathsf{a})$ are eventually equal. Thus $f \in \mathcal{R}_\alpha$ by Lemma 44.

We already know $\mathcal{K}_o = \mathcal{R}_o$. On the other hand, $\mathcal{K}_\iota \neq \mathcal{R}_\iota$ since the identity function is uniformly unbounded, hence in $\mathcal{R}_\iota$ but not in $\mathcal{K}_\iota$. When $\mathcal{K}_\beta = \mathcal{R}_\beta$ for a type β we can prove the domains at a higher type are large.

Lemma 46 *Suppose $\mathcal{R}_\alpha \subseteq \mathcal{K}_\alpha$ and β is a type. Then $\mathcal{D}_{\alpha\to\beta} = \mathcal{D}_\beta^{\mathcal{D}_\alpha}$.*

Proof. We know $\mathcal{D}_{\alpha\to\beta} \subseteq \mathcal{D}_\beta^{\mathcal{D}_\alpha}$ since $\mathcal{D}$ is a frame. To check the reverse inclusion, let $g : \mathcal{D}_\alpha \to \mathcal{D}_\beta$ be given. To check $g \in \mathcal{D}_{\alpha\to\beta}$, let $h \in \mathcal{R}_\alpha \subseteq \mathcal{K}_\alpha$ be given. There is some $N \in \mathbb{N}$ and $\mathsf{a} \in \mathcal{D}_\alpha$ such that $h(n) = \mathsf{a}$ for every $n > N$. Hence $g(h(n)) = g(\mathsf{a})$ for every $n > N$. That is, $g \circ h$ is eventually constant and hence $g \circ h \in \mathcal{R}_\beta$ by Lemma 45. Thus $g \in \mathcal{D}_{\alpha\to\beta}$, as desired.

We can use Lemma 46 to prove certain domains $\mathcal{D}_{(\alpha\to o)\to o}$ are the full function space and hence the frame realizes Π^α and Σ^α.

Theorem 47 *Suppose $\mathcal{R}_{\alpha\to o} \subseteq \mathcal{K}_{\alpha\to o}$. Then the frame $\mathcal{D}$ realizes Π^α and Σ^α.*

Proof. We define $\pi^\alpha : \mathcal{D}_{\alpha\to o} \to \mathcal{D}_o$ and $\sigma^\alpha : \mathcal{D}_{\alpha\to o} \to \mathcal{D}_o$ by

$$\pi^\alpha(f) := \begin{cases} \mathtt{T} & \text{if } \forall \mathsf{a} \in \mathcal{D}_\alpha f(\mathsf{a}) = \mathtt{T} \\ \mathtt{F} & \text{otherwise} \end{cases}$$

and

$$\sigma^\alpha(f) := \begin{cases} \mathtt{T} & \text{if } \exists \mathsf{a} \in \mathcal{D}_\alpha f(\mathsf{a}) = \mathtt{T} \\ \mathtt{F} & \text{otherwise} \end{cases}$$

By Lemma 46, we know $\pi^\alpha \in \mathcal{D}_{(\alpha\to o)\to o}$ and $\sigma^\alpha \in \mathcal{D}_{(\alpha\to o)\to o}$ Clearly, $\mathfrak{L}_{\Pi^\alpha}(\pi^\alpha)$ and $\mathfrak{L}_{\Sigma^\alpha}(\sigma^\alpha)$ hold and hence $\mathcal{D}$ realizes Π^α and Σ^α.

At the moment we can only apply Lemma 46 to the type o. This does not allow us to yet use Theorem 47 to conclude certain quantifiers are realized by the frame. Theorem 47 only provides the motivation for determining types α for which $\mathcal{R}_{\alpha\to o} \subseteq \mathcal{K}_{\alpha\to o}$ holds. We now prove $\mathcal{R}_{\iota\to o} \subseteq \mathcal{K}_{\iota\to o}$ holds, allowing us to conclude that the frame realizes Π^ι and Σ^ι.

Lemma 48 *For every $p \in \mathcal{R}_{\iota\to o}$ there exist $M, N \in \mathbb{N}$ such that $p(m)(n) = p(M)(N)$ for every $m > M$ and $n > N$.*

Proof. Let $p \in \mathcal{R}_{\iota\to o}$ be given. Assume for every $m, n \in \mathbb{N}$ there exist $m' > m$ and $n' > n$ such that $p(m')(n') \neq p(m)(n)$. Using this assumption, we can construct increasing sequences $m_0 < m_1 < \cdots$ and $n_0 < n_1 < \cdots$ in $\mathbb{N}$ such that $m_0 = n_0 = 0$ and

$$p(m_{i+1})(n_{i+1}) \neq p(m_i)(n_i).$$

Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(m) := n_I$ where I is the least i such that $m_i \leq m < m_{i+1}$. By Lemma 41, we can prove $f \in \mathcal{R}_\iota$ by verifying it is uniformly unbounded. Given any $j \in \mathbb{N}$, since $(n_i)_{i\in\mathbb{N}}$ is an increasing sequence there is some $k \in \mathbb{N}$ such that $j < n_k$. Since $f(m) > n_k > j$ for each $m > m_{k+1}$, we know f is uniformly unbounded and $f \in \mathcal{R}_\iota$. Hence $S(p, f) \in \mathcal{R}_o$ and is eventually constant by Lemma 39. On the other hand, by construction we have

$$\begin{aligned} S(p, f)(m_{i+1}) &= p(m_{i+1})(f(m_{i+1})) = p(m_{i+1})(n_{i+1}) \neq p(m_i)(n_i) \\ &= p(m_i)(f(m_i)) = S(p, f)(m_i), \end{aligned}$$

a contradiction.

Lemma 49 *Every $p \in \mathcal{R}_{\iota\to o}$ is eventually constant.*

Proof. Let $p \in \mathcal{R}_{\iota\to o}$ be given. By Lemma 48, there exist $M, N \in \mathbb{N}$ such that $p(m)(n) = p(M)(N)$ for every $m > M$ and $n > N$. Consider each $n \in \{0, \dots, N\}$. By Lemma 21, we know the constant function $K(n)$ is in $\mathcal{R}_\iota$. Since $p \in \mathcal{R}_{\iota\to o}$ we know $S(p, K(n)) \in \mathcal{R}_o$ is eventually constant by Lemma 39. Hence there exist M_n and $\mathsf{a}_n \in \mathcal{D}_o$ such that $p(m)(n) = \mathsf{a}_n$ for every $m > M_n$.

Define $M' := \max(M, M_0, \dots, M_N)$ and $g := p(M'+1)$. We will prove $p(m) = g$ for any $m > M'$. Let $n \in \mathbb{N}$ be given. If $n \leq N$, then

$$p(m)(n) = \mathsf{a}_n = p(M'+1)(n) = g(n)$$

since $m > M' \geq M_n$ and $M'+1 > M' \geq M_n$. If $n > N$, then

$$p(m)(n) = p(M)(N) = p(M'+1)(n) = g(n)$$

since $m > M' \geq M$ and $M'+1 > M' \geq M$. Thus p is eventually constant.

From Lemmas 49 and 45 we know $\mathcal{R}_{\iota\to o} = \mathcal{K}_{\iota\to o}$. We can use this to give a simpler characterization of $\mathcal{R}_{\iota\to o}$. Let $p : \mathbb{N} \to \mathcal{D}_o^{\mathbb{N}}$ be any function. By Lemma 39, we know $p : \mathbb{N} \to \mathcal{D}_{\iota\to o}$ iff $p(n)$ is eventually constant for each $n \in \mathbb{N}$. Consequently, p is in $\mathcal{R}_{\iota\to o}$ iff $p(n)$ is eventually constant for each $n \in \mathbb{N}$ and p is itself eventually constant. The picture in Figure 2 shows a typical example of a $p : \mathbb{N} \to (\mathcal{D}_o)^{\mathbb{N}}$ satisfying these conditions. (The painted in portions of the picture indicate values of n and m for which $p(n)(m) = \mathtt{T}$.)

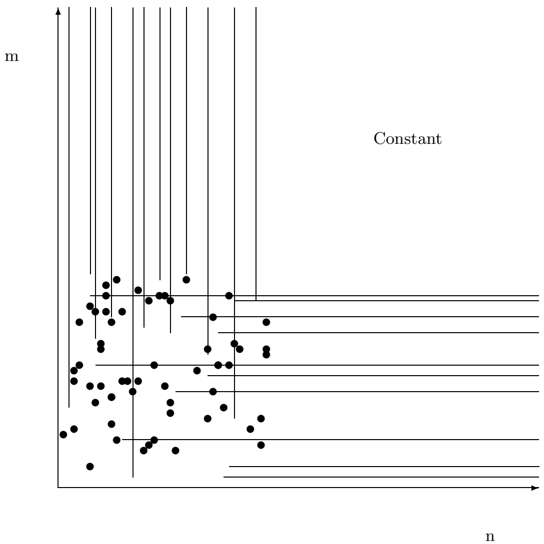

Fig. 2. Members of $\mathcal{R}_{\iota\to o}$

We can now prove $\mathcal{D}_{(\iota\to o)\to\iota}$ is the full function space. This guarantees that an "external" injection $h : \mathcal{D}_{\iota\to o} \to \mathcal{D}_\iota$ is also "internal" in the sense that $h \in \mathcal{D}_{(\iota\to o)\to\iota}$. Hence the frame will induce a model in which injective Cantor fails.

Lemma 50 *For every function $h : \mathcal{D}_{\iota\to o} \to \mathcal{D}_\iota$, $h \in \mathcal{D}_{(\iota\to o)\to\iota}$.*

Proof. By Lemma 49, we know $\mathcal{R}_{\iota\to o} \subseteq \mathcal{K}_{\iota\to o}$. The result follows from Lemma 46.

We can also conclude the frame realizes quantifiers over individuals.

Lemma 51 *$\mathcal{D}$ realizes Π^ι and Σ^ι.*

Proof. By Lemma 49, we know $\mathcal{R}_{\iota\to o} \subseteq \mathcal{K}_{\iota\to o}$. The result follows from Theorem 47.

We can now show the injective Cantor theorem fails in $\mathcal{D}$. Interestingly, we can argue as in Skolem's paradox.

Theorem 52 *There is an injection $\mathsf{h} \in \mathcal{D}_{(\iota\to o)\to\iota}$.*

Proof. The set $\mathcal{B}$ of finite and cofinite subsets of $\mathbb{N}$ is countably infinite. By Lemma 28, we know $\mathcal{D}_{\iota\to o}$ is also countably infinite. Hence both $\mathcal{D}_\iota$ and $\mathcal{D}_{\iota\to o}$ are both countably infinite. Let $\mathsf{h} : \mathcal{D}_{\iota\to o} \to \mathcal{D}_\iota$ be an injection. By Lemma 50, we know $\mathsf{h} \in \mathcal{D}_{(\iota\to o)\to\iota}$. Thus we are done.

Given a calculus and completeness proof relative to such frames, we can conclude injective Cantor is not provable in the fragment of extensional type theory if we are restricted to quantifiers and equalities realized by $\mathcal{D}$.

Note that since the model $\mathcal{D}$ realizes $\neg$, we know surjective Cantor holds. Consequently, while there is an internal injection from $\mathcal{D}_{\iota\to o}$ to $\mathcal{D}_\iota$ in $\mathcal{D}_{(\iota\to o)\to\iota}$, there is no internal surjection from $\mathcal{D}_\iota$ onto $\mathcal{D}_{\iota\to o}$ in $\mathcal{D}_{\iota\to\iota\to o}$.

With more work, we can prove the frame $\mathcal{D}$ realizes additional quantifiers and equalities. We sketch this analysis now. The interested reader can find more details in [Bro04].

Lemma 45 establishes $\mathcal{K}_\alpha$ as a lower bound for $\mathcal{R}_\alpha$. For some function types we can also relate $\mathcal{K}^p_{\alpha\to\beta}$ to $\mathcal{R}_{\alpha\to\beta}$.

Lemma 53 *Let α and β be types.*

1. *If $\mathcal{R}_\beta \subseteq \mathcal{K}_\beta$, then $\mathcal{R}_{\alpha\to\beta} \subseteq \mathcal{K}^p_{\alpha\to\beta}$.*
2. *$\mathcal{R}_{\beta\to o} \subseteq \mathcal{K}^p_{\beta\to o}$.*
3. *If $\mathcal{R}_\alpha \subseteq \mathcal{K}_\alpha$, then $\mathcal{K}^p_{\alpha\to\beta} \subseteq \mathcal{R}_{\alpha\to\beta}$.*

Proof. See [Bro04]

We can use Lemma 53 to establish a kind of continuity result for sequences of certain types.

Lemma 54 *Let α, β and γ be types. Suppose $\mathcal{R}_\alpha \subseteq \mathcal{K}_\alpha$ and $\mathcal{R}_\gamma \subseteq \mathcal{K}_\gamma$. Let $u \in \mathcal{D}_{\alpha\to\beta}$ be a function and $(u^n)_{n\in\mathbb{N}}$ be a sequence of functions $u^n \in \mathcal{D}_{\alpha\to\beta}$ such that $u^n \to^p_n u$. Let $m^0 < m^1 < \cdots$ be an increasing sequence of natural numbers. For each $p \in \mathcal{R}_{(\alpha\to\beta)\to\gamma}$ there is some $\mathsf{c} \in \mathcal{R}_\gamma$ such that $p(m^n)(u) \to_n \mathsf{c}$ and $p(m^n)(u^n) \to_n \mathsf{c}$.*

Proof. See [Bro04]

Lemma 55 *Let α, β and γ be types. Suppose $\mathcal{R}_\alpha \subseteq \mathcal{K}_\alpha$ and $\mathcal{R}_\gamma \subseteq \mathcal{K}_\gamma$. Let $u \in \mathcal{D}_{\alpha\to\beta}$ be a function and $(u^n)_{n\in\mathbb{N}}$ be a sequence of functions $u^n \in \mathcal{D}_{\alpha\to\beta}$ such that $u^n \to^p_n u$. For each $g \in \mathcal{D}_{(\alpha\to\beta)\to\gamma}$, $g(u^n) \to_n g(u)$.*

Proof. Suppose $g \in \mathcal{D}_{(\alpha\to\beta)\to\gamma}$. Applying Lemma 54 to $K(g) \in \mathcal{R}_{(\alpha\to\beta)\to\gamma}$ (cf. Lemma 21), the sequence $(u^n)_{n\in\mathbb{N}}$ and the increasing sequence $(n)_{n\in\mathbb{N}}$ of natural numbers, there is a $\mathsf{c} \in \mathcal{D}_\gamma$ such that $g(u^n) = K(g)(n)(u^n) \to_n \mathsf{c}$ and $g(u) = K(g)(n)(u) \to_n \mathsf{c}$. Hence $g(u) = \mathsf{c}$ and $g(u^n) \to_n g(u)$.

We now turn to the problem of proving the frame $\mathcal{D}$ realizes quantifiers at higher types such as $((\iota \to o) \to o) \to o$. In order to argue generally, we concentrate on studying domains $\mathcal{D}_{(\beta\to o)\to o}$ where $\mathcal{R}_\beta \subseteq \mathcal{K}_\beta$ and $\mathcal{D}_\beta$ is countably infinite. (Note that we already know $\mathcal{R}_{\iota\to o} \subseteq \mathcal{K}_{\iota\to o}$ and $\mathcal{D}_{\iota\to o}$ is countably infinite.) We will determine that every element of such a domain $\mathcal{D}_{(\beta\to o)\to o}$ is in the propositional closure of the set of projection functions. This implies all the elements of such $\mathcal{D}_{(\beta\to o)\to o}$ are definable by terms of the form

$$[\lambda u_{o\beta}[[u\, y_1^1 \wedge \cdots \wedge u\, y_1^{m_1} \wedge \neg u\, z_1^1 \wedge \cdots \wedge \neg u\, z_1^{k_1}] \vee \cdots$$
$$\vee [u\, y_n^1 \wedge \cdots \wedge u\, y_n^{m_n} \wedge \neg u\, z_n^1 \wedge \cdots \wedge \neg u\, z_n^{k_n}]]].$$

Definition 56 *Let A, B and F be sets with $F \subseteq A^B$. For each $\mathsf{b} \in B$, the* projection function *$proj_{\mathsf{b}} : F \to A$ induced by b is defined by $proj_{\mathsf{b}}(\mathsf{u}) := \mathsf{u}(\mathsf{b})$ for each $\mathsf{u} \in F$. We say some $g : F \to A$ is a* projection function *if there is some $\mathsf{b} \in B$ such that $g = proj_{\mathsf{b}}$.*

We can consider $\mathcal{D}_o$ as a topological space with the discrete topology (every set is open). For any type β we can give the function space $\mathcal{D}_o^{\mathcal{D}_\beta}$ the product topology. The product topology is the smallest topology such that for each $\mathsf{b} \in \mathcal{D}_\beta$ the projection function $proj_{\mathsf{b}} : (\mathcal{D}_o^{\mathcal{D}_\beta}) \to \mathcal{D}_o$ (cf. Definition 56) is continuous. As described in [Mun75] the set

$$\mathscr{S}_\beta := \{proj_{\mathsf{b}}^{-1}(U) \mid \mathsf{b} \in \mathcal{D}_\beta, U \text{ open in } \mathcal{D}_o\}$$

is a subbasis for the product topology. Since $\mathcal{D}_o$ has the discrete topology, $\mathscr{S}_\beta$ is the set

$$\{proj_{\mathsf{b}}^{-1}(\mathsf{a}) \mid \mathsf{b} \in \mathcal{D}_\beta, \mathsf{a} \in \mathcal{D}_o\}.$$

Note that for any $\mathsf{b} \in \beta$ the set

$$\{w \in \mathcal{D}_o^{\mathcal{D}_\beta} \mid w(\mathsf{b}) = \mathtt{T}\}$$

is open as a member of the subbasis $\mathscr{S}_\beta$ (since $w(\mathsf{b}) = proj_{\mathsf{b}}(w)$). The complement of this set is

$$\{w \in \mathcal{D}_o^{\mathcal{D}_\beta} \mid w(\mathsf{b}) = \mathtt{F}\}$$

which is also open as a member of the subbasis $\mathscr{S}_\beta$. Hence each member of $\mathscr{S}_\beta$ is both closed and open. Such sets are called *clopen*.

The basis $\mathscr{B}_\beta$ generated by $\mathscr{S}_\beta$ is obtained by taking finite intersections of the sets in $\mathscr{S}_\beta$. Note that finite intersections of clopen sets are also clopen. Hence each basic set in $\mathscr{B}_\beta$ is clopen. One can easily verify the following characterization of basic sets.

Lemma 57 *Let β be a type. For every set $B \subseteq \mathcal{D}_o^{\mathcal{D}_\beta}$ the following are equivalent:*

1. *$B \in \mathscr{B}_\beta$.*
2. *For every $w \in B$ there is a finite set $X \subseteq \mathcal{D}_\beta$ such that for all $u \in \mathcal{D}_o^{\mathcal{D}_\beta}$ we have $u \in B$ iff $u\big|_X = w\big|_X$.*

Proof. See [Bro04]

Lemma 58 *Let β be a type such that $\mathcal{R}_\beta \subseteq \mathcal{K}_\beta$ and $\mathcal{D}_\beta$ is countably infinite. Suppose $p \in \mathcal{R}_{(\beta\to o)\to o}$ and $w \in \mathcal{D}_{o\beta}$. There exists a finite set $X \subseteq \mathcal{D}_\beta$ and an $N \in \mathbb{N}$ such that for every $u \in \mathcal{D}_{o\beta}$ and $n > N$ if $u\big|_X = w\big|_X$, then $p(n)(u) = p(n)(w)$.*

Proof. See [Bro04]

Lemma 59 *Let β be a type such that $\mathcal{R}_\beta \subseteq \mathcal{K}_\beta$ and $\mathcal{D}_\beta$ is countably infinite. Suppose $p \in \mathcal{R}_{(\beta\to o)\to o}$. For every $w \in \mathcal{D}_{o\beta}$ there exist a basic neighborhood $B \in \mathscr{B}_\beta$ of w and an $N \in \mathbb{N}$ such that $p(n)(u) = p(n)(w)$ for every $u \in B$ and $n > N$.*

Proof. This follows easily from Lemmas 58 and 57. See [Bro04]

Using the Tychonoff Theorem (cf. [Mun75]) we know the topological space $\mathcal{D}_o^{\mathcal{D}_\beta}$ is compact. We can conclude $\mathcal{R}_{(\beta\to o)\to o} \subseteq \mathcal{K}_{(\beta\to o)\to o}$ and $\mathcal{D}_{(\beta\to o)\to o}$ is countable whenever $\mathcal{R}_\beta \subseteq \mathcal{K}_\beta$ and $\mathcal{D}_\beta$ is countably infinite using compactness. When $\mathcal{D}_\beta$ is countably infinite, $\mathcal{D}_o^{\mathcal{D}_\beta}$ is a Cantor space (cf. [BS71]).

Lemma 60 *Let β be a type such that $\mathcal{R}_\beta \subseteq \mathcal{K}_\beta$ and $\mathcal{D}_\beta$ is countably infinite. Suppose $p \in \mathcal{R}_{(\beta\to o)\to o}$. Then there exists a finite set $X \subseteq \mathcal{D}_\beta$ and an $N \in \mathbb{N}$ such that for every $u, v \in \mathcal{D}_{o\beta}$ and $n > N$ if $u\big|_X = v\big|_X$, then $p(n)(u) = p(n)(v)$.*

Proof. We apply Lemma 59 to obtain a cover of basic open neighborhoods and then use compactness to reduce this to a finite subcover of basic open sets. See [Bro04]

We can now determine $\mathcal{R}_{(\beta\to o)\to o} \subseteq \mathcal{K}_{(\beta\to o)\to o}$ whenever $\mathcal{R}_\beta \subseteq \mathcal{K}_\beta$ and $\mathcal{D}_\beta$ is countably infinite.

Lemma 61 *Suppose $\mathcal{R}_\beta \subseteq \mathcal{K}_\beta$ and $\mathcal{D}_\beta$ is countably infinite. Then $\mathcal{R}_{(\beta\to o)\to o} \subseteq \mathcal{K}_{(\beta\to o)\to o}$.*

Proof. See [Bro04]

In order to study the cardinality of $\mathcal{D}_{(\beta\to o)\to o}$, we first prove a consequence of Lemma 60.

Lemma 62 *Suppose $\mathcal{R}_\beta \subseteq \mathcal{K}_\beta$, $\mathcal{D}_\beta$ is countably infinite and $g \in \mathcal{D}_{(\beta\to o)\to o}$. There exists a finite set $X \subseteq \mathcal{D}_\beta$ such that for every $u, v \in \mathcal{D}_\beta$, if $u\big|_X = v\big|_X$, then $g(u) = g(v)$.*

Proof. Applying Lemma 60 to $K(g) \in \mathcal{R}_{(\beta\to o)\to o}$ (cf. Lemma 21) we obtain a finite set $X \subseteq \mathcal{D}_\beta$ and an $N \in \mathbb{N}$ such that for every $u, v \in \mathcal{D}_\beta$ and $n > N$ if $u\big|_X = v\big|_X$, then $K(g)(n)(u) = K(g)(n)(v)$. Hence for every $u, v \in \mathcal{D}_\beta$, if $u\big|_X = v\big|_X$, then

$$g(u) = K(g)(N+1)(u) = K(g)(N+1)(v) = g(v).$$

Assuming $\mathcal{R}_\beta \subseteq \mathcal{K}_\beta$ and $\mathcal{D}_\beta$ is countably infinite, Lemma 62 implies every $g \in \mathcal{D}_{(\beta\to o)\to o}$ is determined by a finite set $X \subseteq \mathcal{D}_\beta$ and the values $g(\chi_Y)$ for each $Y \subseteq X$. This is enough to conclude $\mathcal{D}_{(\beta\to o)\to o}$ is countably infinite. In fact, the functions in $\mathcal{D}_{(\beta\to o)\to o}$ are precisely the characteristic functions of clopen sets. We can use the set of projection functions for types o and β to describe $\mathcal{D}_{(\beta\to o)\to o}$ in such a case. Define

$$\mathcal{P}\mathfrak{roj}_\beta := \{proj_{\mathsf{b}} : \mathcal{D}_{o\beta} \to \mathcal{D}_o \mid \mathsf{b} \in \mathcal{D}_\beta\}$$

(cf. Definition 56).

Lemma 63 *Suppose $\mathcal{R}_\beta \subseteq \mathcal{K}_\beta$ and $\mathcal{D}_\beta$ is countably infinite. Then $\mathcal{D}_{(\beta\to o)\to o}$ is the propositional closure of $\mathcal{P}\mathfrak{roj}_\beta$. That is, every function in $\mathcal{D}_{(\beta\to o)\to o}$ can be expressed as a finite combination of projection functions using complement, binary union and binary intersection.*

Proof. See [Bro04]

We know every element of such domains $\mathcal{D}_{(\beta\to o)\to o}$ is definable as a consequence of Lemma 63. We can now prove such domains are countably infinite.

Lemma 64 *Suppose $\mathcal{R}_\beta \subseteq \mathcal{K}_\beta$ and $\mathcal{D}_\beta$ is countably infinite. Then $\mathcal{D}_{(\beta\to o)\to o}$ is countably infinite.*

Proof. Since $\mathcal{D}_\beta$ is countably infinite, the set $\mathcal{P}\mathfrak{roj}_\beta$ is countable. To prove it is infinite, we must verify $proj_{\mathsf{b}} \neq proj_{\mathsf{c}}$ whenever $\mathsf{b} \neq \mathsf{c}$. Suppose $\mathsf{b} \neq \mathsf{c}$. Define $u^{\mathsf{b}} : \mathcal{D}_\beta \to \mathcal{D}_o$ to be the characteristic function for the unit set $\{\mathsf{b}\}$. That is, $u^{\mathsf{b}}(\mathsf{b}) := \mathtt{T}$ and $u^{\mathsf{b}}(\mathsf{a}) := \mathtt{F}$ for all $\mathsf{a} \neq \mathsf{b}$. By Lemma 46, we know $u^{\mathsf{b}} \in \mathcal{D}_{o\beta}$. We compute

$$proj_{\mathsf{b}}(u^{\mathsf{b}}) = u^{\mathsf{b}}(\mathsf{b}) = \mathtt{T} \neq \mathtt{F} = u^{\mathsf{b}}(\mathsf{c}) = proj_{\mathsf{c}}(u^{\mathsf{b}}).$$

Hence $proj_{\mathsf{b}} \neq proj_{\mathsf{c}}$. Thus $\mathcal{P}\mathfrak{roj}_\beta$ is infinite.

The set $\mathcal{D}_{(\beta\to o)\to o}$ is the propositional closure of $\mathcal{P}\mathfrak{roj}_\beta$ by Lemma 63. We can conclude $\mathcal{D}_{(\beta\to o)\to o}$ is countably infinite.

We can inductively define the n'th power type $\mathcal{P}^n(\alpha)$ of a type α as follows. Let $\mathcal{P}^0(\alpha) := \alpha$ and $\mathcal{P}^{n+1}(\alpha) := (\mathcal{P}^n(\alpha) \to o)$.

We can now use induction to prove that the power types alternate between being countably infinite and being the full collection of (characteristic) functions. Likewise, we have an alternation with respect to realization of quantifiers at power types.

Lemma 65 *For each $n \in \mathbb{N}$, $\mathcal{R}_{\mathcal{P}^{2n+1}(\iota)} \subseteq \mathcal{K}_{\mathcal{P}^{2n+1}(\iota)}$ and $\mathcal{D}_{\mathcal{P}^{2n+1}(\iota)}$ is countably infinite.*

Proof. The proof is by induction on $n \in \mathbb{N}$. We know $\mathcal{R}_{\iota\to o} \subseteq \mathcal{K}_{\iota\to o}$ by Lemma 49. Also, the fact that $\mathcal{B}$ is countably infinite implies $\mathcal{D}_{\iota\to o}$ is countably infinite by Lemma 28. Thus the result holds for $n = 0$.

Assume $\mathcal{R}_{\mathcal{P}^{2n+1}(\iota)} \subseteq \mathcal{K}_{\mathcal{P}^{2n+1}(\iota)}$ and $\mathcal{D}_{\mathcal{P}^{2n+1}(\iota)}$ is countably infinite. By Lemma 61, we know $\mathcal{R}_{\mathcal{P}^{2n+3}(\iota)} \subseteq \mathcal{K}_{\mathcal{P}^{2n+3}(\iota)}$. By Lemma 64, we know $\mathcal{D}_{\mathcal{P}^{2n+3}(\iota)}$ is countably infinite. Thus the induction step also holds.

Theorem 66 *For each $n \in \mathbb{N}$, $\mathcal{D}$ realizes $\Pi^{\mathcal{P}^{2n}(\iota)}$ and $\Sigma^{\mathcal{P}^{2n}(\iota)}$.*

Proof. By Lemma 65, we know $\mathcal{R}_{\mathcal{P}^{2n}(\iota)\to o} \subseteq \mathcal{K}_{\mathcal{P}^{2n}(\iota)\to o}$. By Theorem 47, we conclude $\mathcal{D}$ realizes $\Pi^{\mathcal{P}^{2n}(\iota)}$ and $\Sigma^{\mathcal{P}^{2n}(\iota)}$.

We can use Lemma 65 to prove the frame $\mathcal{D}$ does not realize $\Pi^{\mathcal{P}^{2n+1}(\iota)}$ and does not realize $\Sigma^{\mathcal{P}^{2n+1}(\iota)}$ for any n.

Theorem 67 *For each $n \in \mathbb{N}$, $\mathcal{D}$ realizes neither $\Pi^{\mathcal{P}^{2n+1}(\iota)}$ nor $\Sigma^{\mathcal{P}^{2n+1}(\iota)}$.*

Proof. Suppose $\pi^n \in \mathcal{D}_{(\mathcal{P}^{2n+1}(\iota)\to o)\to o}$ satisfies $\mathfrak{L}_{\Pi^{\mathcal{P}^{2n+1}(\iota)}}(\pi^n)$. Let $k^n_{\mathtt{T}}$ be the constant function $k^n_{\mathtt{T}}(\mathsf{b}) := \mathtt{T}$ from $\mathcal{D}_{\mathcal{P}^{2n+1}(\iota)}$ to $\mathcal{D}_o$. Then we have $\pi^n(f) = \mathtt{T}$ iff $f = k^n_{\mathtt{T}}$ for every $f \in \mathcal{D}_{\mathcal{P}^{2n+1}(\iota)\to o}$.

By Lemma 65, we know $\mathcal{R}_{\mathcal{P}^{2n+1}(\iota)} \subseteq \mathcal{K}_{\mathcal{P}^{2n+1}(\iota)}$ and $\mathcal{D}_{\mathcal{P}^{2n+1}(\iota)}$ is countably infinite. Since $\mathcal{D}_{\mathcal{P}^{2n+1}(\iota)}$ is countably infinite, we can enumerate it as $\{\mathsf{b}_0, \mathsf{b}_1, \mathsf{b}_2, \ldots\}$ and define $u^m : \mathcal{D}_{\mathcal{P}^{2n+1}(\iota)} \to \mathcal{D}_o$ by

$$u^m(\mathsf{b}_i) = \begin{cases} \mathtt{T} & \text{if } i < m \\ \mathtt{F} & \text{otherwise} \end{cases}$$

for each $m \in \mathbb{N}$ and $\mathsf{b}_i \in \mathcal{D}_{\mathcal{P}^{2n+1}(\iota)}$. Clearly, $u^m \to^p_m k^n_{\mathtt{T}}$. Since each $u^m \neq k^n_{\mathtt{T}}$, we know $\pi^n(u^m) = \mathtt{F}$. On the other hand, we have $\pi^n(u^m) \to_m \pi^n(k^n_{\mathtt{T}})$ by Lemma 55 since $\mathcal{R}_{\mathcal{P}^{2n+1}(\iota)} \subseteq \mathcal{K}_{\mathcal{P}^{2n+1}(\iota)}$ and $\mathcal{R}_o \subseteq \mathcal{K}_o$. This contradicts $\pi^n(k^n_{\mathtt{T}}) = \mathtt{T}$.

A similar argument proves $\mathcal{D}$ does not realize $\Sigma^{\mathcal{P}^{2n+1}(\iota)}$.

10 More Syntax

Remark 68 (α-conversion) *To define notions such as substitution, reduction and convertibility, one needs to be careful about alphabetic change of bound variables (α-conversion). We consider alphabetic change of bound variables (α-conversion) built into the language. That is, we consider α-convertible terms to be identical.*

We use the notation $[\mathbf{B}_\beta/x_\beta]\mathbf{A}_\alpha$ to denote the result of substituting $\mathbf{B}$ for the free occurrences of the variable x in $\mathbf{A}$. Note that $[\mathbf{B}/x]$ is a typed functional relation from *wff*($\mathcal{S}$) to *wff*($\mathcal{S}$). When applying $[\mathbf{B}/x]$ to $\mathbf{A}$, we will ensure that neither x nor any free variable of $\mathbf{B}$ occur bound in $\mathbf{A}$. As in Remark 68, we let the set of vulnerable variables for $[\mathbf{B}/x]$ be the set $\mathbf{Free}(\mathbf{B}) \cup \{x\}$, define $[\mathbf{B}/x]$ on terms $\mathbf{A}$ such that no variable in $\mathbf{Free}(\mathbf{B}) \cup \{x\}$ occurs bound in $\mathbf{A}$, and extend $[\mathbf{B}/x]$ to all terms up to α-conversion.

We allow for substitutions θ which substitute simultaneously for the variables, parameters and constants in the (finite) domain $\mathbf{Dom}(\theta) \subset \mathcal{V} \cup \mathcal{P} \cup \mathcal{S}$ of θ. Given a substitution θ, we use $\theta, [\mathbf{B}/x]$ to denote the substitution with $\mathbf{Dom}(\theta, [\mathbf{B}/x]) := \mathbf{Dom}(\theta) \cup \{x\}$ such that

$$(\theta, [\mathbf{B}/x])(y) := \begin{cases} \mathbf{B} & \text{if } y = x \\ \theta(y) & \text{otherwise.} \end{cases}$$

Note that while we do allow substitution of terms for constants and parameters, we allow binding only of variables.

Definition 69 *A term of the form $[[\lambda x_\beta\, \mathbf{C}_\alpha]\, \mathbf{D}_\beta]$ is called a β-redex. The β-reduct of this redex is $[\mathbf{D}/x]\mathbf{C}$. We define $\xrightarrow{\beta}_1$ be the congruence closure of the relation consisting of pairs $(\mathbf{A}, \mathbf{B})$ where $\mathbf{A}$ is a β-redex with reduct $\mathbf{B}$. That is, $\mathbf{A} \xrightarrow{\beta}_1 \mathbf{B}$ holds if $\mathbf{B}$ is the result of β-reducing a single β-redex in $\mathbf{A}$. We define $\xrightarrow{\beta}$ to be the reflexive, transitive closure of $\xrightarrow{\beta}_1$. That is, $\mathbf{A} \xrightarrow{\beta} \mathbf{B}$ holds when $\mathbf{A}$ reduces to $\mathbf{B}$ by zero or more β-reductions on subterms. The symmetric transitive closure of the β-reduction relation is β-conversion. We will write $\mathbf{A} \stackrel{\beta}{=} \mathbf{B}$ when $\mathbf{A}$ β-converts to $\mathbf{B}$. We say $\mathbf{A}$ is* β-normal *if no subterm of $\mathbf{A}$ is a β-redex.*

A term of the form $\lambda x_\alpha\, \blacksquare\, \mathbf{F}_{\alpha\to\beta}\, x$ where $x \notin \mathbf{Free}(\mathbf{F})$ is called an η-redex. The η-reduct of this redex is $\mathbf{F}$. We define $\xrightarrow{\eta}_1$ to be the congruence closure of the relation consisting of pairs $(\mathbf{A}, \mathbf{B})$ where $\mathbf{A}$ is an η-redex with reduct $\mathbf{B}$. We define $\xrightarrow{\eta}$ to be the reflexive, transitive closure of $\xrightarrow{\eta}_1$. That is, $\mathbf{A} \xrightarrow{\eta} \mathbf{B}$ holds when $\mathbf{A}$ reduces to $\mathbf{B}$ by zero or more η-reductions on subterms. We will write $\mathbf{A} \stackrel{\eta}{=} \mathbf{B}$ when $\mathbf{A}$ η-converts to $\mathbf{B}$.

Similarly, we let $\xrightarrow{\beta\eta}_1$ be the congruence closure of the relation consisting of pairs $(\mathbf{A}, \mathbf{B})$ where $\mathbf{A}$ is a β-redex with reduct $\mathbf{B}$ or $\mathbf{A}$ is an η-redex with reduct $\mathbf{B}$. Also, we let $\xrightarrow{\beta\eta}$ be the reflexive, transitive closure of $\xrightarrow{\beta\eta}_1$. That is, $\mathbf{A} \xrightarrow{\beta\eta} \mathbf{B}$ holds when $\mathbf{A}$ reduces to $\mathbf{B}$ by zero or more β-reductions and η-reductions on subterms. The symmetric transitive closure of the $\beta\eta$-reduction relation is $\beta\eta$-conversion. We will write $\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B}$ when $\mathbf{A}$ $\beta\eta$-converts to $\mathbf{B}$. We say $\mathbf{A}$ is $\beta\eta$-normal *if no subterm of $\mathbf{A}$ is a β-redex or an η-redex.*

The following result is well known:

Lemma 70 *The reduction relations $\xrightarrow{\beta}$ and $\xrightarrow{\beta\eta}$ above have the strong Church-Rosser property: for every wff $\mathbf{A}$ there is a unique (up to renaming of bound variables) β-normal [$\beta\eta$-normal] $\mathbf{B}$ such that $\mathbf{A} \xrightarrow{\beta} \mathbf{B}$ [$\mathbf{A} \xrightarrow{\beta\eta} \mathbf{B}$].*

Proof. See (for example) [Bar84; Hin97].

We use $\mathbf{A}^{\downarrow\beta}$ [$\mathbf{A}^{\downarrow}$] to denote the β-normal [$\beta\eta$-normal] form of $\mathbf{A}$. In some situations we do not wish to be specific about which normal form we mean. To allow for these situations, we use the notation $\mathbf{A}^{\downarrow *}$ where ${}^{\downarrow *}$ means either ${}^{\downarrow\beta}$ or ${}^{\downarrow}$.

11 Generalizing Semantics

Definition 71 ((Typed) Applicative Structure) *A (*typed*)* applicative structure *is a pair $(\mathcal{D}, @)$ where $\mathcal{D}$ is a typed family of nonempty sets and $@^{\alpha\to\beta} : \mathcal{D}_{\alpha\to\beta} \times \mathcal{D}_\alpha \to \mathcal{D}_\beta$ for each function type $(\alpha \to \beta)$.*

Each (nonempty) set $\mathcal{D}_\alpha$ is called the domain of type α *and the family of functions @ is called the* application operator. *We write simply $\mathsf{f}@\mathsf{b}$ for $\mathsf{f}@^{\alpha\to\beta}\mathsf{b}$ when $\mathsf{f} \in \mathcal{D}_{\alpha\to\beta}$ and $\mathsf{a} \in \mathcal{D}_\alpha$ are clear in context.*

Any frame $\mathcal{D}$ induces an applicative structure $(\mathcal{D}, @)$ taking @ to be ordinary function application.

Definition 72 (Functional/Full/Standard Applicative Structures) *Let $\mathcal{A} := (\mathcal{D}, @)$ be an applicative structure. We say $\mathcal{A}$ is* functional *if for all types α and β and objects $\mathsf{f}, \mathsf{g} \in \mathcal{D}_{\alpha\to\beta}$, $\mathsf{f} = \mathsf{g}$ whenever $\mathsf{f}@\mathsf{a} = \mathsf{g}@\mathsf{a}$ for every $\mathsf{a} \in \mathcal{D}_\beta$.*

Every frame $\mathcal{D}$ (technically, the applicative structure induced by the frame $\mathcal{D}$) is functional.
Proof: Functions are uniquely determined by their values on all arguments.

Suppose $\upsilon : \mathcal{D}_o \to \{\mathtt{T}, \mathtt{F}\}$ is a function. We call such a function a "valuation."

Definition 73 *Let $\mathcal{A} := (\mathcal{D}, @)$ be an applicative structure and $\upsilon\colon \mathcal{D}_o \to \{\mathtt{T}, \mathtt{F}\}$ be a function. For each logical constant c_α and element $\mathsf{a} \in \mathcal{D}_\alpha$, we define the properties $\mathfrak{L}_c(\mathsf{a})$ with respect to υ in Table 2.*

prop.	where	holds when	for all
$\mathfrak{L}_\neg(\mathsf{n})$	$\mathsf{n} \in \mathcal{D}_{o\to o}$	$\upsilon(\mathsf{n}@\mathsf{a}) = \mathtt{T}$ iff $\upsilon(\mathsf{a}) = \mathtt{F}$	$\mathsf{a} \in \mathcal{D}_o$
$\mathfrak{L}_\vee(\mathsf{d})$	$\mathsf{d} \in \mathcal{D}_{o\to o\to o}$	$\upsilon(\mathsf{d}@\mathsf{a}@\mathsf{b}) = \mathtt{T}$ iff $\upsilon(\mathsf{a}) = \mathtt{T}$ or $\upsilon(\mathsf{b}) = \mathtt{T}$	$\mathsf{a}, \mathsf{b} \in \mathcal{D}_o$
$\mathfrak{L}_\wedge(\mathsf{c})$	$\mathsf{c} \in \mathcal{D}_{o\to o\to o}$	$\upsilon(\mathsf{c}@\mathsf{a}@\mathsf{b}) = \mathtt{T}$ iff $\upsilon(\mathsf{a}) = \mathtt{T}$ and $\upsilon(\mathsf{b}) = \mathtt{T}$	$\mathsf{a}, \mathsf{b} \in \mathcal{D}_o$
$\mathfrak{L}_{\Pi^\alpha}(\pi)$	$\pi \in \mathcal{D}_{(\alpha\to o)\to o}$	$\upsilon(\pi@\mathsf{f}) = \mathtt{T}$ iff $\forall \mathsf{a} \in \mathcal{D}_\alpha\ \upsilon(\mathsf{f}@\mathsf{a}) = \mathtt{T}$	$\mathsf{f} \in \mathcal{D}_{\alpha\to o}$
$\mathfrak{L}_{\Sigma^\alpha}(\sigma)$	$\sigma \in \mathcal{D}_{(\alpha\to o)\to o}$	$\upsilon(\sigma@\mathsf{f}) = \mathtt{T}$ iff $\exists \mathsf{a} \in \mathcal{D}_\alpha\ \upsilon(\mathsf{f}@\mathsf{a}) = \mathtt{T}$	$\mathsf{f} \in \mathcal{D}_{\alpha\to o}$
$\mathfrak{L}_{=^\alpha}(\mathsf{q})$	$\mathsf{q} \in \mathcal{D}_{\alpha\to\alpha\to o}$	$\upsilon(\mathsf{q}@\mathsf{a}@\mathsf{b}) = \mathtt{T}$ iff $\mathsf{a} = \mathsf{b}$	$\mathsf{a}, \mathsf{b} \in \mathcal{D}_\alpha$

Table 2. Logical Properties of $\upsilon : \mathcal{D}_o \to \{\mathtt{T}, \mathtt{F}\}$

Definition 74 *Suppose $(\mathcal{D}, @)$ is an applicative structure and $\upsilon : \mathcal{D}_o \to \{\mathtt{T}, \mathtt{F}\}$ is a valuation. We say $(\mathcal{D}, @, \upsilon)$* realizes *a logical constant c_α (or c* is realized by *$\mathcal{D}$) if there is some $\mathsf{a} \in \mathcal{D}_\alpha$ such that $\mathfrak{L}_c(\mathsf{a})$ holds with respect to this υ. We say $(\mathcal{D}, @, \upsilon)$* realizes *a signature $\mathcal{S}$ (or $\mathcal{S}$* is realized by *$\mathcal{D}$) if it realizes every $c \in \mathcal{S}$.*

Definition 75 *Suppose $(\mathcal{D}, @)$ is an applicative structure and $\upsilon : \mathcal{D}_o \to \{\mathtt{T}, \mathtt{F}\}$ is a valuation. We say an applicative structure $(\mathcal{D}, @, \upsilon)$ satisfies property $\mathfrak{q}$ if $(\mathcal{D}, @, \upsilon)$ realizes $=^\alpha$ for every type α.*

Definition 76 (Variable Assignment) *Let $\mathcal{A} := (\mathcal{D}, @)$ be an applicative structure. A typed function $\varphi\colon \mathcal{V} \to \mathcal{D}$ is called a* variable assignment *into $\mathcal{A}$. Given a variable assignment φ, variable X_α, and value $\mathsf{a} \in \mathcal{D}_\alpha$, we use $\varphi, [\mathsf{a}/X]$ to denote the variable assignment with $(\varphi, [\mathsf{a}/X])(X) = \mathsf{a}$ and $(\varphi, [\mathsf{a}/X])(Y) = \varphi(Y)$ for variables Y other than X.*

Definition 77 ($\mathcal{S}$-Evaluation) *Let $\mathcal{E}\colon \mathfrak{F}_\mathcal{T}(\mathcal{V}; \mathcal{D}) \to \mathfrak{F}_\mathcal{T}(wff(\mathcal{S}), \mathcal{D})$ be a total function, where $\mathfrak{F}_\mathcal{T}(\mathcal{V}; \mathcal{D})$ is the set of variable assignments and $\mathfrak{F}_\mathcal{T}(wff(\mathcal{S}), \mathcal{D})$ is the set of typed functions mapping terms into objects in $\mathcal{D}$. We will write the argument of $\mathcal{E}$ as a subscript. So, for each assignment φ, we have a typed function $\mathcal{E}_\varphi\colon wff(\mathcal{S}) \to \mathcal{D}$. $\mathcal{E}$ is called an* evaluation function *for $\mathcal{A}$ if for any assignments φ and ψ into $\mathcal{A}$, we have*

1. *$\mathcal{E}_\varphi\big|_\mathcal{V} = \varphi$.*
2. *$\mathcal{E}_\varphi(\mathbf{FA}) = \mathcal{E}_\varphi(\mathbf{F})@\mathcal{E}_\varphi(\mathbf{A})$ for any $\mathbf{F} \in wff_{\alpha\to\beta}(\mathcal{S})$ and $\mathbf{A} \in wff_\alpha(\mathcal{S})$ and types α and β.*
3. *$\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\psi(\mathbf{A})$ for any type α and $\mathbf{A} \in wff_\alpha(\mathcal{S})$, whenever φ and ψ coincide on $\mathbf{Free}(\mathbf{A})$.*
4. *$\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{A}^{\downarrow\beta})$ for all $\mathbf{A} \in wff_\alpha(\mathcal{S})$.*

We call $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$ a $\mathcal{S}$-evaluation *if $(\mathcal{D}, @)$ is an applicative structure and $\mathcal{E}$ is an evaluation function for $(\mathcal{D}, @)$. We call $\mathcal{E}_\varphi(\mathbf{A}_\alpha) \in \mathcal{D}_\alpha$ the* denotation *of $\mathbf{A}_\alpha$ in $\mathcal{J}$ for φ. (Note that since $\mathcal{E}$ is a function, the denotation in $\mathcal{J}$ is unique. However, for a given applicative structure $\mathcal{A}$, there may be many possible evaluation functions.)*

If $\mathbf{A}$ is a closed formula, then $\mathcal{E}_\varphi(\mathbf{A})$ is independent of φ, since $\mathbf{Free}(\mathbf{A}) = \emptyset$. In these cases we sometimes drop the reference to φ from $\mathcal{E}_\varphi(\mathbf{A})$ and simply write $\mathcal{E}(\mathbf{A})$.

We call a $\mathcal{S}$-evaluation $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$ functional *[*full, standard*] if the applicative structure $(\mathcal{D}, @)$ is* functional *[*full, standard*]. We say $\mathcal{J}$ is a $\mathcal{S}$-evaluation over a frame if $(\mathcal{D}, @)$ is a frame.*

Definition 78 (Weakly Functional Evaluations) *Let $\mathcal{J} = (\mathcal{D}, @, \mathcal{E})$ be a $\mathcal{S}$-evaluation. We say $\mathcal{J}$ is* η-functional *if $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{A}^{\downarrow})$ for any type α, formula $\mathbf{A} \in wff_\alpha(\mathcal{S})$, and assignment φ. We say $\mathcal{J}$ is* ξ-functional *if for all $\alpha, \beta \in \mathcal{T}$, $\mathbf{M}, \mathbf{N} \in wff_\beta(\mathcal{S})$, assignments φ, and variables X_α, $\mathcal{E}_\varphi(\lambda X_\alpha\text{.}\mathbf{M}_\beta) = \mathcal{E}_\varphi(\lambda X_\alpha\text{.}\mathbf{N}_\beta)$ whenever $\mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{M}) = \mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{N})$ for every $\mathsf{a} \in \mathcal{D}_\alpha$.*

Definition 79 ($\mathcal{S}$-Model) *Let $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$ be a $\mathcal{S}$-evaluation. A function $\upsilon\colon \mathcal{D}_o \to \{\mathtt{T}, \mathtt{F}\}$ is called a $\mathcal{S}$*-valuation *for $\mathcal{J}$ if $\mathfrak{L}_\neg(\mathcal{E}(\neg))$ and $\mathfrak{L}_\vee(\mathcal{E}(\vee))$ hold, and for every type α $\mathfrak{L}_{\Pi^\alpha}(\mathcal{E}(\Pi^\alpha))$ holds. In this case, $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ is called a $\mathcal{S}$*-model.

For the case of (the optional) primitive equality, i.e., when $=^\alpha \in \mathcal{S}_{\alpha\to\alpha\to o}$ for all types α, we say $\mathcal{M}$ is a $\mathcal{S}$-model with primitive equality *if $\mathfrak{L}_{=^\alpha}(\mathcal{E}(=^\alpha))$ holds for every type α.*

We say that φ is an assignment into $\mathcal{M}$ if it is an assignment into the underlying applicative structure $(\mathcal{D}, @)$. Furthermore, φ satisfies *a formula $\mathbf{A} \in wff_o(\mathcal{S})$ in $\mathcal{M}$ (we write $\mathcal{M} \models_\varphi \mathbf{A}$) if $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \mathtt{T}$. We say that $\mathbf{A}$ is* valid *in $\mathcal{M}$ (and write $\mathcal{M} \models \mathbf{A}$) if $\mathcal{M} \models_\varphi \mathbf{A}$ for all assignments φ. When $\mathbf{A} \in cwff_o$, we drop the reference to the assignment and use the notation $\mathcal{M} \models \mathbf{A}$. Finally, we say that $\mathcal{M}$ is a $\mathcal{S}$*-model *for a set $\Phi \subseteq cwff_o$ (we write $\mathcal{M} \models \Phi$) if $\mathcal{M} \models \mathbf{A}$ for all $\mathbf{A} \in \Phi$.*

A $\mathcal{S}$-model $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ is called functional *[*full, standard*] if the applicative structure $(\mathcal{D}, @)$ is* functional *[*full, standard*]. Similarly, $\mathcal{M}$ is called* η-functional *[*ξ-functional*] if the evaluation $(\mathcal{D}, @, \mathcal{E})$ is* η-functional *[*ξ-functional*]. We say $\mathcal{M}$ is a $\mathcal{S}$-model over a frame if $(\mathcal{D}, @)$ is a frame.*

We will now introduce semantical properties called $\mathfrak{q}$, η, $\mathfrak{f}$, and $\mathfrak{b}$, which we will use to characterize different classes of $\mathcal{S}$-models.

Definition 80 (Properties $\mathfrak{q}$, η, ξ, $\mathfrak{f}$ and $\mathfrak{b}$) *Given a $\mathcal{S}$-model $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$, we say that $\mathcal{M}$ has* property

$\mathfrak{q}$ *iff for all $\alpha \in \mathcal{T}$ there is some $\mathsf{q}^\alpha \in \mathcal{D}_{\alpha\to\alpha\to o}$ such that $\mathfrak{L}_{=^\alpha}(\mathsf{q}^\alpha)$ holds.*
η *iff $\mathcal{M}$ is η-functional.*
ξ *iff $\mathcal{M}$ is ξ-functional.*
$\mathfrak{f}$ *iff $\mathcal{M}$ is functional. (This is generally associated with functional extensionality.)*
$\mathfrak{b}$ *iff $\mathcal{D}_o$ has at most two elements.*

NOTE: From now on, we restrict to the signature $\mathcal{S}$ being either $\{\neg, \vee\} \cup \{\Pi^\alpha | \alpha \in \mathcal{T}\}$ or $\{\neg, \vee\} \cup \{\Pi^\alpha, =^\alpha | \alpha \in \mathcal{T}\}$.

Definition 81 (Higher-Order Model Classes) *We will denote the class of $\mathcal{S}$-models that satisfy property $\mathfrak{q}$ by $\mathfrak{M}_\beta$, and we will use subclasses of $\mathfrak{M}_\beta$ depending on the validity of the properties η, ξ, $\mathfrak{f}$, and $\mathfrak{b}$. We obtain the specialized classes of $\mathcal{S}$-models $\mathfrak{M}_{\beta\eta}$, $\mathfrak{M}_{\beta\xi}$, $\mathfrak{M}_{\beta\mathfrak{f}}$, $\mathfrak{M}_{\beta\mathfrak{b}}$, $\mathfrak{M}_{\beta\eta\mathfrak{b}}$, $\mathfrak{M}_{\beta\xi\mathfrak{b}}$, and $\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}$ by requiring that the properties specified in the index are valid.*

If primitive equality is in the signature, i.e., if $=^\alpha \in \mathcal{S}_{\alpha\to\alpha\to o}$, then we require the models to be $\mathcal{S}$-models with primitive equality. Note that in this case property $\mathfrak{q}$ is automatically ensured.

Example 82 (Failure of $\mathfrak{b}$ — $\mathcal{M}^{\beta\mathfrak{f}} \in \mathfrak{M}_{\beta\mathfrak{f}} \setminus \mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}$) *Let $(\mathcal{D}, @)$ be the standard frame with $\mathcal{D}_o = \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$ and $\mathcal{D}_\iota = \{0, 1\}$. We define an evaluation function $\mathcal{E}$ using Definition 11 by defining $\mathcal{I} : (\mathcal{P} \cup \mathcal{S}) \to \mathcal{D}$. We define $\mathcal{I}(\neg)$, $\mathcal{I}(\vee)$, and $\mathcal{I}(\Pi^\alpha)$ to be the functions given in the following table:*

$\mathcal{I}(\neg)$	a	b	c
	c	c	a

$\mathcal{I}(\vee)$	a	b	c
a	a	a	a
b	a	a	a
c	a	a	c

$$\mathcal{I}(\Pi^\alpha)@\mathsf{f} = \begin{cases} \mathsf{a}, & \text{if } \mathsf{f}@\mathsf{g} \in \{\mathsf{a}, \mathsf{b}\} \text{ for all } \mathsf{g} \in \mathcal{D}_\alpha \\ \mathsf{c}, & \text{if } \mathsf{f}@\mathsf{g} = \mathsf{c} \text{ for some } \mathsf{g} \in \mathcal{D}_\alpha \end{cases}$$

We can choose $\mathcal{I}(w)$ to be arbitrary for parameters $w \in \mathcal{P}$.

Let the map $\upsilon\colon \mathcal{D}_o \to \{\mathtt{T}, \mathtt{F}\}$ be defined by $\upsilon(\mathsf{a}) := \mathtt{T}$, $\upsilon(\mathsf{b}) := \mathtt{T}$ and $\upsilon(\mathsf{c}) := \mathtt{F}$. It is easy to check that $\mathcal{M}^{\beta\mathfrak{f}} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ is indeed a $\mathcal{S}$-model. Since this is a model over a frame, we automatically know it satisfies property $\mathfrak{f}$. Since the frame is standard, we know property $\mathfrak{q}$ holds. Clearly property $\mathfrak{b}$ fails, so we have $\mathcal{M}^{\beta\mathfrak{f}} \in \mathfrak{M}_{\beta\mathfrak{f}} \setminus \mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}$.

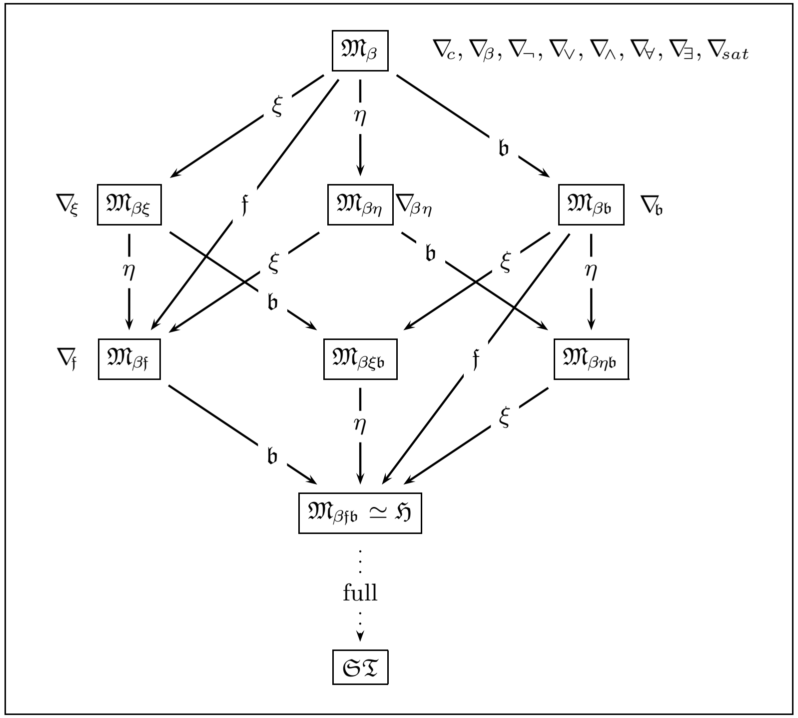

Fig. 3. The landscape of Higher-Order Semantics

12 Informal Exposition: Calculi, Completeness, Model Existence, and Abstract Consistency

For all the notions of model classes from Section 11 (except, of course, for standard models, where such a theorem cannot hold for recursively axiomatizable logical systems) we present model existence theorems tying the differentiating conditions of the models to suitable conditions in the abstract consistency classes (cf. Section 15.3).

A model existence theorem for a logical system $\mathcal{S}$ (i.e., a logical language $\mathcal{L}_\mathcal{S}$ together with a consequence relation $\models_\mathcal{S} \subseteq \mathcal{L}_\mathcal{S} \times \mathcal{L}_\mathcal{S}$) is a theorem of the form:

> *If a set of sentences Φ of $\mathcal{S}$ is a member of an abstract consistency class Γ, then there exists a $\mathcal{S}$-model for Φ.*

For the proof we can use the classical construction in all cases: abstract consistent sets are extended to Hintikka sets (cf. Section 15.2), which induce a valuation on a term structure (cf. Definition 93). We then take a quotient by the congruence induced by Leibniz equality in the term model.

Given a model existence theorem as described above we can show the completeness of a particular calculus $\mathcal{C}$ (i.e., the derivability relation $\vdash_\mathcal{S} \subseteq \mathcal{L}_\mathcal{S} \times \mathcal{L}_\mathcal{S}$) by proving that the class Γ of sets

of sentences Φ that are $\mathcal{C}$-consistent (i.e., cannot be refuted in $\mathcal{C}$) is an abstract consistency class. Then the model existence theorem tells us that $\mathcal{C}$-consistent sets of sentences are satisfiable in $\mathcal{S}$. Now we assume that a sentence $\mathbf{A}$ is valid in $\mathcal{S}$, so $\neg\mathbf{A}$ does not have a $\mathcal{S}$-model and is therefore $\mathcal{C}$-inconsistent. Hence, $\neg\mathbf{A}$ is refutable in $\mathcal{C}$. This shows refutation completeness of $\mathcal{C}$. For many calculi $\mathcal{C}$, this also shows $\mathbf{A}$ is provable, thus establishing completeness of $\mathcal{C}$.

Note that with this argumentation the completeness proof for $\mathcal{C}$ condenses to verifying that Γ is an abstract consistency class, a task that does not refer to $\mathcal{S}$-models. Thus the usefulness of model existence theorems derives from the fact that it replaces the model-theoretic analysis in completeness proofs with the verification of some proof-theoretic conditions. In this respect a model existence theorem is similar to a Herbrand Theorem, but it is easier to generalize to other logic systems like higher-order logic. The technique was developed for first-order logic by Jaakko Hintikka and Raymond Smullyan [Hin55; Smu63; Smu68].

Remark 83 (Assumptions on $\mathcal{S}$) *From now on, we assume that we have infinitely many parameters $p \in \mathcal{P}_\alpha$ for each type α. Furthermore, we assume there is a particular cardinal $\aleph_s$ such that $\mathcal{P}_\alpha$ has cardinality $\aleph_s$ for every type α. Note that there can only be finitely many logical constants in $\mathcal{S}_\alpha$ for each particular type α. Since $\mathcal{V}$ is countable, this implies wff$_\alpha(\mathcal{S})$ and cwff$_\alpha(\mathcal{S})$ have cardinality $\aleph_s$ for each type α. In the countable case, $\aleph_s$ is $\aleph_0$.*

We furthermore assume that the signature $\mathcal{S}$ contains exactly the logical connectives $\neg$, $\vee$, and Π^α (for all types α).

13 Term Structures and Term Models

The most important method for constructing structures (and models) with given properties in this article is well-known for algebraic structures and consists of building a suitable congruence and passing to the quotient structure. We will now develop the formal basis for it.

Definition 84 (Applicative Structure Congruences) *Let $\mathcal{A} := (\mathcal{D}, @)$ be an applicative structure. A typed equivalence relation $\sim$ is called a* congruence *on $\mathcal{A}$ iff for all $\mathsf{f}, \mathsf{f}' \in \mathcal{D}_{\alpha\to\beta}$ and $\mathsf{a}, \mathsf{a}' \in \mathcal{D}_\alpha$ (for any types α and β), $\mathsf{f} \sim \mathsf{f}'$ and $\mathsf{a} \sim \mathsf{a}'$ imply $\mathsf{f}@\mathsf{a} \sim \mathsf{f}'@\mathsf{a}'$.*

The equivalence class *$[\![\mathsf{a}]\!]_\sim$ of $\mathsf{a} \in \mathcal{D}_\alpha$* modulo *$\sim$ is the set of all $\mathsf{a}' \in \mathcal{D}_\alpha$, such that $\mathsf{a} \sim \mathsf{a}'$. A congruence $\sim$ is called* functional *iff for all types α and β and $\mathsf{f}, \mathsf{g} \in \mathcal{D}_{\alpha\to\beta}$, we have $\mathsf{f} \sim \mathsf{g}$ whenever $\mathsf{f}@\mathsf{a} \sim \mathsf{g}@\mathsf{a}$ for every $\mathsf{a} \in \mathcal{D}_\alpha$.*

Lemma 85 *The β-equality and $\beta\eta$-equality relations $=_\beta$ and $\stackrel{\beta\eta}{=}$ are congruences on the applicative structures wff$(\mathcal{S})$ and cwff$(\mathcal{S})$.*

Proof. The congruence properties are a direct consequence of the fact that $\beta\eta$-reduction rules are defined to act on subterm positions.

Definition 86 (Quotient Applicative Structure) *Let $\mathcal{A} := (\mathcal{D}, @)$ be an applicative structure, $\sim$ a congruence on $\mathcal{A}$, and $\mathcal{D}^\sim_\alpha := \{[\![\mathsf{a}]\!]_\sim \mid \mathsf{a} \in \mathcal{D}_\alpha\}$. Furthermore, let $@^\sim$ be defined by $[\![\mathsf{f}]\!]_\sim @^\sim [\![\mathsf{a}]\!]_\sim := [\![\mathsf{f}@\mathsf{a}]\!]_\sim$. (To see that this definition only depends on equivalence classes of $\sim$, consider $\mathsf{f}' \in [\![\mathsf{f}]\!]_\sim$ and $\mathsf{a}' \in [\![\mathsf{a}]\!]_\sim$. Then $\mathsf{f} \sim \mathsf{f}'$ and $\mathsf{a} \sim \mathsf{a}'$ imply $\mathsf{f}@\mathsf{a} \sim \mathsf{f}'@\mathsf{a}'$. Thus, $[\![\mathsf{f}@\mathsf{a}]\!]_\sim = [\![\mathsf{f}'@\mathsf{a}']\!]_\sim$. So, $@^\sim$ is well-defined.) $\mathcal{A}/_\sim := (\mathcal{D}^\sim, @^\sim)$ is also an applicative structure. We call $\mathcal{A}/_\sim$ the* quotient structure *of $\mathcal{A}$ for the relation $\sim$ and the typed function $\pi_\sim \colon \mathcal{A} \to \mathcal{A}/_\sim$ that maps a to $[\![\mathsf{a}]\!]_\sim$ its* canonical projection.

Theorem 87 *Let $\mathcal{A}$ be an applicative structure and let $\sim$ be a congruence on $\mathcal{A}$, then the canonical projection $\pi_\sim$ is a surjective homomorphism. Furthermore, $\mathcal{A}/_\sim$ is functional iff $\sim$ is functional.*

Proof. Let $\mathcal{A} := (\mathcal{D}, @)$ be an applicative structure. To convince ourselves that $\pi_\sim$ is indeed a surjective homomorphism, we note that $\pi_\sim$ is surjective by the definition of $\mathcal{D}^\sim$. To see that $\pi_\sim$ is a homomorphism let $\mathsf{f} \in \mathcal{D}_{\alpha\to\beta}$, and $\mathsf{a} \in \mathcal{D}_\beta$, then $\pi_\sim(\mathsf{f}) @^\sim \pi_\sim(\mathsf{a}) = [\![\mathsf{f}]\!]_\sim @^\sim [\![\mathsf{a}]\!]_\sim = [\![\mathsf{f}@\mathsf{a}]\!]_\sim = \pi_\sim(\mathsf{f}@\mathsf{a})$.

The quotient construction collapses $\sim$ to identity, so functionality of $\sim$ is equivalent to functionality of $\mathcal{A}/_\sim$. Formally, suppose $[\![\mathsf{f}]\!]_\sim$ and $[\![\mathsf{g}]\!]_\sim$ are elements of $\mathcal{D}^\sim_{\alpha\to\beta}$ such that $[\![\mathsf{f}]\!]_\sim @^\sim [\![\mathsf{a}]\!]_\sim = [\![\mathsf{g}]\!]_\sim @^\sim [\![\mathsf{a}]\!]_\sim$ for every $[\![\mathsf{a}]\!]_\sim$ in $\mathcal{D}^\sim_\alpha$. This is equivalent to $[\![\mathsf{f}@\mathsf{a}]\!]_\sim = [\![\mathsf{g}@\mathsf{a}]\!]_\sim$ for every $\mathsf{a} \in \mathcal{D}_\alpha$ and hence $\mathsf{f}@\mathsf{a} \sim \mathsf{g}@\mathsf{a}$ for all $\mathsf{a} \in \mathcal{D}_\alpha$. By functionality of $\sim$, we have $\mathsf{f} \sim \mathsf{g}$. That is, $[\![\mathsf{f}]\!]_\sim = [\![\mathsf{g}]\!]_\sim$.

Lemma 88 *$\stackrel{\beta\eta}{=}$ is a functional congruence on wff($\mathcal{S}$). If $\mathcal{S}_\alpha$ is infinite for all types $\alpha \in \mathcal{T}$, then $\stackrel{\beta\eta}{=}$ is also functional on cwff($\mathcal{S}$).*

Proof. By Lemma 85, $\stackrel{\beta\eta}{=}$ is a congruence relation. To show functionality let $\mathbf{A}, \mathbf{B} \in \mathit{wff}_{\gamma\to\alpha}(\mathcal{S})$ such that $\mathbf{AC} \stackrel{\beta\eta}{=} \mathbf{BC}$ for all $\mathbf{C} \in \mathit{wff}_\gamma(\mathcal{S})$ be given. In particular, for any variable $X \in \mathcal{V}_\gamma$ that is not free in $\mathbf{A}$ or $\mathbf{B}$, we have $\mathbf{A}X \stackrel{\beta\eta}{=} \mathbf{B}X$ and $\lambda X_\blacksquare \mathbf{A}X \stackrel{\beta\eta}{=} \lambda X_\blacksquare \mathbf{B}X$. By definition we have $\mathbf{A} =_\eta \lambda X_{\gamma\blacksquare} \mathbf{A}X \stackrel{\beta\eta}{=} \lambda X_{\gamma\blacksquare} \mathbf{B}X =_\eta \mathbf{B}$.

To show functionality of $\beta\eta$-equality on closed formulae, suppose $\mathbf{A}$ and $\mathbf{B}$ are closed. With the same variable X as above, let $\mathbf{M}$ and $\mathbf{N}$ be the $\beta\eta$-normal forms of $\mathbf{A}X$ and $\mathbf{B}X$, respectively. We cannot conclude that $\mathbf{M} = \mathbf{N}$ since X is not a closed term. Instead, choose a constant $c_\gamma \in \mathcal{S}_\gamma$ that does not occur in $\mathbf{A}$ or $\mathbf{B}$. (Such a constant must exist, since we have assumed that $\mathcal{S}_\gamma$ is infinite.) An easy induction on the length of the $\beta\eta$-reduction sequence from $\mathbf{A}X$ to $\mathbf{M}$ shows that c does not occur in $\mathbf{M}$ and $\mathbf{A}c = [c/X](\mathbf{A}X)$ $\beta\eta$-reduces to $[c/X]\mathbf{M}$. Similarly, c does not occur in $\mathbf{N}$ and $\mathbf{B}c$ $\beta\eta$-reduces to $[c/X]\mathbf{N}$. Since c is a constant, substituting c for X cannot introduce new redexes. So, simple inductions on the sizes of $\mathbf{M}$ and $\mathbf{N}$ show $[c/X]\mathbf{M}$ and $[c/X]\mathbf{N}$ are $\beta\eta$-normal. By assumption, we know $\mathbf{A}c \stackrel{\beta\eta}{=} \mathbf{B}c$. Since normal forms are unique, we must have $[c/X]\mathbf{M} = [c/X]\mathbf{N}$. Using the fact that c does not occur in either $\mathbf{M}$ or $\mathbf{N}$, an induction on the size of $\mathbf{M}$ readily shows $\mathbf{M} = \mathbf{N}$. So, we have $\mathbf{A} =_\eta \lambda X_{\gamma\blacksquare} \mathbf{A}X \stackrel{\beta\eta}{=} \lambda X_{\gamma\blacksquare} \mathbf{M} = \lambda X_{\gamma\blacksquare} \mathbf{N} \stackrel{\beta\eta}{=} \lambda X_{\gamma\blacksquare} \mathbf{B}X =_\eta \mathbf{B}$

Remark 89 *Suppose we have a single constant c_ι. In this case, c is the only closed $\beta\eta$-normal form of type ι. Since $\lambda X_\blacksquare X \neq_{\beta\eta} \lambda X_\blacksquare c$ even though $(\lambda X_\blacksquare X)c \stackrel{\beta\eta}{=} c \stackrel{\beta\eta}{=} (\lambda X_\blacksquare c)c$ we have a counterexample to functionality of $\stackrel{\beta\eta}{=}$ on cwff($\mathcal{S}$). The problem here is that we do not have another constant d_ι to distinguish the two functions. In wff($\mathcal{S}$) we could always use a variable.*

Lemma 90 *Let $\mathcal{J} = (\mathcal{D}, @, \mathcal{E})$ be a $\mathcal{S}$-evaluation, φ and ψ assignments into $\mathcal{J}$ with $\varphi \sim \psi$, and $\sim$ a congruence on $\mathcal{J}$. For every formula $\mathbf{A}$, we have $\mathcal{E}_\varphi(\mathbf{A}) \sim \mathcal{E}_\psi(\mathbf{A})$.*

Proof. [1] Assume that the free variables of A are $X_1, \ldots, X_n$:
$\mathcal{E}_\varphi(A) \sim \mathcal{E}_\varphi((\lambda X_{1\blacksquare} \ldots \lambda X_{n\blacksquare} A)X_1 \ldots X_n)$
$\sim \mathcal{E}_\varphi(\lambda X_{1\blacksquare} \ldots \lambda X_{n\blacksquare} A) @ \varphi(X_1) \ldots @ \varphi(X_n)$
$\sim \mathcal{E}_\psi(\lambda X_{1\blacksquare} \ldots \lambda X_{n\blacksquare} A) @ \varphi(X_1) \ldots @ \varphi(X_n)$ (since $(\lambda X_{1\blacksquare} \ldots \lambda X_{n\blacksquare} A)$ is closed)
$\sim \mathcal{E}_\psi(\lambda X_{1\blacksquare} \ldots \lambda X_{n\blacksquare} A) @ \psi(X_1) ... @ \psi(X_n)$ (since $\varphi \sim \psi$ by assumption)
$\sim \mathcal{E}_\psi((\lambda X_{1\blacksquare} \ldots \lambda X_{n\blacksquare} A)X_1 \ldots X_n)$
$\sim \mathcal{E}_\psi(A)$

Definition 91 (Quotient $\mathcal{S}$-Evaluation) *Let $\mathcal{J} = (\mathcal{D}, @, \mathcal{E})$ be a $\mathcal{S}$-evaluation, $\sim$ a congruence on $\mathcal{J}$ and let $(\mathcal{D}^\sim, @^\sim)$ be the quotient applicative structure of $(\mathcal{D}, @)$ with respect to $\sim$.*

For each $\mathsf{A} \in \mathcal{D}^\sim_\alpha$, we choose a representative $\mathsf{A}^* \in \mathsf{A}$. So, $[\![\mathsf{A}^*]\!]_\sim = \mathsf{A}$. Note that $[\![\mathsf{a}]\!]^*_\sim \sim \mathsf{a}$ for every $\mathsf{a} \in \mathcal{D}_\alpha$. For any assignment φ into $\mathcal{J}/_\sim$, let φ^* be the assignment into $\mathcal{J}$ given by $\varphi^*(X) := \varphi(X)^*$. Note that $\varphi = \pi_\sim \circ \varphi^*$. So we can define $\mathcal{E}^\sim_\varphi$ as $\pi_\sim \circ \mathcal{E}_{\varphi^*}$, and call $\mathcal{J}/_\sim := (\mathcal{D}^\sim, @^\sim, \mathcal{E}^\sim)$ the quotient $\mathcal{S}$-evaluation *of $\mathcal{J}$ modulo $\sim$. (This definition of $\mathcal{E}^\sim$ does not depend on the choice of representatives.)*

This definition is justified by the following theorem.

Theorem 92 (Quotient $\mathcal{S}$-Evaluation Theorem) *If $\mathcal{J}$ is a $\mathcal{S}$-evaluation and $\sim$ is a congruence on $\mathcal{J}$, then $\mathcal{J}/_\sim$ is a $\mathcal{S}$-evaluation.*

[1] Thanks to Andreas Abel.

Proof. We prove that $\mathcal{E}^\sim$ is an evaluation function by verifying the conditions in Definition 77. For any assignment φ into the quotient applicative structure, let φ^* be the assignment with $\varphi = \pi_\sim \circ \varphi^*$ as in Definition 91. First, we compute $\mathcal{E}^\sim_\varphi\big|_\mathcal{V} = (\pi_\sim \circ \mathcal{E}_{\varphi^*})\big|_\mathcal{V} = \pi_\sim \circ \mathcal{E}_{\varphi^*}\big|_\mathcal{V} = \pi_\sim \circ \varphi^* = \varphi$. Since $\pi_\sim$ is a homomorphism we have

$$\begin{aligned}
\mathcal{E}^\sim_\varphi(\mathbf{FA}) &= \pi_\sim(\mathcal{E}_{\varphi^*}(\mathbf{FA})) \\
&= \pi_\sim(\mathcal{E}_{\varphi^*}(\mathbf{F}) @ \mathcal{E}_{\varphi^*}(\mathbf{A})) \\
&= \pi_\sim(\mathcal{E}_{\varphi^*}(\mathbf{F})) @^\sim \pi_\sim(\mathcal{E}_{\varphi^*}(\mathbf{A})) \\
&= \mathcal{E}^\sim_\varphi(\mathbf{F}) @^\sim \mathcal{E}^\sim_\varphi(\mathbf{A}).
\end{aligned}$$

If φ and ψ coincide on $\mathbf{Free}(\mathbf{A})$, then $\mathcal{E}^\sim_\varphi(\mathbf{A}) = [\![\mathcal{E}_{\varphi^*}(\mathbf{A})]\!]_\sim = [\![\mathcal{E}_{\psi^*}(\mathbf{A})]\!]_\sim = \mathcal{E}^\sim_\psi(\mathbf{A})$ since this entails that φ^* and ψ^* coincide on $\mathbf{Free}(\mathbf{A})$ too (as we have chosen particular representatives for each equivalence class). Finally, $\mathcal{E}^\sim_\varphi(\mathbf{A}) = [\![\mathcal{E}_{\varphi^*}(\mathbf{A})]\!]_\sim = [\![\mathcal{E}_{\varphi^*}(\mathbf{A}{\downarrow_\beta})]\!]_\sim = \mathcal{E}^\sim_\varphi(\mathbf{A}{\downarrow_\beta})$.

Definition 93 (Term Evaluations for $\mathcal{S}$) *Let cwff($\mathcal{S}$)$\big\downarrow_\beta$ be the collection of closed well-formed formulae in β-normal form and $\mathbf{A} @^\beta \mathbf{B}$ be $(\mathbf{AB})\big\downarrow_\beta$. For the definition of an evaluation function let φ be an assignment into cwff($\mathcal{S}$)$\big\downarrow_\beta$. Note that $\sigma := \varphi\big|_{\mathbf{Free}(\mathbf{A})}$ is a substitution, since $\mathbf{Free}(\mathbf{A})$ is finite. Thus we can choose $\mathcal{E}^\beta_\varphi(\mathbf{A}) := \sigma(\mathbf{A})\big\downarrow_\beta$. We call $\mathcal{TS}(\mathcal{S})^\beta := (\mathit{cwff}(\mathcal{S})\big\downarrow_\beta, @^\beta, \mathcal{E}^\beta)$ the* β-term evaluation *for $\mathcal{S}$.*

Analogously, we can define $\mathcal{TS}(\mathcal{S})^{\beta\eta} := (\mathit{cwff}(\mathcal{S})\big\downarrow_{\beta\eta}, @^{\beta\eta}, \mathcal{E}^{\beta\eta})$ the $\beta\eta$-term evaluation *for $\mathcal{S}$.*

The name *term evaluation* in the previous definition is justified by the following lemma.

Lemma 94 *$\mathcal{TS}(\mathcal{S})^\beta$ is a $\mathcal{S}$-evaluation and $\mathcal{TS}(\mathcal{S})^{\beta\eta}$ is a functional $\mathcal{S}$-evaluation.*

Proof. The fact that $(\mathit{cwff}(\mathcal{S})\big\downarrow_\beta, @^\beta)$ is an applicative structure is immediate: For each type α, $\mathit{cwff}_\alpha(\mathcal{S})\big\downarrow_\beta$ is non-empty (by the assumption in Remark 83) and

$$@^\beta\colon\ \mathit{cwff}_{\alpha\to\beta}(\mathcal{S})\big\downarrow_\beta \times \mathit{cwff}_\alpha(\mathcal{S})\big\downarrow_\beta \to \mathit{cwff}_\beta(\mathcal{S})\big\downarrow_\beta$$

We next check that $\mathcal{E}^\beta$ is an evaluation function.

1. $\mathcal{E}^\beta_\varphi(X) = \varphi\big|_{\mathbf{Free}(X)}(X) = \varphi(X)$.
2. $\mathcal{E}^\beta_\varphi$ respects application since $\sigma(\mathbf{FA})\big\downarrow_\beta = \Big(\sigma(\mathbf{F})\big\downarrow_\beta \sigma(\mathbf{A})\big\downarrow_\beta\Big)\Big\downarrow_\beta$ where $\sigma = \varphi\big|_{\mathbf{Free}(\mathbf{A})}$.
3. $\mathcal{E}^\beta_\varphi(\mathbf{A}) = \Big(\varphi\big|_{\mathbf{Free}(\mathbf{A})}(\mathbf{A})\Big)\Big\downarrow_\beta = \Big(\varphi'\big|_{\mathbf{Free}(\mathbf{A})}(\mathbf{A})\Big)\Big\downarrow_\beta = \mathcal{E}^\beta_{\varphi'}(\mathbf{A})$ whenever φ and φ' coincide on $\mathbf{Free}(\mathbf{A})$.
4. $\mathcal{E}^\beta_\varphi(\mathbf{A}) = \sigma(\mathbf{A})\big\downarrow_\beta = \sigma(\mathbf{A}{\downarrow_\beta})\big\downarrow_\beta = \mathcal{E}^\beta_\varphi(\mathbf{A}{\downarrow_\beta})$ where $\sigma = \varphi\big|_{\mathbf{Free}(\mathbf{A})}$.

A similar argument shows that $\mathcal{TS}(\mathcal{S})^{\beta\eta}$ is a $\mathcal{S}$-evaluation. Also, one can show $\mathcal{TS}(\mathcal{S})^{\beta\eta}$ is functional using an argument similar to Lemma 88 since $\mathcal{S}$ is infinite at all types by Remark 83. (Alternatively, one can simply apply Lemma 88 and Theorem 87 to note that the applicative structure $\mathit{cwff}(\mathcal{S})/_{\stackrel{\beta\eta}{=}}$ is functional. The applicative structure $\mathit{cwff}(\mathcal{S})/_{\stackrel{\beta\eta}{=}}$ is isomorphic to the applicative structure $(\mathit{cwff}(\mathcal{S})\big\downarrow_{\beta\eta}, @^{\beta\eta})$. One can easily show that functionality is preserved under isomorphism.)

Remark 95 *Note that $\mathcal{TS}(\mathcal{S})^\beta$ is not a functional $\mathcal{S}$-evaluation since, for instance, for any constant $h_{\gamma\to\delta} \in \mathcal{S}$*

$$(\lambda X_{\gamma\blacksquare} h_{\gamma\to\delta} X) @^\beta \mathbf{C}_\gamma = h @^\beta \mathbf{C}$$

for all $\mathbf{C}$ in $\mathcal{TS}_\gamma(\mathcal{S})^\beta$ but $\lambda X_\blacksquare hX \neq h$.

Now let us extend the notion of a quotient evaluation to $\mathcal{S}$-models.

Definition 96 ($\mathcal{S}$-Model Congruences) *A* congruence *on a $\mathcal{S}$-model $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ is a congruence on the underlying $\mathcal{S}$-evaluation $(\mathcal{D}, @, \mathcal{E})$ such that $\upsilon(\mathsf{a}) = \upsilon(\mathsf{b})$ for all $\mathsf{a}, \mathsf{b} \in \mathcal{D}_o$ with $\mathsf{a} \sim \mathsf{b}$.*

Definition 97 (Quotient $\mathcal{S}$-Model) *Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a $\mathcal{S}$-model, $\sim$ be a congruence on $\mathcal{M}$, and $(\mathcal{D}^\sim, @^\sim, \mathcal{E}^\sim)$ be the quotient $\mathcal{S}$-evaluation of $(\mathcal{D}, @, \mathcal{E})$ with respect to $\sim$ (cf. Definition 91). Using the notation for representatives $\mathsf{A}^* \in \mathsf{A}$ for $\mathsf{A} \in \mathcal{D}_\alpha^\sim$ as in Definition 91, we define $\upsilon^\sim \colon \mathcal{D}_o^\sim \to \{\mathtt{T}, \mathtt{F}\}$ by $\upsilon^\sim(\mathsf{A}) := \upsilon(\mathsf{A}^*)$ for every $\mathsf{A} \in \mathcal{D}_o^\sim$. (Since $\upsilon(\mathsf{a}) = \upsilon(\mathsf{b})$ whenever $\mathsf{a} \sim \mathsf{b}$ in $\mathcal{D}_o$, this definition of $\upsilon^\sim$ does not depend on the choice of representatives and $\upsilon^\sim([\![\mathsf{a}]\!]_\sim) = \upsilon(\mathsf{a})$ for every $\mathsf{a} \in \mathcal{D}_o$.) We call $\mathcal{M}/_\sim := (\mathcal{D}^\sim, @^\sim, \mathcal{E}^\sim, \upsilon^\sim)$ the* quotient $\mathcal{S}$-model *of $\mathcal{M}$ with respect to $\sim$.*

Theorem 98 (Quotient $\mathcal{S}$-Model Theorem) *Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a $\mathcal{S}$-model and $\sim$ be a congruence on $\mathcal{M}$. The quotient $\mathcal{M}/_\sim$ is a $\mathcal{S}$-model.*

Proof. We check the conditions of Definition 79, again using the A^* notation for representatives. To check condition $\mathfrak{L}_\neg(\mathcal{E}^\sim(\neg))$ for $\upsilon^\sim$, for all $\mathsf{A} \in \mathcal{D}_o^\sim$ we need to show that $\upsilon^\sim(\mathcal{E}^\sim(\neg)@^\sim\mathsf{A}) = \mathtt{T}$ iff $\upsilon^\sim(\mathsf{A}) = \mathtt{F}$. Let $\mathsf{A} \in \mathcal{D}_o^\sim$ be given. Since $\mathcal{M}$ is a $\mathcal{S}$-model we have $\upsilon(\mathcal{E}(\neg)@\mathsf{A}^*) = \mathtt{T}$ iff $\upsilon(\mathsf{A}^*) = \mathtt{F}$. Since $[\![\mathsf{A}^*]\!]_\sim = \mathsf{A}$ and $[\![\mathcal{E}(\neg)@\mathsf{A}^*]\!]_\sim = \mathcal{E}^\sim(\neg)@^\sim\mathsf{A}$, we have $\upsilon^\sim(\mathcal{E}^\sim(\neg)@^\sim\mathsf{A}) = \mathtt{T}$ iff $\upsilon^\sim(\mathsf{A}) = \mathtt{F}$. Checking condition $\mathfrak{L}_\vee(\mathcal{E}^\sim(\vee))$ for $\upsilon^\sim$ is analogous.

To check condition $\mathfrak{L}_{\Pi^\alpha}(\mathcal{E}^\sim(\Pi^\alpha))$ for $\upsilon^\sim$, suppose we have $\mathsf{G} \in \mathcal{D}_{\alpha\to o}^\sim$. For every $\mathsf{A} \in \mathcal{D}_\alpha^\sim$, $\upsilon^\sim(\mathsf{G}@^\sim\mathsf{A}) = \upsilon(\mathsf{G}^*@\mathsf{A}^*)$. So, if $\upsilon^\sim(\mathsf{G}@^\sim\mathsf{A}) = \mathtt{T}$ for every $\mathsf{A} \in \mathcal{D}_\alpha^\sim$, then $\upsilon(\mathsf{G}^*@\mathsf{a}) = \upsilon(\mathsf{G}^*@[\![\mathsf{a}]\!]_\sim^*) = \mathtt{T}$ for every $\mathsf{a} \in \mathcal{D}_\alpha$, and we conclude $\upsilon(\mathcal{E}(\Pi^\alpha)@\mathsf{G}^*) = \mathtt{T}$. Hence, $\upsilon^\sim(\mathcal{E}^\sim(\Pi^\alpha)@^\sim\mathsf{G}) = \mathtt{T}$. Conversely, suppose $\upsilon^\sim(\mathcal{E}^\sim(\Pi^\alpha)@\mathsf{G}) = \mathtt{T}$. Then $\upsilon(\mathcal{E}(\Pi^\alpha)@\mathsf{G}^*) = \mathtt{T}$ and hence $\upsilon^\sim(\mathsf{G}@\mathsf{A}) = \upsilon(\mathsf{G}^*@\mathsf{A}^*) = \mathtt{T}$ for every $\mathsf{A} \in \mathcal{D}_\alpha^\sim$.

We can define properties of a congruence analogous to those defined for models in Definition 80.

Definition 99 (Properties η, ξ, $\mathfrak{f}$ and $\mathfrak{b}$ for Congruences) *Given a $\mathcal{S}$-model $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ and a congruence $\sim$ on $\mathcal{M}$, we say $\sim$ has* property

- η *iff $\mathcal{E}_\varphi(\mathbf{A}) \sim \mathcal{E}_\varphi(\mathbf{A}{\downarrow_{\beta\eta}})$ for any type α, $\mathbf{A} \in wff_\alpha(\mathcal{S})$, and assignment φ.*
- ξ *iff for all $\alpha, \beta \in \mathcal{T}$, $\mathbf{M}, \mathbf{N} \in wff_\beta(\mathcal{S})$, assignment φ, and variables X_α, $\mathcal{E}_\varphi(\lambda X_\alpha \mathbf{.} \mathbf{M}_\beta) \sim \mathcal{E}_\varphi(\lambda X_\alpha \mathbf{.} \mathbf{N}_\beta)$ whenever $\mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{M}) \sim \mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{N})$ for every $\mathsf{a} \in \mathcal{D}_\alpha$.*
- $\mathfrak{f}$ *iff $\sim$ is functional.*
- $\mathfrak{b}$ *iff $\mathcal{D}_o$ has at most two equivalence classes with respect to $\sim$. (By Remark 107 there are always at least two.)*

Remark 100 *It follows trivially from reflexivity of congruences that if a model satisfies property η, then any congruence on the model satisfies property η. Similarly, if a model has only two elements in $\mathcal{D}_o$, then $\mathcal{D}_o$ can have at most two equivalence classes with respect to any congruence $\sim$. So, if a model satisfies property $\mathfrak{b}$, then any congruence on the model satisfies property $\mathfrak{b}$. This is not true for properties ξ or $\mathfrak{f}$. For an example, we refer to the functional model (satisfying property $\mathfrak{f}$, hence property ξ) constructed by Andrews in [And72a]. Using the results we prove below, one can show Leibniz equality must induce a congruence failing to satisfy properties ξ and $\mathfrak{f}$ on this functional model.*

Lemma 101 *Let $\mathcal{M}$ be a $\mathcal{S}$-model, $\Phi \subseteq cwff_o$, and $\sim$ be a congruence on $\mathcal{M}$. We have $\mathcal{M}/_\sim \models \Phi$ iff $\mathcal{M} \models \Phi$. Furthermore, if $* \in \{\eta, \xi, \mathfrak{f}, \mathfrak{b}\}$ and $\sim$ satisfies property $*$, then $\mathcal{M}/_\sim$ satisfies property $*$.*

Proof. Let $\mathbf{A}_o \in \Phi$. Since $\mathbf{A}$ is closed, $\mathcal{M} \models \mathbf{A}$, iff $\upsilon(\mathcal{E}(\mathbf{A})) = \mathtt{T}$, iff $\upsilon^\sim(\mathcal{E}^\sim(\mathbf{A})) = \mathtt{T}$, iff $\mathcal{M}/_\sim \models \mathbf{A}$. So, $\mathcal{M} \models \Phi$ iff $\mathcal{M}/_\sim \models \Phi$.

Suppose $\sim$ satisfies property η. Let $\mathbf{A} \in wff_\alpha(\mathcal{S})$, and an assignment φ into $\mathcal{M}/_\sim$ be given. Let φ^* be a corresponding assignment into $\mathcal{M}$ (cf. Definition 91). Since $\sim$ satisfies property η, we know $\mathcal{E}_{\varphi^*}(\mathbf{A}) \sim \mathcal{E}_{\varphi^*}(\mathbf{A}{\downarrow_{\beta\eta}})$. Taking equivalence classes, we have $\mathcal{E}_\varphi^\sim(\mathbf{A}) = \mathcal{E}_\varphi^\sim(\mathbf{A}{\downarrow_{\beta\eta}})$.

Suppose $\sim$ satisfies property ξ. Let $\mathbf{M}, \mathbf{N} \in wff_\beta(\mathcal{S})$, a variable X_α and an assignment φ into $\mathcal{M}/_\sim$ be given. Again, let φ^* be a corresponding assignment into $\mathcal{M}$. Suppose $\mathcal{E}_{\varphi,[\mathsf{A}/X]}^\sim(\mathbf{M}) = \mathcal{E}_{\varphi,[\mathsf{A}/X]}^\sim(\mathbf{N})$ for every $\mathsf{A} \in \mathcal{D}_\alpha^\sim$. This means $\mathcal{E}_{\varphi^*,[\mathsf{A}^*/X]}(\mathbf{M}) \sim \mathcal{E}_{\varphi^*,[\mathsf{A}^*/X]}(\mathbf{N})$ for every $\mathsf{A} \in \mathcal{D}_\alpha^\sim$. For any $\mathsf{a} \in \mathcal{D}_\alpha$, using Lemma 90, we know

$$\mathcal{E}_{\varphi^*,[\mathsf{a}/X]}(\mathbf{M}) \sim \mathcal{E}_{\varphi^*,[\mathsf{A}^*/X]}(\mathbf{M}) \sim \mathcal{E}_{\varphi^*,[\mathsf{A}^*/X]}(\mathbf{N}) \sim \mathcal{E}_{\varphi^*,[\mathsf{a}/X]}(\mathbf{N})$$

where $\mathsf{A} \in \mathcal{D}_\alpha^\sim$ is the equivalence class of a. Since $\sim$ satisfies property ξ, we know that $\mathcal{E}_{\varphi^*}(\lambda X \mathbf{.} \mathbf{M}) \sim \mathcal{E}_{\varphi^*}(\lambda X \mathbf{.} \mathbf{N})$. Taking equivalence classes, we see that $\mathcal{E}_\varphi^\sim(\lambda X \mathbf{.} \mathbf{M}) = \mathcal{E}_\varphi^\sim(\lambda X \mathbf{.} \mathbf{N})$.

If $\sim$ is functional (satisfies property $\mathfrak{f}$), we know $\mathcal{M}/_\sim$ is functional (satisfies property $\mathfrak{f}$) by Theorem 87.

Finally, if $\sim$ satisfies property $\mathfrak{b}$, then clearly $\mathcal{D}_o^\sim$ has only two elements. So, $\mathcal{M}/_\sim$ satisfies property $\mathfrak{b}$.

Definition 102 (Congruence Relation $\dot\sim$) *Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a $\mathcal{S}$-model. Let $\mathsf{q}^\alpha \in \mathcal{D}_{\alpha\to\alpha\to o}$ be $\mathcal{E}(\mathbf{Q}^\alpha)$, i.e., the interpretation of Leibniz equality at type α. We define $\mathsf{a} \dot\sim \mathsf{b}$ in $\mathcal{D}_\alpha$ iff $\upsilon(\mathsf{q}^\alpha@\mathsf{a}@\mathsf{b}) = \mathtt{T}$.*

Before checking $\dot\sim$ is a congruence, we first show that it is at least reflexive.

Lemma 103 *Let $\mathcal{M}$ be a $\mathcal{S}$-model. For each type α and $\mathsf{a} \in \mathcal{D}_\alpha$, we have $\mathsf{a} \dot\sim \mathsf{a}$.*

Proof. We need to check $\upsilon(\mathcal{E}(\mathbf{Q}^\alpha)@\mathsf{a}@\mathsf{a}) = \mathtt{T}$. Let X_α be a variable of type α and φ be some assignment with $\varphi(X) = \mathsf{a}$. Let $\mathsf{r} := \mathcal{E}_\varphi(\lambda P_{\alpha\to o} \mathbf{.} \neg(PX) \vee PX))$. For any $\mathsf{p} \in \mathcal{D}_{\alpha\to o}$, since $\mathcal{E}$ is an evaluation function, we have

$$\upsilon(\mathsf{r}@\mathsf{p}) = \upsilon(\mathcal{E}_{\varphi,[\mathsf{p}/P]}(\neg(PX) \vee PX))$$

As $\mathcal{M}$ is a $\mathcal{S}$-model, we have $\upsilon(\mathcal{E}_{\varphi,[\mathsf{p}/P]}(\neg(PX) \vee PX)) = \mathtt{T}$ since either

$$\upsilon(\mathcal{E}_{\varphi,[\mathsf{p}/P]}(PX)) = \mathtt{T} \quad \text{or} \quad \upsilon(\mathcal{E}_{\varphi,[\mathsf{p}/P]}(\neg(PX))) = \mathtt{T}$$

So, again since $\mathcal{M}$ is a $\mathcal{S}$-model, $\upsilon(\mathcal{E}(\Pi^{\alpha\to o})@\mathsf{r}) = \mathtt{T}$. By the definitions of r and $\doteq^\alpha$, we have $\upsilon(\mathcal{E}_\varphi(X \doteq^\alpha X)) = \mathtt{T}$. As $X \doteq^\alpha X$ is a β-reduct of $\mathbf{Q}^\alpha XX$, we have $\upsilon(\mathcal{E}_\varphi(\mathbf{Q}^\alpha XX)) = \mathtt{T}$ as well. Using $\varphi(X) = \mathsf{a}$, we see that $\upsilon(\mathcal{E}(\mathbf{Q}^\alpha)@\mathsf{a}@\mathsf{a}) = \mathtt{T}$.

In order to check that $\dot\sim$ is a congruence, it is useful to unwind the definitions to better characterize when $\mathsf{a} \dot\sim \mathsf{b}$ for $\mathsf{a}, \mathsf{b} \in \mathcal{D}_\alpha$.

Lemma 104 (Properties of $\dot\sim$) *Let $\mathcal{M}$ be a $\mathcal{S}$-model. For each type α and $\mathsf{a}, \mathsf{b} \in \mathcal{D}_\alpha$, the following are equivalent:*

1. *$\mathsf{a} \dot\sim \mathsf{b}$.*
2. *For all variables X_α and Y_α and assignments φ such that $\varphi(X) = \mathsf{a}$ and $\varphi(Y) = \mathsf{b}$, we have $\upsilon(\mathcal{E}_\varphi(X \doteq^\alpha Y)) = \mathtt{T}$.*
3. *For every $\mathsf{p} \in \mathcal{D}_{\alpha\to o}$, $\upsilon(\mathsf{p}@\mathsf{a}) = \mathtt{T}$ implies $\upsilon(\mathsf{p}@\mathsf{b}) = \mathtt{T}$.*
4. *For every $\mathsf{p} \in \mathcal{D}_{\alpha\to o}$, $\upsilon(\mathsf{p}@\mathsf{a}) = \upsilon(\mathsf{p}@\mathsf{b})$.*

Proof. At each type α, let $\mathsf{q}^\alpha \in \mathcal{D}_{\alpha\to\alpha\to o}$ be the interpretation $\mathcal{E}(\mathbf{Q}^\alpha)$ of Leibniz equality. By definition, $\mathsf{a} \dot\sim \mathsf{b}$ iff $\upsilon(\mathsf{q}^\alpha@\mathsf{a}@\mathsf{b}) = \mathtt{T}$.

To show (1) implies (2), suppose $\mathsf{a} \dot\sim \mathsf{b}$ and φ is an assignment with $\varphi(X_\alpha) = \mathsf{a}$ and $\varphi(Y_\alpha) = \mathsf{b}$. Since $\upsilon(\mathsf{q}^\alpha@\mathsf{a}@\mathsf{b}) = \mathtt{T}$, we have $\upsilon(\mathcal{E}_\varphi(\mathbf{Q}^\alpha XY)) = \mathtt{T}$. Since $\mathcal{E}$ respects β-equality, we have $\upsilon(\mathcal{E}_\varphi(X \doteq^\alpha Y)) = \mathtt{T}$.

To show (2) implies (3), suppose $\upsilon(\mathcal{E}_\varphi(X \doteq^\alpha Y)) = \mathtt{T}$ whenever φ is an assignment with $\varphi(X) = \mathsf{a}$ and $\varphi(Y) = \mathsf{b}$. Let X and Y be particular distinct variables of type α and φ be any such assignment with $\varphi(X) = \mathsf{a}$ and $\varphi(Y) = \mathsf{b}$. Let $\mathsf{p} \in \mathcal{D}_{\alpha\to o}$ with $\upsilon(\mathsf{p}@\mathsf{a}) = \mathtt{T}$ and a variable $P_{\alpha\to o}$ be given. By assumption, $\upsilon(\mathcal{E}_\varphi(\forall P_{\alpha\to o}.\neg(PX)\vee(PY)))=\mathtt{T}$. Since $\upsilon(\mathcal{E}_{\varphi,[\mathsf{p}/P]}(PX))=\upsilon(\mathsf{p}@\mathsf{a})=\mathtt{T}$, we have $\upsilon(\mathsf{p}@\mathsf{b})=\upsilon(\mathcal{E}_{\varphi,[\mathsf{p}/P]}(PY))=\mathtt{T}$.

To show (3) implies (4), let $\mathsf{p}\in\mathcal{D}_{\alpha\to o}$ be given. If $\upsilon(\mathsf{p}@\mathsf{a})=\mathtt{T}$, then we have $\upsilon(\mathsf{p}@\mathsf{b})=\mathtt{T}$ by assumption. So, $\upsilon(\mathsf{p}@\mathsf{a})=\upsilon(\mathsf{p}@\mathsf{b})$ in this case. Otherwise, we must have $\upsilon(\mathsf{p}@\mathsf{a})=\mathtt{F}$. Let $\mathsf{q}:=\mathcal{E}_\varphi(\lambda X_\alpha\text{▪}\neg(P_{\alpha\to o}X))$ where φ is some assignment with $\varphi(P):=\mathsf{p}$. Since $\mathcal{M}$ is a model, $\upsilon(\mathsf{q}@\mathsf{a})=\upsilon(\mathcal{E}(\neg)@(\mathsf{p}@\mathsf{a}))=\mathtt{T}$. Applying the assumption to q, we have $\upsilon(\mathsf{q}@\mathsf{b})=\mathtt{T}$ and so $\upsilon(\mathcal{E}(\neg)@(\mathsf{p}@\mathsf{b}))=\mathtt{T}$. Thus, $\upsilon(\mathsf{p}@\mathsf{b})=\mathtt{F}$ and $\upsilon(\mathsf{p}@\mathsf{a})=\upsilon(\mathsf{p}@\mathsf{b})$ in this case as well.

To show (4) implies (1), suppose $\upsilon(\mathsf{p}@\mathsf{a})=\upsilon(\mathsf{p}@\mathsf{b})$ for every $\mathsf{p}\in\mathcal{D}_{\alpha\to o}$. In particular, this holds for $\mathsf{p}:=\mathsf{q}^\alpha@\mathsf{a}\in\mathcal{D}_{\alpha\to o}$. Since $\upsilon(\mathsf{q}^\alpha@\mathsf{a}@\mathsf{a})=\mathtt{T}$ by Lemma 103, we must have $\upsilon(\mathsf{q}^\alpha@\mathsf{a}@\mathsf{b})=\mathtt{T}$. That is, $\mathsf{a}\dot{\sim}\mathsf{b}$.

Theorem 105 (Properties of $\mathcal{M}/_{\dot\sim}$) *Let $\mathcal{M}$ be a $\mathcal{S}$-model. Then $\dot\sim$ is a congruence relation on the model $\mathcal{M}$ and $\mathcal{M}/_{\dot\sim}$ satisfies property $\mathfrak{q}$.*

Proof. We first verify that $\dot\sim$ is an equivalence relation on each $\mathcal{D}_\alpha$. Reflexivity was shown in Lemma 103. To check symmetry and transitivity we use condition (4) in Lemma 104. For symmetry, let $\mathsf{a}\dot\sim\mathsf{b}$ in $\mathcal{D}_\alpha$ and $\mathsf{p}\in\mathcal{D}_{\alpha\to o}$ be given. So, $\upsilon(\mathsf{p}@\mathsf{a})=\upsilon(\mathsf{p}@\mathsf{b})$. Generalizing over p, we have $\mathsf{b}\dot\sim\mathsf{a}$. For transitivity, let $\mathsf{a}\dot\sim\mathsf{b}$ and $\mathsf{b}\dot\sim\mathsf{c}$ in $\mathcal{D}_\alpha$ and $\mathsf{p}\in\mathcal{D}_{\alpha\to o}$ be given. So, $\upsilon(\mathsf{p}@\mathsf{a})=\upsilon(\mathsf{p}@\mathsf{b})=\upsilon(\mathsf{p}@\mathsf{c})$. Generalizing over p, we have $\mathsf{a}\dot\sim\mathsf{c}$.

We next verify that $\dot\sim$ is a congruence. Suppose $\mathsf{f}\dot\sim\mathsf{g}$ in $\mathcal{D}_{\alpha\to\beta}$ and $\mathsf{a}\dot\sim\mathsf{b}\in\mathcal{D}_\alpha$. To show $\mathsf{f}@\mathsf{a}\dot\sim\mathsf{g}@\mathsf{b}$ we use condition (3) in Lemma 104. Let $\mathsf{p}\in\mathcal{D}_{\beta\to o}$ with $\upsilon(\mathsf{p}@(\mathsf{f}@\mathsf{a}))=\mathtt{T}$ be given. Let φ be an assignment with $\varphi(P_{\beta\to o})=\mathsf{p}$, $\varphi(X_\alpha)=\mathsf{a}$ and $\varphi(G_{\alpha\to\beta})=\mathsf{g}$ for variables P, X and G. We can use Lemma 104(3) with $\mathcal{E}_\varphi(\lambda F_{\alpha\to\beta}\text{▪}(P(FX)))$ and $\mathsf{f}\dot\sim\mathsf{g}$ to verify that $\upsilon(\mathsf{p}@(\mathsf{g}@\mathsf{a}))=\mathtt{T}$. Using Lemma 104(3) with $\mathcal{E}_\varphi(\lambda X_\alpha\text{▪}(P(GX)))$ and $\mathsf{a}\dot\sim\mathsf{b}$ verifies $\upsilon(\mathsf{p}@(\mathsf{g}@\mathsf{b}))=\mathtt{T}$. So, $\mathsf{f}@\mathsf{a}\dot\sim\mathsf{g}@\mathsf{b}$.

It remains to check that $\upsilon(\mathsf{a})=\upsilon(\mathsf{b})$ whenever $\mathsf{a}\dot\sim\mathsf{b}$ for $\mathsf{a},\mathsf{b}\in\mathcal{D}_o$. Let $\mathsf{a}\dot\sim\mathsf{b}$ in $\mathcal{D}_o$ be given. Applying Lemma 104(4) to $\mathcal{E}(\lambda X_o\text{▪}X)\in\mathcal{D}_{o\to o}$ we have $\upsilon(\mathsf{a})=\upsilon(\mathcal{E}(\lambda X_o\text{▪}X)@\mathsf{a})=\upsilon(\mathcal{E}(\lambda X_o\text{▪}X)@\mathsf{b})=\upsilon(\mathsf{b})$ as desired. So, $\dot\sim$ is a congruence relation on $\mathcal{M}$.

Now, we show $\mathcal{M}/_{\dot\sim}$ satisfies property $\mathfrak{q}$. At each type α, let $\mathsf{q}^\alpha\in\mathcal{D}_{\alpha\to\alpha\to o}$ be the interpretation $\mathcal{E}(\mathbf{Q}^\alpha)$ of Leibniz equality. To check property $\mathfrak{q}$, we show that $[\![\mathsf{q}^\alpha]\!]_{\dot\sim}$ is the appropriate object in $\mathcal{D}^{\dot\sim}_{\alpha\to\alpha\to o}$ for each $\alpha\in\mathcal{T}$. Let $\mathsf{a},\mathsf{b}\in\mathcal{D}_\alpha$ be given. Note that $[\![\mathsf{a}]\!]_{\dot\sim}=[\![\mathsf{b}]\!]_{\dot\sim}$ is equivalent to $\mathsf{a}\dot\sim\mathsf{b}$.

Also, $\upsilon^{\dot\sim}([\![\mathsf{q}^\alpha]\!]_{\dot\sim}@^{\dot\sim}[\![\mathsf{a}]\!]_{\dot\sim}@^{\dot\sim}[\![\mathsf{b}]\!]_{\dot\sim})=[\![\mathtt{T}]\!]_{\dot\sim}$ is equivalent to $\upsilon(\mathsf{q}^\alpha@\mathsf{a}@\mathsf{b})=\mathtt{T}$. So, we need to show that $\upsilon(\mathsf{q}^\alpha@\mathsf{a}@\mathsf{b})=\mathtt{T}$ iff $\mathsf{a}\dot\sim\mathsf{b}$. But this is precisely the definition of $\dot\sim$.

14 Defined Logical Connectives in $\mathcal{S}$-Models

Lemma 106 (Truth and Falsity in $\mathcal{S}$-Models) *Let $\mathcal{M}:=(\mathcal{D},@,\mathcal{E},\upsilon)$ be a $\mathcal{S}$-model and φ an assignment. Let $\mathbf{T}_o:=\forall P_o.P\vee\neg P$ and $\mathbf{F}_o:=\neg\mathbf{T}_o$. Then $\upsilon(\mathcal{E}_\varphi(\mathbf{T}_o))=\mathtt{T}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{F}_o))=\mathtt{F}$.*

Proof. Let P be a variable of type o. We have $\upsilon(\mathcal{E}_\varphi(\mathbf{T}_o))=\mathtt{T}$, iff $\upsilon(\mathcal{E}_\varphi(P\vee\neg P))=\mathtt{T}$ for every assignment φ. The properties of υ show that this statement is equivalent to $\upsilon(\varphi(P))=\mathtt{T}$ or $\upsilon(\varphi(P))=\mathtt{F}$, which is always true since υ maps into $\{\mathtt{T},\mathtt{F}\}$. Note further that $\upsilon(\mathcal{E}_\varphi(\mathbf{F}_o))=\mathtt{F}$ since $\upsilon(\mathcal{E}_\varphi(\mathbf{T}_o))=\mathtt{T}$.

Remark 107 *Let $\mathcal{M}:=(\mathcal{D},@,\mathcal{E},\upsilon)$ be a $\mathcal{S}$-model. By Lemma 106, $\mathcal{D}_o$ must have at least the two elements $\mathcal{E}_\varphi(\mathbf{T}_o)$ and $\mathcal{E}_\varphi(\mathbf{F}_o)$, and υ must be surjective.*

Lemma 108 (Equivalence) *Let $\mathcal{M}:=(\mathcal{D},@,\mathcal{E},\upsilon)$ be a $\mathcal{S}$-model, φ an assignment into $\mathcal{M}$, and $\mathbf{A},\mathbf{B}\in wff_o(\mathcal{S})$. $\upsilon(\mathcal{E}_\varphi(\mathbf{A}\Leftrightarrow\mathbf{B}))=\mathtt{T}$ iff $\upsilon(\mathcal{E}_\varphi(\mathbf{A}))=\upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.*

Proof. Suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A}\Leftrightarrow\mathbf{B}))=\mathtt{T}$. This implies $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{A}\vee\mathbf{B}))=\mathtt{T}$ and $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{B}\vee\mathbf{A}))=\mathtt{T}$. If $\upsilon(\mathcal{E}_\varphi(\mathbf{A}))=\mathtt{T}$, then $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{A}\vee\mathbf{B}))=\mathtt{T}$ implies $\upsilon(\mathcal{E}_\varphi(\mathbf{B}))=\mathtt{T}$, so $\upsilon(\mathcal{E}_\varphi(\mathbf{A}))=\mathtt{T}=\upsilon(\mathcal{E}_\varphi(\mathbf{B}))$. If $\upsilon(\mathcal{E}_\varphi(\mathbf{A}))=\mathtt{F}$, then $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{B}\vee\mathbf{A}))=\mathtt{T}$ implies $\upsilon(\mathcal{E}_\varphi(\mathbf{B}))=\mathtt{F}$, so $\upsilon(\mathcal{E}_\varphi(\mathbf{A}))=\mathtt{F}=\upsilon(\mathcal{E}_\varphi(\mathbf{B}))$. Since these are the only two possible values for $\upsilon(\mathcal{E}_\varphi(\mathbf{A}))$, we have $\upsilon(\mathcal{E}_\varphi(\mathbf{A}))=\upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A}))=\upsilon(\mathcal{E}_\varphi(\mathbf{B}))$. Either $\upsilon(\mathcal{E}_\varphi(\mathbf{A}))=\upsilon(\mathcal{E}_\varphi(\mathbf{B}))=\mathtt{T}$ or $\upsilon(\mathcal{E}_\varphi(\mathbf{A}))=\upsilon(\mathcal{E}_\varphi(\mathbf{B}))=\mathtt{F}$. An easy consideration of both cases verifies $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{A}\vee\mathbf{B}))=\mathtt{T}$ and $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{B}\vee\mathbf{A}))=\mathtt{T}$. Hence, $\upsilon(\mathcal{E}_\varphi(\mathbf{A}\Leftrightarrow\mathbf{B}))=\mathtt{T}$.

Definition 109 (Extensionality for Leibniz Equality) *We call a formula of the form*

$$\mathrm{EXT}_{\doteq}^{\alpha\to\beta}:=\forall F_{\alpha\to\beta}.\forall G_{\alpha\to\beta}.(\forall X_\alpha.FX\doteq^\beta GX)\Rightarrow F\doteq^{\alpha\to\beta}G$$

an axiom of (strong) functional extensionality for Leibniz equality*, and refer to the set*

$$\mathrm{EXT}_{\doteq}^{\to}:=\{\mathrm{EXT}_{\doteq}^{\alpha\to\beta}\mid\alpha,\beta\in\mathcal{T}\}$$

as the axioms of (strong) functional extensionality for Leibniz equality*. Note that $\mathrm{EXT}_{\doteq}^{\to}$ specifies functionality of the relation corresponding to Leibniz equality $\doteq$. We call the formula*

$$\mathrm{EXT}_{\doteq}^{o}:=\forall A_o.\forall B_o.(A\Leftrightarrow B)\Rightarrow A\doteq^o B$$

the axiom of Boolean extensionality*. We call the set $\mathrm{EXT}_{\doteq}^{\to}\cup\{\mathrm{EXT}_{\doteq}^{o}\}$ the* axioms of (strong) extensionality for Leibniz equality.

Lemma 110 (Leibniz Equality in $\mathcal{S}$-models) *Let $\mathcal{M}:=(\mathcal{D},@,\mathcal{E},\upsilon)$ be a $\mathcal{S}$-model, φ be an assignment, $\alpha\in\mathcal{T}$, and $\mathbf{A},\mathbf{B}\in wff_\alpha(\mathcal{S})$.*

1. *If $\mathcal{E}_\varphi(\mathbf{A})=\mathcal{E}_\varphi(\mathbf{B})$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A}\doteq^\alpha\mathbf{B}))=\mathtt{T}$.*
2. *If $\mathcal{M}$ satisfies property $\mathfrak{q}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{A}\doteq^\alpha\mathbf{B}))=\mathtt{T}$, then $\mathcal{E}_\varphi(\mathbf{A})=\mathcal{E}_\varphi(\mathbf{B})$.*

Proof. Let φ be any assignment into $\mathcal{M}$. For the first part, suppose $\mathcal{E}_\varphi(\mathbf{A})=\mathcal{E}_\varphi(\mathbf{B})$. Given $\mathsf{r}\in\mathcal{D}_{\alpha\to o}$, we have either $\upsilon(\mathsf{r}@\mathcal{E}_\varphi(\mathbf{A}))=\upsilon(\mathsf{r}@\mathcal{E}_\varphi(\mathbf{B}))=\mathtt{F}$ or $\upsilon(\mathsf{r}@\mathcal{E}_\varphi(\mathbf{B}))=\upsilon(\mathsf{r}@\mathcal{E}_\varphi(\mathbf{A}))=\mathtt{T}$. In either case, for any variable $P_{\alpha\to o}$ not in $\mathbf{Free}(\mathbf{A})\cup\mathbf{Free}(\mathbf{B})$, we have $\upsilon(\mathcal{E}_{\varphi,[\mathsf{r}/P]}(\neg(P\mathbf{A})\vee P\mathbf{B}))=\mathtt{T}$. So, we have $\mathcal{E}_\varphi(\mathbf{A}\doteq^\alpha\mathbf{B})=\mathtt{T}$.

To show the second part, suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A}\doteq^\alpha\mathbf{B}))=\mathtt{T}$. By property $\mathfrak{q}$, there is some $\mathsf{q}^\alpha\in\mathcal{D}_{\alpha\to\alpha\to o}$ such that for $\mathsf{a},\mathsf{b}\in\mathcal{D}_\alpha$ we have $\upsilon(\mathsf{q}^\alpha@\mathsf{a}@\mathsf{b})=\mathtt{T}$ iff $\mathsf{a}=\mathsf{b}$. Let $\mathsf{r}=\mathsf{q}^\alpha@\mathcal{E}_\varphi(\mathbf{A})$. From $\upsilon(\mathcal{E}_\varphi(\mathbf{A}\doteq^\alpha\mathbf{B}))=\mathtt{T}$, we obtain $\mathcal{E}_{\varphi,[\mathsf{r}/P]}(\neg P\mathbf{A}\vee P\mathbf{B})=\mathtt{T}$ (where $P_{\alpha\to o}\notin\mathbf{Free}(\mathbf{A})\cup\mathbf{Free}(\mathbf{B})$). Since $\mathcal{E}_{\varphi,[\mathsf{r}/P]}(P\mathbf{A})=\mathsf{q}^\alpha@\mathcal{E}_\varphi(\mathbf{A})@\mathcal{E}_\varphi(\mathbf{A})=\mathtt{T}$, we must have $\upsilon(\mathcal{E}_{\varphi,[\mathsf{r}/P]}(P\mathbf{B}))=\mathtt{T}$. That is, $\upsilon(\mathsf{q}^\alpha@\mathcal{E}_\varphi(\mathbf{A})@\mathcal{E}_\varphi(\mathbf{B}))=\mathtt{T}$. By the choice of q^α, we have $\mathcal{E}_\varphi(\mathbf{A})=\mathcal{E}_\varphi(\mathbf{B})$.

Theorem 111 (Extensionality in $\mathcal{S}$-models) *Let $\mathcal{M}=(\mathcal{D},@,\mathcal{E},\upsilon)$ be a $\mathcal{S}$-model.*

1. *If $\mathcal{M}$ satisfies property $\mathfrak{q}$ but not property $\mathfrak{f}$, then $\mathcal{M}\not\models\mathrm{EXT}_{\doteq}^{\to}$.*
2. *If $\mathcal{M}$ satisfies property $\mathfrak{q}$ but not property $\mathfrak{b}$, then $\mathcal{M}\not\models\mathrm{EXT}_{\doteq}^{o}$.*
3. *If $\mathcal{M}$ satisfies properties $\mathfrak{q}$ and $\mathfrak{f}$, then $\mathcal{M}\models\mathrm{EXT}_{\doteq}^{\to}$.*
4. *If $\mathcal{M}$ satisfies property $\mathfrak{b}$, then $\mathcal{M}\models\mathrm{EXT}_{\doteq}^{o}$.*

Thus we can characterize the different semantical structures with respect to Boolean and functional extensionality by the table in Figure 4.[2]

in	$\mathfrak{M}_\beta$, $\mathfrak{M}_{\beta\eta}$, $\mathfrak{M}_{\beta\xi}$		$\mathfrak{M}_{\beta\mathfrak{f}}$		$\mathfrak{M}_{\beta\mathfrak{b}}$, $\mathfrak{M}_{\beta\eta\mathfrak{b}}$, $\mathfrak{M}_{\beta\xi\mathfrak{b}}$		$\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}$	
formula	valid?	by	valid?	by	valid?	by	valid?	by
$\mathrm{EXT}_{\doteq}^{\to}$	—	1.	+	3.	—	1.	+	3.
$\mathrm{EXT}_{\doteq}^{o}$	—	2.	—	2.	+	4.[2]	+	4.[2]

Fig. 4. Extensionality in $\mathcal{S}$-models

[2] The cases in the figure corresponding to Theorem 111(4) are actually special cases. In Theorem 111(4), we can infer a model satisfies $\mathrm{EXT}_{\doteq}^{o}$ even if property $\mathfrak{q}$ does not hold. However, the models in $\mathfrak{M}_{\beta\mathfrak{b}}$, $\mathfrak{M}_{\beta\eta\mathfrak{b}}$, $\mathfrak{M}_{\beta\xi\mathfrak{b}}$ and $\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}$ do satisfy property $\mathfrak{q}$ by the definition of these model classes.

Proof. Suppose $\mathcal{M}$ satisfies property q but does not satisfy property $\mathfrak{f}$. Then there must be types α and β and objects $\mathsf{f}, \mathsf{g} \in \mathcal{D}_{\alpha\to\beta}$ such that $\mathsf{f} \neq \mathsf{g}$ but $\mathsf{f}@\mathsf{a} = \mathsf{g}@\mathsf{a}$ for every $\mathsf{a} \in \mathcal{D}_\alpha$. Let $F_{\alpha\to\beta}, G_{\alpha\to\beta} \in \mathcal{V}_{\alpha\to\beta}$ be distinct variables, $X_\alpha \in \mathcal{V}_\alpha$, and φ be any assignment with $\varphi(F) = \mathsf{f}$ and $\varphi(G) = \mathsf{g}$. For any $\mathsf{a} \in \mathcal{D}_\alpha$, $\mathsf{f}@\mathsf{a} = \mathsf{g}@\mathsf{a}$ implies $\upsilon(\mathcal{E}_{\varphi,[\mathsf{a}/X]}(FX \doteq^\beta GX)) = \mathtt{T}$ by Lemma 110(1). Using the fact that υ is a valuation, we have $\upsilon(\mathcal{E}_\varphi(\forall X.(FX \doteq^\beta GX))) = \mathtt{T}$. On the other hand, since $\mathsf{f} \neq \mathsf{g}$ and $\mathcal{M}$ satisfies property q, we have $\upsilon(\mathcal{E}_\varphi(F \doteq^{\alpha\to\beta} G)) = \mathtt{F}$ by contraposition of Lemma 110(2). This implies $\mathcal{M} \not\models \mathrm{EXT}_{\doteq}^{\alpha\to\beta}$.

Suppose $\mathcal{M}$ satisfies property q but does not satisfy property $\mathfrak{b}$. Then, there must be at least three elements in $\mathcal{D}_o$. Since υ maps into a two element set, there must be two distinct elements $\mathsf{a}, \mathsf{b} \in \mathcal{D}_o$ such that $\upsilon(\mathsf{a}) = \upsilon(\mathsf{b})$. Let $A_o, B_o \in \mathcal{V}_o$ be distinct variables and φ be any assignment into $\mathcal{M}$ with $\varphi(A) = \mathsf{a}$ and $\varphi(B) = \mathsf{b}$. By Lemma 108, we know $\upsilon(\mathcal{E}_\varphi(A \Leftrightarrow B)) = \mathtt{T}$. Since $\mathsf{a} \neq \mathsf{b}$ and property q holds, by contraposition of Lemma 110(2), we know $\upsilon(\mathcal{E}_\varphi(A \doteq^o B)) = \mathtt{F}$. It follows that $\mathcal{M} \not\models \mathrm{EXT}_{\doteq}^o$.

Let φ be any assignment into $\mathcal{M}$. From $\upsilon(\mathcal{E}_\varphi(\forall X_\alpha.FX \doteq GX)) = \mathtt{T}$ we know $\upsilon(\mathcal{E}_{\varphi,[\mathsf{a}/X]}(FX \doteq GX)) = \mathtt{T}$ holds for all $\mathsf{a} \in \mathcal{D}_\alpha$. By Lemma 110(2) we can conclude that $\mathcal{E}_{\varphi,[\mathsf{a}/X]}(FX) = \mathcal{E}_{\varphi,[\mathsf{a}/X]}(GX)$ for all $\mathsf{a} \in \mathcal{D}_\alpha$ and hence $\mathcal{E}_{\varphi,[\mathsf{a}/X]}(F)@\mathcal{E}_{\varphi,[\mathsf{a}/X]}(X) = \mathcal{E}_{\varphi,[\mathsf{a}/X]}(G)@\mathcal{E}_{\varphi,[\mathsf{a}/X]}(X)$ for all $\mathsf{a} \in \mathcal{D}_\alpha$. That is, $\mathcal{E}_{\varphi,[\mathsf{a}/X]}(F)@\mathsf{a} = \mathcal{E}_{\varphi,[\mathsf{a}/X]}(G)@\mathsf{a}$ for all $\mathsf{a} \in \mathcal{D}_\alpha$. Since X does not occur free in F or G, by property $\mathfrak{f}$ and Definition 77(3) we obtain $\mathcal{E}_\varphi(F) = \mathcal{E}_\varphi(G)$. This finally gives us that $\upsilon(\mathcal{E}_\varphi(F \doteq^{\alpha\to\beta} G)) = \mathtt{T}$ with Lemma 110(1). It follows that $\mathcal{M} \models \mathrm{EXT}_{\doteq}^{\alpha\to\beta}$ and $\mathcal{M} \models \mathrm{EXT}_{\doteq}^{\to}$, since α and β were chosen arbitrarily. Note that we certainly need the assumption that $\mathcal{M}$ satisfies property q (which is employed within the application of Lemma 110(2). (There is a functional model in which property q fails and $\mathrm{EXT}_{\doteq}^{\iota\to\iota}$ is not valid.)

Let $A_o, B_o \in \mathcal{V}_o$ be distinct variables and φ be any assignment into $\mathcal{M}$. Since property $\mathfrak{b}$ holds, we can assume $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$ and υ is the identity function. Suppose $\upsilon(\mathcal{E}_\varphi(A \Leftrightarrow B)) = \mathtt{T}$. By Lemma 108, we have $\mathcal{E}_\varphi(A) = \upsilon(\mathcal{E}_\varphi(A)) = \upsilon(\mathcal{E}_\varphi(B)) = \mathcal{E}_\varphi(B)$. By Lemma 110(1), we have $\upsilon(\mathcal{E}_\varphi(A \doteq^o B)) = \mathtt{T}$. It follows that $\mathcal{M} \models \mathrm{EXT}_{\doteq}^o$.

15 Model Existence

We now present model existence theorems for the different semantical notions introduced in Section 11. The model existence theorems have the following form, where $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$:

> **Theorem (Model Existence)**: *For a given abstract consistency class $\mathscr{C} \in \mathfrak{Acc}_*$ (cf. Definition 116) and a set $\Phi \in \mathscr{C}$ there is a $\mathcal{S}$-model $\mathcal{M}$ of Φ, such that $\mathcal{M} \in \mathfrak{M}_*$ (cf. Definition 81).*

The most important tools used in the proofs of the model existence theorems are the so-called $\mathcal{S}$-Hintikka sets. These sets allow computations that resemble those in the considered semantical structures (e.g., Henkin models) and allow us to construct appropriate valuations for the term evaluation $\mathcal{TS}(\mathcal{S})^\beta$ defined in Definition 93. The key step in the proof of the model existence theorems is an extension lemma, which guarantees a $\mathcal{S}$-Hintikka set $\mathcal{H}$ for any sufficiently pure set of sentences Φ in $\mathscr{C}$.

15.1 Abstract Consistency

Let us now review a few technicalities that we will need for the proofs of the model existence theorems.

Definition 112 (Compactness) *Let $\mathcal{C}$ be a class of sets.*

1. *$\mathcal{C}$ is called* closed under subsets *if for any sets S and T, $S \in \mathcal{C}$ whenever $S \subseteq T$ and $T \in \mathcal{C}$.*
2. *$\mathcal{C}$ is called* compact *if for every set S we have $S \in \mathcal{C}$ iff every finite subset of S is a member of $\mathcal{C}$.*

Lemma 113 *If $\mathcal{C}$ is compact, then $\mathcal{C}$ is closed under subsets.*

Proof. Suppose $S \subseteq T$ and $T \in \mathcal{C}$. Every finite subset A of S is a finite subset of T, and since $\mathcal{C}$ is compact we know that $A \in \mathcal{C}$. Thus $S \in \mathcal{C}$.

We will now introduce a technical side-condition that ensures that we always have enough witness constants.

Definition 114 (Sufficiently Pure) *Let Φ be a set of $\mathcal{S}$-sentences. Φ is called* sufficiently pure *if for each type α there is a set $\mathcal{Q}_\alpha \subseteq \mathcal{P}_\alpha$ of parameters with equal cardinality to $\mathit{wff}_\alpha(\mathcal{S})$, such that the elements of $\mathcal{Q}_\alpha$ do not occur in the sentences of Φ.*

This can be obtained in practice by enriching the signature with spurious parameters.

Notation: For reasons of legibility we will write $S * a$ for $S \cup \{a\}$, where S is a set. We will use this notation with the convention that $*$ associates to the left.

Definition 115 (Properties for Abstract Consistency Classes) *Let $\mathscr{C}$ be a class of sets of $\mathcal{S}$-sentences. We define the following properties of $\mathscr{C}$, where $\Phi \in \mathscr{C}$, $\alpha, \beta \in \mathcal{T}$, $\mathbf{A}, \mathbf{B} \in \mathit{cwff}_o(\mathcal{S})$, $\mathbf{F} \in \mathit{cwff}_{\alpha\to o}(\mathcal{S})$, and $\mathbf{G}, \mathbf{H}, (\lambda X_\alpha \centerdot \mathbf{M}), (\lambda X_\alpha \centerdot \mathbf{N}) \in \mathit{cwff}_{\alpha\to\beta}(\mathcal{S})$ are arbitrary.*

∇_c *If $\mathbf{A}$ is atomic, then $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$.*
$\nabla_\neg$ *If $\neg\neg\mathbf{A} \in \Phi$, then $\Phi * \mathbf{A} \in \mathscr{C}$.*
∇_β *If $\mathbf{A} =_\beta \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{B} \in \mathscr{C}$.*
$\nabla_{\beta\eta}$ *If $\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{B} \in \mathscr{C}$.*
$\nabla_\vee$ *If $\mathbf{A} \vee \mathbf{B} \in \Phi$, then $\Phi * \mathbf{A} \in \mathscr{C}$ or $\Phi * \mathbf{B} \in \mathscr{C}$.*
$\nabla_\wedge$ *If $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, then $\Phi * \neg\mathbf{A} * \neg\mathbf{B} \in \mathscr{C}$.*
$\nabla_\forall$ *If $\Pi^\alpha\mathbf{F} \in \Phi$, then $\Phi * \mathbf{FW} \in \mathscr{C}$ for each $\mathbf{W} \in \mathit{cwff}_\alpha(\mathcal{S})$.*
$\nabla_\exists$ *If $\neg\Pi^\alpha\mathbf{F} \in \Phi$, then $\Phi * \neg(\mathbf{F}w) \in \mathscr{C}$ for any parameter $w_\alpha \in \mathcal{S}_\alpha$ which does not occur in any sentence of Φ.*
$\nabla_\mathfrak{b}$ *If $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$, then $\Phi * \mathbf{A} * \neg\mathbf{B} \in \mathscr{C}$ or $\Phi * \neg\mathbf{A} * \mathbf{B} \in \mathscr{C}$.*
∇_ξ *If $\neg(\lambda X_\alpha \centerdot \mathbf{M} \doteq^{\alpha\to\beta} \lambda X_\alpha \centerdot \mathbf{N}) \in \Phi$, then $\Phi * \neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N}) \in \mathscr{C}$ for any parameter $w_\alpha \in \mathcal{S}_\alpha$ which does not occur in any sentence of Φ.*
$\nabla_\mathfrak{f}$ *If $\neg(\mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H}) \in \Phi$, then $\Phi * \neg(\mathbf{G}w \doteq^\beta \mathbf{H}w) \in \mathscr{C}$ for any parameter $w_\alpha \in \mathcal{S}_\alpha$ which does not occur in any sentence of Φ.*
∇_{sat} *Either $\Phi * \mathbf{A} \in \mathscr{C}$ or $\Phi * \neg\mathbf{A} \in \mathscr{C}$.*

Definition 116 (Abstract Consistency Classes) *Let $\mathcal{S}$ be a signature and $\mathscr{C}$ be a class of sets of $\mathcal{S}$-sentences that is closed under subsets. If $\nabla_c, \nabla_\neg, \nabla_\beta, \nabla_\vee, \nabla_\wedge, \nabla_\forall$ and $\nabla_\exists$ are valid for $\mathscr{C}$, then $\mathscr{C}$ is called an* abstract consistency class *for $\mathcal{S}$-models.*

We will denote the collection of abstract consistency classes by $\mathfrak{Acc}_\beta$. Similarly, we introduce the following collections of specialized abstract consistency classes: $\mathfrak{Acc}_{\beta\eta}, \mathfrak{Acc}_{\beta\xi}, \mathfrak{Acc}_{\beta\mathfrak{f}}, \mathfrak{Acc}_{\beta\mathfrak{b}}, \mathfrak{Acc}_{\beta\eta\mathfrak{b}}, \mathfrak{Acc}_{\beta\xi\mathfrak{b}}, \mathfrak{Acc}_{\mathfrak{f}\mathfrak{b}}$, where we indicate by indices which additional properties from $\{\nabla_{\beta\eta}, \nabla_\xi, \nabla_\mathfrak{f}, \nabla_\mathfrak{b}\}$ are required.

Remark 117 *$\mathfrak{Acc}_\beta$ corresponds to the abstract consistency property discussed by Andrews in [And71]. The only (technical) differences correspond to $\alpha\beta$-conversion. In [And71], α-conversion is handled in the ∇_β rule using α-standardized forms. Also, we have defined the ∇_β rule to work with β-conversion instead of β-reduction. We prefer this stronger version of ∇_β over the weaker option "If $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{A}{\downarrow_\beta} \in \mathscr{C}$" since it helps to avoid the use of ∇_{sat} in several proofs below. (Note that ∇_β follows from the weaker option and ∇_{sat}.) Furthermore, in practical applications, e.g., proving completeness of calculi, the stronger property is typically as easy to validate as the weaker one. An analogous argument applies to $\nabla_{\beta\eta}$.*

Remark 118 *While the work presented in this article is based on the choice of the primitive logical connectives* $\neg, \vee$, *and* Π^α, *a means to generalize the framework over the concrete choice of logical primitives is provided by the uniform notation approach as, for instance, given in [Fit96]. It is clearly possible to achieve such a generalization for our framework as well. This can be done in straightforward manner:* $\nabla_\wedge$ *becomes an* α*-property,* $\nabla_\vee$ *becomes a* β*-property,* $\nabla_\forall$ *becomes a* γ*-property, and* $\nabla_\exists$ *becomes a* δ*-property. Thus they will have the following form:*

α**-case** *If* $\alpha \in \Phi$, *then* $\Phi * \alpha_1 * \alpha_2 \in \mathscr{C}$.
β**-case** *If* $\beta \in \Phi$, *then* $\Phi * \beta_1 \in \mathscr{C}$ *or* $\Phi * \beta_2 \in \mathscr{C}$.
γ**-case** *If* $\gamma \in \Phi$, *then* $\Phi * \gamma\mathbf{W} \in \mathscr{C}$ *for each* $\mathbf{W} \in cwff_\alpha(\mathcal{S})$.
δ**-case** *If* $\delta \in \Phi$, *then* $\Phi * \delta w \in \mathscr{C}$ *for any parameter* $w_\alpha \in \mathcal{S}$ *which does not occur in any sentence of* Φ.

We may also want to add cases for primitive equality. For more details on this see [BBK04; Ben99].

We often refer to property ∇_c as "atomic consistency". The next lemma shows that we also have the corresponding property for non-atoms.

Lemma 119 (Non-atomic Consistency) *Let* $\mathscr{C}$ *be an abstract consistency class and* $\mathbf{A} \in cwff_o$, *then for all* $\Phi \in \mathscr{C}$ *we have* $\mathbf{A} \notin \Phi$ *or* $\neg\mathbf{A} \notin \Phi$.

Proof. (following a similar argument in [And71], Lemma 3.3.3)
If for some $\Phi \in \mathscr{C}$ and $\mathbf{A} \in cwff_o$ we have $\mathbf{A} \in \Phi$ and $\neg\mathbf{A} \in \Phi$, then $\{\mathbf{A}, \neg\mathbf{A}\} \in \mathscr{C}$ since $\mathscr{C}$ is closed under subsets. Furthermore, using ∇_β and closure under subsets we can assume such an $\mathbf{A}$ is β-normal. We prove $\{\mathbf{A}, \neg\mathbf{A}\} \notin \mathscr{C}$ for any β-normal $\mathbf{A} \in cwff_o$ by induction on the number of logical constants in $\mathbf{A}$.

If $\mathbf{A}$ is atomic, this follows immediately from ∇_c. Suppose $\mathbf{A} = \neg\mathbf{B}$ for some $\mathbf{B} \in cwff_o$ and $\{\neg\mathbf{B}, \neg\neg\mathbf{B}\} \in \mathscr{C}$. By $\nabla_\neg$ and closure under subsets, we have $\{\neg\mathbf{B}, \mathbf{B}\} \in \mathscr{C}$, contradicting the induction hypothesis for $\mathbf{B}$. Suppose $\mathbf{A} = \mathbf{B} \vee \mathbf{C}$ for some $\mathbf{B}, \mathbf{C} \in cwff_o$ and $\{\mathbf{B} \vee \mathbf{C}, \neg(\mathbf{B} \vee \mathbf{C})\} \in \mathscr{C}$. By $\nabla_\vee$, $\nabla_\wedge$ and closure under subsets, we have either $\{\mathbf{B}, \neg\mathbf{B}\} \in \mathscr{C}$ or $\{\mathbf{C}, \neg\mathbf{C}\} \in \mathscr{C}$, contradicting the induction hypotheses for $\mathbf{B}$ and $\mathbf{C}$. Suppose $\mathbf{A} = \Pi^\alpha\mathbf{B}$ for some $\mathbf{B} \in cwff_{\alpha\to o}(\mathcal{S})$ and $\{\Pi^\alpha\mathbf{B}, \neg(\Pi^\alpha\mathbf{B})\} \in \mathscr{C}$. Since $\mathcal{S}_\alpha$ is assumed to be infinite (we assume that for each type α we have infinitely many parameters), there is a parameter $w_\alpha \in \mathcal{S}_\alpha$ which does not occur in $\mathbf{A}$. Since w is a parameter, the sentence $\mathbf{B}w$ clearly has one less logical constant than $\Pi^\alpha\mathbf{B}$. However, we cannot directly apply the induction hypothesis as $\mathbf{B}w$ may not be β-normal. Since $\mathbf{B}$ is β-normal, the only way $\mathbf{B}w$ can fail to be β-normal is if $\mathbf{B}$ has the form $\lambda X_\alpha \mathbf{.} \mathbf{C}$ for some $\mathbf{C} \in wff_o(\mathcal{S})$ where $\mathbf{Free}(\mathbf{C}) \subseteq \{X_\alpha\}$. In this case, it is easy to show that the reduct $[w/X]\mathbf{C}$ is β-normal and contains the same number of logical constants as $\mathbf{B}$. In either case, we can let $\mathbf{N}$ be the β-normal form of $\mathbf{B}w$ and apply the induction hypothesis to obtain $\{\mathbf{N}, \neg\mathbf{N}\} \notin \mathscr{C}$. On the other hand, $\nabla_\exists$, $\nabla_\forall$, ∇_β and closure under subsets implies $\{\mathbf{N}, \neg\mathbf{N}\} \in \mathscr{C}$, a contradiction.

Remark 120 *Note that for the connectives* $\vee$ *and* Π^α *there is a positive and a negative condition given in the definition above, namely* $\nabla_\vee/\nabla_\wedge$ *for* $\vee$ *and* $\nabla_\forall/\nabla_\exists$ *for* Π^α. *For* $\doteq^o$ *and* $\doteq^{\alpha\to\beta}$ *the situation is different since we need only conditions for the negative cases. Positive counterparts can be inferred by expanding the Leibniz definition of equality (cf. Lemma 121).*

Lemma 121 (Leibniz Equality) *Let* $\mathscr{C}$ *be an abstract consistency class. The following properties are valid for all* $\Phi \in \mathscr{C}$, $\mathbf{A}, \mathbf{B} \in cwff_o(\mathcal{S})$, $\mathbf{C} \in cwff_\alpha(\mathcal{S})$ *and* $\mathbf{F}, \mathbf{G} \in cwff_{\alpha\to\beta}(\mathcal{S})$.

$\nabla^r_{\doteq}$ $\neg(\mathbf{C} \doteq^\alpha \mathbf{C}) \notin \Phi$.
$\nabla^{\to}_{\doteq}$ *If* $\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G} \in \Phi$, *then* $\Phi * \mathbf{FW} \doteq^\beta \mathbf{GW} \in \mathscr{C}$ *for any closed* $\mathbf{W} \in cwff_\alpha(\mathcal{S})$.
$\nabla^o_{\doteq}$ *If* $\mathbf{A} \doteq^o \mathbf{B} \in \Phi$, *then* $\Phi * \mathbf{A} * \mathbf{B} \in \mathscr{C}$ *or* $\Phi * \neg\mathbf{A} * \neg\mathbf{B} \in \mathscr{C}$.

Proof. To show $\nabla^r_{\doteq}$, assume $\neg(\mathbf{C} \doteq \mathbf{C}) \in \Phi$. By subset closure $\{\neg(\mathbf{C} \doteq \mathbf{C})\} \in \mathscr{C}$ and by $\nabla_\exists$ with some parameter p which does not occur in $\mathbf{C}$ and ∇_β we get $\{\neg(\mathbf{C} \doteq \mathbf{C}), \neg(\neg p\mathbf{C} \vee p\mathbf{C})\} \in \mathscr{C}$. The contradiction follows by $\nabla_\wedge$, $\nabla_\neg$ and ∇_c. So, $\nabla^r_{\doteq}$ holds.

To show $\nabla^{\to}_{\doteq}$, suppose $\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G} \in \Phi$. By application of $\nabla_\forall$ with $\lambda X_{\alpha\to\beta} \mathbf{.} \mathbf{FW} \doteq X\mathbf{W}$ and ∇_β we have $\Phi * (\neg(\mathbf{FW} \doteq \mathbf{FW}) \vee \mathbf{FW} \doteq \mathbf{GW}) \in \mathscr{C}$. By $\nabla_\vee$ and subset closure we get $\Phi * \neg(\mathbf{FW} \doteq \mathbf{FW}) \in \mathscr{C}$ or $\Phi * \mathbf{FW} \doteq \mathbf{GW} \in \mathscr{C}$. The latter proves the assertion since the first option is ruled out by $\nabla^r_{\doteq}$ (shown above).

To show $\nabla^o_{\doteq}$, suppose $\mathbf{A} \doteq^o \mathbf{B} \in \Phi$. Applying $\nabla_\forall$ with $\lambda Y \mathbf{.} Y$ we have $\Phi * (\lambda P_{o\to o} \mathbf{.} \neg P\mathbf{A} \vee P\mathbf{B})(\lambda Y \mathbf{.} Y) \in \mathscr{C}$. By ∇_β and subset closure we get $\Phi * \neg\mathbf{A} \vee \mathbf{B} \in \mathscr{C}$. Similarly, we further derive by $\nabla_\forall$ with $\lambda Y \mathbf{.} \neg Y$, ∇_β, and subset closure that $\Phi * \neg\mathbf{A} \vee \mathbf{B} * \neg\neg\mathbf{A} \vee \neg\mathbf{B} \in \mathscr{C}$. By applying $\nabla_\vee$ twice and subset closure we get the following four options: (i) $\Phi * \neg\mathbf{A} * \neg\neg\mathbf{A} \in \mathscr{C}$, (ii) $\Phi * \neg\mathbf{A} * \neg\mathbf{B} \in \mathscr{C}$, (iii) $\Phi * \mathbf{B} * \neg\neg\mathbf{A} \in \mathscr{C}$, or (iv) $\Phi * \mathbf{B} * \neg\mathbf{B} \in \mathscr{C}$. Cases (i) and (iv) are ruled out by non-atomic consistency. In case (iii) we furthermore get by $\nabla_\neg$ and subset closure that $\Phi * \mathbf{B} * \mathbf{A} \in \mathscr{C}$. Thus, $\Phi * \neg\mathbf{A} * \neg\mathbf{B} \in \mathscr{C}$ or $\Phi * \mathbf{B} * \mathbf{A} \in \mathscr{C}$.

We could easily add respective properties for symmetry, transitivity, and congruence to the previous lemma. They can be shown analogously, i.e., they also follow from the properties of Leibniz equality.

In contrast to [And71], we work with saturated abstract consistency classes in order to simplify the proofs of the model existence theorems.

Definition 122 (Saturatedness) *We call an abstract consistency class* $\mathscr{C}$ *saturated if it satisfies* ∇_{sat}.

Remark 123 *Clearly, not all abstract consistency classes are saturated, since the empty set is one that is not (*$cwff_o$ *is certainly non-empty since* $\forall P_o \mathbf{.} P \in cwff_o$*).*

Remark 124 *The saturation condition* ∇_{sat} *can be very difficult to verify in practice. For example, showing that an abstract consistency class induced from a sequent calculus (as in [And71]) is saturated corresponds to showing cut-elimination (cf. [BBK02]). Since Andrews [And71] did not use saturation, he could use his results to give a model-theoretic proof of cut-elimination for a sequent calculus. We cannot use the results of this article to obtain similar cut-elimination results.*

Lemma 125 (Compactness of abstract consistency classes) *For each abstract consistency class* $\mathscr{C}$ *there exists a compact abstract consistency class* $\mathscr{C}'$ *satisfying the same* ∇_* *properties such that* $\mathscr{C} \subseteq \mathscr{C}'$.

Proof. (following and extending [And02b], Proposition 2506)
We choose $\mathscr{C}' := \{\Phi \subseteq cwff_o(\mathcal{S}) \mid \text{every finite subset of } \Phi \text{ is in } \mathscr{C}\}$. Now suppose that $\Phi \in \mathscr{C}$. $\mathscr{C}$ is closed under subsets, so every finite subset of Φ is in $\mathscr{C}$ and thus $\Phi \in \mathscr{C}'$. Hence $\mathscr{C} \subseteq \mathscr{C}'$.

Next let us show that $\mathscr{C}'$ is compact. Suppose $\Phi \in \mathscr{C}'$ and Ψ is an arbitrary finite subset of Φ. By definition of $\mathscr{C}'$ all finite subsets of Φ are in $\mathscr{C}$ and therefore $\Psi \in \mathscr{C}'$. Thus all finite subsets of Φ are in $\mathscr{C}'$ whenever Φ is in $\mathscr{C}'$. On the other hand, suppose all finite subsets of Φ are in $\mathscr{C}'$. Then by the definition of $\mathscr{C}'$ the finite subsets of Φ are also in $\mathscr{C}$, so $\Phi \in \mathscr{C}'$. Thus $\mathscr{C}'$ is compact. Note that by Lemma 113 we have that $\mathscr{C}'$ is closed under subsets.

Next we show that if $\mathscr{C}$ satisfies ∇_*, then $\mathscr{C}'$ satisfies ∇_*.

∇_c Let $\Phi \in \mathscr{C}'$ and suppose there is an atom $\mathbf{A}$, such that $\{\mathbf{A}, \neg\mathbf{A}\} \subseteq \Phi$. $\{\mathbf{A}, \neg\mathbf{A}\}$ is clearly a finite subset of Φ and hence $\{\mathbf{A}, \neg\mathbf{A}\} \in \mathscr{C}$ contradicting ∇_c for $\mathscr{C}$.
$\nabla_\neg$ Let $\Phi \in \mathscr{C}'$, $\neg\neg\mathbf{A} \in \Phi$, Ψ be any finite subset of $\Phi * \mathbf{A}$, and $\Theta := (\Psi \setminus \{\mathbf{A}\}) * \neg\neg\mathbf{A}$. Θ is a finite subset of Φ, so $\Theta \in \mathscr{C}$. Since $\mathscr{C}$ is an abstract consistency class and $\neg\neg\mathbf{A} \in \Theta$, we get $\Theta * \mathbf{A} \in \mathscr{C}$ by $\nabla_\neg$ for $\mathscr{C}$. We know that $\Psi \subseteq \Theta * \mathbf{A}$ and $\mathscr{C}$ is closed under subsets, so $\Psi \in \mathscr{C}$. Thus every finite subset Ψ of $\Phi * \mathbf{A}$ is in $\mathscr{C}$ and therefore by definition $\Phi * \mathbf{A} \in \mathscr{C}'$.
$\nabla_\beta, \nabla_{\beta\eta}, \nabla_\vee, \nabla_\wedge, \nabla_\forall, \nabla_\exists$ Analogous to $\nabla_\neg$.

∇_ξ Let $\Phi \in \mathcal{C}'$, $\neg(\lambda X_\alpha{\scriptscriptstyle\blacksquare}\mathbf{M} \doteq^{\alpha\to\beta} \lambda X{\scriptscriptstyle\blacksquare}\mathbf{N}) \in \Phi$ and Ψ be any finite subset of $\Phi * \neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N})$, where $w \in \mathcal{S}_\alpha$ is a parameter that does not occur in any sentence of Φ. We show that $\Psi \in \mathcal{C}$. Clearly $\Theta := (\Psi \setminus \{\neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N})\}) * \neg(\lambda X{\scriptscriptstyle\blacksquare}\mathbf{M} \doteq^{\alpha\to\beta} \lambda X{\scriptscriptstyle\blacksquare}\mathbf{N})$ is a finite subset of Φ and therefore $\Theta \in \mathcal{C}$. Since $\mathcal{C}$ satisfies ∇_ξ and $\neg(\lambda X{\scriptscriptstyle\blacksquare}\mathbf{M} \doteq^{\alpha\to\beta} \lambda X{\scriptscriptstyle\blacksquare}\mathbf{N}) \in \Theta$, we have $\Theta * \neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N}) \in \mathcal{C}$. Furthermore, $\Psi \subseteq \Theta * \neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N})$ and $\mathcal{C}$ is closed under subsets, so $\Psi \in \mathcal{C}$. Thus every finite subset Ψ of $\Phi * \neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N})$ is in $\mathcal{C}$, and therefore by definition we have $\Phi * \neg([w/X]\mathbf{M} \doteq^\alpha [w/X]\mathbf{N}) \in \mathcal{C}'$.

$\nabla_\mathfrak{f}$ Analogous to ∇_ξ.

$\nabla_\mathfrak{b}$ Let $\Phi \in \mathcal{C}'$ with $\neg(\mathbf{A} \doteq \mathbf{B}) \in \Phi$. Assume $\Phi * \mathbf{A} * \neg\mathbf{B} \notin \mathcal{C}$ and $\Phi * \neg\mathbf{A} * \mathbf{B} \notin \mathcal{C}$. Then there exists finite subsets Φ_1 and Φ_2 of Φ, such that $\Phi_1 * \mathbf{A} * \neg\mathbf{B} \notin \mathcal{C}$ and $\Phi_2 * \neg\mathbf{A} * \mathbf{B} \notin \mathcal{C}$. Now we choose $\Phi_3 := \Phi_1 \cup \Phi_2 * \neg(\mathbf{A} \doteq \mathbf{B})$. Obviously Φ_3 is a finite subset of Φ and therefore $\Phi_3 \in \mathcal{C}$. Since $\mathcal{C}$ satisfies $\nabla_\mathfrak{b}$, we have that $\Phi_3 * \mathbf{A} * \neg\mathbf{B} \in \mathcal{C}$ or $\Phi_3 * \neg\mathbf{A} * \mathbf{B} \in \mathcal{C}$. From this and the fact that $\mathcal{C}$ is closed under subsets we get that $\Phi_1 * \mathbf{A} * \neg\mathbf{B} \in \mathcal{C}$ or $\Phi_2 * \neg\mathbf{A} * \mathbf{B} \in \mathcal{C}$, which contradicts our assumption.

∇_{sat} Let $\Phi \in \mathcal{C}'$. Assume neither $\Phi * \mathbf{A}$ nor $\Phi * \neg\mathbf{A}$ is in $\mathcal{C}'$. Then there are finite subsets Φ_1 and Φ_2 of Φ, such that $\Phi_1 * \mathbf{A} \notin \mathcal{C}$ and $\Phi_2 * \neg\mathbf{A} \notin \mathcal{C}$. As $\Psi := \Phi_1 \cup \Phi_2$ is a finite subset of Φ, we have $\Psi \in \mathcal{C}$. Furthermore, $\Psi * \mathbf{A} \in \mathcal{C}$ or $\Psi * \neg\mathbf{A} \in \mathcal{C}$ because $\mathcal{C}$ is saturated. $\mathcal{C}$ is closed under subsets, so $\Phi_1 * \mathbf{A} \in \mathcal{C}$ or $\Phi_2 * \neg\mathbf{A} \in \mathcal{C}$. This is a contradiction, so we can conclude that if $\Phi \in \mathcal{C}'$, then $\Phi * \mathbf{A} \in \mathcal{C}'$ or $\Phi * \neg\mathbf{A} \in \mathcal{C}'$.

15.2 Hintikka Sets

Hintikka sets connect syntax with semantics as they provide the basis for the model constructions in the model existence theorems. We have defined eight different notions of abstract consistency classes by first defining properties ∇_*, then specifying which should hold in $\mathfrak{Acc}_*$. Similarly, we define Hintikka sets by first defining the desired properties.

Definition 126 ($\mathcal{S}$-Hintikka Properties) *Let $\mathcal{H}$ be a set of sentences. We define the following properties which $\mathcal{H}$ may satisfy, where $\mathbf{A}, \mathbf{B} \in cwff_o(\mathcal{S})$, $\mathbf{C}, \mathbf{D} \in cwff_\alpha(\mathcal{S})$, $\mathbf{F} \in cwff_{\alpha\to o}(\mathcal{S})$, and $(\lambda X_\alpha{\scriptscriptstyle\blacksquare}\mathbf{M}), (\lambda X{\scriptscriptstyle\blacksquare}\mathbf{N}), \mathbf{G}, \mathbf{H} \in cwff_{\alpha\to\beta}(\mathcal{S})$:*

$\boldsymbol{\nabla}_c$ *$\mathbf{A} \notin \mathcal{H}$ or $\neg\mathbf{A} \notin \mathcal{H}$.*
$\boldsymbol{\nabla}_\neg$ *If $\neg\neg\mathbf{A} \in \mathcal{H}$, then $\mathbf{A} \in \mathcal{H}$.*
$\boldsymbol{\nabla}_\beta$ *If $\mathbf{A} \in \mathcal{H}$ and $\mathbf{A} =_\beta \mathbf{B}$, then $\mathbf{B} \in \mathcal{H}$.*
$\boldsymbol{\nabla}_{\beta\eta}$ *If $\mathbf{A} \in \mathcal{H}$ and $\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B}$, then $\mathbf{B} \in \mathcal{H}$.*
$\boldsymbol{\nabla}_\vee$ *If $\mathbf{A} \vee \mathbf{B} \in \mathcal{H}$, then $\mathbf{A} \in \mathcal{H}$ or $\mathbf{B} \in \mathcal{H}$.*
$\boldsymbol{\nabla}_\wedge$ *If $\neg(\mathbf{A} \vee \mathbf{B}) \in \mathcal{H}$, then $\neg\mathbf{A} \in \mathcal{H}$ and $\neg\mathbf{B} \in \mathcal{H}$.*
$\boldsymbol{\nabla}_\forall$ *If $\Pi^\alpha\mathbf{F} \in \mathcal{H}$, then $\mathbf{FW} \in \mathcal{H}$ for each $\mathbf{W} \in cwff_\alpha(\mathcal{S})$.*
$\boldsymbol{\nabla}_\exists$ *If $\neg\Pi^\alpha\mathbf{F} \in \mathcal{H}$, then there is a parameter $w_\alpha \in \mathcal{S}_\alpha$ such that $\neg(\mathbf{F}w) \in \mathcal{H}$.*
$\boldsymbol{\nabla}_\mathfrak{b}$ *If $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \mathcal{H}$, then $\{\mathbf{A}, \neg\mathbf{B}\} \subseteq \mathcal{H}$ or $\{\neg\mathbf{A}, \mathbf{B}\} \subseteq \mathcal{H}$.*
$\boldsymbol{\nabla}_\xi$ *If $\neg(\lambda X_\alpha{\scriptscriptstyle\blacksquare}\mathbf{M} \doteq^{\alpha\to\beta} \lambda X{\scriptscriptstyle\blacksquare}\mathbf{N}) \in \mathcal{H}$, then there is a parameter $w_\alpha \in \mathcal{S}_\alpha$ such that $\neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N}) \in \mathcal{H}$.*
$\boldsymbol{\nabla}_\mathfrak{f}$ *If $\neg(\mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H}) \in \mathcal{H}$, then there is a parameter $w_\alpha \in \mathcal{S}_\alpha$ such that $\neg(\mathbf{G}w \doteq^\beta \mathbf{H}w) \in \mathcal{H}$.*
$\boldsymbol{\nabla}_{sat}$ *Either $\mathbf{A} \in \mathcal{H}$ or $\neg\mathbf{A} \in \mathcal{H}$.*

Definition 127 ($\mathcal{S}$-Hintikka Set) *A set $\mathcal{H}$ of sentences is called a $\mathcal{S}$-*Hintikka set *if it satisfies $\boldsymbol{\nabla}_c$, $\boldsymbol{\nabla}_\neg$, $\boldsymbol{\nabla}_\beta$, $\boldsymbol{\nabla}_\vee$, $\boldsymbol{\nabla}_\wedge$, $\boldsymbol{\nabla}_\forall$ and $\boldsymbol{\nabla}_\exists$.*

We define the following collections of Hintikka sets: $\mathfrak{Hint}_\beta$, $\mathfrak{Hint}_{\beta\eta}$, $\mathfrak{Hint}_{\beta\xi}$, $\mathfrak{Hint}_{\beta\mathfrak{f}}$, $\mathfrak{Hint}_{\beta\mathfrak{b}}$, $\mathfrak{Hint}_{\beta\eta\mathfrak{b}}$, $\mathfrak{Hint}_{\beta\xi\mathfrak{b}}$, and $\mathfrak{Hint}_{\beta\mathfrak{f}\mathfrak{b}}$, where we indicate by indices which additional properties from $\{\boldsymbol{\nabla}_{\beta\eta}, \boldsymbol{\nabla}_\xi, \boldsymbol{\nabla}_\mathfrak{f}, \boldsymbol{\nabla}_\mathfrak{b}\}$ are required.

We will construct Hintikka sets as maximal elements of abstract consistency classes. To obtain a Hintikka set, we must explicitly show the property $\boldsymbol{\nabla}_\exists$ (and $\boldsymbol{\nabla}_\xi$ or $\boldsymbol{\nabla}_\mathfrak{f}$ when appropriate). This will ensure that Hintikka sets have enough parameters which act as witnesses.

Lemma 128 (Hintikka Lemma) *Let $\mathcal{C}$ be an abstract consistency class in $\mathfrak{Acc}_*$. Suppose a set $\mathcal{H} \in \mathcal{C}$ satisfies the following properties:*

1. *$\mathcal{H}$ is subset-maximal in $\mathcal{C}$ (i.e., for each sentence $\mathbf{D} \in cwff_o(\mathcal{S})$ such that $\mathcal{H} * \mathbf{D} \in \mathcal{C}$, we already have $\mathbf{D} \in \mathcal{H}$).*
2. *$\mathcal{H}$ satisfies $\boldsymbol{\nabla}_\exists$.*
3. *If $* \in \{\beta\xi, \beta\xi\mathfrak{b}\}$, then $\boldsymbol{\nabla}_\xi$ holds in $\mathcal{H}$.*
4. *If $* \in \{\beta\mathfrak{f}, \beta\mathfrak{f}\mathfrak{b}\}$, then $\boldsymbol{\nabla}_\mathfrak{f}$ holds in $\mathcal{H}$.*

Then, $\mathcal{H} \in \mathfrak{Hint}_*$. Furthermore, if $\mathcal{C}$ is saturated, then $\mathcal{H}$ satisfies $\boldsymbol{\nabla}_{sat}$.

Proof. $\mathcal{H}$ satisfies $\boldsymbol{\nabla}_\exists$ by assumption. Also, if $* \in \{\beta\xi, \beta\xi\mathfrak{b}\}$ ($* \in \{\beta\mathfrak{f}, \beta\mathfrak{f}\mathfrak{b}\}$), then we have explicitly assumed $\mathcal{H}$ satisfies $\boldsymbol{\nabla}_\xi$ ($\boldsymbol{\nabla}_\mathfrak{f}$). The fact that $\mathcal{H} \in \mathcal{C}$ satisfies $\boldsymbol{\nabla}_c$ follows directly from non-atomic consistency (Lemma 119).

Every other $\boldsymbol{\nabla}_*$ property follows directly from the corresponding ∇_* property and maximality of $\mathcal{H}$ in $\mathcal{C}$. For example, to show $\boldsymbol{\nabla}_\neg$, suppose $\neg\neg\mathbf{A} \in \mathcal{H}$. By $\nabla_\neg$, we know $\mathcal{H} * \mathbf{A} \in \mathcal{C}$. By maximality of $\mathcal{H}$, we have $\mathbf{A} \in \mathcal{H}$. Checking $\boldsymbol{\nabla}_\beta$, $\boldsymbol{\nabla}_{\beta\eta}$ (if $* \in \{\beta\eta, \beta\eta\mathfrak{b}\}$), $\boldsymbol{\nabla}_\wedge$, and $\boldsymbol{\nabla}_\forall$ hold for $\mathcal{H}$ follows exactly this same pattern. Checking $\boldsymbol{\nabla}_\vee$, $\boldsymbol{\nabla}_\mathfrak{b}$ (if $* \in \{\beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$) and $\boldsymbol{\nabla}_{sat}$ (if $\mathcal{C}$ is saturated) follows a similar pattern, but with a simple case analysis. For example, to check $\boldsymbol{\nabla}_{sat}$, given $\mathbf{A} \in cwff_o$, ∇_{sat} implies $\mathcal{H} * \mathbf{A} \in \mathcal{C}$ or $\mathcal{H} * \neg\mathbf{A} \in \mathcal{C}$. So, either $\mathbf{A} \in \mathcal{H}$ or $\neg\mathbf{A} \in \mathcal{H}$.

Lemma 129 *Suppose $\mathcal{H}$ is a Hintikka set. For any $\mathbf{F}, \mathbf{G} \in cwff_{\alpha\to\beta}(\mathcal{S})$ and $\mathbf{A}, \mathbf{B}, \mathbf{C} \in cwff_\alpha(\mathcal{S})$ (for types α and β), we have the following:*

$\boldsymbol{\nabla}^r_{\doteq}$ *$\neg(\mathbf{A} \doteq^\alpha \mathbf{A}) \notin \mathcal{H}$.*
$\boldsymbol{\nabla}^{tr}_{\doteq}$ *If $\mathbf{A} \doteq^\alpha \mathbf{B} \in \mathcal{H}$ and $\mathbf{B} \doteq^\alpha \mathbf{C} \in \mathcal{H}$, then $\mathbf{A} \doteq^\alpha \mathbf{C} \in \mathcal{H}$.*
$\boldsymbol{\nabla}^{\to}_{\doteq}$ *If $(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G}) \in \mathcal{H}$ and $(\mathbf{A} \doteq^\alpha \mathbf{B}) \in \mathcal{H}$, then $(\mathbf{FA} \doteq^\beta \mathbf{GB}) \in \mathcal{H}$.*

Proof. To show $\boldsymbol{\nabla}^r_{\doteq}$, suppose $\neg(\mathbf{A} \doteq^\alpha \mathbf{A}) \in \mathcal{H}$. By $\boldsymbol{\nabla}_\exists$ and $\boldsymbol{\nabla}_\beta$, there must be some parameter $q_{\alpha\to o}$ such that $\neg(\neg q\mathbf{A} \vee q\mathbf{A}) \in \mathcal{H}$. By $\boldsymbol{\nabla}_\wedge$, we have $\neg\neg q\mathbf{A} \in \mathcal{H}$ and $\neg q\mathbf{A} \in \mathcal{H}$, contradicting $\boldsymbol{\nabla}_c$.

To show $\boldsymbol{\nabla}^{tr}_{\doteq}$, suppose $\mathbf{A} \doteq^\alpha \mathbf{B} \in \mathcal{H}$ and $\mathbf{B} \doteq^\alpha \mathbf{C} \in \mathcal{H}$. Let $\mathbf{Q}_{\alpha\to o}$ be the closed formula $(\lambda X_\alpha{\scriptscriptstyle\blacksquare}\mathbf{A} \doteq^\alpha X)$. Applying $\boldsymbol{\nabla}_\forall$ to $\mathbf{B} \doteq^\alpha \mathbf{C} \in \mathcal{H}$ and $\mathbf{Q}$, we know $\neg(\mathbf{QB}) \vee \mathbf{QC} \in \mathcal{H}$. By $\boldsymbol{\nabla}_\vee$, we know $\neg(\mathbf{QB}) \in \mathcal{H}$ or $\mathbf{QC} \in \mathcal{H}$. If $\neg(\mathbf{QB}) \in \mathcal{H}$, then $\neg(\mathbf{A} \doteq^\alpha \mathbf{B}) \in \mathcal{H}$ by $\boldsymbol{\nabla}_\beta$, contradicting $\boldsymbol{\nabla}_c$. So, $\mathbf{QC} \in \mathcal{H}$ and hence $\mathbf{A} \doteq^\alpha \mathbf{C} \in \mathcal{H}$ as desired.

To show $\boldsymbol{\nabla}^{\to}_{\doteq}$, let $\mathbf{P}_{(\alpha\to\beta)\to o}$ be the closed formula $(\lambda H_{\alpha\to\beta}{\scriptscriptstyle\blacksquare}\mathbf{FA} \doteq^\beta H\mathbf{A})$, Applying $\boldsymbol{\nabla}_\forall$ to $(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G}) \in \mathcal{H}$ and $\mathbf{P}$, we have $\neg(\mathbf{PF}) \vee \mathbf{PG} \in \mathcal{H}$. By $\boldsymbol{\nabla}_\vee$, we know $\neg(\mathbf{PF}) \in \mathcal{H}$ or $\mathbf{PG} \in \mathcal{H}$. If $\neg(\mathbf{PF}) \in \mathcal{H}$, then $\neg(\mathbf{FA} \doteq^\beta \mathbf{FA}) \in \mathcal{H}$ by $\boldsymbol{\nabla}_\beta$, which contradicts $\boldsymbol{\nabla}^r_{\doteq}$. So, we must have $\mathbf{PG} \in \mathcal{H}$ and hence $(\mathbf{FA} \doteq^\beta \mathbf{GA}) \in \mathcal{H}$. Let $\mathbf{Q}_{\alpha\to o}$ be the closed formula $(\lambda X_\alpha{\scriptscriptstyle\blacksquare}\mathbf{FA} \doteq^\beta \mathbf{G}X)$. Applying $\boldsymbol{\nabla}_\forall$ and $\boldsymbol{\nabla}_\vee$ to $(\mathbf{A} \doteq^\alpha \mathbf{B}) \in \mathcal{H}$, we know $\neg(\mathbf{QA}) \in \mathcal{H}$ or $\mathbf{QB} \in \mathcal{H}$. If $\neg(\mathbf{QA}) \in \mathcal{H}$, then $\neg(\mathbf{FA} \doteq^\beta \mathbf{GA}) \in \mathcal{H}$ by $\boldsymbol{\nabla}_\beta$, contradicting $\boldsymbol{\nabla}_c$. So, $\mathbf{QB} \in \mathcal{H}$ and hence $(\mathbf{FA} \doteq^\beta \mathbf{GB}) \in \mathcal{H}$ as desired.

Whenever a Hintikka set satisfies $\boldsymbol{\nabla}_{sat}$, we can prove far more closure properties. For example, we can prove converses of $\boldsymbol{\nabla}_\neg$, $\boldsymbol{\nabla}_\beta$, $\boldsymbol{\nabla}_\vee$, $\boldsymbol{\nabla}_\wedge$, $\boldsymbol{\nabla}_\forall$, and $\boldsymbol{\nabla}_\exists$. Also, if any of $\boldsymbol{\nabla}_{\beta\eta}$, $\boldsymbol{\nabla}_\mathfrak{b}$, $\boldsymbol{\nabla}_\xi$ or $\boldsymbol{\nabla}_\mathfrak{f}$ hold, we can prove the corresponding converse. (We could call these properties $\overleftarrow{\nabla}_*$.) The proofs of the stronger properties $\overline{\nabla}_\neg$ and $\overline{\nabla}_\vee$ in Lemma 131 indicate how one would prove any of these converse properties.

Definition 130 (Saturated Set) *We say a set of sentences $\mathcal{H}$ is* saturated *if it satisfies $\boldsymbol{\nabla}_{sat}$.*

By Lemma 128, any Hintikka set constructed as a maximal member of a saturated abstract consistency class will be saturated. However, it is also possible for a maximal member of an abstract consistency class $\mathcal{C}$ to be saturated without $\mathcal{C}$ being saturated.

Lemma 131 (Saturated Sets Lemma) *Suppose $\mathcal{H}$ is a saturated Hintikka set. Then we have the following properties for every $\mathbf{A}, \mathbf{B} \in cwff_o$, $\mathbf{F} \in cwff_{\alpha\to o}(\mathcal{S})$, and $\mathbf{C} \in cwff_\alpha(\mathcal{S})$ (for any type α):*

$\overline{\nabla}_\neg$ $\neg\mathbf{A} \in \mathcal{H}$ *iff* $\mathbf{A} \notin \mathcal{H}$.
$\overline{\nabla}_\vee$ $(\mathbf{A} \vee \mathbf{B}) \in \mathcal{H}$ *iff* $\mathbf{A} \in \mathcal{H}$ *or* $\mathbf{B} \in \mathcal{H}$.
$\overline{\nabla}_\forall$ $(\Pi^\alpha \mathbf{F}) \in \mathcal{H}$ *iff* $\mathbf{FD} \in \mathcal{H}$ *for every* $\mathbf{D} \in cwff_\alpha(\mathcal{S})$.
$\overline{\nabla}_\forall^\beta$ $(\Pi^\alpha \mathbf{F}) \in \mathcal{H}$ *iff* $(\mathbf{FD})\!\downarrow_\beta \in \mathcal{H}$ *for every* $\mathbf{D} \in cwff_\alpha(\mathcal{S})\!\downarrow_\beta$.
$\overline{\nabla}_r$ $(\mathbf{C} \doteq^\alpha \mathbf{C}) \in \mathcal{H}$.

Proof. If $\neg\mathbf{A} \in \mathcal{H}$, then $\mathbf{A} \notin \mathcal{H}$ by $\boldsymbol{\nabla}_c$. If $\mathbf{A} \notin \mathcal{H}$, then $\neg\mathbf{A} \in \mathcal{H}$ since $\mathcal{H}$ is saturated. So, $\overline{\nabla}_\neg$ holds.

If $(\mathbf{A} \vee \mathbf{B}) \in \mathcal{H}$, then $\mathbf{A} \in \mathcal{H}$ or $\mathbf{B} \in \mathcal{H}$ by $\boldsymbol{\nabla}_\vee$. We prove the converse by contraposition. Suppose $(\mathbf{A} \vee \mathbf{B}) \notin \mathcal{H}$. By saturation we have $\neg(\mathbf{A} \vee \mathbf{B}) \in \mathcal{H}$, and by $\boldsymbol{\nabla}_\wedge$ we get $\neg\mathbf{A} \in \mathcal{H}$ and $\neg\mathbf{B} \in \mathcal{H}$. So, by $\boldsymbol{\nabla}_c$, $\mathbf{A} \notin \mathcal{H}$ and $\mathbf{B} \notin \mathcal{H}$. Thus, $\overline{\nabla}_\vee$ holds.

One direction of $\overline{\nabla}_\forall$ is $\boldsymbol{\nabla}_\forall$. For one direction of $\overline{\nabla}_\forall^\beta$, note that if $(\Pi^\alpha \mathbf{F}) \in \mathcal{H}$, then for any $\mathbf{D} \in cwff_\alpha(\mathcal{S})\!\downarrow_\beta$ we have $(\mathbf{FD})\!\downarrow_\beta \in \mathcal{H}$ by $\boldsymbol{\nabla}_\forall$ and $\boldsymbol{\nabla}_\beta$.

Suppose $(\Pi^\alpha \mathbf{F}) \notin \mathcal{H}$. By saturation, $\neg(\Pi^\alpha \mathbf{F}) \in \mathcal{H}$. By $\boldsymbol{\nabla}_\exists$, there is a parameter $w_\alpha \in \mathcal{S}_\alpha$ such that $\neg(\mathbf{F}w) \in \mathcal{H}$. By $\boldsymbol{\nabla}_c$, we know $(\mathbf{F}w) \notin \mathcal{H}$. This shows the other direction of $\overline{\nabla}_\forall$. Furthermore, by $\boldsymbol{\nabla}_\beta$ we know $\neg(\mathbf{F}w)\!\downarrow_\beta \in \mathcal{H}$ and so $(\mathbf{F}w)\!\downarrow_\beta \notin \mathcal{H}$. Since w is β-normal, we also have the other direction of $\overline{\nabla}_\forall^\beta$.

Finally, $\overline{\nabla}_r$ follows directly from saturation and $\boldsymbol{\nabla}_{\doteq}^r$.

Lemma 132 (Saturated Sets Lemma for $\mathfrak{b}$) *Suppose $\mathcal{H} \in \mathfrak{Hint}_*$ where $* \in \{\beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{fb}\}$. If $\mathcal{H}$ is saturated, then the following property holds for all $\mathbf{A}, \mathbf{B} \in cwff_o$.*

$\overline{\nabla}_\mathfrak{b}$ $\mathbf{A} \doteq^o \mathbf{B} \in \mathcal{H}$ *or* $\mathbf{A} \doteq^o \neg\mathbf{B} \in \mathcal{H}$.

Proof. Suppose $(\mathbf{A} \doteq^o \mathbf{B}) \notin \mathcal{H}$ and $(\mathbf{A} \doteq^o \neg\mathbf{B}) \notin \mathcal{H}$. By saturation, $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \mathcal{H}$ and $\neg(\mathbf{A} \doteq^o \neg\mathbf{B}) \in \mathcal{H}$. By $\boldsymbol{\nabla}_\mathfrak{b}$, we must have $\{\mathbf{A}, \neg\mathbf{B}\} \subseteq \mathcal{H}$ or $\{\neg\mathbf{A}, \mathbf{B}\} \subseteq \mathcal{H}$. We must also have $\{\mathbf{A}, \neg\neg\mathbf{B}\} \subseteq \mathcal{H}$ or $\{\neg\mathbf{A}, \neg\mathbf{B}\} \subseteq \mathcal{H}$. Each of the four cases leads to an immediate contradiction to $\boldsymbol{\nabla}_c$.

Lemma 133 (Saturated Sets Lemma for η) *Suppose $\mathcal{H} \in \mathfrak{Hint}_*$ where $* \in \{\beta\eta, \beta\eta\mathfrak{b}\}$. If $\mathcal{H}$ is saturated, then the following property holds for every type α and $\mathbf{A} \in cwff_\alpha(\mathcal{S})$:*

$\overline{\nabla}_\eta$ $(\mathbf{A} \doteq^\alpha \mathbf{A}\!\downarrow_{\beta\eta}) \in \mathcal{H}$.

Proof. If $(\mathbf{A} \doteq \mathbf{A}\!\downarrow_{\beta\eta}) \notin \mathcal{H}$, then by saturation $\neg(\mathbf{A} \doteq \mathbf{A}\!\downarrow_{\beta\eta}) \in \mathcal{H}$. So, by $\boldsymbol{\nabla}_{\beta\eta}$ we have $\neg(\mathbf{A}\!\downarrow_{\beta\eta} \doteq^\alpha \mathbf{A}\!\downarrow_{\beta\eta}) \in \mathcal{H}$. But this contradicts $\boldsymbol{\nabla}_{\doteq}^r$.

Lemma 134 (Saturated Sets Lemma for ξ) *Suppose $\mathcal{H} \in \mathfrak{Hint}_*$ where $* \in \{\beta\xi, \beta\xi\mathfrak{b}\}$. If $\mathcal{H}$ is saturated, then the following properties hold for all $\alpha, \beta \in \mathcal{T}$ and $(\lambda X_\alpha \blacksquare \mathbf{M}), (\lambda X \blacksquare \mathbf{N}) \in cwff_{\alpha\to\beta}(\mathcal{S})$:*

$\overline{\nabla}_\xi$ $(\lambda X \blacksquare \mathbf{M} \doteq^{\alpha\to\beta} \lambda X \blacksquare \mathbf{N}) \in \mathcal{H}$ *iff* $([\mathbf{A}/X]\mathbf{M} \doteq^\beta [\mathbf{A}/X]\mathbf{N}) \in \mathcal{H}$ *for every* $\mathbf{A} \in cwff_\alpha(\mathcal{S})$.
$\overline{\nabla}_\xi^\beta$ $(\lambda X \blacksquare \mathbf{M} \doteq^{\alpha\to\beta} \lambda X \blacksquare \mathbf{N}) \in \mathcal{H}$ *iff* $([\mathbf{A}/X]\mathbf{M} \doteq^\beta [\mathbf{A}/X]\mathbf{N})\!\downarrow_\beta \in \mathcal{H}$ *for every* $\mathbf{A} \in cwff_\alpha(\mathcal{S})^{\downarrow\beta}$.

Proof. Suppose $(\lambda X \blacksquare \mathbf{M} \doteq^{\alpha\to\beta} \lambda X \blacksquare \mathbf{N}) \in \mathcal{H}$ and $\mathbf{A} \in cwff_\alpha(\mathcal{S})$. We can apply $\boldsymbol{\nabla}_\forall$ and $\boldsymbol{\nabla}_\beta$ using the closed formula $(\lambda K_{\alpha\to\beta} \blacksquare [\mathbf{A}/X]\mathbf{M} \doteq^\beta K\mathbf{A})$ to obtain

$$(\neg([\mathbf{A}/X]\mathbf{M} \doteq^\beta [\mathbf{A}/X]\mathbf{M}) \vee [\mathbf{A}/X]\mathbf{M} \doteq^\beta [\mathbf{A}/X]\mathbf{N}) \in \mathcal{H}$$

Since $\neg([\mathbf{A}/X]\mathbf{M} \doteq^\beta [\mathbf{A}/X]\mathbf{M}) \notin \mathcal{H}$ (by $\boldsymbol{\nabla}_{\doteq}^r$), we know $([\mathbf{A}/X]\mathbf{M} \doteq^\beta [\mathbf{A}/X]\mathbf{N}) \in \mathcal{H}$. This shows one direction of $\overline{\nabla}_\xi$. By $\boldsymbol{\nabla}_\beta$ we have $([\mathbf{A}/X]\mathbf{M} \doteq^\beta [\mathbf{A}/X]\mathbf{N})^{\downarrow\beta} \in \mathcal{H}$. Since this holds in particular for any $\mathbf{A} \in cwff_\alpha(\mathcal{S})\!\downarrow_\beta$, this shows one direction of $\overline{\nabla}_\xi^\beta$.

Suppose $(\lambda X \blacksquare \mathbf{M} \doteq^{\alpha\to\beta} \lambda X \blacksquare \mathbf{N}) \notin \mathcal{H}$. We show that there is a (β-normal) $\mathbf{A} \in cwff_\alpha(\mathcal{S})$ with $[\mathbf{A}/X]\mathbf{M} \doteq^\beta [\mathbf{A}/X]\mathbf{N} \notin \mathcal{H}$. By saturation, $\neg(\lambda X \blacksquare \mathbf{M} \doteq^{\alpha\to\beta} \lambda X \blacksquare \mathbf{N}) \in \mathcal{H}$. By $\boldsymbol{\nabla}_\xi$, there is a parameter $w_\alpha \in \mathcal{S}_\alpha$ such that $\neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N}) \in \mathcal{H}$. By $\boldsymbol{\nabla}_c$, $[w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N} \notin \mathcal{H}$. Choosing $\mathbf{A} := w$ we have the other direction of $\overline{\nabla}_\xi$. Since w is β-normal and $([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N})^{\downarrow\beta} \notin \mathcal{H}$ (using $\boldsymbol{\nabla}_\beta$), we have the other direction of $\overline{\nabla}_\xi^\beta$.

Lemma 135 (Saturated Sets Lemma for $\mathfrak{f}$) *Suppose $\mathcal{H} \in \mathfrak{Hint}_*$ where $* \in \{\beta\mathfrak{f}, \beta\mathfrak{fb}\}$. If $\mathcal{H}$ is saturated, then the following property holds for any types α and β and $\mathbf{G}, \mathbf{H} \in cwff_{\alpha\to\beta}(\mathcal{S})$.*

$\overline{\nabla}_\mathfrak{f}$ $\mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H} \in \mathcal{H}$ *iff* $\mathbf{GA} \doteq^\beta \mathbf{HA} \in \mathcal{H}$ *for every* $\mathbf{A} \in cwff_\alpha(\mathcal{S})$.
$\overline{\nabla}_\mathfrak{f}^\beta$ $\mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H} \in \mathcal{H}$ *iff* $(\mathbf{GA} \doteq^\beta \mathbf{HA})^{\downarrow\beta} \in \mathcal{H}$ *for every* $\mathbf{A} \in cwff_\alpha(\mathcal{S})^{\downarrow\beta}$.

Proof. Suppose $(\mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H}) \in \mathcal{H}$ and $\mathbf{A} \in cwff_\alpha(\mathcal{S})$. Since $(\mathbf{A} \doteq^\alpha \mathbf{A}) \in \mathcal{H}$ by $\overline{\nabla}_r$ we have $(\mathbf{GA} \doteq^\beta \mathbf{HA}) \in \mathcal{H}$ by $\boldsymbol{\nabla}_{\doteq}^{\to}$ (cf. Lemma 129). This shows one direction of $\overline{\nabla}_\mathfrak{f}$. By $\boldsymbol{\nabla}_\beta$ we have $(\mathbf{GA} \doteq^\beta \mathbf{HA})^{\downarrow\beta} \in \mathcal{H}$. Since this holds in particular for any $\mathbf{A} \in cwff_\alpha(\mathcal{S})\!\downarrow_\beta$, this shows one direction of $\overline{\nabla}_\mathfrak{f}^\beta$.

Suppose $(\mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H}) \notin \mathcal{H}$. By saturation, $\neg(\mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H}) \in \mathcal{H}$. By $\boldsymbol{\nabla}_\mathfrak{f}$, there is a parameter $w_\alpha \in \mathcal{S}_\alpha$ such that $\neg(\mathbf{G}w \doteq^\beta \mathbf{H}w) \in \mathcal{H}$. By $\boldsymbol{\nabla}_c$, $(\mathbf{G}w \doteq^\beta \mathbf{H}w) \notin \mathcal{H}$. Choosing $\mathbf{A} := w$ we have the other direction of $\overline{\nabla}_\mathfrak{f}$. Since w is β-normal and $(\mathbf{G}w \doteq^\beta \mathbf{H}w)^{\downarrow\beta} \notin \mathcal{H}$ (using $\boldsymbol{\nabla}_\beta$), we have the other direction of $\overline{\nabla}_\mathfrak{f}^\beta$.

Theorem 137 compares $\boldsymbol{\nabla}_{\beta\eta}$, $\boldsymbol{\nabla}_\xi$, and $\boldsymbol{\nabla}_\mathfrak{f}$ as properties of Hintikka sets. Showing $\boldsymbol{\nabla}_\mathfrak{f}$ implies $\boldsymbol{\nabla}_{\beta\eta}$ requires saturation and must be shown in several steps reflected by Lemma 136.

Lemma 136 *Let $\mathcal{H}$ be a saturated Hintikka set satisfying $\boldsymbol{\nabla}_\mathfrak{f}$.*

1. *For all $\mathbf{F} \in cwff_{\alpha\to\beta}(\mathcal{S})$ we have $(\lambda X_\alpha \blacksquare \mathbf{F}X) \doteq^{\alpha\to\beta} \mathbf{F} \in \mathcal{H}$.*
2. *For all $\mathbf{A}, \mathbf{B} \in cwff_\alpha(\mathcal{S})$, if $\mathbf{A}$ η-reduces to $\mathbf{B}$ in one step, then $\mathbf{A} \doteq^\alpha \mathbf{B} \in \mathcal{H}$.*
3. *For all $\mathbf{A} \in cwff_\alpha(\mathcal{S})$, $\mathbf{A} \doteq^\alpha \mathbf{A}\!\downarrow_{\beta\eta} \in \mathcal{H}$.*
4. *For all $\mathbf{A} \in cwff_o(\mathcal{S})$, if $\mathbf{A} \in \mathcal{H}$, then $\mathbf{A}\!\downarrow_{\beta\eta} \in \mathcal{H}$.*

Proof. To show part (1), suppose $(\lambda X_\alpha \blacksquare \mathbf{F}X) \doteq^{\alpha\to\beta} \mathbf{F} \notin \mathcal{H}$. By saturation, $\neg((\lambda X_\alpha \blacksquare \mathbf{F}X) \doteq^{\alpha\to\beta} \mathbf{F}) \in \mathcal{H}$. By $\boldsymbol{\nabla}_\mathfrak{f}$, there is a parameter w_α such that $\neg(((\lambda X_\alpha \blacksquare \mathbf{F}X)w) \doteq^\beta (\mathbf{F}w)) \in \mathcal{H}$. By $\boldsymbol{\nabla}_\beta$, $\neg((\mathbf{F}w) \doteq^\beta (\mathbf{F}w)) \in \mathcal{H}$, which contradicts $\boldsymbol{\nabla}_{\doteq}^r$ (cf. Lemma 129).

We prove part (2) by induction on the position of the η-redex in $\mathbf{A}$. If $\mathbf{A}$ is the η-redex reduced to obtain $\mathbf{B}$, then this follows from part (1). Suppose $\mathbf{A} = (\mathbf{F}_{\gamma\to\alpha}\mathbf{C}_\gamma)$ and $\mathbf{B} = (\mathbf{G}_{\gamma\to\alpha}\mathbf{C})$ where $\mathbf{F}$ η-reduces to $\mathbf{G}$ in one step. By induction, we know $\mathbf{F} \doteq^{\gamma\to\alpha} \mathbf{G} \in \mathcal{H}$. By $\overline{\nabla}_r$, $\mathbf{C} \doteq^\gamma \mathbf{C} \in \mathcal{H}$. By $\boldsymbol{\nabla}_{\doteq}^{\to}$, we have $(\mathbf{FC}) \doteq^\alpha (\mathbf{GC}) \in \mathcal{H}$ as desired. The case in which $\mathbf{A} = (\mathbf{F}_{\gamma\to\alpha}\mathbf{C}_\gamma)$ and $\mathbf{B} = (\mathbf{FD}_\gamma)$ where $\mathbf{C}$ η-reduces to $\mathbf{D}$ in one step is analogous.

Suppose $\mathbf{A} = (\lambda Y_\beta \blacksquare \mathbf{C}_\gamma)$ and $\mathbf{B} = (\lambda Y_\beta \blacksquare \mathbf{D}_\gamma)$ where $\mathbf{C}$ η-reduces to $\mathbf{D}$ in one step. Let p be the position of the redex in $\mathbf{C}$. Assume $\mathbf{A} \doteq^{\beta\to\gamma} \mathbf{B} \notin \mathcal{H}$. By saturation, $\neg(\mathbf{A} \doteq^{\beta\to\gamma} \mathbf{B}) \in \mathcal{H}$. By $\boldsymbol{\nabla}_\mathfrak{f}$, there is some parameter w_β such that $\neg(\mathbf{A}w \doteq^\gamma \mathbf{B}w) \in \mathcal{H}$. By $\boldsymbol{\nabla}_\beta$, we know $\neg([w/Y]\mathbf{C} \doteq^\gamma [w/Y]\mathbf{D}) \in \mathcal{H}$. Note that $[w/Y]\mathbf{C}$ η-reduces to $[w/Y]\mathbf{D}$ in one step by reducing the redex at position p in $[w/Y]\mathbf{C}$. So, by the induction hypothesis, $[w/Y]\mathbf{C} \doteq^\gamma [w/Y]\mathbf{D} \in \mathcal{H}$, contradicting $\boldsymbol{\nabla}_c$.

Part (3) follows by induction on the number of $\beta\eta$-reductions from $\mathbf{A}$ to $\mathbf{A}\!\downarrow_{\beta\eta}$. If $\mathbf{A}$ is $\beta\eta$-normal, we have $\mathbf{A} \doteq^\alpha \mathbf{A} \in \mathcal{H}$ by $\overline{\nabla}_r$. If $\mathbf{A}$ reduces to $\mathbf{A}\!\downarrow_{\beta\eta}$ in $n+1$ steps, then there is some $\mathbf{B}_\alpha$ such that $\mathbf{A}$ reduces to $\mathbf{B}$ in one step and $\mathbf{B}$ reduces to $\mathbf{A}\!\downarrow_{\beta\eta}$ in n steps. By induction, we have $\mathbf{B} \doteq^\alpha \mathbf{A}\!\downarrow_{\beta\eta} \in \mathcal{H}$. If $\mathbf{A}$ β-reduces to $\mathbf{B}$ in one step, then $\mathbf{A} \doteq^\alpha \mathbf{B} \in \mathcal{H}$ by $\overline{\nabla}_r$ and $\boldsymbol{\nabla}_\beta$. If $\mathbf{A}$ η-reduces to $\mathbf{B}$ in one step, then $\mathbf{A} \doteq^\alpha \mathbf{B} \in \mathcal{H}$ by part (2). Using $\boldsymbol{\nabla}_{\doteq}^{tr}$, $\mathbf{A} \doteq^\alpha \mathbf{B} \in \mathcal{H}$ and $\mathbf{B} \doteq^\alpha \mathbf{A}\!\downarrow_{\beta\eta} \in \mathcal{H}$ imply $\mathbf{A} \doteq^\alpha \mathbf{A}\!\downarrow_{\beta\eta} \in \mathcal{H}$ as desired.

Finally, to show part (4), suppose $\mathbf{A} \in \mathcal{H}$. By part (3), $\mathbf{A} \doteq^o \mathbf{A}\!\downarrow_{\beta\eta} \in \mathcal{H}$. By $\boldsymbol{\nabla}_\forall$, $\neg(\lambda X_o \blacksquare X)\mathbf{A} \vee (\lambda X_o \blacksquare X)\,\mathbf{A}\!\downarrow_{\beta\eta} \in \mathcal{H}$. By $\boldsymbol{\nabla}_\beta$ and $\boldsymbol{\nabla}_\vee$, we have $\neg\mathbf{A} \in \mathcal{H}$ (contradicting $\boldsymbol{\nabla}_c$) or $\mathbf{A}\!\downarrow_{\beta\eta} \in \mathcal{H}$. Hence, $\mathbf{A}\!\downarrow_{\beta\eta} \in \mathcal{H}$.

Theorem 137 *Let $\mathcal{H}$ be a Hintikka set.*

1. *If $\mathcal{H}$ satisfies $\boldsymbol{\nabla}_{\beta\eta}$ and $\boldsymbol{\nabla}_\xi$, then $\mathcal{H}$ satisfies $\boldsymbol{\nabla}_\mathfrak{f}$.*

2. *If $\mathcal{H}$ satisfies $\boldsymbol{\nabla}_\mathfrak{f}$, then $\mathcal{H}$ satisfies $\boldsymbol{\nabla}_\xi$.*
3. *If $\mathcal{H}$ is saturated and satisfies $\boldsymbol{\nabla}_\mathfrak{f}$, then $\mathcal{H}$ satisfies $\boldsymbol{\nabla}_{\beta\eta}$.*

Proof. Suppose $\mathcal{H}$ satisfies $\boldsymbol{\nabla}_{\beta\eta}$ and $\boldsymbol{\nabla}_\xi$. Assume $\neg(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G}) \in \mathcal{H}$. By $\boldsymbol{\nabla}_{\beta\eta}$, $\neg((\lambda X_\alpha{\blacksquare}\mathbf{F}X) \doteq^{\alpha\to\beta} (\lambda X{\blacksquare}\mathbf{G}X)) \in \mathcal{H}$. By $\boldsymbol{\nabla}_\xi$, there is a parameter w_α such that $\neg((\mathbf{F}w) \doteq^\beta (\mathbf{G}w)) \in \mathcal{H}$. Thus, $\boldsymbol{\nabla}_\mathfrak{f}$ holds.

Suppose $\mathcal{H}$ satisfies $\boldsymbol{\nabla}_\mathfrak{f}$ and $\neg(\lambda X_\alpha{\blacksquare}\mathbf{M} \doteq^{\alpha\to\beta} \lambda X{\blacksquare}\mathbf{N}) \in \mathcal{H}$. By $\boldsymbol{\nabla}_\mathfrak{f}$, there is a parameter w_α such that $\neg((\lambda X_\alpha{\blacksquare}\mathbf{M})w \doteq^\beta (\lambda X{\blacksquare}\mathbf{N})w) \in \mathcal{H}$. By $\boldsymbol{\nabla}_\beta$, we have $\neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N}) \in \mathcal{H}$. Thus, $\boldsymbol{\nabla}_\xi$ holds.

Suppose $\mathcal{H}$ is saturated and satisfies $\boldsymbol{\nabla}_\mathfrak{f}$. Assume $\mathbf{A} \in \mathcal{H}$, $\mathbf{B} \in \mathit{cwff}_o(\mathcal{S})$, $\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B}$ and $\mathbf{B} \notin \mathcal{H}$. By saturation, we know $\neg\mathbf{B} \in \mathcal{H}$. By Lemma 136(4), we know $\mathbf{A}{\downarrow_{\beta\eta}} \in \mathcal{H}$ and $\neg\, \mathbf{B}{\downarrow_{\beta\eta}} \in \mathcal{H}$. Since $\mathbf{A}{\downarrow_{\beta\eta}} = \mathbf{B}{\downarrow_{\beta\eta}}$, this contradicts $\boldsymbol{\nabla}_c$.

15.3 Model Existence Theorems

We shall now present the proof of the abstract extension lemma, which will nearly immediately yield the model existence theorems. For the proof we adapt the construction of Henkin's completeness proof from [Hen50; Hen96].

Lemma 138 (Abstract Extension Lemma) *Let $\mathcal{S}$ be a signature, $\mathscr{C}$ be a compact abstract consistency class in $\mathfrak{Acc}_*$, where $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$, and let $\Phi \in \mathscr{C}$ be sufficiently pure. Then there exists a $\mathcal{S}$-Hintikka set $\mathcal{H} \in \mathfrak{Hint}_*$, such that $\Phi \subseteq \mathcal{H}$. Furthermore, if $\mathscr{C}$ is saturated, then $\mathcal{H}$ is saturated.*

Proof. In the following argument, note that α, β, and γ are types as usual, while δ, ϵ, σ and τ are ordinals.

Note that there is an infinite cardinal $\aleph_s$ which is the cardinality of $\mathcal{P}_\alpha$ for each type α. This easily implies $\mathit{cwff}_\alpha(\mathcal{S})$ is of cardinality $\aleph_s$ for each type α. Let ϵ be the first ordinal of this cardinality. (In the countable case, ϵ is ω.) Since the cardinality of cwff_o is $\aleph_s$, we can use the well-ordering principle to enumerate cwff_o as $(\mathbf{A}^\delta)_{\delta<\epsilon}$.

Let α be a type. For each $\delta < \epsilon$, let U_α^δ be the set of constants of type α which occur in a sentence in the set $\{\mathbf{A}^\sigma \mid \sigma \le \delta\}$. Since $\delta < \epsilon$, the set $\{\mathbf{A}^\sigma \mid \sigma \le \delta\}$ has cardinality less than $\aleph_s$. Hence, U_α^δ has cardinality less than $\aleph_s$. By sufficient purity, we know there is a set of parameters $P_\alpha \subseteq \mathcal{S}_\alpha$ of cardinality $\aleph_s$ such that the parameters in P_α do not occur in the sentences of Φ. So, $P_\alpha \setminus U_\alpha^\delta$ must have cardinality $\aleph_s$ for any $\delta < \epsilon$. Using the axiom of choice, we can find a sequence $(w_\alpha^\delta)_{\delta<\epsilon}$ where for each $\delta < \epsilon$, $w_\alpha^\delta \in P_\alpha \setminus (U_\alpha^\delta \cup \{w_\alpha^\sigma | \sigma < \delta\})$. That is, for each type α, we know w_α^δ is a parameter of type α which does not occur in any sentence in $\Phi \cup \{\mathbf{A}^\sigma \mid \sigma \le \delta\}$. As a consequence, if w_α^δ occurs in $\mathbf{A}^\sigma$, then $\delta < \sigma$. Also, we have ensured that if $w_\alpha^\delta = w_\alpha^\sigma$, then $\delta = \sigma$ for any $\delta, \sigma < \epsilon$.

The parameters w_α^δ are intended to serve as witnesses. To ease the argument, we define two sequences of witnessing sentences related to the sequence $(\mathbf{A}^\delta)_{\delta<\epsilon}$. For each $\delta < \epsilon$, let $\mathbf{E}^\delta := \neg(\mathbf{B}w_\alpha^\delta)$ if $\mathbf{A}^\delta$ is of the form $\neg(\Pi^\alpha\mathbf{B})$, and let $\mathbf{E}^\delta := \mathbf{A}^\delta$ otherwise. If $* \in \{\beta\mathfrak{f}, \beta\mathfrak{f}\mathfrak{b}\}$ and $\mathbf{A}^\delta$ is of the form $\neg(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G})$, let $\mathbf{X}^\delta := \neg(\mathbf{F}w_\alpha^\delta \doteq^\beta \mathbf{G}w_\alpha^\delta)$. If $* \in \{\beta\xi, \beta\xi\mathfrak{b}\}$ and $\mathbf{A}^\delta$ is of the form $\neg((\lambda X_\alpha{\blacksquare}\mathbf{M}) \doteq^{\alpha\to\beta} (\lambda X{\blacksquare}\mathbf{N}))$, let $\mathbf{X}^\delta := \neg([w_\alpha^\delta/X]\mathbf{M} \doteq^\beta [w_\alpha^\delta/X]\mathbf{N})$. Otherwise, let $\mathbf{X}^\delta := \mathbf{A}^\delta$. (Notice that any sentence $\neg(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G})$ is also of the form $\neg(\Pi^\gamma\mathbf{B})$, where γ is $(\alpha \to \beta) \to o$. So, whenever $\mathbf{X}^\delta \neq \mathbf{A}^\delta$, we must also have $\mathbf{E}^\delta \neq \mathbf{A}^\delta$.)

We construct $\mathcal{H}$ by inductively constructing a transfinite sequence $(\mathcal{H}^\delta)_{\delta<\epsilon}$ such that $\mathcal{H}^\delta \in \mathscr{C}$ for each $\delta < \epsilon$. Then the $\mathcal{S}$-Hintikka set is $\mathcal{H} := \bigcup_{\delta<\epsilon} \mathcal{H}^\delta$. We define $\mathcal{H}^0 := \Phi$. For limit ordinals δ, we define $\mathcal{H}^\delta := \bigcup_{\sigma<\delta} \mathcal{H}^\sigma$.

In the successor case, if $\mathcal{H}^\delta * \mathbf{A}^\delta \in \mathscr{C}$, then we let $\mathcal{H}^{\delta+1} := \mathcal{H}^\delta * \mathbf{A}^\delta * \mathbf{E}^\delta * \mathbf{X}^\delta$. If $\mathcal{H}^\delta * \mathbf{A}^\delta \notin \mathscr{C}$, we let $\mathcal{H}^{\delta+1} := \mathcal{H}^\delta$.

We show by induction that for every $\delta < \epsilon$, type α and parameter w_α^τ which occurs in some sentence in $\mathcal{H}^\delta$, we have $\tau < \delta$. The base case holds since no w_α^τ occurs in any sentence in $\mathcal{H}^0 = \Phi$.

For any limit ordinal δ, if w_α^τ occurs in some sentence in $\mathcal{H}^\delta$, then by definition of $\mathcal{H}^\delta$, w_α^τ already occurs in some sentence in $\mathcal{H}^\sigma$ for some $\sigma < \delta$. So, $\tau < \sigma < \delta$.

For any successor ordinal $\delta + 1$, suppose w_α^τ occurs in some sentence in $\mathcal{H}^{\delta+1}$. If it already occurred in a sentence in $\mathcal{H}^\delta$, then we have $\tau < \delta < \delta + 1$ by the inductive assumption. So, we need only consider the case where w_α^τ occurs in a sentence in $\mathcal{H}^{\delta+1} \setminus \mathcal{H}^\delta$. Note that $(\mathcal{H}^{\delta+1} \setminus \mathcal{H}^\delta) \subseteq \{\mathbf{A}^\delta, \mathbf{E}^\delta, \mathbf{X}^\delta\}$. In any case, note that if τ is δ, then we are done, since $\delta < \delta + 1$. If w_α^τ is any parameter with $\tau \neq \delta$ and occurs in $\mathbf{E}^\delta$ or $\mathbf{X}^\delta$, then it must also occur in $\mathbf{A}^\delta$ (by noting that $w_\alpha^\tau \neq w_\alpha^\delta$ and inspecting the possible definitions of $\mathbf{E}^\delta$ and $\mathbf{X}^\delta$), in which case $\tau < \delta < \delta + 1$.

In particular, we now know w_α^δ does not occur in any sentence of $\mathcal{H}^\delta$ for any $\delta < \epsilon$ and type α.

Next we show by induction that $\mathcal{H}^\delta \in \mathscr{C}$ for all $\delta < \epsilon$. The base case holds by the assumption that $\mathcal{H}^0 = \Phi \in \mathscr{C}$. For any limit ordinal δ, assume $\mathcal{H}^\sigma \in \mathscr{C}$ for every $\sigma < \delta$. We have $\mathcal{H}^\delta = \bigcup_{\sigma<\delta} \mathcal{H}^\sigma \in \mathscr{C}$ by compactness, since any finite subset of $\mathcal{H}^\delta$ is a subset of $\mathcal{H}^\sigma$ for some $\sigma < \delta$.

For any successor ordinal $\delta + 1$, we assume $\mathcal{H}^\delta \in \mathscr{C}$. We have to show that $\mathcal{H}^{\delta+1} \in \mathscr{C}$. This is trivial in case $\mathcal{H}^\delta * \mathbf{A}^\delta \notin \mathscr{C}$ (for all abstract consistency classes) since $\mathcal{H}^{\delta+1} = \mathcal{H}^\delta$. Suppose $\mathcal{H}^\delta * \mathbf{A}^\delta \in \mathscr{C}$. We consider three sub-cases: (i) If $\mathbf{E}^\delta = \mathbf{A}^\delta$ and $\mathbf{X}^\delta = \mathbf{A}^\delta$, then $\mathcal{H}^\delta * \mathbf{A}^\delta * \mathbf{E}^\delta * \mathbf{X}^\delta \in \mathscr{C}$ since $\mathcal{H}^\delta * \mathbf{A}^\delta \in \mathscr{C}$. (ii) If $\mathbf{E}^\delta \neq \mathbf{A}^\delta$ and $\mathbf{X}^\delta = \mathbf{A}^\delta$, then $\mathbf{A}^\delta$ is of the form $\neg\Pi^\alpha\mathbf{B}$ and $\mathbf{E}^\delta = \neg\mathbf{B}w_\alpha^\delta$. We conclude that $\mathcal{H}^\delta * \mathbf{A}^\delta * \mathbf{E}^\delta \in \mathscr{C}$ by $\nabla_\exists$ since w_α^δ does not occur in $\mathbf{A}^\delta$ or any sentence of $\mathcal{H}^\delta$. Since $\mathbf{X}^\delta = \mathbf{A}^\delta$, this is the same as concluding $\mathcal{H}^\delta * \mathbf{A}^\delta * \mathbf{E}^\delta * \mathbf{X}^\delta \in \mathscr{C}$. (iii) If $\mathbf{X}^\delta \neq \mathbf{A}^\delta$, then $* \in \{\beta\xi, \beta\mathfrak{f}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ (by the definition of $\mathbf{X}^\delta$). $\mathcal{H}^\delta * \mathbf{A}^\delta * \mathbf{E}^\delta \in \mathscr{C}$ by $\nabla_\exists$ since $w_{(\alpha\to\beta)\to o}^\delta$ does not occur in $\mathbf{A}^\delta$ or any sentence in $\mathcal{H}^\delta$. Now, w_α^δ (which is different from $w_{(\alpha\to\beta)\to o}^\delta$ since it has a different type) does not occur in any sentence in $\mathcal{H}^\delta * \mathbf{A}^\delta * \mathbf{E}^\delta$. We have $\mathcal{H}^\delta * \mathbf{A}^\delta * \mathbf{E}^\delta * \mathbf{X}^\delta \in \mathcal{H}$ by ∇_ξ (if $* \in \{\beta\xi, \beta\xi\mathfrak{b}\}$) or by $\nabla_\mathfrak{f}$ (if $* \in \{\beta\mathfrak{f}, \beta\mathfrak{f}\mathfrak{b}\}$).

Since $\mathscr{C}$ is compact, we also have $\mathcal{H} \in \mathscr{C}$.

Now we know that our inductively defined set $\mathcal{H}$ is indeed in $\mathscr{C}$ and that $\Phi \subseteq \mathcal{H}$. In order to apply Lemma 128, we must check $\mathcal{H}$ is maximal, satisfies $\boldsymbol{\nabla}_\exists$, $\boldsymbol{\nabla}_\xi$ (if $* \in \{\beta\xi, \beta\xi\mathfrak{b}\}$), and $\boldsymbol{\nabla}_\mathfrak{f}$ (if $* \in \{\beta\mathfrak{f}, \beta\mathfrak{f}\mathfrak{b}\}$). It is immediate from the construction that $\boldsymbol{\nabla}_\exists$ holds since if $\neg(\Pi^\alpha\mathbf{F}) \in \mathcal{H}$, then $\neg(\mathbf{F}w_\alpha^\delta) \in \mathcal{H}$ where δ is the ordinal such that $\mathbf{A}^\delta = \neg(\Pi^\alpha\mathbf{F})$. If $* \in \{\beta\xi, \beta\xi\mathfrak{b}\}$, then we have ensured $\boldsymbol{\nabla}_\xi$ holds since $\neg([w_\alpha^\delta/X]\mathbf{M} \doteq^\beta [w_\alpha^\delta/X]\mathbf{N}) \in \mathcal{H}$ whenever $\neg((\lambda X_\alpha{\blacksquare}\mathbf{M}) \doteq^{\alpha\to\beta} (\lambda X{\blacksquare}\mathbf{N})) \in \mathcal{H}$ where δ is the ordinal such that $\mathbf{A}^\delta = \neg((\lambda X_\alpha{\blacksquare}\mathbf{M}) \doteq^{\alpha\to\beta} (\lambda X{\blacksquare}\mathbf{N}))$. Similarly, we have ensured $\boldsymbol{\nabla}_\mathfrak{f}$ holds when $* \in \{\beta\mathfrak{f}, \beta\mathfrak{f}\mathfrak{b}\}$ since $\neg(\mathbf{F}w_\alpha^\delta \doteq^\beta \mathbf{G}w_\alpha^\delta) \in \mathcal{H}$ whenever $\neg(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G}) \in \mathcal{H}$ where δ is the ordinal such that $\mathbf{A}^\delta = \neg(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G})$.

It only remains to show that $\mathcal{H}$ is maximal in $\mathscr{C}$. So, let $\mathbf{A} \in \mathit{cwff}_o(\mathcal{S})$ and $\mathcal{H} * \mathbf{A} \in \mathscr{C}$ be given. Note that $\mathbf{A} = \mathbf{A}^\delta$ for some $\delta < \epsilon$. Since $\mathcal{H}$ is closed under subsets we know that $\mathcal{H}^\delta * \mathbf{A}^\delta \in \mathscr{C}$. By definition of $\mathcal{H}^{\delta+1}$ we conclude that $\mathbf{A}^\delta \in \mathcal{H}^{\delta+1}$ and hence $\mathbf{A} \in \mathcal{H}$.

So, Lemma 128 implies $\mathcal{H} \in \mathfrak{Hint}_*$ and $\mathcal{H}$ is saturated if $\mathscr{C}$ is saturated.

We now use the $\mathcal{S}$-Hintikka sets, guaranteed by Lemma 138, to construct a $\mathcal{S}$-valuation for the $\mathcal{S}$-term evaluation that turns it into a model.

Theorem 139 (Model Existence Theorem for Saturated Sets) *For all $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ we have: If $\mathcal{H}$ is a saturated Hintikka set in $\mathfrak{Hint}_*$ (cf. Definition 127), then there exists a model $\mathcal{M} \in \mathfrak{M}_*$ (cf. Definition 81) that satisfies $\mathcal{H}$. Furthermore, each domain $\mathcal{D}_\alpha$ of $\mathcal{M}$ has cardinality at most $\aleph_s$.*

Proof. We start with the construction of a $\mathcal{S}$-model $\mathcal{M}_1^\mathcal{H}$ for $\mathcal{H}$ based on the term evaluation $\mathcal{TS}(\mathcal{S})^\beta$. This model may not be in the model class $\mathfrak{M}_*$ as it may not satisfy property $\mathfrak{q}$. However, we will be able to use Theorem 105 to obtain a model of $\mathcal{H}$ which is.

Note that since $\mathcal{H}$ is saturated, by Lemma 131, $\mathcal{H}$ satisfies $\overline{\nabla}_\neg$, $\overline{\nabla}_\vee$, and $\overline{\nabla}_\forall^\beta$.

The domain of type α of the evaluation $\mathcal{TS}(\mathcal{S})^\beta$ (cf. Definition 93 and Lemma 94) is $\mathit{cwff}_\alpha(\mathcal{S}){\downarrow_\beta}$, which has cardinality $\aleph_s$. To construct $\mathcal{M}_1^\mathcal{H}$, we simply need to give a valuation function for this evaluation. This valuation function should be a function $\upsilon\colon\ \mathit{cwff}_o{\downarrow_\beta} \to \{\mathtt{T}, \mathtt{F}\}$. We define

$$\upsilon(\mathbf{A}) := \begin{cases} \mathtt{T} \text{ if } \mathbf{A} \in \mathcal{H} \\ \mathtt{F} \text{ if } \mathbf{A} \notin \mathcal{H} \end{cases}$$

To show υ is a valuation, we must check the logical constants are interpreted appropriately. For each $\mathbf{A} \in cwff_o\downarrow_\beta$, we have $\upsilon(\neg\mathbf{A}) = \mathtt{T}$ iff $\upsilon(\mathbf{A}) = \mathtt{F}$ since $\neg\mathbf{A} \in \mathcal{H}$ iff $\mathbf{A} \notin \mathcal{H}$ by $\overline{\nabla}_\neg$. For each $\mathbf{A}, \mathbf{B} \in cwff_o\downarrow_\beta$, we have $\upsilon(\mathbf{A} \vee \mathbf{B}) = \mathtt{T}$ iff $\upsilon(\mathbf{A}) = \mathtt{T}$ or $\upsilon(\mathbf{B}) = \mathtt{T}$, since $(\mathbf{A} \vee \mathbf{B}) \in \mathcal{H}$ iff $\mathbf{A} \in \mathcal{H}$ or $\mathbf{B} \in \mathcal{H}$ by $\overline{\nabla}_\vee$. Finally, for each type α and $\mathbf{F} \in cwff_{\alpha\to o}(\mathcal{S})\downarrow_\beta$, $\overline{\nabla}_\forall^\beta$ implies $(\Pi^\alpha\mathbf{F}) \in \mathcal{H}$ iff $(\mathbf{F}\mathbf{A})\downarrow_\beta \in \mathcal{H}$ for every $\mathbf{A} \in cwff_\alpha(\mathcal{S})\downarrow_\beta$. Thus, we have $\upsilon(\Pi^\alpha\mathbf{F}) = \mathtt{T}$ iff $\upsilon(\mathbf{F}@^\beta\mathbf{A}) = \mathtt{T}$ for every $\mathbf{A} \in cwff_\alpha(\mathcal{S})\downarrow_\beta$.

This verifies $\mathcal{M}_1^\mathcal{H} := (cwff(\mathcal{S})\downarrow_\beta, @^\beta, \mathcal{E}^\beta, \upsilon)$ is a $\mathcal{S}$-model. Clearly, $\mathcal{M}_1^\mathcal{H} \models \mathcal{H}$ since $\upsilon(\mathbf{A}) = \mathtt{T}$ for every $\mathbf{A} \in \mathcal{H}$ by definition.

By Theorem 105, we have a congruence relation $\dot\sim$ on $\mathcal{M}_1^\mathcal{H}$ induced by Leibniz equality. Note that by Lemma 104 in the term model $\mathcal{M}_1^\mathcal{H}$, for every type α and every $\mathbf{A}, \mathbf{B} \in cwff_\alpha(\mathcal{S})\downarrow_\beta$, we have $\mathbf{A}_\alpha \dot\sim \mathbf{B}_\alpha$, iff $\upsilon(\mathbf{A} \doteq \mathbf{B}) = \mathtt{T}$, iff $(\mathbf{A} \doteq^\alpha \mathbf{B}) \in \mathcal{H}$. We have $\mathbf{A} \dot\sim \mathbf{B}$, iff $(\mathbf{A} \doteq^\alpha \mathbf{B}) \in \mathcal{H}$, iff (by $\overline{\nabla}_=^{\doteq}$) $(\mathbf{A} =^\alpha \mathbf{B}) \in \mathcal{H}$, iff $\upsilon(\mathcal{E}(=^\alpha)@\mathbf{A}@\mathbf{B}) = \mathtt{T}$.

Let $\mathcal{M} := \mathcal{M}_1^\mathcal{H}/_{\dot\sim}$. Each domain of this model has cardinality at most $\aleph_s$ as it is the quotient of a set of cardinality $\aleph_s$. By Theorem 105, we know the quotient model $\mathcal{M}$ models $\mathcal{H}$, satisfies property $\mathfrak{q}$, and is a model with primitive equality (if primitive equality is in the signature). Hence, $\mathcal{M} \in \mathfrak{M}_\beta$. Now, we can use Lemma 101 to check $\mathcal{M} \in \mathfrak{M}_*$ by checking certain properties of $\dot\sim$.

When $* \in \{\beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$, we must check that $\dot\sim$ has only two equivalence classes in $\mathcal{D}_o^\beta$. To show this, first note that $\overline{\nabla}_\mathfrak{b}$ holds for $\mathcal{H}$ by Lemma 132. Choose any β-normal $\mathbf{B} \in \mathcal{H}$. By $\boldsymbol{\nabla}_c$, $\neg\mathbf{B} \notin \mathcal{H}$. By $\overline{\nabla}_\mathfrak{b}$, for every $\mathbf{A} \in cwff_o\downarrow_\beta$ either $(\mathbf{A} \doteq^o \mathbf{B})$ or $(\mathbf{A} \doteq^o \neg\mathbf{B})$. That is, in $\mathcal{M}_1^\mathcal{H}$, for every $\mathbf{A} \in cwff_o\downarrow_\beta$ we either have $\mathbf{A} \dot\sim \mathbf{B}$ or $\mathbf{A} \dot\sim \neg\mathbf{B}$. So, we know $\mathcal{M}$ satisfies property $\mathfrak{b}$.

When $* \in \{\beta\eta, \beta\eta\mathfrak{b}\}$, the fact that $\dot\sim$ satisfies property η follows from $\overline{\nabla}_\eta$ which holds for $\mathcal{H}$ by Lemma 133.

When $* \in \{\beta\xi, \beta\xi\mathfrak{b}\}$, we must show that $\dot\sim$ satisfies property ξ. Let $\mathbf{M}, \mathbf{N} \in wff_\beta(\mathcal{S})$, an assignment φ and a variable X_α be given. Suppose $\mathcal{E}_{\varphi,[\mathbf{A}/X]}(\mathbf{M}) \dot\sim \mathcal{E}_{\varphi,[\mathbf{A}/X]}(\mathbf{N})$ for every $\mathbf{A} \in cwff_\alpha(\mathcal{S})\downarrow_\beta$. Let θ be the substitution defined by $\theta(Y) := \varphi(Y)$ for each variable $Y \in (\mathbf{Free}(\mathbf{M}) \cup \mathbf{Free}(\mathbf{N})) \setminus \{X\}$. So, for each $\mathbf{A} \in cwff_\alpha(\mathcal{S})\downarrow_\beta$,

$$([\mathbf{A}/X]\theta(\mathbf{M}))\downarrow_\beta = \mathcal{E}_{\varphi,[\mathbf{A}/X]}(\mathbf{M}) \dot\sim \mathcal{E}_{\varphi,[\mathbf{A}/X]}(\mathbf{N}) = ([\mathbf{A}/X]\theta(\mathbf{N}))\downarrow_\beta$$

That is, $([\mathbf{A}/X]\theta(\mathbf{M}) \doteq^\beta [\mathbf{A}/X]\theta(\mathbf{N}))^{\downarrow\beta} \in \mathcal{H}$ for every $\mathbf{A} \in cwff_\alpha(\mathcal{S})\downarrow_\beta$. By $\overline{\nabla}_\xi^\beta$ (Lemma 134), we have $((\lambda X_\bullet\theta(\mathbf{M})) \doteq^{\alpha\to\beta} \lambda X_\bullet\theta(\mathbf{N}))^{\downarrow\beta} \in \mathcal{H}$. So,

$$\mathcal{E}_\varphi(\lambda X_\bullet\mathbf{M}) = (\lambda X_\bullet\theta(\mathbf{M}))\downarrow_\beta \dot\sim (\lambda X_\bullet\theta(\mathbf{N}))\downarrow_\beta = \mathcal{E}_\varphi(\lambda X_\bullet\mathbf{N}).$$

Thus, $\dot\sim$ satisfies ξ as desired.

When $* \in \{\beta\mathfrak{f}, \beta\mathfrak{f}\mathfrak{b}\}$, we must show $\dot\sim$ is functional. Let α and β be types and $\mathbf{G}, \mathbf{H} \in cwff_{\alpha\to\beta}(\mathcal{S})\downarrow_\beta$. We need to show $\mathbf{G} \dot\sim \mathbf{H}$ iff $(\mathbf{G}\mathbf{A})\downarrow_\beta \dot\sim (\mathbf{H}\mathbf{A})\downarrow_\beta$ for every $\mathbf{A} \in cwff_\alpha(\mathcal{S})\downarrow_\beta$. This follows directly from $\overline{\nabla}_\mathfrak{f}^\beta$.

This verifies the fact that $\mathcal{M} \in \mathfrak{M}_*$ whenever $\mathcal{H} \in \mathfrak{Hint}_*$.

Theorem 140 (Model Existence Theorem) *Let $\mathscr{C}$ be a saturated abstract consistency class and let $\Phi \in \mathscr{C}$ be a sufficiently pure set of sentences. For all $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ we have: If $\mathscr{C}$ is an $\mathfrak{Acc}_*$ (cf. Definition 116), then there exists a model $\mathcal{M} \in \mathfrak{M}_*$ (cf. Definition 81) that satisfies Φ. Furthermore, each domain of $\mathcal{M}$ has cardinality at most $\aleph_s$.*

Proof. Let $\mathscr{C}$ be an abstract consistency class. We can assume without loss of generality (cf. Lemma 125) that $\mathscr{C}$ is compact, so the preconditions of Lemma 138 are met. Therefore, there exists a saturated Hintikka set $\mathcal{H} \in \mathfrak{Hint}_*$ with $\Phi \subseteq \mathcal{H}$. The proof is completed by a simple appeal to the Theorem 139.

Remark 141 *The model existence theorems show there are "enough" models in each class $\mathfrak{M}_*$ to model sufficiently pure sets in saturated abstract consistency classes in $\mathfrak{Acc}_*$. These results are abstract forms of completeness. To complete the analysis, we can show abstract forms of soundness. One way to show this is to define a class of sentences*

$$\mathscr{C}^* := \{\Phi \subseteq cwff_o(\mathcal{S}) | \exists \mathcal{M} \in \mathfrak{M}_{*\blacksquare}\mathcal{M} \models \Phi\}$$

for each $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ and show $\mathscr{C}^*$ is a (saturated) $\mathfrak{Acc}_*$. We only sketch the proof here.

The fact that each $\mathscr{C}^*$ satisfy ∇_c, ∇_β, $\nabla_\neg$, $\nabla_\vee$, $\nabla_\wedge$, $\nabla_\forall$, and ∇_{sat} is straightforward. The proof that $\nabla_\exists$ holds has the technical difficulty that one must modify the evaluation of a parameter. Showing $\nabla_\mathfrak{b}$ [$\nabla_{\beta\eta}$] holds when considering models with property $\mathfrak{b}$ [η] is also easy.

When showing $\nabla_\mathfrak{f}$ holds in $\mathscr{C}^{\beta\mathfrak{f}}$ [$\mathscr{C}^{\beta\mathfrak{f}\mathfrak{b}}$], one sees the importance of assuming property $\mathfrak{q}$ holds. Suppose $\Phi \in \mathscr{C}^{\beta\mathfrak{f}}$ [$\mathscr{C}^{\beta\mathfrak{f}\mathfrak{b}}$] and $\neg(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G}) \in \Phi$. Then there is a model $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_{\beta\mathfrak{f}}$ [$\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}$] such that $\mathcal{M} \models \Phi$. This implies $\mathcal{M} \models \neg(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G})$. Without using property $\mathfrak{q}$, it follows by Lemma 110(1) that $\mathcal{E}(\mathbf{F}) \neq \mathcal{E}(\mathbf{G})$. By functionality, there is an $\mathsf{a} \in \mathcal{D}_\alpha$ such that $\mathcal{E}(\mathbf{F})@\mathsf{a} \neq \mathcal{E}(\mathbf{G})@\mathsf{a}$. Let φ be any assignment into $\mathcal{M}$. Then $\mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{F}X) \neq \mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{G}X)$. Now, using property $\mathfrak{q}$, we can conclude $\mathcal{M}_{\varphi,[\mathsf{a}/X]} \models \neg((\mathbf{F}X) \doteq^\beta (\mathbf{G}X))$ by Lemma 110(2). Let $w_\alpha \in \mathcal{S}$ be a parameter that does not occur in any sentence of Φ. With some technical work which we omit, one can change the evaluation function to $\mathcal{E}'$ so that $\mathcal{E}'(\mathbf{A}) = \mathcal{E}(\mathbf{A})$ for all $\mathbf{A} \in \Phi$, and $\mathcal{E}'(w) = \mathsf{a}$. In the new model $\mathcal{M}' = (\mathcal{D}, @, \mathcal{E}', \upsilon)$, we have $\mathcal{M}' \models \Phi$ and $\mathcal{M}' \models \neg(\mathbf{F}w \doteq^\beta \mathbf{G}w)$. Also, $\mathcal{M}' \in \mathfrak{Acc}_{\beta\mathfrak{f}}$ [$\mathfrak{Acc}_{\mathfrak{f}\mathfrak{b}}$]. This shows $\Phi * \neg(\mathbf{F}w \doteq^\beta \mathbf{G}w) \in \mathscr{C}^{\beta\mathfrak{f}}$ [$\mathscr{C}^{\beta\mathfrak{f}\mathfrak{b}}$]. The proof that ∇_ξ holds in $\mathscr{C}^{\beta\xi}$ [$\mathscr{C}^{\beta\xi\mathfrak{b}}$] is analogous.

We have now established a set of proof-theoretic conditions that are sufficient to guarantee the existence of a model.

16 Higher-Order Natural Deduction Calculi

Higher-order natural deduction calculi form the logical basis for semi-automated theorem proving systems such as HOL [GM93], Isabelle [NPW02], or Ωmega [Sea02].

We define different higher-order natural deduction calculi, link them to the higher-order model classes defined before, and analyse their soundness and completeness. For the latter we will employ the abstract consistency method and model existence theorems which will formally introduce later.

Definition 142 (The Calculi $\mathfrak{NK}_*$) *The calculus $\mathfrak{NK}_\beta$ consists of the inference rules[3] in Figure 5 for the provability judgment $\Vdash$ between sets of sentences Φ and sentences $\mathbf{A}$. (We write $\Vdash \mathbf{A}$ for $\emptyset \Vdash \mathbf{A}$.) The rule $\mathfrak{NK}(\beta)$ incorporates β-equality into $\Vdash$. The others characterize the semantics of the connectives and quantifiers.*

For $* \in \{\beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ we obtain the calculus $\mathfrak{NK}_*$ by adding the rules shown in Figure 6 when specified in $*$.

Remark 143 *It is worth noting that there is a derivation of $\Vdash \mathbf{T}_o$ (i.e., $\Vdash \forall P_o.P \vee \neg P$) which only uses the rules in Figure 5. Let p be a parameter of type o. A derivation of $\neg(p \vee \neg p) \Vdash (p \vee \neg p)$ is shown in Figure 7. Using $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\neg E)$, we obtain $\neg(p \vee \neg p) \Vdash \mathbf{F}_o$. So, we can conclude $\Vdash (p \vee \neg p)$ using $\mathfrak{NK}(Contr)$. Finally, we obtain a derivation of $\Vdash \mathbf{T}_o$ using $\mathfrak{NK}(\Pi I)^p$. Hence, $\Vdash \mathbf{T}_o$ is derivable in each calculus $\mathfrak{NK}_*$ where $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Also, we can apply the rule $\mathfrak{NK}(\Pi E)$ to the end of this derivation with any sentence $\mathbf{A}$ to derive $\Vdash (\mathbf{A} \vee \neg\mathbf{A})$.*

Note that $\mathfrak{NK}_\beta$ and $\mathfrak{NK}_{\beta\mathfrak{f}\mathfrak{b}}$ correspond to the extremes of the model classes in our landscape of model classes. Standard models do not admit (recursively axiomatizable) calculi that are sound and complete, $\mathfrak{NK}_{\beta\mathfrak{f}\mathfrak{b}}$ is complete for Henkin models, and $\mathfrak{NK}_\beta$ is complete for $\mathfrak{M}_\beta$. We will now show soundness and completeness of each $\mathfrak{NK}_*$ with respect to each corresponding model class $\mathfrak{M}_*$ by using the model existence theorems in Section 15.

[3] $\mathbf{F}_o$ is defined to be $\neg(\forall P_{o\bullet}(P \vee \neg P))$ and it is easy to show that $\mathcal{M} \not\models \mathbf{F}_o$ for each $\mathcal{S}$-model $\mathcal{M}$

$$\dfrac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(Hyp) \qquad \dfrac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\beta)$$

$$\dfrac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg\mathbf{A}}\,\mathfrak{NK}(\neg I) \qquad \dfrac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\neg E)$$

$$\dfrac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\,\mathfrak{NK}(\vee I_L) \qquad \dfrac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\,\mathfrak{NK}(\vee I_R)$$

$$\dfrac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\vee E)$$

$$\dfrac{\Phi \Vdash \mathbf{G}w_\alpha \quad w \text{ parameter not occurring in } \Phi \text{ or } \mathbf{G}}{\Phi \Vdash \Pi^\alpha \mathbf{G}}\,\mathfrak{NK}(\Pi I)^w$$

$$\dfrac{\Phi \Vdash \Pi^\alpha \mathbf{G}}{\Phi \Vdash \mathbf{G}\mathbf{A}}\,\mathfrak{NK}(\Pi E) \qquad \dfrac{\Phi * \neg\mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(Contr)$$

Fig. 5. Inference Rules for $\mathfrak{NK}_\beta$

$$\dfrac{\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\eta) \qquad \dfrac{\Phi \Vdash \forall X_\alpha.\mathbf{M} \doteq^\beta \mathbf{N}}{\Phi \Vdash (\lambda X_\alpha.\mathbf{M}) \doteq^{\alpha\to\beta} (\lambda X_\alpha.\mathbf{N})}\,\mathfrak{NK}(\xi)$$

$$\dfrac{\Phi \Vdash \forall X_\alpha.\mathbf{G}X \doteq^\beta \mathbf{H}X}{\Phi \Vdash \mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H}}\,\mathfrak{NK}(\mathfrak{f})$$

$$\dfrac{\Phi * \mathbf{A} \Vdash \mathbf{B} \quad \Phi * \mathbf{B} \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \doteq^o \mathbf{B}}\,\mathfrak{NK}(\mathfrak{b})$$

Fig. 6. Extensional Inference Rules

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(p \vee \neg p), p \Vdash \neg(p \vee \neg p)}\,\mathfrak{NK}(Hyp) \qquad \dfrac{\dfrac{}{\neg(p \vee \neg p), p \Vdash p}\,\mathfrak{NK}(Hyp)}{\neg(p \vee \neg p), p \Vdash (p \vee \neg p)}\,\mathfrak{NK}(\vee I_L)}{\neg(p \vee \neg p), p \Vdash \mathbf{F}_o}\,\mathfrak{NK}(\neg E)}{\neg(p \vee \neg p) \Vdash \neg p}\,\mathfrak{NK}(\neg I)}{\neg(p \vee \neg p) \Vdash (p \vee \neg p)}\,\mathfrak{NK}(\vee I_R)$$

Fig. 7. Derivation of $\neg(p \vee \neg p) \Vdash (p \vee \neg p)$

Theorem 144 (Soundness) *$\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*$ such that $\mathcal{M} \models \Phi$.*

Proof. This can be shown by a simple induction on the derivation of $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$. We distinguish based on the last rule of the derivation. The only base case is $\mathfrak{NK}(Hyp)$, which is trivial since $\mathcal{M} \models \mathbf{C}$ whenever $\mathcal{M} \models \Phi$ and $\mathbf{C} \in \Phi$.

$\mathfrak{NK}(\beta)$ Suppose $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash \mathbf{A}$ and $\mathbf{A} =_\beta \mathbf{C}$. Let $\mathcal{M} \in \mathfrak{M}_*$ be a model of Φ. By induction, we know $\mathcal{M} \models \mathbf{A}$ and so $\mathcal{M} \models \mathbf{C}$ since $\mathcal{S}$-Evaluations respect β-Equality.

$\mathfrak{NK}(Contr)$ Suppose $\mathcal{M} \in \mathfrak{M}_*$, $\mathcal{M} \models \Phi$ and $\Phi \Vdash \mathbf{C}$ follows from $\Phi * \neg\mathbf{C} \Vdash \mathbf{F}_o$. By Lemma 106, $\mathcal{M} \not\models \mathbf{F}_o$. So, we must have $\mathcal{M} \not\models \neg\mathbf{C}$. Hence, $\mathcal{M} \models \mathbf{C}$.

$\mathfrak{NK}(\neg I)$ Analogous to $\mathfrak{NK}(Contr)$.

$\mathfrak{NK}(\neg E)$ Suppose $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash \neg\mathbf{A}$ and $\Phi \Vdash \mathbf{A}$. By induction, any model in $\mathfrak{M}_*$ of Φ would have to model both $\mathbf{A}$ and $\neg\mathbf{A}$. So, there is no such model of Φ and we are done.

$\mathfrak{NK}(\vee I_L)$ Suppose $\mathcal{M} \in \mathfrak{M}_*$, $\mathcal{M} \models \Phi$, $\mathbf{C}$ is $(\mathbf{A} \vee \mathbf{B})$ and $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash \mathbf{A}$. By induction, $\mathcal{M} \models \mathbf{A}$ and so $\mathcal{M} \models (\mathbf{A} \vee \mathbf{B})$.

$\mathfrak{NK}(\vee I_R)$ Analogous to $\mathfrak{NK}(\vee I_L)$.

$\mathfrak{NK}(\vee E)$ Suppose $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash (\mathbf{A} \vee \mathbf{B})$, $\Phi * \mathbf{A} \Vdash \mathbf{C}$ and $\Phi * \mathbf{B} \Vdash \mathbf{C}$. Let $\mathcal{M} \in \mathfrak{M}_*$ be a model of Φ. By induction, $\mathcal{M} \models \mathbf{A} \vee \mathbf{B}$. If $\mathcal{M} \models \mathbf{A}$, then by induction $\mathcal{M} \models \mathbf{C}$ since $\Phi * \mathbf{A} \Vdash \mathbf{C}$. If $\mathcal{M} \models \mathbf{B}$, then by induction $\mathcal{M} \models \mathbf{C}$ since $\Phi * \mathbf{B} \Vdash \mathbf{C}$. In either case, $\Phi \Vdash \mathbf{C}$.

$\mathfrak{NK}(\Pi I)$ Suppose $\mathbf{C}$ is $(\Pi^\alpha \mathbf{G})$ and $\Phi \Vdash (\Pi^\alpha \mathbf{G})$ follows from $\Phi \Vdash \mathbf{G}w$ where w_α is a parameter which does not occur in any sentence of Φ or in $\mathbf{G}$. Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_*$ be a model of Φ. Assume $\mathcal{M} \not\models \Pi^\alpha \mathbf{G}$. Then there must be some $\mathsf{a} \in \mathcal{D}_\alpha$ such that $\upsilon(\mathcal{E}(\mathbf{G})@\mathsf{a}) = \mathtt{F}$. From the evaluation function $\mathcal{E}$, one can define another evaluation function $\mathcal{E}'$ such that $\mathcal{E}'(w) = \mathsf{a}$ and $\mathcal{E}'_\varphi(\mathbf{A}_\alpha) = \mathcal{E}_\varphi(\mathbf{A}_\alpha)$ if w does not occur in $\mathbf{A}$. Let $\mathcal{M}' := (\mathcal{D}, @, \mathcal{E}', \upsilon)$. One can check $\mathcal{M}' \in \mathfrak{M}_*$ using the fact that $\mathcal{M} \in \mathfrak{M}_*$. Since $\mathcal{M}' \models \Phi$, by induction we have $\mathcal{M}' \models \mathbf{G}w$. This contradicts $\upsilon(\mathcal{E}'(\mathbf{G})@\mathsf{a}) = \upsilon(\mathcal{E}(\mathbf{G})@\mathsf{a}) = \mathtt{F}$. Thus, $\mathcal{M} \models \Pi^\alpha \mathbf{G}$.

$\mathfrak{NK}(\Pi E)$ Suppose $\mathbf{C}$ is $(\mathbf{G}\mathbf{A})$ and $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash (\Pi^\alpha \mathbf{G})$. Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_*$ be a model of Φ. By induction, $\mathcal{M} \models (\Pi^\alpha \mathbf{G})$ and thus $\upsilon(\mathcal{E}(\mathbf{G}))@\mathsf{a} = \mathtt{T}$ for every $\mathsf{a} \in \mathcal{D}_\alpha$. In particular, $\mathcal{M} \models \mathbf{G}\mathbf{A}$.

We now check soundness of the rules in Figure 6 with respect to their model classes:

$\mathfrak{NK}(\eta)$ Analogous to $\mathfrak{NK}(\beta)$ using property η.

$\mathfrak{NK}(\xi)$ Suppose $\mathbf{C}$ is $(\lambda X_\alpha.\mathbf{M}) \doteq^{\alpha\to\beta} (\lambda X_\alpha.\mathbf{N})$ and $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash \forall X_\alpha.\mathbf{M} \doteq^\beta \mathbf{N}$. Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_*$ be a model of Φ. By induction, we have $\mathcal{M} \models \forall X_\alpha.\mathbf{M} \doteq^\beta \mathbf{N}$. So, for any assignment φ and $\mathsf{a} \in \mathcal{D}_\alpha$, $\mathcal{M} \models_{\varphi,[\mathsf{a}/X]} \mathbf{M} \doteq^\beta \mathbf{N}$. Note that property $\mathfrak{q}$ holds in $\mathcal{M}$ since $\mathcal{M} \in \mathfrak{M}_*$ (cf. Definition 81). By Lemma 110(2), $\mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{M}) = \mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{N})$. By property ξ, $\mathcal{E}_\varphi(\lambda X_\alpha.\mathbf{M}) = \mathcal{E}_\varphi(\lambda X_\alpha.\mathbf{N})$ and thus $\mathcal{M} \models \mathbf{C}$ by Lemma 110(1).

$\mathfrak{NK}(\mathfrak{f})$ Suppose $\mathbf{C}$ is $\mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H}$ and $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash \forall X_\alpha.\mathbf{G}X \doteq^\beta \mathbf{H}X$. Let $\mathcal{M} \in \mathfrak{M}_*$ be a model of Φ. By induction, we know $\mathcal{M} \models \forall X_\alpha.\mathbf{G}X \doteq^\beta \mathbf{H}X$. Note that property $\mathfrak{q}$ holds for $\mathcal{M}$ since $\mathcal{M} \in \mathfrak{M}_*$. By Theorem 111(3), we must have $\mathcal{M} \models (\mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H})$.

$\mathfrak{NK}(\mathfrak{b})$ Suppose $\mathbf{C}$ is $\mathbf{A} \doteq^o \mathbf{B}$ and $\Phi \Vdash \mathbf{C}$ follows from $\Phi * \mathbf{A} \Vdash \mathbf{B}$ and $\Phi * \mathbf{B} \Vdash \mathbf{A}$. Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_*$ be a model of Φ. If $\mathcal{M} \models \mathbf{A}$, then $\mathcal{M} \models \mathbf{B}$ by induction. If $\mathcal{M} \models \mathbf{B}$, then $\mathcal{M} \models \mathbf{A}$ by induction. These facts imply $\upsilon(\mathcal{E}(\mathbf{A})) = \upsilon(\mathcal{E}(\mathbf{B}))$. By Lemma 108, we have $\mathcal{M} \models (\mathbf{A} \Leftrightarrow \mathbf{B})$. By Theorem 111(4), we must have $\mathcal{M} \models (\mathbf{A} \doteq^o \mathbf{B})$.

Definition 145 ($\mathfrak{NK}_*$-Consistent) *A set of sentences Φ is* $\mathfrak{NK}_*$-inconsistent *if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{F}_o$, and* $\mathfrak{NK}_*$-consistent *otherwise.*

Now, we use the model existence theorems for $\mathcal{HOL}$ to give short and elegant proofs of completeness for $\mathfrak{NK}_*$.

Lemma 146 *The class $\mathscr{C}^* := \{\Phi \subseteq cwff_o(\mathcal{S}) \mid \Phi \text{ is } \mathfrak{NK}_*\text{-consistent}\}$ is a saturated $\mathfrak{Acc}_*$.*

Proof. Obviously $\mathscr{C}^*$ is closed under subsets, since any subset of an $\mathfrak{NK}_*$-consistent set is $\mathfrak{NK}_*$-consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

- ∇_c Suppose $\mathbf{A}, \neg\mathbf{A} \in \Phi$. We have $\Phi \Vdash \mathbf{F}_o$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\neg E)$.
- ∇_β Let $\mathbf{A} \in \Phi$, $\mathbf{A} =_\beta \mathbf{B}$ and $\Phi * \mathbf{B}$ be $\mathfrak{NK}_*$-inconsistent. That is, $\Phi * \mathbf{B} \Vdash \mathbf{F}_o$. By $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{B}$. Since $\mathbf{A} \in \Phi$, we know $\Phi \Vdash \mathbf{B}$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\beta)$. Using $\mathfrak{NK}(\neg E)$, we know $\Phi \Vdash \mathbf{F}_o$ and hence Φ is $\mathfrak{NK}_*$-inconsistent.
- $\nabla_\neg$ Suppose $\neg\neg\mathbf{A} \in \Phi$ and $\Phi * \mathbf{A}$ is $\mathfrak{NK}_*$-inconsistent. From $\Phi * \mathbf{A} \Vdash \mathbf{F}_o$ and $\mathfrak{NK}(\neg I)$, we have $\Phi \Vdash \neg\mathbf{A}$. Since $\neg\neg\mathbf{A} \in \Phi$, we can apply $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\neg E)$ to obtain $\Phi \Vdash \mathbf{F}_o$.
- $\nabla_\vee$ Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\vee E)$, we have $\Phi \Vdash \mathbf{F}_o$.
- $\nabla_\wedge$ Suppose $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and $\Phi * \neg\mathbf{A} * \neg\mathbf{B}$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_R)$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{F}_o$. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_L)$, we have $\Phi \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, Φ is $\mathfrak{NK}_*$-inconsistent.
- $\nabla_\forall$ Suppose $(\Pi^\alpha \mathbf{G}) \in \Phi$ and $\Phi * (\mathbf{GA})$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(\neg I)$, $\Phi \Vdash \neg(\mathbf{GA})$. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\Pi E)$, $\Phi \Vdash \mathbf{GA}$. Finally, $\mathfrak{NK}(\neg E)$ implies $\Phi \Vdash \mathbf{F}_o$.
- $\nabla_\exists$ Suppose $\neg(\Pi^\alpha \mathbf{G}) \in \Phi$, w_α is a parameter which does not occur in Φ, and $\Phi * \neg(\mathbf{G}w)$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Contr)$, $\Phi \Vdash \mathbf{G}w$. By $\mathfrak{NK}(\Pi I)^w$, $\Phi \Vdash (\Pi^\alpha \mathbf{G})$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\Pi^\alpha \mathbf{G}) \in \Phi$, Φ is $\mathfrak{NK}_*$-inconsistent.
- ∇_{sat} Let $\Phi * \mathbf{A}$ and $\Phi * \neg\mathbf{A}$ be $\mathfrak{NK}_*$-inconsistent. We show that Φ is $\mathfrak{NK}_*$-inconsistent. Using $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{A}$ and $\Phi \Vdash \neg\neg\mathbf{A}$. By $\mathfrak{NK}(\neg E)$, we have $\Phi \Vdash \mathbf{F}_o$.

Thus we have shown that $\mathscr{C}^\beta$ is saturated and in $\mathfrak{Acc}_\beta$. Now let us check the conditions for the additional properties η, ξ, $\mathfrak{f}$, and $\mathfrak{b}$.

- $\nabla_{\beta\eta}$ If $*$ includes η, then the proof proceeds as in ∇_β above, but with the rule $\mathfrak{NK}(\eta)$.
- ∇_ξ Suppose $*$ includes ξ, $\neg(\lambda X_\blacksquare \mathbf{M} \doteq^{\alpha\to\beta} \lambda X_\blacksquare \mathbf{N}) \in \Phi$, and $\Phi * \neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N})$ is $\mathfrak{NK}_*$-inconsistent for some parameter w_α which does not occur in any sentence of Φ. By $\mathfrak{NK}(Contr)$, we have $\Phi \Vdash ([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N})$. By $\mathfrak{NK}(\beta)$, we have $\Phi \Vdash ((\lambda X_\blacksquare \mathbf{M} \doteq^\beta \mathbf{N})w)$. By $\mathfrak{NK}(\Pi I)$, $\Phi \Vdash (\forall X.\mathbf{M} \doteq^\beta \mathbf{N})$. By $\mathfrak{NK}(\xi)$, $\Phi \Vdash (\lambda X_\blacksquare \mathbf{M} \doteq^{\alpha\to\beta} \lambda X_\blacksquare \mathbf{N})$. By $\mathfrak{NK}(\neg E)$, Φ is $\mathfrak{NK}_*$-inconsistent.
- $\nabla_\mathfrak{f}$ This case is analogous to the previous one, generalizing $\lambda X_\blacksquare \mathbf{M} \doteq \lambda X_\blacksquare \mathbf{N}$ to arbitrary $\mathbf{G} \doteq \mathbf{H}$ and using the extensionality rule $\mathfrak{NK}(\mathfrak{f})$ instead of $\mathfrak{NK}(\xi)$.
- $\nabla_\mathfrak{b}$ Suppose $*$ includes $\mathfrak{b}$. Assume that $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$ but both $\Phi * \neg\mathbf{A} * \mathbf{B} \notin \mathscr{C}^*$ and $\Phi * \mathbf{A} * \neg\mathbf{B} \notin \mathscr{C}^*$. So both are $\mathfrak{NK}_*$-inconsistent and we have $\Phi * \mathbf{A} \Vdash \mathbf{B}$ and $\Phi * \mathbf{B} \Vdash \mathbf{A}$ by $\mathfrak{NK}(Contr)$. By $\mathfrak{NK}(\mathfrak{b})$, we have $\Phi \Vdash (\mathbf{A} \doteq^o \mathbf{B})$. Since $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$, Φ is $\mathfrak{NK}_*$-inconsistent.

Theorem 147 (Henkin's Theorem for $\mathfrak{NK}_*$) *Let $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Every sufficiently pure $\mathfrak{NK}_*$-consistent set of sentences has an $\mathfrak{M}_*$-model.*

Proof. Let Φ be a sufficiently pure $\mathfrak{NK}_*$-consistent set of sentences. By Theorem 146 we know that the class of sets of $\mathfrak{NK}_*$-consistent sentences constitute a saturated $\mathfrak{Acc}_*$, thus the Model Existence Theorem (Theorem 140) guarantees an $\mathfrak{M}_*$ model for Φ.

Corollary 148 (Completeness Theorem for $\mathfrak{NK}_*$) *Let Φ be a sufficiently pure set of sentences, $\mathbf{A}$ be a sentence, and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. If $\mathbf{A}$ is valid in all models $\mathcal{M} \in \mathfrak{M}_*$ that satisfy Φ, then $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{A}$.*

Proof. Let $\mathbf{A}$ be given such that $\mathbf{A}$ is valid in all $\mathfrak{M}_*$ models that satisfy Φ. So, $\Phi * \neg\mathbf{A}$ is unsatisfiable in $\mathfrak{M}_*$. Since only finitely many constants occur in $\neg\mathbf{A}$, $\Phi * \neg\mathbf{A}$ is sufficiently pure. So, $\Phi * \neg\mathbf{A}$ must be $\mathfrak{NK}_*$-inconsistent by Henkin's theorem above. Thus, $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{A}$ by $\mathfrak{NK}(Contr)$.

Finally we can use the completeness theorems obtained so far to prove a compactness theorem for our semantics.

Corollary 149 (Compactness Theorem for $\mathfrak{NK}_*$) *Let Φ be a sufficiently pure set of sentences and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Φ has an $\mathfrak{M}_*$-model iff every finite subset of Φ has an $\mathfrak{M}_*$-model.*

Proof. If Φ has no $\mathfrak{M}_*$-model, then by Theorem 147 Φ is $\mathfrak{NK}_*$-inconsistent. Since every $\mathfrak{NK}_*$-proof is finite, this means some finite subset Ψ of Φ is $\mathfrak{NK}_*$-inconsistent. Hence, Ψ has no $\mathfrak{M}_*$-model.

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