

Semantic Techniques for Cut-Elimination in Higher-Order Logics

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CHRISTOPH BENZMÜLLER, CHAD E. BROWN, AND MICHAEL KOHLHASE

Abstract. This paper is part of an ongoing effort to examine the role of extensionality in higher-order logic and provide tools for analyzing higher-order calculi.

In an earlier paper, we have presented eight classes of higher order models with respect to various combinations of Boolean extensionality and three forms of functional extensionality. Furthermore, we have developed a methodology of abstract consistency methods (by providing the necessary model existence theorems) needed to analyze completeness of higher-order calculi with respect to these model classes. This framework, employs a strong saturation criterion which prevents analysis of, e.g., the deductive power of machine-oriented calculi.

In this paper we extend our saturated abstract consistency approach and obtain analogous model existence results without assuming saturation. For this, we replace the saturation conditions by a set of weaker *acceptability conditions* which are sufficient to prove model existence. We further show that saturation can be as hard to prove as cut elimination. We apply our extended abstract consistency approach to show completeness of five different sequent calculi (with varying strength regarding Boolean and functional extensionality reasoning) with respect to five of our eight model classes. We conclude that cut-elimination holds for each of these five calculi.

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§1. Motivation. In [4], we have introduced and studied eight different model classes (including Henkin models) for classical type theory which generalize the notion of standard models and which allow for complete (recursively axiomatizable) calculi. Unfortunately, the model existence theorems presented there are too weak to support completeness proofs for many higher-order machine-oriented calculi, such as higher-order resolution [9, 5] and tableau-based calculi [3, 10], since they assume saturation of abstract consistency classes. The *saturation* condition is fulfilled by an abstract consistency class Γ_Σ if for all $\Phi \in \Gamma_\Sigma$ and for all (closed) formulae \mathbf{A} we have $\Phi \cup \{\mathbf{A}\} \in \Gamma_\Sigma$ or $\Phi \cup \{\neg\mathbf{A}\} \in \Gamma_\Sigma$. We will show that saturation corresponds to admissibility of cut (which is hard to show for higher-order calculi). In fact, we will show the stronger result that admissibility of cut for certain sequent calculi is equivalent to the existence of saturated extensions of abstract consistency classes.

Cut-elimination states that every proof in a sequent calculus \mathcal{C} that employs a cut rule can be transformed into an alternative \mathcal{C} proof without cut. Using induction, cut-elimination can easily be reduced to the problem admissibility of the cut rule (essentially eliminating one application of the cut rule). In [1], Peter Andrews applies his Unifying Principle to cut-elimination in a non-extensional sequent calculus: He proves the calculus complete (relative to a Hilbert style calculus \mathfrak{T}_β) without the cut rule and concludes that cut-elimination holds for this calculus. Because of the saturation condition we cannot obtain extensional cut-elimination theorems (as in [15, 16]) using the model existence theorem of [4].

In this paper we present strong model existence theorems for some of the model classes presented in [4] using extensions of Peter Andrews' *v*-complexes [1]. For this we weaken the saturation assumption from [4] to certain *acceptability* conditions for abstract consistency classes. We use sequent calculi to demonstrate that the acceptability conditions correspond to reasonable rules in a cut-free, machine-oriented calculus. Using the stronger versions of the model existence theorems, we can show the cut-free sequent calculi are complete. Hence cut-elimination holds for each sequent calculus. In general, to show completeness of a higher-order machine-oriented calculus \mathcal{C} , one can consider the class Γ of \mathcal{C} -irrefutable sets of sentences and show that it is an acceptable abstract consistency class.

If we restrict our attention to five of the eight cases, then we can use the model existence theorem to show that every acceptable abstract consistency class can be extended to a saturated one. In fact, we can use each model class to define a single saturated abstract consistency class which extends all acceptable abstract consistency classes. We give examples to show that without acceptability there are abstract consistency classes which have no saturated extension (in spite of the fact that the saturation condition does hold with respect to atomic formulae).

In Section 2 we will review the necessary material from [4] to make this article self-contained. Section 3 explores the connection between saturation and cut elimination for sequent calculi. In Section 4, we motivate and present our set of *acceptability* conditions that are weaker than saturation, but sufficient to prove model existence. In Section 5 we define cut-free sequent calculi which we will show complete. In Section 6 we introduce acceptable Hintikka sets as a basis for

model construction and we define and study the notion of Hintikka compatibility. We then introduce methods for constructing models in Section 7. This framework is then exploited in Section 8 for the proof of the Strong Model Existence Theorem that forms a primary result of this paper. Together with the material in Section 5, this yields prototypical machine-oriented calculi (sound, complete, and cut-free) for all of the semantic notations identified in [4]. Finally, in Section 9 we prove another important result in the paper: the Saturated Extension Theorem.

§2. Abstract Consistency and Saturation in Higher-Order Logic. We review the fundamental framework from [4] (which can be consulted for details).

2.1. Higher-Order Logic (\mathcal{HOL}). As in [7], we formulate higher-order logic (\mathcal{HOL}) based on the simply typed λ -calculus. The set of simple types \mathcal{T} is freely generated from basic types o and ι using the function type constructor \rightarrow .

We start with a set \mathcal{V} of (typed) variables (denoted by $X_\alpha, Y, Z, X_\beta^1, X_\gamma^2, \dots$) and a signature Σ of (typed) constants (denoted by $c_\alpha, f_{\alpha \rightarrow \beta}, \dots$). We let \mathcal{V}_α (Σ_α) denote the set of variables (constants) of type α . The signature Σ of constants includes the logical constants $\neg_{o \rightarrow o}$, $\vee_{o \rightarrow o \rightarrow o}$ and $\Pi_{(\alpha \rightarrow o) \rightarrow o}^\alpha$ for each type α . All other constants in Σ are called parameters. As in [4], we assume there is an infinite cardinal \aleph_s such that the cardinality of Σ_α is \aleph_s for each type α (cf. Remark 3.16 in [4]). The set of \mathcal{HOL} -formulae (or terms) are constructed from typed variables and constants using application and λ -abstraction. We let $wff_\alpha(\Sigma)$ be the set of all terms of type α and $wff(\Sigma)$ be the set of all terms.

We use vector notation to abbreviate k -fold applications and abstractions as $\overline{\mathbf{A}}\overline{\mathbf{U}}^k$ and $\lambda\overline{\mathbf{X}}^k.\mathbf{A}$, respectively. We also use Church's dot notation so that $\mathbf{A}.$ stands for a (missing) left bracket whose mate is as far to the right as possible (consistent with given brackets). We use infix notation $\mathbf{A} \vee \mathbf{B}$ for $((\vee \mathbf{A})\mathbf{B})$ and binder notation $\forall X_\alpha.\mathbf{A}$ for $(\Pi^\alpha \lambda X_\alpha.\mathbf{A}_o)$. We further use $\mathbf{A} \wedge \mathbf{B}$, $\mathbf{A} \Rightarrow \mathbf{B}$, $\mathbf{A} \Leftrightarrow \mathbf{B}$ and $\exists X_\alpha.\mathbf{A}$ as shorthand for formulae defined in terms of \neg , \vee and Π^α (cf. [4]). Finally, we let $(\mathbf{A}_\alpha \doteq^\alpha \mathbf{B}_\alpha)$ denote the Leibniz equation $\forall P_{\alpha \rightarrow o} (P\mathbf{A} \Rightarrow P\mathbf{B})$.

Each occurrence of a variable in a term is either bound by a λ or free. We use $free(\mathbf{A})$ to denote the set of free variables of \mathbf{A} (i.e., variables with a free occurrence in \mathbf{A}). We consider two terms to be equal if the terms are the same up to the names of bound variables (i.e., we consider α -conversion implicitly). A term \mathbf{A} is closed if $free(\mathbf{A})$ is empty. We let $cwff_\alpha(\Sigma)$ denote the set of closed terms of type α and $cwff(\Sigma)$ denote the set of all closed terms. Each term $\mathbf{A} \in wff_o(\Sigma)$ is called a proposition and each term $\mathbf{A} \in cwff_o(\Sigma)$ is called a sentence.

We denote substitution of a term \mathbf{A}_α for a variable X_α in a term \mathbf{B}_β by $[\mathbf{A}/X]\mathbf{B}$. Since we consider α -conversion implicitly, we assume the bound variables of \mathbf{B} avoid variable capture. Similarly, we consider simultaneous substitutions σ for finitely many free variables. A substitution $\sigma, [\mathbf{A}/X]$ is the substitution such that $(\sigma, [\mathbf{A}/X])(X) \equiv \mathbf{A}$ and $(\sigma, [\mathbf{A}/X])(Y) \equiv \sigma(Y)$ for variables Y other than X .

Two common relations on terms are given by β -reduction and η -reduction. A β -redex $(\lambda X.\mathbf{A})\mathbf{B}$ β -reduces to $[\mathbf{B}/X]\mathbf{A}$. An η -redex $(\lambda X.CX)$ (where $X \notin$

$free(\mathbf{C})$) η -reduces to \mathbf{C} . For $\mathbf{A}, \mathbf{B} \in wff_\alpha(\Sigma)$, we write $\mathbf{A} \equiv_\beta \mathbf{B}$ to mean \mathbf{A} can be converted to \mathbf{B} by a series of β -reductions and expansions. Similarly, $\mathbf{A} \equiv_{\beta\eta} \mathbf{B}$ means \mathbf{A} can be converted to \mathbf{B} using both β and η . For each $\mathbf{A} \in wff(\Sigma)$ there is a unique β -normal form (denoted $\mathbf{A} \downarrow_\beta$) and a unique $\beta\eta$ -normal form (denoted $\mathbf{A} \downarrow_{\beta\eta}$). From this fact we know $\mathbf{A} \equiv_\beta \mathbf{B}$ ($\mathbf{A} \equiv_{\beta\eta} \mathbf{B}$) iff $\mathbf{A} \downarrow_\beta \equiv \mathbf{B} \downarrow_\beta$ ($\mathbf{A} \downarrow_{\beta\eta} \equiv \mathbf{B} \downarrow_{\beta\eta}$).

A non-atomic formula in $wff_o(\Sigma)$ is any formula whose β -normal form is of the form $[c\overline{\mathbf{A}^n}]$ where c is a logical constant. An atomic formula is any other formula in $wff_o(\Sigma)$.

2.2. Semantics for \mathcal{HOL} . A model of \mathcal{HOL} is given by four objects: a typed collection of nonempty sets $(\mathcal{D}_\alpha)_{\alpha \in \mathcal{T}}$, an application operator $@: \mathcal{D}_{\alpha \rightarrow \beta} \times \mathcal{D}_\alpha \rightarrow \mathcal{D}_\beta$, an evaluation function \mathcal{E} for terms and a valuation function $v: \mathcal{D}_o \rightarrow \{\mathbf{T}, \mathbf{F}\}$. A pair $(\mathcal{D}, @)$ is called a Σ -applicative structure (cf. Definition 3.1 in [4]). If \mathcal{E} is an evaluation function for $(\mathcal{D}, @)$ (cf. Definition 3.18 in [4]), then we call the triple $(\mathcal{D}, @, \mathcal{E})$ a Σ -evaluation. If v satisfies appropriate properties, then we call the tuple $(\mathcal{D}, @, \mathcal{E}, v)$ a Σ -model (cf. Definitions 3.40 and 3.41 in [4]).

Given an applicative structure $(\mathcal{D}, @)$, an assignment φ is a (typed) function from \mathcal{V} to \mathcal{D} . An evaluation function \mathcal{E} maps an assignment φ and a term $\mathbf{A}_\alpha \in wff_\alpha(\Sigma)$ to an element $\mathcal{E}_\varphi(\mathbf{A}) \in \mathcal{D}_\alpha$. Evaluations \mathcal{E} are required to satisfy four properties (cf. Definition 3.18 in [4]):

1. $\mathcal{E}_\varphi|_{\mathcal{V}} \equiv \varphi$.
2. $\mathcal{E}_\varphi(\mathbf{FA}) \equiv \mathcal{E}_\varphi(\mathbf{F}) @ \mathcal{E}_\varphi(\mathbf{A})$ for any $\mathbf{F} \in wff_{\alpha \rightarrow \beta}(\Sigma)$ and $\mathbf{A} \in wff_\alpha(\Sigma)$ and types α and β .
3. $\mathcal{E}_\varphi(\mathbf{A}) \equiv \mathcal{E}_\psi(\mathbf{A})$ for any type α and $\mathbf{A} \in wff_\alpha(\Sigma)$, whenever φ and ψ coincide on $free(\mathbf{A})$.
4. $\mathcal{E}_\varphi(\mathbf{A}) \equiv \mathcal{E}_\varphi(\mathbf{A} \downarrow_\beta)$ for all $\mathbf{A} \in wff_\alpha(\Sigma)$.

If \mathbf{A} is closed, then we can simply write $\mathcal{E}(\mathbf{A})$ since the value $\mathcal{E}_\varphi(\mathbf{A})$ cannot depend on φ .

Given an evaluation $(\mathcal{D}, @, \mathcal{E})$, Figure 1 shows the definition of several properties a function $v: \mathcal{D}_o \rightarrow \{\mathbf{T}, \mathbf{F}\}$ may satisfy (cf. Definition 3.40 in [4]). A valuation $v: \mathcal{D}_o \rightarrow \{\mathbf{T}, \mathbf{F}\}$ is required to satisfy $\mathfrak{L}_\neg(\mathcal{E}(\neg))$, $\mathfrak{L}_\vee(\mathcal{E}(\vee))$ and $\mathfrak{L}_\forall^\alpha(\mathcal{E}(\Pi^\alpha))$ for every type α .

prop.	where	holds when	for all
$\mathfrak{L}_\neg(\mathbf{n})$	$\mathbf{n} \in \mathcal{D}_{o \rightarrow o}$	$v(\mathbf{n} @ \mathbf{a}) \equiv \mathbf{T}$ iff $v(\mathbf{a}) \equiv \mathbf{F}$	$\mathbf{a} \in \mathcal{D}_o$
$\mathfrak{L}_\vee(\mathbf{d})$	$\mathbf{d} \in \mathcal{D}_{o \rightarrow o \rightarrow o}$	$v(\mathbf{d} @ \mathbf{a} @ \mathbf{b}) \equiv \mathbf{T}$ iff $v(\mathbf{a}) \equiv \mathbf{T}$ or $v(\mathbf{b}) \equiv \mathbf{T}$	$\mathbf{a}, \mathbf{b} \in \mathcal{D}_o$
$\mathfrak{L}_\forall^\alpha(\pi)$	$\pi \in \mathcal{D}_{(\alpha \rightarrow o) \rightarrow o}$	$v(\pi @ \mathbf{f}) \equiv \mathbf{T}$ iff $\forall \mathbf{a} \in \mathcal{D}_\alpha v(\mathbf{f} @ \mathbf{a}) \equiv \mathbf{T}$	$\mathbf{f} \in \mathcal{D}_{\alpha \rightarrow o}$
$\mathfrak{L}_\equiv^\alpha(\mathbf{q})$	$\mathbf{q} \in \mathcal{D}_{\alpha \rightarrow \alpha \rightarrow o}$	$v(\mathbf{q} @ \mathbf{a} @ \mathbf{b}) \equiv \mathbf{T}$ iff $\mathbf{a} \equiv \mathbf{b}$	$\mathbf{a}, \mathbf{b} \in \mathcal{D}_\alpha$

FIGURE 1. Logical Properties in Σ -Models

Given a model $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, v)$, an assignment φ and a proposition \mathbf{A} (or set of propositions Φ), we say \mathcal{M} satisfies \mathbf{A} (or Φ) and write $\mathcal{M} \models_\varphi \mathbf{A}$ (or $\mathcal{M} \models_\varphi \Phi$) if $v(\mathcal{E}_\varphi(\mathbf{A})) \equiv \mathbf{T}$ (or $v(\mathcal{E}_\varphi(\mathbf{A})) \equiv \mathbf{T}$ for each $\mathbf{A} \in \Phi$). If \mathbf{A} is closed (or every member of Φ is closed), then we simply write $\mathcal{M} \models \mathbf{A}$ (or $\mathcal{M} \models \Phi$) and say \mathcal{M} is a model of \mathbf{A} (or Φ).

In order to define model classes \mathfrak{M}_* which correspond to different notions of extensionality, we define five properties of models (cf. Definitions 3.46, 3.21 and 3.5 in [4]). Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, v)$ be a model. We say \mathcal{M} has property

- q**: iff for all $\alpha \in \mathcal{T}$ there is some $\mathbf{q}^\alpha \in \mathcal{D}_{\alpha \rightarrow \alpha \rightarrow o}$ such that $\mathcal{L}_{=}^\alpha(\mathbf{q}^\alpha)$ holds.
- η** : iff $(\mathcal{D}, @, \mathcal{E})$ is η -functional (i.e., for each $\mathbf{A} \in \text{wff}_\alpha(\Sigma)$ and assignment φ , $\mathcal{E}_\varphi(\mathbf{A}) \equiv \mathcal{E}_\varphi(\mathbf{A}|_{\beta\eta})$)
- ξ** : iff $(\mathcal{D}, @, \mathcal{E})$ is ξ -functional (i.e., for each $\mathbf{M}, \mathbf{N} \in \text{wff}_\beta(\Sigma)$, $X \in \mathcal{V}_\alpha$ and assignment φ , $\mathcal{E}_\varphi(\lambda X_\alpha. \mathbf{M}_\beta) \equiv \mathcal{E}_\varphi(\lambda X_\alpha. \mathbf{N}_\beta)$ whenever $\mathcal{E}_{\varphi, [a/X]}(\mathbf{M}) \equiv \mathcal{E}_{\varphi, [a/X]}(\mathbf{N})$ for every $\mathbf{a} \in \mathcal{D}_\alpha$).
- f**: iff $(\mathcal{D}, @)$ is functional (i.e., for each $\mathbf{f}, \mathbf{g} \in \mathcal{D}_{\alpha \rightarrow \beta}$, $\mathbf{f} \equiv \mathbf{g}$ whenever $\mathbf{f}@a \equiv \mathbf{g}@a$ for every $\mathbf{a} \in \mathcal{D}_\alpha$).
- b**: iff \mathcal{D}_o has at most two elements.

For each $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathbf{f}, \beta\mathbf{b}, \beta\eta\mathbf{b}, \beta\xi\mathbf{b}, \beta\mathbf{f}\mathbf{b}\}$ we define \mathfrak{M}_* to be the class of all Σ -models \mathcal{M} such that \mathcal{M} satisfies property **q** and each of the additional properties $\{\eta, \xi, \mathbf{f}, \mathbf{b}\}$ indicated in the subscript $*$ (cf. Definition 3.49 in [4]).

NOTATION 2.1. Let \mathfrak{B} denote the set $\{\beta, \beta\eta, \beta\xi, \beta\mathbf{f}, \beta\mathbf{b}, \beta\eta\mathbf{b}, \beta\xi\mathbf{b}, \beta\mathbf{f}\mathbf{b}\}$. Let \mathfrak{D} denote the set $\{\beta, \beta\eta, \beta\xi, \beta\mathbf{f}, \beta\mathbf{b}\}$ of five indices on which we will often focus.

We always include β in the subscript $* \in \mathfrak{B}$ to indicate that β -equal terms are always interpreted as identical elements. We do not include property **q** as an explicit subscript; **q** is treated as a basic, implicit requirement for all model classes. See Remark 3.52 in [4] for a discussion on why we require property **q**. Since we are varying four properties, one would expect to obtain 16 model classes. However, we showed in [4] that **f** is equivalent to the conjunction of ξ and η . Hence we obtain 8 model classes. These model classes are depicted as a cube in Figure 2.

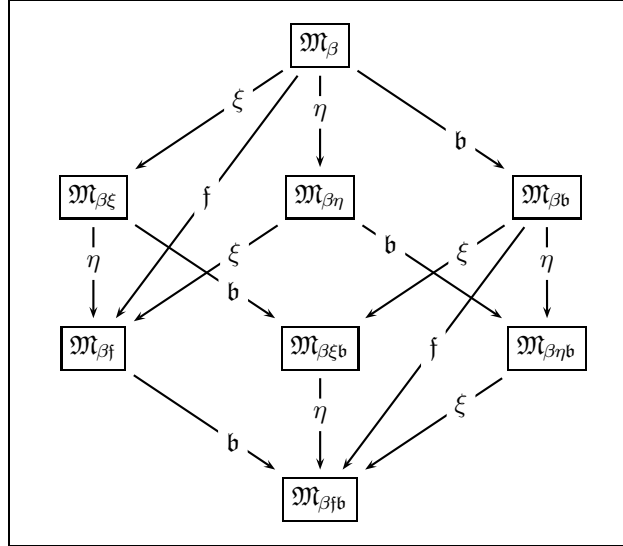


FIGURE 2. The landscape of Higher-Order Semantics

Special cases of Σ -models are Henkin models and standard models (cf. Definitions 3.50 and 3.51 in [4]). A Henkin model is a model in $\mathfrak{M}_{\beta\text{fb}}$ such that the applicative structure $(\mathcal{D}, @)$ is a frame, i.e., $\mathcal{D}_{\alpha \rightarrow \beta}$ is a subset of the function space $(\mathcal{D}_\beta)^{\mathcal{D}_\alpha}$ for each $\alpha, \beta \in \mathcal{T}$ and $@$ is function application. A standard model is a Henkin model in which $\mathcal{D}_{\alpha \rightarrow \beta}$ is the full function space $(\mathcal{D}_\beta)^{\mathcal{D}_\alpha}$. Every model in $\mathfrak{M}_{\beta\text{fb}}$ is isomorphic to a Henkin model (see the discussion following Theorem 3.68 in [4]).

2.3. Abstract Consistency, Hintikka Sets and Model Existence. Finally, we review the model existence theorems proved in [4]. There are three stages to obtaining a model in our framework. First, we obtain an abstract consistency class Γ_Σ (usually defined as the class of irrefutable sets of sentences with respect to some calculus). Second, given a (sufficiently pure) set of sentences Φ in the abstract consistency class Γ_Σ we construct a Hintikka set \mathcal{H} extending Φ . Third, we construct a model of this Hintikka set (hence a model of Φ).

A Σ -abstract consistency class Γ_Σ is a class of sets of Σ -sentences. An abstract consistency class is always required to be closed under subsets (cf. Definition 6.1 in [4]). Sometimes we require the stronger property that Γ_Σ is compact, i.e., a set Φ is in Γ_Σ iff every finite subset of Φ is in Γ_Σ (cf. Definition 6.1 and Lemma 6.2 in [4]).

To describe the remaining properties of an abstract consistency class, we use the notation $S * a$ for $S \cup \{a\}$ as in [4]. The following is a list of properties a class Γ_Σ of sets of sentences can satisfy with respect to arbitrary $\Phi \in \Gamma_\Sigma$ (cf. Definition 6.5 in [4]):

- ∇_c : If \mathbf{A} is atomic, then $\mathbf{A} \notin \Phi$ or $\neg \mathbf{A} \notin \Phi$.
- ∇_{\neg} : If $\neg \neg \mathbf{A} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$.
- ∇_β : If $\mathbf{A} \equiv_\beta \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{B} \in \Gamma_\Sigma$.
- ∇_η : If $\mathbf{A} \equiv_{\beta\eta} \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{B} \in \Gamma_\Sigma$.
- ∇_\vee : If $\mathbf{A} \vee \mathbf{B} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$ or $\Phi * \mathbf{B} \in \Gamma_\Sigma$.
- ∇_\wedge : If $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, then $\Phi * \neg \mathbf{A} * \neg \mathbf{B} \in \Gamma_\Sigma$.
- ∇_\forall : If $\Pi^\alpha \mathbf{F} \in \Phi$, then $\Phi * \mathbf{F}\mathbf{W} \in \Gamma_\Sigma$ for each $\mathbf{W} \in \text{cwf}_\alpha(\Sigma)$.
- ∇_\exists : If $\neg \Pi^\alpha \mathbf{F} \in \Phi$, then $\Phi * \neg(\mathbf{F}w) \in \Gamma_\Sigma$ for any parameter $w_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ .
- ∇_b : If $\neg(\mathbf{A} \dot{=}^o \mathbf{B}) \in \Phi$, then $\Phi * \mathbf{A} * \neg \mathbf{B} \in \Gamma_\Sigma$ or $\Phi * \neg \mathbf{A} * \mathbf{B} \in \Gamma_\Sigma$.
- ∇_ξ : If $\neg(\lambda X_\alpha. \mathbf{M} \dot{=}^{\alpha \rightarrow \beta} \lambda X_\alpha. \mathbf{N}) \in \Phi$, then $\Phi * \neg([w/X]\mathbf{M} \dot{=}^\beta [w/X]\mathbf{N}) \in \Gamma_\Sigma$ for any parameter $w_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ .
- ∇_f : If $\neg(\mathbf{G} \dot{=}^{\alpha \rightarrow \beta} \mathbf{H}) \in \Phi$, then $\Phi * \neg(\mathbf{G}w \dot{=}^\beta \mathbf{H}w) \in \Gamma_\Sigma$ for any parameter $w_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ .
- ∇_{sat} : Either $\Phi * \mathbf{A} \in \Gamma_\Sigma$ or $\Phi * \neg \mathbf{A} \in \Gamma_\Sigma$.

We say Γ_Σ is an abstract consistency class if it is closed under subsets and satisfies $\nabla_c, \nabla_{\neg}, \nabla_\beta, \nabla_\vee, \nabla_\wedge, \nabla_\forall$ and ∇_\exists . We let \mathfrak{Acc}_β denote the collection of all abstract consistency classes. For each $* \in \mathfrak{A}$ we refine \mathfrak{Acc}_β to a collection \mathfrak{Acc}_* where the additional properties $\{\nabla_\eta, \nabla_\xi, \nabla_f, \nabla_b\}$ indicated by $*$ are required (cf. Definition 6.7 in [4]). We say an abstract consistency class Γ_Σ is saturated if ∇_{sat} holds.

Using ∇_c (atomic consistency) and the fact that there are infinitely many parameters at each type, we can show every abstract consistency class satisfies

non-atomic consistency. That is, for every abstract consistency class \mathbb{I}_Σ , $\mathbf{A} \in \text{cuff}_o(\Sigma)$ and $\Phi \in \mathbb{I}_\Sigma$, we have either $\mathbf{A} \notin \Phi$ or $\neg \mathbf{A} \notin \Phi$ (cf. Lemma 6.10 in [4]).

In order to obtain a Hintikka set extending a set Φ , we must have parameters which will act as witnesses. For this we require sufficient purity of Φ . A set Φ of Σ -sentences is called sufficiently Σ -pure (cf. Definition 6.3 in [4]) if for each type α there is a set \mathcal{P}_α of parameters of type α with cardinality \aleph_s (the cardinality of $\text{wff}_\alpha(\Sigma)$) and such that no parameter in \mathcal{P} occurs in a sentence in Φ . Note that since Σ is assumed to have infinite cardinality \aleph_s for each type, every finite set of Σ -sentences is sufficiently Σ -pure.

A Hintikka set is a set of sentences satisfying certain properties. The following is a list of properties a set \mathcal{H} of sentences may satisfy (cf. Definition in [4]):

- $\vec{\nabla}_c$: $\mathbf{A} \notin \mathcal{H}$ or $\neg \mathbf{A} \notin \mathcal{H}$.
- $\vec{\nabla}_\neg$: If $\neg \neg \mathbf{A} \in \mathcal{H}$, then $\mathbf{A} \in \mathcal{H}$.
- $\vec{\nabla}_\beta$: If $\mathbf{A} \in \mathcal{H}$ and $\mathbf{A} \equiv_\beta \mathbf{B}$, then $\mathbf{B} \in \mathcal{H}$.
- $\vec{\nabla}_\eta$: If $\mathbf{A} \in \mathcal{H}$ and $\mathbf{A} \equiv_{\beta\eta} \mathbf{B}$, then $\mathbf{B} \in \mathcal{H}$.
- $\vec{\nabla}_\vee$: If $\mathbf{A} \vee \mathbf{B} \in \mathcal{H}$, then $\mathbf{A} \in \mathcal{H}$ or $\mathbf{B} \in \mathcal{H}$.
- $\vec{\nabla}_\wedge$: If $\neg(\mathbf{A} \vee \mathbf{B}) \in \mathcal{H}$, then $\neg \mathbf{A} \in \mathcal{H}$ and $\neg \mathbf{B} \in \mathcal{H}$.
- $\vec{\nabla}_\forall$: If $\Pi^\alpha \mathbf{F} \in \mathcal{H}$, then $\mathbf{F}\mathbf{W} \in \mathcal{H}$ for each $\mathbf{W} \in \text{cuff}_\alpha(\Sigma)$.
- $\vec{\nabla}_\exists$: If $\neg \Pi^\alpha \mathbf{F} \in \mathcal{H}$, then there is a parameter $w_\alpha \in \Sigma_\alpha$ such that $\neg(\mathbf{F}w) \in \mathcal{H}$.
- $\vec{\nabla}_b$: If $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \mathcal{H}$, then $\{\mathbf{A}, \neg \mathbf{B}\} \subseteq \mathcal{H}$ or $\{\neg \mathbf{A}, \mathbf{B}\} \subseteq \mathcal{H}$.
- $\vec{\nabla}_\xi$: If $\neg(\lambda X_\alpha. \mathbf{M} \doteq^{\alpha \rightarrow \beta} \lambda X. \mathbf{N}) \in \mathcal{H}$, then there is a parameter $w_\alpha \in \Sigma_\alpha$ such that $\neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N}) \in \mathcal{H}$.
- $\vec{\nabla}_f$: If $\neg(\mathbf{G} \doteq^{\alpha \rightarrow \beta} \mathbf{H}) \in \mathcal{H}$, then there is a parameter $w_\alpha \in \Sigma_\alpha$ such that $\neg(\mathbf{G}w \doteq^\beta \mathbf{H}w) \in \mathcal{H}$.
- $\vec{\nabla}_{sat}$: Either $\mathbf{A} \in \mathcal{H}$ or $\neg \mathbf{A} \in \mathcal{H}$.

A set \mathcal{H} of sentences is called a Σ -Hintikka set if $\vec{\nabla}_c, \vec{\nabla}_\neg, \vec{\nabla}_\beta, \vec{\nabla}_\vee, \vec{\nabla}_\wedge, \vec{\nabla}_\forall$ and $\vec{\nabla}_\exists$ hold. We define the following collections of Hintikka sets: $\mathfrak{Hint}_\beta, \mathfrak{Hint}_{\beta\eta}, \mathfrak{Hint}_{\beta\xi}, \mathfrak{Hint}_{\beta f}, \mathfrak{Hint}_{\beta b}, \mathfrak{Hint}_{\beta\eta b}, \mathfrak{Hint}_{\beta\xi b}$, and $\mathfrak{Hint}_{\beta f b}$, where we indicate by indices which additional properties from $\{\vec{\nabla}_\eta, \vec{\nabla}_\xi, \vec{\nabla}_f, \vec{\nabla}_b\}$ are required (cf. Definition 6.20 in [4]). We call a Hintikka set \mathcal{H} *saturated* if $\vec{\nabla}_{sat}$ holds (cf. Definition 6.24 in [4]).

One of the main theorems of [4] is the Model Existence Theorem for Saturated Sets (cf. Theorem 6.33) which states the following:

THEOREM: *For all $* \in \mathfrak{G}$ we have: If \mathcal{H} is a saturated Hintikka set in \mathfrak{Hint}_* , then there exists a model $\mathcal{M} \in \mathfrak{M}_*$ that satisfies \mathcal{H} . Furthermore, each domain \mathcal{D}_α of \mathcal{M} has cardinality at most \aleph_s .*

Since saturated abstract consistency classes give rise to saturated Hintikka sets, we conclude a corresponding model existence theorem for saturated abstract consistency classes (cf. Theorem 6.34):

THEOREM: *For all $* \in \mathfrak{G}$, if \mathbb{I}_Σ is a saturated abstract consistency class in \mathfrak{Acc}_* and $\Phi \in \mathbb{I}_\Sigma$ is a sufficiently Σ -pure set of sentences, then there exists a model $\mathcal{M} \in \mathfrak{M}_*$ that satisfies Φ . Furthermore, each domain of \mathcal{M} has cardinality at most \aleph_s .*

However, in this paper we consider abstract consistency classes which may not be saturated. When we obtain a Hintikka set \mathcal{H} using such an abstract

consistency class, \mathcal{H} may or may not be saturated. If \mathcal{H} is saturated, then we can (and will) apply Theorem 6.33 in [4] to obtain a model of \mathcal{H} . Hence we primarily focus on constructing models for a Hintikka set \mathcal{H} which is not saturated.

The importance of model existence theorems for abstract consistency classes is that one can check completeness for a calculus by verifying proof-theoretic conditions instead of performing model-theoretic analysis (for historical background of the method in first-order logic, cf. [8, 12, 13]).

For each $*$ in \mathfrak{A} we can use the model existence theorem for saturated abstract consistency classes (cf. Theorem 6.34 in [4]) to show the completeness of a particular calculus \mathcal{C} as follows: We first show the class Γ_Σ of sets of sentences Φ that are \mathcal{C} -consistent (i.e., cannot be refuted in \mathcal{C}) is a saturated abstract consistency class in \mathfrak{Acc}_* . Assume a sentence \mathbf{A} is valid in every model in \mathfrak{M}_* . Then $\neg\mathbf{A}$ does not have a model. By the model existence theorem, $\{\neg\mathbf{A}\} \notin \Gamma_\Sigma$ and so $\neg\mathbf{A}$ is \mathcal{C} -inconsistent. This shows refutation completeness of \mathcal{C} . For many calculi \mathcal{C} , this also shows \mathbf{A} is provable, thus establishing completeness of \mathcal{C} .

Unfortunately, the saturation condition is very difficult to prove for machine-oriented calculi (indeed as we will see in Section 3 it is equivalent to cut elimination), so Theorem 6.34 in [4] cannot be used for this purpose directly. In the Section 4, we will motivate and present a set of conditions we call *acceptability* conditions that are weaker than saturation, but sufficient to prove model existence.

§3. Saturation and Cut Elimination for Higher-Order Sequent Calculi. Since the task of showing an abstract consistency class is saturated appears to be difficult in many cases, we instead investigate abstract consistency classes which possess a saturated extension. The main purpose of an abstract consistency class Γ_Σ is to ensure the existence of models for every sufficiently Σ -pure $\Phi \in \Gamma_\Sigma$ (cf. Definition 6.3 in [4]). As this is the case, we only require that an extension of Γ_Σ contains all the sufficiently Σ -pure sets in Γ_Σ .

DEFINITION 3.1 (Saturated Extension). Let $*$ in \mathfrak{A} and $\Gamma_\Sigma, \Gamma'_\Sigma \in \mathfrak{Acc}_*$ be abstract consistency classes. We say Γ'_Σ is an *extension* of Γ_Σ if $\Phi \in \Gamma'_\Sigma$ for every sufficiently Σ -pure $\Phi \in \Gamma_\Sigma$. We say Γ'_Σ is a *saturated extension* of Γ_Σ if Γ'_Σ is saturated and an extension of Γ_Σ .

We now look at two examples demonstrating that not every abstract consistency class can be extended to a saturated one. Later we will use these examples to motivate the acceptability conditions.

EXAMPLE 3.2 (Unsaturated $\mathfrak{Acc}_{\beta b}$ without a Saturated Extension).

Let $a_o, b_o, q_{o \rightarrow o} \in \Sigma$ and $\Phi = \{a, b, (qa), \neg(qb)\}$. We construct an abstract consistency class Γ_Σ from Φ by first building the closure Φ' of Φ under relation $\equiv_{\beta\eta}$ and then taking the powerset of Φ' . (We must close Φ at least under β -conversion to ensure ∇_β in the definition of abstract consistency classes.) It is easy to check that this Γ_Σ is in $\mathfrak{Acc}_{\beta fb}, \mathfrak{Acc}_{\beta\eta b}, \mathfrak{Acc}_{\beta\zeta b}$ and $\mathfrak{Acc}_{\beta b}$. Suppose we have a saturated extension Γ'_Σ of Γ_Σ in $\mathfrak{Acc}_{\beta b}$. (Note that Γ'_Σ is in $\mathfrak{Acc}_{\beta b}$ whenever it is in $\mathfrak{Acc}_{\beta fb}, \mathfrak{Acc}_{\beta\eta b}$ or $\mathfrak{Acc}_{\beta\zeta b}$.) Then $\Phi \in \Gamma'_\Sigma$ since Φ is finite (hence sufficiently pure). By saturation, $\Phi * (a \doteq^o b) \in \Gamma'_\Sigma$ or $\Phi * \neg(a \doteq^o b) \in \Gamma'_\Sigma$. In the first case,

applying ∇_v with the constant q , ∇_v and ∇_c contradicts $(qa), \neg(qb) \in \Phi$. In the second case, ∇_b and ∇_c contradict $a, b \in \Phi$. One can also extend this example by letting $\Phi := \{\neg(q \rightarrow_o b_o)\} \cup \{\mathbf{A} \mid \mathbf{A} \text{ atomic, } \beta\text{-normal and } \mathbf{A} \neq (qb)\}$ to obtain a member of $\mathfrak{Acc}_{\beta\mathfrak{fb}}$ which is atomically saturated (saturated with respect to atomic formulae) and yet has no saturated extension.

EXAMPLE 3.3 (Unsaturated $\mathfrak{Acc}_{\beta\mathfrak{f}}$ without a Saturated Extension).

Similar to the previous example we assume $q_{(\iota \rightarrow \iota) \rightarrow o}, g_{\iota \rightarrow \iota} \in \Sigma$ and choose a Γ_Σ constructed from $\Phi = \{\neg q(\lambda X. gX), qg\}$ by closing under \equiv_β (not $\equiv_{\beta\eta}$) and taking the power set. Again it is easy to check that this is in $\mathfrak{Acc}_{\beta\mathfrak{fb}}$, hence in $\mathfrak{Acc}_{\beta\mathfrak{f}}$. Suppose we have a saturated extension Γ'_Σ of Γ_Σ in $\mathfrak{Acc}_{\beta\mathfrak{f}}$. Then we must either have $\Phi * ((\lambda X. gX) \doteq^{\iota \rightarrow \iota} g) \in \Gamma'_\Sigma$ or $\Phi * \neg((\lambda X. gX) \doteq^{\iota \rightarrow \iota} g) \in \Gamma'_\Sigma$ by saturation. In the first case, we obtain a contradiction to $\neg(q(\lambda X. gX)), (qg) \in \Phi$ using ∇_v with constant q , ∇_v and ∇_c . In the second case, we obtain $\Phi * \neg((\lambda X. gX)w \doteq^{\iota \rightarrow \iota} gw) \in \Gamma'_\Sigma$ for some new $w \in \Sigma_\iota$ by ∇_f and hence $\Phi * \neg(gw \doteq^{\iota \rightarrow \iota} gw) \in \Gamma'_\Sigma$ by ∇_β , contradicting ∇_β^r (cf. Lemma 6.12 in [4]).

Let us now consider the relationship between abstract consistency classes and sequent calculi. We consider a sequent to be a finite set Δ of sentences from $\text{cuff}_o(\Sigma)$. A sequent calculus \mathcal{G} provides an inductive definition for when $\Vdash_{\mathcal{G}} \Delta$ holds. We say a sequent calculus rule

$$\frac{\Delta_1 \quad \cdots \quad \Delta_n}{\Delta}$$

is *admissible* if $\Vdash_{\mathcal{G}} \Delta$ holds whenever $\Vdash_{\mathcal{G}} \Delta_i$ for all $1 \leq i \leq n$. Given a sequent Δ and a model \mathcal{M} , we say Δ is *valid for* \mathcal{M} if $\mathcal{M} \models \mathbf{D}$ for some $\mathbf{D} \in \Delta$. For a class \mathfrak{M} of models, we say Δ is *valid for* \mathfrak{M} if Δ is valid for every $\mathcal{M} \in \mathfrak{M}$. As for sets in abstract consistency classes, we use the notation $\Delta * \mathbf{A}$ to denote the set $\Delta \cup \{\mathbf{A}\}$ (which is simply Δ if $\mathbf{A} \in \Delta$).

We will consider sequent calculi \mathcal{G} in general before we specialize to consider sequent calculi corresponding to our eight model classes in Figure 2. In Figure 3 we show basic rules which we will often assume are admissible in a sequent calculus \mathcal{G} . When we define particular sequent calculi, these rules will be included explicitly. In Figure 4 we show rules for β -reduction and $\beta\eta$ -reduction. In particular cases, we will explicitly include either $\mathcal{G}(\beta)$ or $\mathcal{G}(\beta\eta)$. In Figure 5 we show inversion rules for negation, β -reduction and $\beta\eta$ -reduction and in Figure 6 we show weakening and cut rules. We will not explicitly include the rules in Figure 5 or Figure 6 in the definitions of our sequent calculi, but admissible of these rules will be important. Finally, we consider various extensionality rules in Figure 7.

For any sequent calculus \mathcal{G} we can define a class $\Gamma_\Sigma^{\mathcal{G}}$ of sets of formulas. Under certain assumptions, $\Gamma_\Sigma^{\mathcal{G}}$ is an abstract consistency class. First we adopt the notation $\neg\Phi$ for the set $\{\neg\mathbf{A} \mid \mathbf{A} \in \Phi\}$ where $\Phi \subseteq \text{cuff}_o(\Sigma)$. Furthermore, we assume this use of \neg binds more strongly than \cup or $*$, so that $\neg\Phi \cup \Delta$ means $(\neg\Phi) \cup \Delta$ and $\neg\Phi * \mathbf{A}$ means $(\neg\Phi) * \mathbf{A}$.

DEFINITION 3.4. Let \mathcal{G} be a sequent calculus. We define $\Gamma_\Sigma^{\mathcal{G}}$ to be the class of all finite $\Phi \subset \text{cuff}_o(\Sigma)$ such that $\Vdash_{\mathcal{G}} \neg\Phi$ does not hold.

$$\boxed{
\begin{array}{cc}
\frac{\mathbf{A} \text{ atomic}}{\Delta * \mathbf{A} * \neg \mathbf{A}} \mathcal{G}(init) & \frac{\Delta * \mathbf{A}}{\Delta * \neg \neg \mathbf{A}} \mathcal{G}(\neg) \\
\\
\frac{\Delta * \neg \mathbf{A} \quad \Delta * \neg \mathbf{B}}{\Delta * \neg(\mathbf{A} \vee \mathbf{B})} \mathcal{G}(\vee_-) & \frac{\Delta * \mathbf{A} * \mathbf{B}}{\Delta * (\mathbf{A} \vee \mathbf{B})} \mathcal{G}(\vee_+) \\
\\
\frac{\Delta * \neg \mathbf{A} \mathbf{C} \quad \mathbf{C} \in cwf_{\alpha}^{\varepsilon}(\Sigma)}{\Delta * \neg \Pi^{\alpha} \mathbf{A}} \mathcal{G}(\Pi_{-}^{\mathbf{C}}) & \frac{\Delta * \mathbf{A} c \quad c_{\alpha} \in \Sigma \text{ new}}{\Delta * \Pi^{\alpha} \mathbf{A}} \mathcal{G}(\Pi_{+}^c)
\end{array}
}$$

FIGURE 3. Basic Higher-Order Sequent Calculus Rules

$$\boxed{
\begin{array}{cc}
\frac{\Delta * \mathbf{A} \downarrow_{\beta}}{\Delta * \mathbf{A}} \mathcal{G}(\beta) & \frac{\Delta * \mathbf{A} \downarrow_{\beta \eta}}{\Delta * \mathbf{A}} \mathcal{G}(\beta \eta)
\end{array}
}$$

FIGURE 4. Conversion Rules

$$\boxed{
\begin{array}{ccc}
\frac{\Delta * \neg \neg \mathbf{A}}{\Delta * \mathbf{A}} \mathcal{G}(Inv^{\neg}) & \frac{\Delta * \mathbf{A}}{\Delta * \mathbf{A} \downarrow_{\beta}} \mathcal{G}(\beta^{\downarrow}) & \frac{\Delta * \mathbf{A}}{\Delta * \mathbf{A} \downarrow_{\beta \eta}} \mathcal{G}(\beta \eta^{\downarrow})
\end{array}
}$$

FIGURE 5. Inversion Rules

$$\boxed{
\begin{array}{cc}
\frac{\Delta}{\Delta \cup \Delta'} \mathcal{G}(weak) & \frac{\Delta * \mathbf{C} \quad \Delta * \neg \mathbf{C}}{\Delta} \mathcal{G}(cut)
\end{array}
}$$

FIGURE 6. Weakening and Cut Rules

Before relating admissibility of rules to abstract consistency conditions, we show a helpful lemma.

LEMMA 3.5. *Let \mathcal{G} be a sequent calculus such that $\mathcal{G}(Inv^{\neg})$ is admissible. For any finite sets Φ and Δ of sentences, if $\Phi \cup \neg \Delta \notin \Gamma_{\Sigma}^{\mathcal{G}}$, then $\Vdash_{\mathcal{G}} \neg \Phi \cup \Delta$ holds.*

PROOF. Suppose $\Phi \cup \neg \Delta \notin \Gamma_{\Sigma}^{\mathcal{G}}$. By definition, $\Vdash_{\mathcal{G}} \neg \Phi \cup \neg \neg \Delta$ holds. Applying $\mathcal{G}(Inv^{\neg})$ to each member of Δ , we have $\Vdash_{\mathcal{G}} \neg \Phi \cup \Delta$. \dashv

We now consider abstract consistency conditions satisfied by $\Gamma_{\Sigma}^{\mathcal{G}}$.

THEOREM 3.6. *Let \mathcal{G} be a sequent calculus such that $\mathcal{G}(Inv^{\neg})$, $\mathcal{G}(\neg)$, $\mathcal{G}(\beta)$ and $\mathcal{G}(\beta^{\downarrow})$ are admissible.*

$\frac{\Delta * \neg(\forall X_{\alpha} \mathbf{M} \dot{=}^{\beta} \mathbf{N})}{\Delta * (\lambda X_{\alpha} \mathbf{M} \dot{=}^{\alpha \rightarrow \beta} \lambda X_{\alpha} \mathbf{N})} \mathcal{G}(\xi)$	$\frac{\Delta * \neg(\forall X_{\alpha} \mathbf{A} X \dot{=}^{\beta} \mathbf{B} X)}{\Delta * \neg(\mathbf{A} \dot{=}^{\alpha \rightarrow \beta} \mathbf{B})} \mathcal{G}(\mathfrak{f})$
$\frac{\Delta * \neg \mathbf{A} * \mathbf{B} \quad \Delta * \neg \mathbf{B} * \mathbf{A}}{\Delta * (\mathbf{A} \dot{=}^o \mathbf{B})} \mathcal{G}(\mathfrak{b})$	

FIGURE 7. Extensionality Rules

1. If the rules $\mathcal{G}(\text{weak})$, $\mathcal{G}(\text{init})$, $\mathcal{G}(\vee_-)$, $\mathcal{G}(\vee_+)$, $\mathcal{G}(\Pi_-^c)$ and $\mathcal{G}(\Pi_+^c)$ are admissible in \mathcal{G} , then $\Gamma_{\Sigma}^{\mathcal{G}} \in \mathfrak{A}cc_{\beta}$.
2. If $\mathcal{G}(\beta\eta)$ and $\mathcal{G}(\beta\eta^{\perp})$ are admissible, then $\Gamma_{\Sigma}^{\mathcal{G}}$ satisfies ∇_{η} .
3. If $\mathcal{G}(\xi)$ and $\mathcal{G}(\Pi_+^c)$ are admissible, then $\Gamma_{\Sigma}^{\mathcal{G}}$ satisfies ∇_{ξ} .
4. If $\mathcal{G}(\mathfrak{f})$ and $\mathcal{G}(\Pi_+^c)$ are admissible, then $\Gamma_{\Sigma}^{\mathcal{G}}$ satisfies $\nabla_{\mathfrak{f}}$.
5. If $\mathcal{G}(\mathfrak{b})$ is admissible, then $\Gamma_{\Sigma}^{\mathcal{G}}$ satisfies $\nabla_{\mathfrak{b}}$.

PROOF. First, assume $\mathcal{G}(\text{weak})$, $\mathcal{G}(\text{init})$, $\mathcal{G}(\vee_-)$, $\mathcal{G}(\vee_+)$, $\mathcal{G}(\Pi_-^c)$ and $\mathcal{G}(\Pi_+^c)$ are admissible in \mathcal{G} . We prove $\Gamma_{\Sigma}^{\mathcal{G}}$ is closed under subsets and satisfies ∇_c , ∇_{\neg} , ∇_{\wedge} and ∇_{β} . The remaining conditions are proven analogously.

Suppose $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$. If $\Phi_0 \subseteq \Phi$ and $\Phi_0 \notin \Gamma_{\Sigma}^{\mathcal{G}}$, then $\Vdash_{\mathcal{G}} \neg \Phi_0$ and so $\Vdash_{\mathcal{G}} \neg \Phi$ by admissibility of $\mathcal{G}(\text{weak})$. Hence $\Gamma_{\Sigma}^{\mathcal{G}}$ is closed under subsets.

Suppose $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$ and $\mathbf{A}, \neg \mathbf{A} \in \Phi$ where \mathbf{A} is atomic. By admissibility of $\mathcal{G}(\text{init})$, $\Vdash_{\mathcal{G}} \neg \Phi * \mathbf{A}$ since $\neg \mathbf{A} \in \neg \Phi$. By admissibility of $\mathcal{G}(\neg)$, $\Vdash_{\mathcal{G}} \neg \Phi$ since $\neg \mathbf{A} \in \neg \Phi$, contradicting $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$. Thus ∇_c holds.

Suppose $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$, $\neg \neg \mathbf{A} \in \Phi$ and $\Phi * \mathbf{A} \notin \Gamma_{\Sigma}^{\mathcal{G}}$. Hence $\Vdash_{\mathcal{G}} \neg \Phi * \neg \mathbf{A}$ and so $\Vdash_{\mathcal{G}} \neg \Phi * \neg \neg \mathbf{A}$ by admissibility of $\mathcal{G}(\neg)$. Since $\neg \neg \mathbf{A} \in \Phi$, we know $\neg \Phi$ is equal to $\neg \Phi * \neg \neg \mathbf{A}$. Hence $\Vdash_{\mathcal{G}} \neg \Phi$, contradicting $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$. Thus ∇_{\neg} holds. By similar arguments, admissibility of $\mathcal{G}(\vee_-)$ implies ∇_{\vee} and admissibility of $\mathcal{G}(\Pi_-^c)$ implies ∇_{\vee} .

Suppose $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$, $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and $\Phi * \neg \mathbf{A} * \neg \mathbf{B} \notin \Gamma_{\Sigma}^{\mathcal{G}}$. By Lemma 3.5, $\Vdash_{\mathcal{G}} \neg \Phi * \mathbf{A} * \mathbf{B}$. Applying $\mathcal{G}(\vee_+)$, we have $\Vdash_{\mathcal{G}} \neg \Phi * (\mathbf{A} \vee \mathbf{B})$. Applying $\mathcal{G}(\neg)$, we have $\Vdash_{\mathcal{G}} \neg \Phi$ since $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, contradicting $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$. Thus ∇_{\wedge} holds. By a similar argument, admissibility of $\mathcal{G}(\Pi_+^c)$, $\mathcal{G}(\text{Inv}^{\neg})$ and $\mathcal{G}(\neg)$ imply ∇_{\exists} .

Suppose $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$, $\mathbf{A} \in \Phi$, $\mathbf{A} \equiv_{\beta} \mathbf{B}$ and $\Phi * \mathbf{B} \notin \Gamma_{\Sigma}^{\mathcal{G}}$. Hence $\Vdash_{\mathcal{G}} \neg \Phi * \neg \mathbf{B}$, $\Vdash_{\mathcal{G}} \neg \Phi * \neg \mathbf{B} \downarrow_{\beta}$ (by $\mathcal{G}(\beta^{\perp})$) and $\Vdash_{\mathcal{G}} \neg \Phi * \neg \mathbf{A}$ (by $\mathcal{G}(\beta)$). This contradicts $\mathbf{A} \in \Phi$ and $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$. Thus ∇_{β} holds.

Now, we turn to the various extensionality conditions. In all of these cases we continue to assume $\mathcal{G}(\text{Inv}^{\neg})$, $\mathcal{G}(\neg)$, $\mathcal{G}(\beta)$ and $\mathcal{G}(\beta^{\perp})$ are admissible.

If $\mathcal{G}(\beta\eta)$ and $\mathcal{G}(\beta\eta^{\perp})$ are admissible, then one can prove ∇_{η} analogously to the ∇_{β} case.

Assume $\mathcal{G}(\xi)$ and $\mathcal{G}(\Pi_+^c)$ are admissible. Suppose $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$, $\neg((\lambda X_{\alpha} \mathbf{M}) \dot{=}^{\alpha \rightarrow \beta} \lambda X_{\alpha} \mathbf{N}) \in \Phi$, w_{β} is a parameter which does not occur in any sentence in Φ and $\Phi * \neg([w/X] \mathbf{M} \dot{=}^{\beta} [w/X] \mathbf{N}) \notin \Gamma_{\Sigma}^{\mathcal{G}}$. By Lemma 3.5, $\Vdash_{\mathcal{G}} \neg \Phi * ([w/X] \mathbf{M} \dot{=}^{\beta} [w/X] \mathbf{N})$.

Applying $\mathcal{G}(\beta^\perp)$ and $\mathcal{G}(\beta)$, we have $\Vdash_{\mathcal{G}} \neg\Phi * ((\lambda X.(\mathbf{M} \dot{=}^\beta \mathbf{N}))w)$. Applying $\mathcal{G}(\Pi_+^c)$ and $\mathcal{G}(\xi)$, we obtain $\Vdash_{\mathcal{G}} \neg\Phi * (\forall X.\mathbf{M} \dot{=}^\beta \mathbf{N})$ and $\Vdash_{\mathcal{G}} \neg\Phi * (\lambda X.\mathbf{M}) \dot{=}^{\alpha \rightarrow \beta} \lambda X.\mathbf{N}$. Applying $\mathcal{G}(\neg)$, we conclude $\Vdash_{\mathcal{G}} \neg\Phi$ since $\neg((\lambda X.\mathbf{M}) \dot{=}^{\alpha \rightarrow \beta} \lambda X.\mathbf{N}) \in \Phi$, contradicting $\Phi \in \Gamma_\Sigma^{\mathcal{G}}$. Thus ∇_ξ holds.

Assume the rules $\mathcal{G}(\mathbf{f})$ and $\mathcal{G}(\Pi_+^c)$ are admissible. This is analogous to the ∇_ξ case. If $\neg(\mathbf{G} \dot{=}^{\alpha \rightarrow \beta} \mathbf{H}) \in \Phi$ and $\Vdash_{\mathcal{G}} \neg\Phi * (\mathbf{G}w \dot{=}^\beta \mathbf{H}w)$ (with w_α new) holds, then we can show $\Vdash_{\mathcal{G}} \neg\Phi$ holds using $\mathcal{G}(\beta)$, $\mathcal{G}(\beta^\perp)$, $\mathcal{G}(\Pi_+^c)$ and $\mathcal{G}(\mathbf{f})$.

Assume the rule $\mathcal{G}(\mathbf{b})$ is admissible. Suppose $\Phi \in \Gamma_\Sigma^{\mathcal{G}}$, $\neg(\mathbf{A} \dot{=}^o \mathbf{B}) \in \Phi$, $\Phi * \mathbf{A} * \neg\mathbf{B} \notin \Gamma_\Sigma^{\mathcal{G}}$ and $\Phi * \neg\mathbf{A} * \mathbf{B} \notin \Gamma_\Sigma^{\mathcal{G}}$. By Lemma 3.5, $\Vdash_{\mathcal{G}} \neg\Phi * \neg\mathbf{A} * \mathbf{B}$ and $\Vdash_{\mathcal{G}} \neg\Phi * \mathbf{A} * \neg\mathbf{B}$. Applying $\mathcal{G}(\mathbf{b})$, $\Vdash_{\mathcal{G}} \neg\Phi * (\mathbf{A} \dot{=}^o \mathbf{B}) \in \Phi$. Applying $\mathcal{G}(\neg)$, $\Vdash_{\mathcal{G}} \neg\Phi$ since $\neg(\mathbf{A} \dot{=}^o \mathbf{B}) \in \Phi$, contradicting $\Phi \in \Gamma_\Sigma^{\mathcal{G}}$. Thus ∇_b holds. \dashv

We can also show $\Gamma_\Sigma^{\mathcal{G}}$ is saturated whenever $\mathcal{G}(\text{cut})$ is admissible in \mathcal{G} (without assuming admissibility of any other rule).

LEMMA 3.7. *Let \mathcal{G} be a sequent calculus such that $\mathcal{G}(\text{cut})$ is admissible. Then $\Gamma_\Sigma^{\mathcal{G}}$ satisfies ∇_{sat} .*

PROOF. Suppose $\Phi \in \Gamma_\Sigma^{\mathcal{G}}$, $\mathbf{A} \in \text{cwf}_o(\Sigma)$, $\Phi * \mathbf{A} \notin \Gamma_\Sigma^{\mathcal{G}}$ and $\Phi * \neg\mathbf{A} \notin \Gamma_\Sigma^{\mathcal{G}}$. Hence $\Vdash_{\mathcal{G}} \neg\Phi * \neg\mathbf{A}$ and $\Vdash_{\mathcal{G}} \neg\Phi * \neg\neg\mathbf{A}$. Using $\mathcal{G}(\text{cut})$, we have $\Vdash_{\mathcal{G}} \neg\Phi$, contradicting $\Phi \in \Gamma_\Sigma^{\mathcal{G}}$. \dashv

We can also show a partial converse.

LEMMA 3.8. *Let \mathcal{G} be a sequent calculus such that $\mathcal{G}(\neg)$ and $\mathcal{G}(\text{Inv}^\neg)$ are admissible. If $\Gamma_\Sigma^{\mathcal{G}}$ satisfies ∇_{sat} , then $\mathcal{G}(\text{cut})$ is admissible in \mathcal{G} .*

PROOF. Suppose $\Vdash_{\mathcal{G}} \Delta * \mathbf{C}$ and $\Vdash_{\mathcal{G}} \Delta * \neg\mathbf{C}$ hold but $\Vdash_{\mathcal{G}} \Delta$ does not. Applying $\mathcal{G}(\neg)$ to every member of Δ and to \mathbf{C} we have $\Vdash_{\mathcal{G}} \neg\neg\Delta * \neg\neg\mathbf{C}$ and $\Vdash_{\mathcal{G}} \neg\neg\Delta * \neg\mathbf{C}$. By Lemma 3.5, we know $\neg\Delta \in \Gamma_\Sigma^{\mathcal{G}}$. By saturation, we must have $\neg\Delta * \mathbf{C} \in \Gamma_\Sigma^{\mathcal{G}}$ or $\neg\Delta * \neg\mathbf{C} \in \Gamma_\Sigma^{\mathcal{G}}$. The first case contradicts $\Vdash_{\mathcal{G}} \neg\neg\Delta * \neg\mathbf{C}$ while the second case contradicts $\Vdash_{\mathcal{G}} \neg\neg\Delta * \neg\neg\mathbf{C}$. \dashv

Assuming soundness of the sequent calculus, we can show a stronger version of the converse.

DEFINITION 3.9 (Soundness). Let $\mathfrak{M} \subseteq \mathfrak{M}_\beta$ be a class of models \mathcal{G} be a sequent calculus. We say \mathcal{G} is *sound for the model class \mathfrak{M}* if Δ is valid for every $\mathcal{M} \in \mathfrak{M}$ whenever $\Vdash_{\mathcal{G}} \Delta$.

Existence of any saturated extension of a sound sequent calculus implies admissibility of cut.

THEOREM 3.10. *Let $*$ be in \mathfrak{A} and \mathcal{G} be a sequent calculus which is sound for \mathfrak{M}_* . If $\Gamma_\Sigma^{\mathcal{G}}$ has a saturated extension $\Gamma'_\Sigma \in \mathfrak{Acc}_*$, then $\mathcal{G}(\text{cut})$ is admissible in \mathcal{G} .*

PROOF. Suppose $\Gamma'_\Sigma \in \mathfrak{Acc}_*$ is a saturated extension of $\Gamma_\Sigma^{\mathcal{G}}$. Assume $\Vdash_{\mathcal{G}} \Delta * \mathbf{C}$ and $\Vdash_{\mathcal{G}} \Delta * \neg\mathbf{C}$ hold and $\Vdash_{\mathcal{G}} \Delta$ does not. By Lemma 3.5, we know $\neg\Delta \in \Gamma_\Sigma^{\mathcal{G}}$. Since $\neg\Delta$ is finite (hence sufficiently pure), $\neg\Delta \in \Gamma'_\Sigma$. By the model existence theorem for saturated abstract consistency classes (cf. Theorem 6.34 in [4]), there is a model $\mathcal{M} \in \mathfrak{M}_*$ such that $\mathcal{M} \models \neg\Delta$. By soundness of Γ'_Σ , we know both $\Delta * \mathbf{C}$ and $\Delta * \neg\mathbf{C}$ must be valid in \mathcal{M} . Since $\mathcal{M} \models \neg\Delta$, we must have $\mathcal{M} \models \mathbf{C}$ and $\mathcal{M} \models \neg\mathbf{C}$, a contradiction. \dashv

Therefore, showing an abstract consistency class $\Gamma_{\Sigma}^{\mathcal{G}}$ has a saturated extension in one of our eight classes \mathfrak{Acc}_* is as hard as showing admissibility of cut in \mathcal{G} .

REMARK 3.11 (Resolution Calculi). One can also do a similar analysis for resolution calculi. In that case, showing saturation corresponds to showing that one can derive the empty clause from a set of clauses Φ whenever one can derive the empty clause from $\Phi * (\mathbf{A} \vee \neg \mathbf{A})$.

We now define a higher-order sequent calculus \mathcal{G}_*^- for each $* \in \mathfrak{A}$. Each \mathcal{G}_*^- will be sound with respect to the model class \mathfrak{M}_* , but only three will be complete. Later we will introduce additional rules to obtain complete calculi for five of the eight model classes in Figure 2.

DEFINITION 3.12 (Sequent Calculi \mathcal{G}_*^-). For each $* \in \mathfrak{A}$, the sequent calculus \mathcal{G}_*^- is defined by the rules in Figure 3 and the additional rules listed in Figure 8.

for $* =$	\mathcal{G}_*^- includes the following rules
β	$\mathcal{G}(\beta)$
$\beta\eta$	$\mathcal{G}(\beta\eta)$
$\beta\xi$	$\mathcal{G}(\beta)$ and $\mathcal{G}(\xi)$
$\beta\mathfrak{f}$	$\mathcal{G}(\beta)$ and $\mathcal{G}(\mathfrak{f})$
$\beta\mathfrak{b}$	$\mathcal{G}(\beta)$ and $\mathcal{G}(\mathfrak{b})$
$\beta\eta\mathfrak{b}$	$\mathcal{G}(\beta\eta)$ and $\mathcal{G}(\mathfrak{b})$
$\beta\xi\mathfrak{b}$	$\mathcal{G}(\beta)$, $\mathcal{G}(\xi)$ and $\mathcal{G}(\mathfrak{b})$
$\beta\mathfrak{f}\mathfrak{b}$	$\mathcal{G}(\beta)$, $\mathcal{G}(\mathfrak{f})$ and $\mathcal{G}(\mathfrak{b})$

FIGURE 8. Extensional Higher-Order Sequent Calculi \mathcal{G}_*^-

We will not discuss soundness of these calculi (though each is sound with respect to the corresponding model class). Instead, we show the calculi $\mathcal{G}_{\beta\mathfrak{f}}^-$, $\mathcal{G}_{\beta\mathfrak{b}}^-$, $\mathcal{G}_{\beta\eta\mathfrak{b}}^-$, $\mathcal{G}_{\beta\xi\mathfrak{b}}^-$ and $\mathcal{G}_{\beta\mathfrak{f}\mathfrak{b}}^-$ are incomplete. The examples showing incompleteness directly correspond to Examples 3.2 and 3.3.

EXAMPLE 3.13. Let $* \in \{\beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ and Δ be $\{\neg a, \neg b, \neg(qa), (qb)\}$ where $a_o, b_o, q_{o \rightarrow o} \in \Sigma$ are parameters. For any $\mathcal{M} \equiv (\mathcal{D}, @, \mathcal{E}, v) \in \mathfrak{M}_*$, either $v(\mathcal{E}(a)) \equiv \mathbf{F}$, $v(\mathcal{E}(b)) \equiv \mathbf{F}$ or $\mathcal{E}(a) \equiv \mathcal{E}(b)$ by property \mathfrak{b} . Hence the sequent Δ is valid for every $\mathcal{M} \in \mathfrak{M}_*$. However, $\Vdash_{\mathcal{G}_*^-} \Delta$ does not hold. To prove this, suppose \mathcal{D} is a derivation of $\Vdash_{\mathcal{G}_*^-} \Delta$ which uses a minimal number of rule applications. By inspection, Δ cannot be the conclusion of any rule used to define \mathcal{G}_*^- except $\mathcal{G}(\beta)$ or $\mathcal{G}(\beta\eta)$. Assume \mathcal{D}_1 is a derivation of the premise of this rule. Since every member of Δ is $\beta\eta$ -normal, \mathcal{D}_1 is also a derivation of Δ , contradicting minimality of \mathcal{D} . Furthermore, $\mathcal{G}(\text{cut})$ is not admissible in \mathcal{G}_*^- (for $* \in \{\beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$). In particular, one can show $\Vdash_{\mathcal{G}_*^-} \Delta * (a \dot{=}^o b)$ and $\Vdash_{\mathcal{G}_*^-} \Delta * \neg(a \dot{=}^o b)$.

EXAMPLE 3.14. Let Δ be the sequent $\{q(\lambda X_{\iota}.gX), \neg(qg)\}$ where $q_{(\iota \rightarrow \iota) \rightarrow o}$ and $g_{\iota \rightarrow \iota}$ are parameters. By Lemma 3.24 in [4], $\mathcal{E}(\lambda X_{\iota}.gX) \equiv \mathcal{E}(g)$ for every $\mathcal{M} \equiv (\mathcal{D}, @, \mathcal{E}, v) \in \mathfrak{M}_{\beta\mathfrak{f}}$. Hence Δ is valid for every $\mathcal{M} \in \mathfrak{M}_{\beta\mathfrak{f}}$. However, $\Vdash_{\mathcal{G}_{\beta\mathfrak{f}}^-} \Delta$

does not hold. Suppose \mathcal{D} is a derivation of $\Vdash_{\mathcal{G}_{\beta f}}^- \Delta$ using a minimal number of rule applications. By inspection, the last rule of \mathcal{D} must be $\mathcal{G}(\beta)$. Since every member of Δ is β -normal (though not $\beta\eta$ -normal), the derivation of the premise is a derivation of Δ , contradicting minimality of \mathcal{D} . Again, $\mathcal{G}(\text{cut})$ is not admissible in $\mathcal{G}_{\beta f}$ since one can show $\Vdash \Delta * ((\lambda X.gX) \doteq g)$ and $\Vdash \Delta * \neg((\lambda X.gX) \doteq g)$.

On the other hand, one can show $\Gamma_{\Sigma}^{\mathcal{G}^-} \in \mathfrak{Acc}_*$ for each $* \in \mathfrak{Q}$. We conclude there is no hope for a general model existence theorem for any $\Gamma_{\Sigma} \in \mathfrak{Acc}_*$. Instead, we must ensure $\Gamma_{\Sigma} \in \mathfrak{Acc}_*$ satisfies extra properties in order to obtain model existence.

§4. Acceptable Abstract Consistency Classes. As illustrated by the Examples 3.2 and 3.3 we need some extra abstract consistency properties to ensure the existence of saturated extensions. We call these extra properties *acceptability conditions*.

DEFINITION 4.1 (Acceptability Conditions). Let Γ_{Σ} be an abstract consistency class in \mathfrak{Acc}_* . We define the following properties:

- ∇_m If $\mathbf{A}, \mathbf{B} \in \text{cuff}_o(\Sigma)$ are atomic and $\mathbf{A}, \neg\mathbf{B} \in \Phi$, then $\Phi * \neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Gamma_{\Sigma}$.
- ∇_d If $\neg(h\overline{\mathbf{A}^n} \doteq^{\beta} h\overline{\mathbf{B}^n}) \in \Phi$ for some types α_i where $\beta \in \{o, \iota\}$ and $h_{\alpha^n \rightarrow \beta} \in \Sigma$ is a parameter, then there is an i with $1 \leq i \leq n$ such that $\Phi * \neg(\mathbf{A}^i \doteq^{\alpha_i} \mathbf{B}^i) \in \Gamma_{\Sigma}$.

DEFINITION 4.2 (Acceptable Classes). Every abstract consistency class Γ_{Σ} in $\mathfrak{Acc}_{\beta} [\mathfrak{Acc}_{\beta\eta}, \mathfrak{Acc}_{\beta\xi}]$ is called *acceptable* in $\mathfrak{Acc}_{\beta} [\mathfrak{Acc}_{\beta\eta}, \mathfrak{Acc}_{\beta\xi}]$. For $* \in \{\beta f, \beta fb\}$, an abstract consistency class Γ_{Σ} is called *acceptable* in \mathfrak{Acc}_* if it satisfies the conditions in Figure 9.

for $* =$	Γ_{Σ} is <i>acceptable</i> in \mathfrak{Acc}_* if it satisfies
βf	∇_{η} .
βfb	∇_m and ∇_d .

FIGURE 9. Acceptability Conditions for \mathfrak{Acc}_*

A main result of this paper will be presented as Theorem 9.3 in Section 9. It is called the **Saturated Extension Theorem** and states:

For each $ \in \mathfrak{Q}$, there is a saturated abstract consistency class in \mathfrak{Acc}_* that is an extension of all acceptable Γ_{Σ} in \mathfrak{Acc}_* .*

By Theorem 3.10, we know the existence of a saturated extension implies cut-elimination for certain sequent calculi. Thus (for $* \in \mathfrak{Q}$) we can conclude that for any sequent calculus \mathcal{G} (sound for \mathfrak{M}_*), if $\Gamma_{\Sigma}^{\mathcal{G}}$ is acceptable, then $\mathcal{G}(\text{cut})$ is admissible in \mathcal{G} . It is well-known that strong proof methods are required to show cut-elimination for a higher-order logic such as \mathcal{G}_{β} since cut-elimination implies consistency of analysis (see [2] and the references cited there). Consequently, we can conclude that the Saturated Extension Theorem requires strong proof methods as well.

We next extend Lemma 6.18 in [4] to include the new acceptability conditions.

LEMMA 4.3 (Compactness of abstract consistency classes).

For each abstract consistency class Γ_Σ there exists a compact abstract consistency class Γ'_Σ satisfying the same ∇_ properties such that $\Gamma_\Sigma \subseteq \Gamma'_\Sigma$.*

PROOF. (following and extending Lemma 6.18 in [4])

We choose $\Gamma'_\Sigma := \{\Phi \subseteq \text{cwff}_o(\Sigma) \mid \text{every finite subset of } \Phi \text{ is in } \Gamma_\Sigma\}$. Now suppose that $\Phi \in \Gamma_\Sigma$. Γ_Σ is closed under subsets, so every finite subset of Φ is in Γ_Σ and thus $\Phi \in \Gamma'_\Sigma$. Hence $\Gamma_\Sigma \subseteq \Gamma'_\Sigma$.

We have shown in the proof of Lemma 6.18 in [4] that Γ'_Σ is compact and hence closed under subsets.

Next we show that if Γ_Σ satisfies ∇_* , then Γ'_Σ satisfies ∇_* for the acceptability conditions in Definition 4.1. All the others have been covered in Lemma 6.18 in [4].

∇_m : Let $\Phi \in \Gamma'_\Sigma$ and $\mathbf{A}, \neg\mathbf{B} \in \Phi$ for atomic $\mathbf{A}, \mathbf{B} \in \text{cwff}_o(\Sigma)$. Furthermore, let Ψ be any finite subset of $\Phi * \neg(\mathbf{A} \dot{=}^o \mathbf{B})$ and $\Theta := (\Psi \setminus \{\neg(\mathbf{A} \dot{=}^o \mathbf{B})\}) \cup \{\mathbf{A}, \neg\mathbf{B}\}$. Θ is a finite subset of Φ , hence $\Theta \in \Gamma_\Sigma$. Since Γ_Σ is an abstract consistency class and $\mathbf{A}, \neg\mathbf{B} \in \Theta$, we get $\Theta * \neg(\mathbf{A} \dot{=}^o \mathbf{B}) \in \Gamma_\Sigma$ by ∇_m for Γ_Σ . We know that $\Psi \subseteq \Theta * \neg(\mathbf{A} \dot{=}^o \mathbf{B})$ and Γ_Σ is closed under subsets, hence $\Psi \in \Gamma_\Sigma$. Thus every finite subset Ψ of $\Phi * \neg(\mathbf{A} \dot{=}^o \mathbf{B})$ is in Γ_Σ and therefore by definition $\Phi * \neg(\mathbf{A} \dot{=}^o \mathbf{B}) \in \Gamma'_\Sigma$.

∇_d : Let $\Phi \in \Gamma'_\Sigma$ and $\neg(h\overline{\mathbf{A}^n} \dot{=}^\beta h\overline{\mathbf{B}^n}) \in \Phi$, where $\beta \in \{o, \iota\}$ and $h_{\overline{\alpha^n} \rightarrow \beta} \in \Sigma$ is a parameter for some types α_i . Assume $\Phi * \neg(\mathbf{A}^i \dot{=}^{\alpha_i} \mathbf{B}^i) \notin \Gamma'_\Sigma$ for all $1 \leq i \leq n$. Then there are finite subsets Ψ_i of Φ such that $\Psi_i * \neg(\mathbf{A}^i \dot{=}^{\alpha_i} \mathbf{B}^i) \notin \Gamma_\Sigma$ for all $1 \leq i \leq n$. Let $\Psi := (\bigcup_{1 \leq i \leq n} \Psi_i) * \neg(h\overline{\mathbf{A}^n} \dot{=}^\beta h\overline{\mathbf{B}^n})$. Ψ is a finite subset of Φ and thus $\Psi \in \Gamma_\Sigma$. By ∇_d^h we get $\Psi * \neg(\mathbf{A}^i \dot{=}^{\alpha_i} \mathbf{B}^i) \in \Gamma_\Sigma$ for some i with $1 \leq i \leq n$. Γ_Σ is closed under subsets, so $\Psi_i * \neg(\mathbf{A}^i \dot{=}^{\alpha_i} \mathbf{B}^i) \in \Gamma_\Sigma$ for some i with $1 \leq i \leq n$. This is a contradiction, and we conclude that $\Phi * \neg(\mathbf{A}^i \dot{=}^{\alpha_i} \mathbf{B}^i) \in \Gamma'_\Sigma$ for some $1 \leq i \leq n$.

⊥

§5. Five Complete Higher-Order Sequent Calculi. In order to obtain complete higher-order sequent calculi, we must include rules corresponding to the new acceptability conditions. We show such rules in Figure 10.

DEFINITION 5.1 (Sequent Calculi \mathcal{G}_*). For each $* \in \{\beta, \beta\eta, \beta\xi\}$, let \mathcal{G}_* be the sequent calculus \mathcal{G}_*^- . Let $\mathcal{G}_{\beta\mathfrak{f}}$ be the sequent calculus defined by the rules in Figure 3 along with the rules $\mathcal{G}(\beta\eta)$ and $\mathcal{G}(\mathfrak{f})$. Let $\mathcal{G}_{\beta\mathfrak{fb}}$ be the sequent calculus defined by the rules for $\mathcal{G}_{\beta\mathfrak{fb}}^-$ along with $\mathcal{G}(\text{Init}^\dot{=})$ and $\mathcal{G}(d)$.

For each $* \in \mathfrak{B}$ other than $* \equiv \beta\mathfrak{f}$, the sequent calculus \mathcal{G}_* is defined to include at least the rules for \mathcal{G}_*^- . Hence for these calculi $\vdash_{\mathcal{G}_*} \Delta$ whenever $\vdash_{\mathcal{G}_*^-} \Delta$. We have used $\mathcal{G}(\beta\eta)$ instead of $\mathcal{G}(\beta)$ in the definition of $\mathcal{G}_{\beta\mathfrak{f}}$. This is necessary for completeness. We will show $\mathcal{G}(\beta)$ is admissible in $\mathcal{G}_{\beta\mathfrak{f}}$ (cf. Lemma 5.2) and hence $\vdash_{\mathcal{G}_{\beta\mathfrak{f}}} \Delta$ whenever $\vdash_{\mathcal{G}_{\beta\mathfrak{f}}^-} \Delta$ as well. We could also replace $\mathcal{G}(\beta)$ with $\mathcal{G}(\beta\eta)$ in the definition of $\mathcal{G}_{\beta\mathfrak{fb}}$, but this is not necessary for completeness.

$\frac{\Delta * (\mathbf{A} \doteq^o \mathbf{B}) \quad \mathbf{A} \text{ and } \mathbf{B} \text{ atomic}}{\Delta * \neg \mathbf{A} * \mathbf{B}} \mathcal{G}(\text{Init}^{\doteq})$	
$\frac{\Delta * (\mathbf{A}^1 \doteq^{\alpha_1} \mathbf{B}^1) \quad \dots \quad \Delta * (\mathbf{A}^n \doteq^{\alpha_n} \mathbf{B}^n) \quad (\dagger)}{\Delta * (h\overline{\mathbf{A}}^n \doteq^\beta h\overline{\mathbf{B}}^n)} \mathcal{G}(d)$	
$(\dagger) \quad n \geq 0, \beta \in \{o, \iota\}, h_{\alpha^n \rightarrow \beta} \in \Sigma \text{ parameter}$	

FIGURE 10. Rules for Completeness

We note the necessary additional rules are admissible for each sequent calculus in order to conclude $\Gamma_{\Sigma}^{\mathcal{G}_*} \in \mathfrak{A}\mathfrak{C}\mathfrak{C}_*$.

LEMMA 5.2. *Let $*$ be given. We have the following:*

1. *The rule $\mathcal{G}(\text{weak})$ is admissible in \mathcal{G}_* .*
2. *If $*$ is $\{\beta\eta, \beta\mathfrak{f}\}$, then the rules $\mathcal{G}(\beta\eta^\downarrow)$ and $\mathcal{G}(\beta)$ are admissible in \mathcal{G}_* .*
3. *The rule $\mathcal{G}(\beta^\downarrow)$ is admissible in \mathcal{G}_* .*
4. *The rule $\mathcal{G}(\text{Inv}^\neg)$ is admissible in \mathcal{G}_* .*

PROOF. The primary proof technique to prove these is induction on derivations. To prove admissibility of $\mathcal{G}(\text{weak})$ one must generalize to allow renaming of witness parameters (due to the rule $\mathcal{G}(\Pi_+^c)$ which requires a new parameter c).

Suppose $*$ is $\{\beta\eta, \beta\mathfrak{f}\}$ and so $\mathcal{G}(\beta\eta)$ is a rule in \mathcal{G}_* . One can show by induction that if $\Vdash_{\mathcal{G}_*} \Delta$, then $\Vdash_{\mathcal{G}_*} \{\mathbf{A} \downarrow_{\beta\eta} \mid \mathbf{A} \in \Delta\}$. Once one knows this, the rule $\mathcal{G}(\beta\eta)$ can be used to $\beta\eta$ -expand every formula except the principal formula of $\mathcal{G}(\beta\eta^\downarrow)$. Given $\mathcal{G}(\beta\eta)$ and admissibility of $\mathcal{G}(\beta\eta^\downarrow)$, one can easily show $\mathcal{G}(\beta)$ and $\mathcal{G}(\beta^\downarrow)$ are admissible.

If $*$ is $\{\beta\eta, \beta\mathfrak{f}\}$, then we already know $\mathcal{G}(\beta^\downarrow)$ is admissible. Otherwise, the rule $\mathcal{G}(\beta)$ is included in \mathcal{G}_* . As above, prove $\Vdash_{\mathcal{G}_*} \{\mathbf{A} \downarrow_{\beta} \mid \mathbf{A} \in \Delta\}$ whenever $\Vdash_{\mathcal{G}_*} \Delta$.

Finally, to show $\mathcal{G}(\text{Inv}^\neg)$ is admissible, we can use $\mathcal{G}(\beta^\downarrow)$ (or $\mathcal{G}(\beta\eta^\downarrow)$ if $*$ is $\{\beta\eta, \beta\mathfrak{f}\}$) to assume the principal formula is normal. Then a simple induction proof shows admissibility where $\mathcal{G}(\neg)$ is the only significant case. \dashv

THEOREM 5.3 (Soundness).

\mathcal{G}_* is sound for \mathfrak{M}_* where $*$ is \mathfrak{A} . That is, if $\Vdash_{\mathcal{G}_*} \Delta$ is derivable, then the sequent Δ is valid in \mathfrak{M}_* .

PROOF. This result can be shown by induction on the derivation of $\Vdash_{\mathcal{G}_*} \Delta$. The essential task is to show soundness of the individual inference rules, i.e. to show that they preserve validity with respect to the model class \mathfrak{M}_* . In fact, for every rule other than $\mathcal{G}(\Pi_+^c)$ (where we may need to change the interpretation of the parameter c), we can show the rule preserves validity with respect to each model $\mathcal{M} \in \mathfrak{M}_*$. The cases of the rules from Figure 3 are absolutely standard, so we omit their proof in the interest of brevity and concentrate on the new rules.

The rules $\mathcal{G}(\text{Init}^\pm)$ and $\mathcal{G}(d)$ Figure 10 are sound with respect to any $\mathcal{M} \in \mathfrak{M}_\beta$. Let $\mathcal{M} \equiv (\mathcal{D}, @, \mathcal{E}, v) \in \mathfrak{M}_\beta$ and φ be an arbitrary assignment. Note that by Lemma 4.2 in [4] (since $\mathcal{M} \in \mathfrak{M}_\beta$ satisfies property \mathfrak{q}) we know $\mathcal{E}_\varphi(\mathbf{C}) \equiv \mathcal{E}_\varphi(\mathbf{D})$ iff $\mathcal{M} \models_\varphi (\mathbf{C} \dot{=}^\alpha \mathbf{D})$ for any $\mathbf{C}, \mathbf{D} \in \text{wff}_\alpha(\Sigma)$.

$\mathcal{G}(\text{Init}^\pm)$: If $\mathcal{M} \models \mathbf{A}$ and $\mathcal{M} \models \mathbf{A} \dot{=}^o \mathbf{B}$, then $v(\mathcal{E}(\mathbf{B})) \equiv v(\mathcal{E}(\mathbf{A})) \equiv \mathbf{T}$ and so $\mathcal{M} \models \mathbf{B}$. Using this fact, we can show soundness of the rule $\mathcal{G}(\text{Init}^\pm)$ as follows: Assume the premise $\Delta * (\mathbf{A} \dot{=}^o \mathbf{B})$ is valid for \mathcal{M} but $\Delta * \neg \mathbf{A} * \mathbf{B}$ is not valid for \mathcal{M} . Then $\mathcal{M} \models \mathbf{A}$, $\mathcal{M} \not\models \mathbf{B}$ and $\mathcal{M} \not\models \mathbf{D}$ for each $\mathbf{D} \in \Delta$. By validity of the premise for \mathcal{M} , we know $\mathcal{M} \models \mathbf{A} \dot{=}^o \mathbf{B}$. However, we have shown $\mathcal{M} \models \mathbf{B}$ whenever $\mathcal{M} \models \mathbf{A}$ and $\mathcal{M} \models \mathbf{A} \dot{=}^o \mathbf{B}$, a contradiction. (For the remaining rules, except $\mathcal{G}(\mathfrak{b})$, the final verification of soundness of the rule is omitted as it follows using this same pattern.)

$\mathcal{G}(d)$: Suppose $\mathcal{M} \models \mathbf{A}^i \dot{=} \mathbf{B}^i$ and so $\mathcal{E}(\mathbf{A}^i) \equiv \mathcal{E}(\mathbf{B}^i)$ for each $i \in \{1, \dots, n\}$. Let $\mathbf{c}^i := \mathcal{E}(\mathbf{A}^i)$ and choose a (new) variable $Y_{\alpha^i}^i$ for each $i \in \{1, \dots, n\}$. Let ψ be the variable assignment $\varphi, [\mathbf{c}^1/Y^1], \dots, [\mathbf{c}^n/Y^n]$. Using the Substitution-Value Lemma (cf. Lemma 3.20 in [4]) n times and the fact that \mathcal{E} respects β -conversion, we compute

$$\mathcal{E}(h\overline{\mathbf{A}^n}) \equiv \mathcal{E}_\psi(h\overline{Y^n}) \equiv \mathcal{E}(h\overline{\mathbf{B}^n}).$$

This concludes the cases, where we could argue for any model in $\mathcal{M} \in \mathfrak{M}_\beta$. We now check soundness of the $\mathcal{G}(\beta\eta)$ rule and the rules in Figure 7 with respect to models in the corresponding model classes. Let $\mathcal{M} \equiv (\mathcal{D}, @, \mathcal{E}, v) \in \mathfrak{M}_*$ and φ be arbitrary.

$\mathcal{G}(\beta\eta)$: The rule $\mathcal{G}(\beta\eta)$ is only included when $*$ $\in \{\beta\eta, \beta\mathfrak{f}\}$. In each case, $\mathcal{M} \in \mathfrak{M}_*$ satisfies property η (either by definition or by Lemma 3.24 in [4]).

Hence $\mathcal{E}(\mathbf{A}) \equiv \mathcal{E}(\mathbf{A} \downarrow_{\beta\eta})$ and so $\mathcal{M} \models \mathbf{A}$ iff $\mathcal{M} \models \mathbf{A} \downarrow_{\beta\eta}$.

$\mathcal{G}(\mathfrak{f})$: We check soundness of $\mathcal{G}(\mathfrak{f})$ when $*$ is $\beta\mathfrak{f}\mathfrak{b}$ and so \mathcal{M} satisfies \mathfrak{f} . We show that $\mathcal{M} \models \mathbf{G} \dot{=}^{\alpha \rightarrow \beta} \mathbf{H}$ whenever $\mathcal{M} \models \forall X_{\alpha}. (\mathbf{G}X) \dot{=}^\beta (\mathbf{H}X)$. Suppose $\mathcal{M} \models \forall X_{\alpha}. (\mathbf{G}X) \dot{=}^\beta (\mathbf{H}X)$. Hence $\mathcal{M} \models_{\varphi, [a/X]} \mathbf{G}X \dot{=}^\beta \mathbf{H}X$ and so $\mathcal{E}_\varphi(\mathbf{G})@a \equiv \mathcal{E}_\varphi(\mathbf{H})@a$ for all $a \in \mathcal{D}_\alpha$. By \mathfrak{f} we conclude $\mathcal{E}_\varphi(\mathbf{G}) \equiv \mathcal{E}_\varphi(\mathbf{H})$ and so $\mathcal{M} \models \mathbf{G} \dot{=}^{\alpha \rightarrow \beta} \mathbf{H}$.

$\mathcal{G}(\mathfrak{b})$: We check soundness of $\mathcal{G}(\mathfrak{b})$ when $*$ is $\beta\mathfrak{f}\mathfrak{b}$ and so \mathcal{M} satisfies \mathfrak{b} . Suppose $\Delta * \neg \mathbf{A} * \mathbf{B}$ and $\Delta * \neg \mathbf{B} * \mathbf{A}$ are valid for \mathcal{M} . Assume $\Delta * (\mathbf{A} \dot{=}^o \mathbf{B})$ is not valid for \mathcal{M} . Hence $\mathcal{M} \not\models (\mathbf{A} \dot{=}^o \mathbf{B})$ and $\mathcal{M} \not\models \mathbf{D}$ for each $\mathbf{D} \in \Delta$. Either $\mathcal{M} \models \mathbf{A}$ or $\mathcal{M} \not\models \mathbf{A}$. First, assume $\mathcal{M} \models \mathbf{A}$. By validity of $\Delta * \neg \mathbf{A} * \mathbf{B}$ for \mathcal{M} , we have $\mathcal{M} \models \mathbf{B}$ and so $v(\mathcal{E}(\mathbf{A})) \equiv \mathbf{T} \equiv v(\mathcal{E}(\mathbf{B}))$. By property \mathfrak{b} , $\mathcal{E}(\mathbf{A}) \equiv \mathcal{E}(\mathbf{B})$ and so $\mathcal{M} \models (\mathbf{A} \dot{=}^o \mathbf{B})$, a contradiction. Next, assume $\mathcal{M} \not\models \mathbf{A}$. In this case, $\mathcal{M} \not\models \mathbf{B}$ (by validity of $\Delta * \neg \mathbf{B} * \mathbf{A}$) and so $v(\mathcal{E}(\mathbf{A})) \equiv \mathbf{F} \equiv v(\mathcal{E}(\mathbf{B}))$. Again, by property \mathfrak{b} , we have $\mathcal{E}(\mathbf{A}) \equiv \mathcal{E}(\mathbf{B})$ and $\mathcal{M} \models (\mathbf{A} \dot{=}^o \mathbf{B})$, a contradiction.

⊥

We now show $\Gamma_\Sigma^{\mathcal{G}*}$ is an acceptable abstract consistency class.

THEOREM 5.4. *For each $*$ $\in \mathfrak{D}$, $\Gamma_\Sigma^{\mathcal{G}*} \in \mathfrak{Acc}_*$ and $\Gamma_\Sigma^{\mathcal{G}*}$ is acceptable in \mathfrak{Acc}_* .*

PROOF. We immediately conclude $\Gamma_\Sigma^{\mathcal{G}*} \in \mathfrak{Acc}_*$ from Lemma 5.2 and Theorem 3.6. We only must check acceptability.

There is nothing to check if $*$ $\in \{\beta, \beta\eta, \beta\xi\}$. We know ∇_η holds for $\Gamma_\Sigma^{\mathcal{G}_{\beta\eta}}$ by Theorem 3.6 since $\mathcal{G}(\beta\eta)$ and $\mathcal{G}(\beta\eta^\perp)$ are admissible (by Lemma 5.2). Hence $\Gamma_\Sigma^{\mathcal{G}_{\beta\eta}}$ is acceptable in $\mathfrak{A}\mathfrak{c}\mathfrak{c}_*$.

We now consider the remaining conditions when $*$ is $\beta\mathfrak{f}\mathfrak{b}$.

∇_m : Assume $\Phi \in \Gamma_\Sigma^{\mathcal{G}_*}$, $\mathbf{A}, \neg\mathbf{B} \in \Phi$ are atomic and $\Phi * \neg(\mathbf{A} \doteq^o \mathbf{B}) \notin \Gamma_\Sigma^{\mathcal{G}_*}$. By Lemmas 3.5 and 5.2, $\Vdash_{\mathcal{G}_*} \neg\Phi * (\mathbf{A} \doteq \mathbf{B})$. Applying $\mathcal{G}(\text{Init}^\doteq)$, $\Vdash_{\mathcal{G}_*} \neg\Phi * \mathbf{B}$ (since $\neg\mathbf{A} \in \neg\Phi$). Applying $\mathcal{G}(\neg)$, $\Vdash_{\mathcal{G}_*} \neg\Phi$ (since $\neg\neg\mathbf{B} \in \neg\Phi$), contradicting $\Phi \in \Gamma_\Sigma^{\mathcal{G}_*}$.

∇_d : Assume $\Phi \in \Gamma_\Sigma^{\mathcal{G}_*}$, $h_{\alpha^n \rightarrow \beta} \in \Sigma$ is a parameter, $\beta \in \{o, \iota\}$, $\neg(h\overline{\mathbf{A}^n} \doteq^\beta h\overline{\mathbf{B}^n}) \in \Phi$ and $\Phi * \neg(\mathbf{A}^i \doteq^{\alpha^i} \mathbf{B}^i) \notin \Gamma_\Sigma^{\mathcal{G}_*}$ for each $1 \leq i \leq n$. By Lemmas 3.5 and 5.2, $\Vdash_{\mathcal{G}_*} \neg\Phi * (\mathbf{A}^i \doteq \mathbf{B}^i)$ for each $1 \leq i \leq n$. Applying $\mathcal{G}(d)$, $\Vdash_{\mathcal{G}_*} \neg\Phi * (h\overline{\mathbf{A}^n} \doteq h\overline{\mathbf{B}^n})$. Applying $\mathcal{G}(\neg)$, $\Vdash_{\mathcal{G}_*} \neg\Phi$ (since $\neg(h\overline{\mathbf{A}^n} \doteq h\overline{\mathbf{B}^n}) \in \Phi$), contradicting $\Phi \in \Gamma_\Sigma^{\mathcal{G}_*}$.

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§6. Acceptable Hintikka Sets and Hintikka-Compatibility. In this section we will study the notion of acceptable Hintikka sets. Hintikka sets provide the basis for the model constructions in the model existence theorems (see [4]). We define the Hintikka equivalents for the acceptability conditions and provide a version of the *abstract extension lemma* for them: In the absence of saturation, we will need Hintikka sets to satisfy appropriate conditions to ensure the existence of models.

DEFINITION 6.1 (Hintikka Acceptability Conditions). Let \mathcal{H} be a Hintikka set in $\mathfrak{H}\text{int}_*$. We define the following properties:

- $\vec{\nabla}_m$ If $\mathbf{A}, \mathbf{B} \in \text{cuff}_o(\Sigma)$ are atomic and $\mathbf{A}, \neg\mathbf{B} \in \mathcal{H}$, then $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \mathcal{H}$.
- $\vec{\nabla}_d$ If $\neg(h\overline{\mathbf{A}^n} \doteq^\beta h\overline{\mathbf{B}^n}) \in \mathcal{H}$ where $\beta \in \{o, \iota\}$ and h is a parameter, then there is an i with $1 \leq i \leq n$ such that $\neg(\mathbf{A}^i \doteq \mathbf{B}^i) \in \mathcal{H}$.

These conditions directly correspond to the ones in Definition 4.1, except they are closure conditions for \mathcal{H} instead of extensibility conditions for members of Γ_Σ .

Every Hintikka set \mathcal{H} in $\mathfrak{H}\text{int}_\beta$ [$\mathfrak{H}\text{int}_{\beta\eta}$, $\mathfrak{H}\text{int}_{\beta\mathfrak{f}}$] is *acceptable* in $\mathfrak{H}\text{int}_\beta$ [$\mathfrak{H}\text{int}_{\beta\eta}$, $\mathfrak{H}\text{int}_{\beta\mathfrak{f}}$]. For $*$ $\in \{\beta\mathfrak{f}, \beta\mathfrak{f}\mathfrak{b}\}$, we say a Hintikka set \mathcal{H} is *acceptable* in $\mathfrak{H}\text{int}_*$ if it obeys the conditions in Figure 11.

for $*$ =	\mathcal{H} is <i>acceptable</i> in $\mathfrak{H}\text{int}_*$ if it satisfies
$\beta\mathfrak{f}$	$\vec{\nabla}_\eta$.
$\beta\mathfrak{f}\mathfrak{b}$	$\vec{\nabla}_m$ and $\vec{\nabla}_d$.

FIGURE 11. Acceptability Conditions for $\mathfrak{H}\text{int}_*$

In [4] we constructed Hintikka sets as maximal elements of abstract consistency classes (cf. Lemma 6.21 in [4]). In order to account for the new acceptability

conditions, we show maximal sets in abstract consistency classes satisfy the corresponding acceptability conditions.

LEMMA 6.2 (Acceptable Hintikka Lemma). *Let $*$ $\in \mathfrak{A}$ and Γ_Σ be an abstract consistency class in \mathfrak{Acc}_* . Suppose a set $\mathcal{H} \in \Gamma_\Sigma$ is maximal in Γ_Σ with respect to subset (i.e., for each sentence $\mathbf{D} \in \text{cwff}_o(\Sigma)$ such that $\mathcal{H} * \mathbf{D} \in \Gamma_\Sigma$, we already have $\mathbf{D} \in \mathcal{H}$). If Γ_Σ satisfies $\nabla_m [\nabla_d]$, then \mathcal{H} satisfies $\vec{\nabla}_m [\vec{\nabla}_d]$.*

PROOF. Suppose Γ_Σ satisfies ∇_m , $\mathbf{A}, \mathbf{B} \in \text{cwff}_o(\Sigma)$ are atomic and $\mathbf{A}, \neg \mathbf{B} \in \mathcal{H}$. By ∇_m , $\mathcal{H} * \neg(\mathbf{A} \dot{=}^o \mathbf{B}) \in \Gamma_\Sigma$. Since \mathcal{H} is maximal in Γ_Σ , $\neg(\mathbf{A} \dot{=} \mathbf{B}) \in \mathcal{H}$ as desired. The ∇_d follows similarly. \dashv

We can now modify the abstract extension lemma from [4] to include the acceptability conditions.

THEOREM 6.3 (Abstract Extension Lemma). *Let Γ_Σ be a compact abstract consistency class in \mathfrak{Acc}_* and let $\Phi \in \Gamma_\Sigma$ be sufficiently Σ -pure. Then there exists a Σ -Hintikka set $\mathcal{H} \in \mathfrak{Hint}_*$ such that $\Phi \subseteq \mathcal{H}$. Furthermore, if $*$ $\in \mathfrak{A}$ and Γ_Σ is acceptable in \mathfrak{Acc}_* , then \mathcal{H} is acceptable in \mathfrak{Hint}_* .*

PROOF. The lemma extends the abstract extension lemma (Lemma 6.32 in [4]) by the acceptability consideration in the last sentence. The Hintikka set constructed in the proof of Lemma 6.32 in [4] is maximal in Γ_Σ (as is explicitly proven). If Γ_Σ is acceptable in \mathfrak{Acc}_* , then we can conclude \mathcal{H} is acceptable in \mathfrak{Hint}_* by Lemma 6.2. \dashv

When constructing models in \mathfrak{M}_* of a Hintikka set \mathcal{H} , we must verify property q. For this purpose, the assumption that \mathcal{H} contains no Leibniz equations is very helpful.

DEFINITION 6.4. Let \mathcal{H} be a set of formulae. We say \mathcal{H} is *Leibniz-free* if there do not exist terms $\mathbf{A}_\alpha, \mathbf{B}_\alpha$ such that $(\mathbf{A} \dot{=}^\alpha \mathbf{B}) \in \mathcal{H}$.

We can now show every Hintikka set is either saturated (in which case we have already constructed models in [4]) or Leibniz-free. Hence we will only need to construct models for Leibniz-free Hintikka sets.

THEOREM 6.5 (Impredicativity Gap). *Let \mathcal{H} be a Hintikka set. Either \mathcal{H} is saturated or \mathcal{H} is Leibniz-free.*

PROOF. Suppose \mathcal{H} is not Leibniz-free. Then $(\mathbf{A} \dot{=}^\alpha \mathbf{B}) \in \mathcal{H}$ for some $\mathbf{A}_\alpha, \mathbf{B}_\alpha$. We show \mathcal{H} satisfies $\vec{\nabla}_{sat}$. Let \mathbf{C}_o be a closed formula. Since $(\forall Q_{\alpha \rightarrow o}. Q\mathbf{A} \Rightarrow Q\mathbf{B}) \in \mathcal{H}$, we know $(\neg \mathbf{C} \vee \mathbf{C}) \in \mathcal{H}$ by $\vec{\nabla}_\forall$ (with the term $\lambda X_\alpha. \mathbf{C}$) and $\vec{\nabla}_\beta$. By $\vec{\nabla}_\forall$, either $\neg \mathbf{C} \in \mathcal{H}$ or $\mathbf{C} \in \mathcal{H}$. \dashv

With a similar argument we could show that Hintikka sets are either saturated or free of sentences such as $\forall P_o. P \Rightarrow P$ (a generalized tautology) and $\forall P_{\alpha \rightarrow o}. (Pz_\alpha \wedge (\forall X_\alpha. PX \Rightarrow P(f_{\alpha \rightarrow \alpha} X))) \Rightarrow \forall Y_\alpha. PY$ (an induction axiom).

We now introduce the notion of Hintikka-compatibility (\mathcal{H} -Compatibility) and prove some useful lemmas. We begin by defining when two closed terms are weakly compatible relative to a Hintikka set \mathcal{H} .

DEFINITION 6.6 (\mathcal{H} -Weak Compatibility). Let \mathcal{H} be a Hintikka set. We say two closed terms $\mathbf{A}, \mathbf{B} \in \text{cwff}_\alpha(\Sigma)$ are \mathcal{H} -weakly compatible (written $\mathbf{A} \check{\jmath} \mathbf{B}$) if $\neg(\mathbf{A} \dot{=}^\alpha \mathbf{B}) \notin \mathcal{H}$ and $\neg(\mathbf{B} \dot{=}^\alpha \mathbf{A}) \notin \mathcal{H}$.

We next strengthen weak compatibility to compatibility.

DEFINITION 6.7 (\mathcal{H} -Compatibility). Let \mathcal{H} be a Hintikka set. We define when two closed terms $\mathbf{A}, \mathbf{B} \in \text{cwff}_\alpha(\Sigma)$ are \mathcal{H} -compatible (written $\mathbf{A} \parallel \mathbf{B}$) by induction on the type α .

- o : $\mathbf{A}_o \parallel \mathbf{B}_o$ if $\{\mathbf{A}, \neg \mathbf{B}\} \not\subseteq \mathcal{H}$ and $\{\neg \mathbf{A}, \mathbf{B}\} \not\subseteq \mathcal{H}$.
- ι : $\mathbf{A}_\iota \parallel \mathbf{B}_\iota$ if $\mathbf{A} \check{\parallel} \mathbf{B}$.
- $\beta \rightarrow \gamma$: $\mathbf{A}_{\beta \rightarrow \gamma} \parallel \mathbf{B}_{\beta \rightarrow \gamma}$ if $\mathbf{AC} \parallel \mathbf{BD}$ whenever $\mathbf{C}, \mathbf{D} \in \text{cwff}_\beta(\Sigma)$ and $\mathbf{C} \parallel \mathbf{D}$.

We say a set $S \subseteq \text{cwff}_\alpha(\Sigma)$ is \mathcal{H} -compatible if $\mathbf{A} \parallel \mathbf{B}$ for every pair $\mathbf{A}, \mathbf{B} \in S$.

Intuitively, if two terms are \mathcal{H} -compatible, then they *might* be equal in a model of \mathcal{H} . Notice that \mathcal{H} -compatibility is not an equivalence relation, as it need not be transitive.

LEMMA 6.8. *Let \mathcal{H} be a Hintikka set. For any $\mathbf{A}, \mathbf{B} \in \text{cwff}_\alpha(\Sigma)$, $\mathbf{A} \check{\parallel} \mathbf{B}$ iff $\mathbf{A} \downarrow_\beta \check{\parallel} \mathbf{B} \downarrow_\beta$. Also, $\mathbf{A} \parallel \mathbf{B}$ iff $\mathbf{A} \downarrow_\beta \parallel \mathbf{B} \downarrow_\beta$.*

PROOF. We know the result for $\check{\parallel}$ since $\neg(\mathbf{A} \dot{=}^\alpha \mathbf{B}) \in \mathcal{H}$ iff $\neg(\mathbf{A} \downarrow_\beta \dot{=}^\alpha \mathbf{B} \downarrow_\beta) \in \mathcal{H}$ for all $\mathbf{A}, \mathbf{B} \in \text{cwff}_\alpha(\Sigma)$ by $\vec{\nabla}_\beta$. The result for \parallel follows by an easy induction on the type α . In particular, for $\mathbf{A}, \mathbf{B} \in \text{cwff}_\iota(\Sigma)$, the equivalence follows from the equivalence for $\check{\parallel}$.

By $\vec{\nabla}_\beta$, we know $\{\mathbf{A}, \neg \mathbf{B}\} \subseteq \mathcal{H}$ iff $\{\mathbf{A} \downarrow_\beta, \neg \mathbf{B} \downarrow_\beta\} \subseteq \mathcal{H}$ for all $\mathbf{A}, \mathbf{B} \in \text{cwff}_o(\Sigma)$. Hence $\mathbf{A} \parallel \mathbf{B}$ iff $\mathbf{A} \downarrow_\beta \parallel \mathbf{B} \downarrow_\beta$ for all $\mathbf{A}, \mathbf{B} \in \text{cwff}_o(\Sigma)$.

At type $\beta \rightarrow \gamma$, let $\mathbf{C}, \mathbf{D} \in \text{cwff}_\beta(\Sigma)$ with $\mathbf{C} \parallel \mathbf{D}$ be given. By induction, we know $(\mathbf{AC}) \parallel (\mathbf{BD})$ iff $(\mathbf{AC}) \downarrow_\beta \parallel (\mathbf{BD}) \downarrow_\beta$ iff $(\mathbf{A} \downarrow_\beta \mathbf{C}) \parallel (\mathbf{B} \downarrow_\beta \mathbf{D})$. We can simply generalize over \mathbf{C} and \mathbf{D} to obtain $\mathbf{A} \parallel \mathbf{B}$ iff $\mathbf{A} \downarrow_\beta \parallel \mathbf{B} \downarrow_\beta$. \dashv

Without certain closure conditions on \mathcal{H} , there may be constants which are not compatible with themselves (consider q in Example 3.2). The acceptability conditions on Hintikka sets in $\mathfrak{Hint}_{\beta\text{fb}}$ will guarantee that constants (indeed all closed terms) are compatible with themselves.

LEMMA 6.9. *Let \mathcal{H} be an acceptable Hintikka set in $\mathfrak{Hint}_{\beta\text{fb}}$.*

1. *For any $\mathbf{A}, \mathbf{B} \in \text{cwff}_\alpha(\Sigma)$, if $\mathbf{A} \parallel \mathbf{B}$, then $\mathbf{A} \check{\parallel} \mathbf{B}$.*
2. *For any $n \geq 0$ and parameter $h_{\alpha_1 \rightarrow \dots \rightarrow \alpha_n \rightarrow \alpha}$, if $\mathbf{C}^i, \mathbf{D}^i \in \text{cwff}_{\alpha_i}(\Sigma)$ and $\mathbf{C}^i \check{\parallel} \mathbf{D}^i$ for $1 \leq i \leq n$, then $(h\overline{\mathbf{C}^n}) \parallel (h\overline{\mathbf{D}^n})$.*

PROOF. We can prove this by mutual induction on the type α .

- ι : We know $\mathbf{A} \check{\parallel} \mathbf{B}$ if $\mathbf{A} \parallel \mathbf{B}$ by definition.
- o : Assume $\mathbf{A} \parallel \mathbf{B}$ holds and $\mathbf{A} \check{\parallel} \mathbf{B}$ fails. Then $\neg(\mathbf{A} \dot{=}^o \mathbf{B}) \in \mathcal{H}$ or $\neg(\mathbf{B} \dot{=}^o \mathbf{A}) \in \mathcal{H}$. By $\vec{\nabla}_b$, either $\{\mathbf{A}, \neg \mathbf{B}\} \subseteq \mathcal{H}$ or $\{\mathbf{B}, \neg \mathbf{A}\} \subseteq \mathcal{H}$, contradicting $\mathbf{A} \parallel \mathbf{B}$.
- $\beta \rightarrow \gamma$: Assume $\mathbf{A} \parallel \mathbf{B}$ holds and $\mathbf{A} \check{\parallel} \mathbf{B}$ fails. Then $\neg(\mathbf{A} \dot{=}^{\beta \rightarrow \gamma} \mathbf{B}) \in \mathcal{H}$ or $\neg(\mathbf{B} \dot{=}^{\beta \rightarrow \gamma} \mathbf{A}) \in \mathcal{H}$. By $\vec{\nabla}_f$, there is a parameter w_β such that $\neg((\mathbf{A}w) \dot{=}^\gamma (\mathbf{B}w)) \in \mathcal{H}$ or $\neg((\mathbf{B}w) \dot{=}^\gamma (\mathbf{A}w)) \in \mathcal{H}$. Hence $(\mathbf{A}w) \check{\parallel} (\mathbf{B}w)$ fails. On the other hand, applying the induction hypothesis for part 2 at type β to w (with no arguments), we know $w \parallel w$. Thus $(\mathbf{A}w) \parallel (\mathbf{B}w)$ holds since $\mathbf{A} \parallel \mathbf{B}$. This contradicts the induction hypothesis for part 1 at type γ .

We now turn to the second statement.

- ι : Suppose $(h\overline{\mathbf{C}^n})\|(h\overline{\mathbf{D}^n})$ fails. Then $(h\overline{\mathbf{C}^n}) \not\parallel (h\overline{\mathbf{D}^n})$ fails and so either $\neg((h\overline{\mathbf{C}^n}) \dot{=}^L (h\overline{\mathbf{D}^n})) \in \mathcal{H}$ or $\neg((h\overline{\mathbf{D}^n}) \dot{=}^L (h\overline{\mathbf{C}^n})) \in \mathcal{H}$. Since \mathcal{H} is acceptable, $\vec{\nabla}_d$ holds and so there is some i such that $1 \leq i \leq n$ and either $\neg(\mathbf{C}^i \dot{=} \mathbf{D}^i) \in \mathcal{H}$ or $\neg(\mathbf{D}^i \dot{=} \mathbf{C}^i) \in \mathcal{H}$. Either case contradicts $\mathbf{C}^i \not\parallel \mathbf{D}^i$.
- \circ : Suppose $\{(h\overline{\mathbf{C}^n}), \neg(h\overline{\mathbf{D}^n})\} \subseteq \mathcal{H}$ or $\{\neg(h\overline{\mathbf{C}^n}), (h\overline{\mathbf{D}^n})\} \subseteq \mathcal{H}$. Since \mathcal{H} is acceptable, $\vec{\nabla}_m$ and $\vec{\nabla}_d$ hold. By $\vec{\nabla}_m$, either $\neg((h\overline{\mathbf{C}^n}) \dot{=} (h\overline{\mathbf{D}^n})) \in \mathcal{H}$ or $\neg((h\overline{\mathbf{D}^n}) \dot{=} (h\overline{\mathbf{C}^n})) \in \mathcal{H}$. By $\vec{\nabla}_d$, there is some i such that $1 \leq i \leq n$ and either $\neg(\mathbf{C}^i \dot{=} \mathbf{D}^i) \in \mathcal{H}$ or $\neg(\mathbf{D}^i \dot{=} \mathbf{C}^i) \in \mathcal{H}$, contradicting $\mathbf{C}^i \not\parallel \mathbf{D}^i$. Hence $(h\overline{\mathbf{C}^n})\|(h\overline{\mathbf{D}^n})$.
- $\beta \rightarrow \gamma$: To show $(h\overline{\mathbf{C}^n})\|(h\overline{\mathbf{D}^n})$, let $\mathbf{A}, \mathbf{B} \in \text{cwff}_\beta(\Sigma)$ with $\mathbf{A} \parallel \mathbf{B}$ be given. By the inductive hypothesis for part 1, we have $\mathbf{A} \not\parallel \mathbf{B}$. So, we can apply the induction hypothesis to the two terms $h\mathbf{C}^1 \dots \mathbf{C}^n \mathbf{A}$ and $h\mathbf{D}^1 \dots \mathbf{D}^n \mathbf{B}$ at type γ to obtain $(h\mathbf{C}^1 \dots \mathbf{C}^n \mathbf{A})\|(h\mathbf{D}^1 \dots \mathbf{D}^n \mathbf{B})$.

⊥

We can now show reflexivity of \parallel by induction on terms.

LEMMA 6.10. *Let \mathcal{H} be an acceptable Hintikka set in $\mathfrak{Hint}_{\beta\text{fb}}$. Then, for every closed term \mathbf{A}_α , $\mathbf{A} \parallel \mathbf{A}$.*

PROOF. We prove the stronger statement that given any term $\mathbf{A}_\alpha \in \text{wff}_\alpha$ and substitutions θ and ψ defined on the free variables of \mathbf{A} such that $\theta(X) \parallel \psi(X)$ for every $X_\beta \in \text{free}(\mathbf{A})$ then $\theta(\mathbf{A}) \parallel \psi(\mathbf{A})$. We prove this by induction on the term \mathbf{A} . Since \mathcal{H} is acceptable in $\mathfrak{Hint}_{\beta\text{fb}}$, it satisfies $\vec{\nabla}_m$, $\vec{\nabla}_d$, $\vec{\nabla}_f$ and $\vec{\nabla}_b$.

If \mathbf{A} is a variable, the assertion follows directly from the assumption. We can apply Lemma 6.9(2) to determine \mathbf{A} is \mathcal{H} -compatible with itself if \mathbf{A} is a parameter. We next consider logical constants.

The most interesting logical constants are the Π^β for each type β . If Π^β is \mathcal{H} -incompatible with itself, there must be two \mathcal{H} -compatible closed terms $\mathbf{B}, \mathbf{C} \in \text{cwff}_{\beta \rightarrow o}(\Sigma)$, such that $\Pi\mathbf{B}$ and $\Pi\mathbf{C}$ are \mathcal{H} -incompatible. Without loss of generality, we have $\Pi\mathbf{B}, \neg(\Pi\mathbf{C}) \in \mathcal{H}$. By $\vec{\nabla}_\exists$, there is a parameter w_β with $\neg\mathbf{C}w \in \mathcal{H}$. By $\vec{\nabla}_\forall$, $\mathbf{B}w \in \mathcal{H}$. Hence $\mathbf{B}w$ and $\mathbf{C}w$ are \mathcal{H} -incompatible by definition. Since we know $w \parallel w$ for parameters already, this contradicts $\mathbf{B} \parallel \mathbf{C}$.

To check \neg , suppose \neg is not \mathcal{H} -compatible with itself. There must be \mathcal{H} -compatible $\mathbf{B}, \mathbf{C} \in \text{cwff}_o(\Sigma)$ where $\neg\mathbf{B}$ and $\neg\mathbf{C}$ are \mathcal{H} -incompatible. Without loss of generality, we have $\{\neg\mathbf{B}, \neg\neg\mathbf{C}\} \subseteq \mathcal{H}$. Using $\vec{\nabla}_\neg$, we have $\{\neg\mathbf{B}, \mathbf{C}\} \subseteq \mathcal{H}$, contradicting $\mathbf{B} \parallel \mathbf{C}$.

To check \vee , suppose \vee is not \mathcal{H} -compatible with itself. There exist $\mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E} \in \text{cwff}_o(\Sigma)$ with $\mathbf{B} \parallel \mathbf{C}$, $\mathbf{D} \parallel \mathbf{E}$ such that $\mathbf{B} \vee \mathbf{D}$ and $\mathbf{C} \vee \mathbf{E}$ are \mathcal{H} -incompatible. Without loss of generality, $\{(\mathbf{B} \vee \mathbf{D}), \neg(\mathbf{C} \vee \mathbf{E})\} \subseteq \mathcal{H}$. By $\vec{\nabla}_\wedge$ and $\vec{\nabla}_\vee$, either $\{\mathbf{B}, \neg\mathbf{C}, \neg\mathbf{E}\} \subseteq \mathcal{H}$ (contradicting $\mathbf{B} \parallel \mathbf{C}$) or $\{\mathbf{D}, \neg\mathbf{C}, \neg\mathbf{E}\} \subseteq \mathcal{H}$ (contradicting $\mathbf{D} \parallel \mathbf{E}$).

For the application case let \mathbf{A} be of the form $\mathbf{G}\mathbf{B}$. By induction, we have $\theta(\mathbf{G}) \parallel \psi(\mathbf{G})$ and $\theta(\mathbf{B}) \parallel \psi(\mathbf{B})$. By definition of \mathcal{H} -compatibility at function types, we have $\theta(\mathbf{G}\mathbf{B}) \equiv \theta(\mathbf{G})\theta(\mathbf{B}) \parallel \psi(\mathbf{G})\psi(\mathbf{B}) \equiv \psi(\mathbf{G}\mathbf{B})$.

Suppose \mathbf{A} is of the form $\lambda X_\beta. \mathbf{D}_\gamma$. Let $\mathbf{B}, \mathbf{C} \in \text{cwff}_\beta(\Sigma)$ be \mathcal{H} -compatible terms. We must show $\theta(\mathbf{A})\mathbf{B}$ and $\psi(\mathbf{A})\mathbf{C}$ are \mathcal{H} -compatible. Let $\theta' := \theta, [\mathbf{B}/X]$

and $\psi' := \psi, [\mathbf{C}/X]$ be the obvious substitutions extending θ and ψ . By induction, we have $\theta'(\mathbf{D}) \parallel \psi'(\mathbf{D})$. Since $\theta'(\mathbf{D}) \downarrow_\beta \equiv_\beta (\theta(\mathbf{A})\mathbf{B}) \downarrow_\beta$ and $\psi'(\mathbf{D}) \downarrow_\beta \equiv (\psi(\mathbf{A})\mathbf{C}) \downarrow_\beta$, we know by Lemma 6.8 that $\theta(\mathbf{A})\mathbf{B} \parallel \psi(\mathbf{A})\mathbf{C}$. Generalizing over \mathbf{B} and \mathbf{C} , we know $\theta(\mathbf{A}) \parallel \psi(\mathbf{A})$. \dashv

§7. Constructing Models. We now introduce techniques for constructing models. These techniques will be used to prove a model existence theorem (cf. Theorem 8.1) for acceptable abstract consistency classes.

7.1. Per Evaluations. When constructing models below, we will make use of quotients obtained from partial equivalence relations (pers), i.e., symmetric, transitive relations. This is more general than taking quotients by a total equivalence relations, but for the quotient to be well-defined requires more assumptions. We develop a theory of per quotients. A similar theory is developed in [6].

Since we need the domains of the quotient to be nonempty, we are typically interested in nonempty pers. A per is nonempty if there is some \mathbf{a} and \mathbf{b} such that $\mathbf{a} \sim \mathbf{b}$.

DEFINITION 7.1 (Σ -Per Evaluation). A Σ -per evaluation is a triple $\mathcal{P} := (\mathcal{J}, \sim, v)$ where $\mathcal{J} \equiv (\mathcal{D}, @, \mathcal{E})$ is a Σ -evaluation, $v: \mathcal{D}_o \rightarrow \{\mathbf{T}, \mathbf{F}\}$ and \sim_α is a nonempty per on \mathcal{D}_α for each type α . (We usually write simply \sim instead of \sim_α .)

Given any typed collection of domains \mathcal{D} and typed binary relation \sim , we use the notation $\overline{\mathcal{D}}_\alpha^\sim$ for $\{\mathbf{a} \in \mathcal{D}_\alpha \mid \mathbf{a} \sim \mathbf{a}\}$. Furthermore, for any two typed functions φ and ψ from \mathcal{V} to \mathcal{D} , we use the notation $\varphi \sim \psi$ to indicate that $\varphi(X) \sim \psi(X)$ for every variable X .

LEMMA 7.2. If $\mathcal{P} := (\mathcal{J}, \sim, v)$ is a Σ -per evaluation where $\mathcal{J} \equiv (\mathcal{D}, @, \mathcal{E})$ is a Σ -evaluation, then $\overline{\mathcal{D}}_\alpha^\sim$ is nonempty for every type α .

PROOF. Let α be a type. By Definition 7.1, \sim_α is nonempty on \mathcal{D}_α and so there exist $\mathbf{a}, \mathbf{b} \in \mathcal{D}_\alpha$ such that $\mathbf{a} \sim \mathbf{b}$. By symmetry and transitivity, $\mathbf{a} \sim \mathbf{b} \sim \mathbf{a}$ and so $\mathbf{a} \sim \mathbf{a}$. Thus $\mathbf{a} \in \overline{\mathcal{D}}_\alpha^\sim$. \dashv

We now consider a number of conditions a per evaluation may satisfy.

DEFINITION 7.3 (Σ -Per Evaluation Properties). Let $\mathcal{P} := (\mathcal{J}, \sim, v)$ be a Σ -per evaluation with $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$. We define the following properties for \mathcal{P} :

- ∂^Σ : For every constant $c_\alpha \in \Sigma$, $\mathcal{E}(c) \in \overline{\mathcal{D}}_\alpha^\sim$ (i.e. $\mathcal{E}(c) \sim \mathcal{E}(c)$ in \mathcal{D}_α).
- $\partial^@$: For every $\mathbf{g}, \mathbf{h} \in \mathcal{D}_{\alpha \rightarrow \beta}$ and $\mathbf{a}, \mathbf{b} \in \mathcal{D}_\alpha$, if $\mathbf{g} \sim \mathbf{h}$ and $\mathbf{a} \sim \mathbf{b}$, then $\mathbf{g}@\mathbf{a} \sim \mathbf{h}@\mathbf{b}$.
- ∂^{sub} : For every type α , every $\mathbf{A} \in \text{wff}_\alpha(\Sigma)$, and all assignments φ and ψ , if $\varphi \sim \psi$, then $\mathcal{E}_\varphi(\mathbf{A}) \sim \mathcal{E}_\psi(\mathbf{A})$.
- ∂^v : For all $\mathbf{a}, \mathbf{b} \in \mathcal{D}_o$, $\mathbf{a} \sim \mathbf{b}$ implies $v(\mathbf{a}) \equiv v(\mathbf{b})$.
- ∂^{TF} : There exist $\mathbf{t}, \mathbf{f} \in \mathcal{D}_o$ such that $\mathbf{t} \sim \mathbf{t}$, $\mathbf{f} \sim \mathbf{f}$, $v(\mathbf{t}) \equiv \mathbf{T}$ and $v(\mathbf{f}) \equiv \mathbf{F}$.
- ∂^- : For all $\mathbf{a} \in \overline{\mathcal{D}}_o^\sim$, $v(\mathcal{E}(\neg)@\mathbf{a}) \equiv \mathbf{T}$ iff $v(\mathbf{a}) \equiv \mathbf{F}$.
- ∂^\vee : For all $\mathbf{a}, \mathbf{b} \in \overline{\mathcal{D}}_o^\sim$, $v(\mathcal{E}(\vee)@\mathbf{a}@\mathbf{b}) \equiv \mathbf{T}$ iff $v(\mathbf{a}) \equiv \mathbf{T}$ or $v(\mathbf{b}) \equiv \mathbf{T}$.
- ∂^Π : For all $\mathbf{f} \in \overline{\mathcal{D}}_{\alpha \rightarrow o}^\sim$, $v(\mathcal{E}(\Pi^\alpha)@\mathbf{f}) \equiv \mathbf{T}$ iff $v(\mathbf{f}@\mathbf{a}) \equiv \mathbf{T}$ for each $\mathbf{a} \in \overline{\mathcal{D}}_\alpha^\sim$.
- ∂^q : At each type $\alpha \in \mathcal{T}$, there is an element $\mathbf{q}^\alpha \in \overline{\mathcal{D}}_{\alpha \rightarrow \alpha \rightarrow o}^\sim$, such that for all $\mathbf{a}, \mathbf{b} \in \overline{\mathcal{D}}_\alpha^\sim$ we have $v(\mathbf{q}^\alpha @ \mathbf{a} @ \mathbf{b}) \equiv \mathbf{T}$ iff $\mathbf{a} \sim \mathbf{b}$.
- ∂^η : For every type α , $\mathbf{A} \in \text{wff}_\alpha(\Sigma)$, and assignment φ , we have $\mathcal{E}_\varphi(\mathbf{A}) \sim \mathcal{E}_\varphi(\mathbf{A} \downarrow_{\beta\eta})$.

- ∂^ξ : For all $\alpha, \beta \in \mathcal{T}$, $\mathbf{M}, \mathbf{N} \in \text{wff}_\beta(\Sigma)$, assignments φ with $\varphi \sim \varphi$, and variables X_α , $\mathcal{E}_\varphi(\lambda X_\alpha. \mathbf{M}_\beta) \sim \mathcal{E}_\varphi(\lambda X_\alpha. \mathbf{N}_\beta)$ whenever $\mathcal{E}_{\varphi, [a/X]}(\mathbf{M}) \sim \mathcal{E}_{\varphi, [a/X]}(\mathbf{N})$ for every $a \in \overline{\mathcal{D}_\alpha^\sim}$.
- ∂^f : For every $\mathbf{g}, \mathbf{h} \in \mathcal{D}_{\alpha \rightarrow \beta}$, $\mathbf{g} \sim \mathbf{h}$ whenever for every $\mathbf{a}, \mathbf{b} \in \mathcal{D}_\alpha$ $\mathbf{a} \sim \mathbf{b}$ implies $\mathbf{g} @ \mathbf{a} \sim \mathbf{h} @ \mathbf{b}$.
- ∂^b : There are only two \sim -equivalence classes on \mathcal{D}_o .

Note that the properties ∂^\oplus and ∂^f determine \sim on function types. We can also show ∂^Σ , ∂^\oplus and ∂^f imply ∂^{sub} .

LEMMA 7.4. *Let $\mathcal{P} := (\mathcal{J}, \sim, v)$ be a Σ -per evaluation with $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$. Suppose \mathcal{P} satisfies ∂^Σ , ∂^\oplus and ∂^f . Then \mathcal{P} satisfies ∂^{sub} .*

PROOF. One can easily show $\mathcal{E}_\varphi(\mathbf{A}) \sim \mathcal{E}_\psi(\mathbf{A})$ for all assignments φ and ψ and every $\mathbf{A} \in \text{wff}_\alpha(\Sigma)$ by induction on \mathbf{A} . If \mathbf{A} is a variable x_α , this follows from the assumption $\varphi(x) \sim \psi(x)$. The condition ∂^Σ is used if \mathbf{A} is a constant, ∂^\oplus is used if \mathbf{A} is an application, and ∂^f is used if \mathbf{A} is a λ -abstraction. \dashv

THEOREM 7.5. *Let $\mathcal{P} \equiv (\mathcal{J}, \sim, v)$ be a Σ -per evaluation with $\mathcal{J} \equiv (\mathcal{D}, @, \mathcal{E})$. Suppose \mathcal{P} satisfies ∂^Σ , ∂^\oplus , ∂^{sub} , ∂^v , ∂^{TF} , ∂^\neg , ∂^\vee and ∂^Π . Then, there is a Σ -model $\mathcal{M} \equiv (\mathcal{D}^\sim, @^\sim, \mathcal{E}^\sim, v^\sim)$ such that $v^\sim(\mathcal{E}^\sim(\mathbf{A})) \equiv v(\mathcal{E}(\mathbf{A}))$ for all $\mathbf{A} \in \text{cwff}_o(\Sigma)$. Furthermore, we have:*

1. If \mathcal{P} satisfies ∂^q , then \mathcal{M} satisfies property \mathfrak{q} .
2. If \mathcal{P} satisfies ∂^η , then \mathcal{M} satisfies property η .
3. If \mathcal{P} satisfies ∂^ξ , then \mathcal{M} satisfies property ξ .
4. If \mathcal{P} satisfies ∂^f , then \mathcal{M} satisfies property \mathfrak{f} .
5. If \mathcal{P} satisfies ∂^b , then \mathcal{M} satisfies property \mathfrak{b} .

PROOF. We define the domains \mathcal{D}^\sim of \mathcal{M} as $\mathcal{D}_\alpha^\sim := \{[a]_\sim \mid a \in \overline{\mathcal{D}_\alpha^\sim}\}$. It is helpful to choose representatives $\mathbf{A}^* \in \mathbf{A}$ of the equivalence classes $\mathbf{A} \in \mathcal{D}_\alpha^\sim$. Note that $[a]_\sim^* \sim a$ for every $a \in \overline{\mathcal{D}_\alpha^\sim}$. So, for each $\mathbf{A} \in \mathcal{D}_\alpha^\sim$, we have $\mathbf{A}^* \in \overline{\mathcal{D}_\alpha^\sim}$ and $\mathbf{A} \equiv [\mathbf{A}^*]_\sim$. It follows that if $\mathbf{A}, \mathbf{B} \in \mathcal{D}_\alpha^\sim$ and $\mathbf{A}^* \sim \mathbf{B}^*$, then $\mathbf{A} \equiv [\mathbf{A}^*]_\sim \equiv [\mathbf{B}^*]_\sim \equiv \mathbf{B}$. Also, for any $a \in \overline{\mathcal{D}_\alpha^\sim}$, $a \in [a]_\sim$ and $[a]_\sim^* \in [a]_\sim$, so we have $a \sim [a]_\sim^*$.

We define $@^\sim$ by $\mathbf{G} @^\sim \mathbf{A} := [\mathbf{G}^* @ \mathbf{A}^*]_\sim$ for $\mathbf{G} \in \mathcal{D}_{\alpha \rightarrow \beta}^\sim$ and $\mathbf{A} \in \mathcal{D}_\alpha^\sim$. Note that for any $\mathbf{g} \in \overline{\mathcal{D}_{\alpha \rightarrow \beta}^\sim}$ and $\mathbf{a} \in \overline{\mathcal{D}_\alpha^\sim}$, we have $[\mathbf{g}]_\sim^* @ [\mathbf{a}]_\sim^* \sim \mathbf{g} @ \mathbf{a}$ by ∂^\oplus , $\mathbf{g} \sim [\mathbf{g}]_\sim^*$ and $\mathbf{a} \sim [\mathbf{a}]_\sim^*$. As a result, we have $[\mathbf{g}]_\sim @^\sim [\mathbf{a}]_\sim \equiv [\mathbf{g} @ \mathbf{a}]_\sim$.

By Lemma 7.2, we know $\overline{\mathcal{D}_\alpha^\sim}$ is nonempty and so \mathcal{D}_α^\sim is nonempty. Hence $(\mathcal{D}^\sim, @^\sim)$ is an applicative structure.

For each assignment φ taking variables X_α to \mathcal{D}_α^\sim , we define φ^* to be an assignment taking variables X_α to $\overline{\mathcal{D}_\alpha^\sim}$ by $\varphi^*(X) := \varphi(X)^*$. We next define $\mathcal{E}_\varphi^\sim(\mathbf{A}) := [\mathcal{E}_{\varphi^*}(\mathbf{A})]_\sim$. To check this is well-defined, we need to know $\mathcal{E}_{\varphi^*}(\mathbf{A}_\alpha) \in \overline{\mathcal{D}_\alpha^\sim}$ for each $\mathbf{A} \in \text{wff}_\alpha(\Sigma)$. This follows directly from ∂^{sub} since $\varphi^*(X) \sim \varphi^*(X)$ for every variable X .

We check \mathcal{E}^\sim is an evaluation function.

1. For each variable x_α , $\mathcal{E}_\varphi^\sim(x) \equiv [\varphi^*(x)]_\sim \equiv \varphi(x)$.
2. \mathcal{E}^\sim preserves application since $\mathcal{E}_\varphi^\sim(\mathbf{GC}) \equiv [\mathcal{E}_{\varphi^*}(\mathbf{GC})]_\sim \equiv [\mathcal{E}_{\varphi^*}(\mathbf{G}) @ \mathcal{E}_{\varphi^*}(\mathbf{C})]_\sim \equiv [\mathcal{E}_{\varphi^*}(\mathbf{G})]_\sim @^\sim [\mathcal{E}_{\varphi^*}(\mathbf{C})]_\sim \equiv \mathcal{E}_\varphi^\sim(\mathbf{G}) @^\sim \mathcal{E}_\varphi^\sim(\mathbf{C})$.

3. If φ and ψ coincide on the free variables of \mathbf{A} , then so do φ^* and ψ^* . So, $\mathcal{E}_{\varphi^*}(\mathbf{A}) \equiv \mathcal{E}_{\psi^*}(\mathbf{A})$ since \mathcal{J} is a Σ -evaluation. This directly implies $\mathcal{E}_{\varphi}^{\sim}(\mathbf{A}) \equiv \mathcal{E}_{\psi}^{\sim}(\mathbf{A})$.
4. Let φ be an assignment and $\mathbf{A} \in \text{wff}_{\alpha}(\Sigma)$. Since \mathcal{J} is a Σ -evaluation, we have $\mathcal{E}_{\varphi^*}(\mathbf{A}) \equiv \mathcal{E}_{\varphi^*}(\mathbf{A} \downarrow_{\beta})$. Since $\mathcal{E}_{\varphi^*}(\mathbf{A}) \in \overline{\mathcal{D}_{\alpha}^{\sim}}$, we can pass to equivalence classes and obtain $\mathcal{E}_{\varphi}^{\sim}(\mathbf{A}) \equiv \mathcal{E}_{\varphi}^{\sim}(\mathbf{A} \downarrow_{\beta})$ as desired.

So, \mathcal{E}^{\sim} is an evaluation function

We now define $v^{\sim}: \mathcal{D}_{\alpha}^{\sim} \rightarrow \{\mathbf{T}, \mathbf{F}\}$ by $v^{\sim}(\mathbf{A}) := v(\mathbf{A}^*)$ for $\mathbf{A} \in \mathcal{D}_{\alpha}^{\sim}$. For each $\mathbf{a} \in \overline{\mathcal{D}_{\alpha}^{\sim}}$, from ∂^v we conclude

$$v^{\sim}(\llbracket \mathbf{a} \rrbracket_{\sim}) \equiv v(\llbracket \mathbf{a} \rrbracket_{\sim}^*) \equiv v(\mathbf{a})$$

We can show v^{\sim} is surjective using ∂^{TF} since $v^{\sim}(\llbracket \mathbf{t} \rrbracket_{\sim}) \equiv \mathbf{T}$ and $v^{\sim}(\llbracket \mathbf{f} \rrbracket_{\sim}) \equiv \mathbf{F}$ (where $\mathbf{t}, \mathbf{f} \in \overline{\mathcal{D}_{\alpha}^{\sim}}$ are given by ∂^{TF}).

An important property of v^{\sim} is that if $\mathbf{A} \in \mathcal{D}_{\alpha}^{\sim}$ and $\mathbf{a} \in \mathbf{A}$, then $v^{\sim}(\mathbf{A}) \equiv v(\mathbf{a})$. This follows from ∂^v , since $\mathbf{A}^* \sim \mathbf{a}$. This will be used several times to verify the required properties of v^{\sim} below. In particular, $\mathcal{E}(\mathbf{A}) \in \overline{\mathcal{D}_{\alpha}^{\sim}}$ implies $\mathcal{E}(\mathbf{A}) \in \mathcal{E}^{\sim}(\mathbf{A})$ and so $v^{\sim}(\mathcal{E}^{\sim}(\mathbf{A})) \equiv v(\mathcal{E}(\mathbf{A}))$ as required. We finally check that \mathcal{M} is a model by checking the conditions on v^{\sim} .

1. $v^{\sim}(\mathcal{E}^{\sim}(\neg) @ \sim \mathbf{A}) \equiv \mathbf{T}$, iff $v(\mathcal{E}(\neg) @ \mathbf{A}^*) \equiv \mathbf{T}$, iff (by ∂^{\neg}) $v(\mathbf{A}^*) \equiv \mathbf{F}$, iff $v^{\sim}(\mathbf{A}) \equiv \mathbf{F}$
2. $v^{\sim}(\mathcal{E}^{\sim}(\vee) @ \sim \mathbf{A} @ \sim \mathbf{B}) \equiv \mathbf{T}$, iff $v(\mathcal{E}(\vee) @ \mathbf{A}^* @ \mathbf{B}^*) \equiv \mathbf{T}$, iff (by ∂^{\vee}) $v(\mathbf{A}^*) \equiv \mathbf{T}$ or $v(\mathbf{B}^*) \equiv \mathbf{T}$, iff $v^{\sim}(\mathbf{A}) \equiv \mathbf{T}$ or $v^{\sim}(\mathbf{B}) \equiv \mathbf{T}$,
3. Let $\mathbf{P} \in \mathcal{D}_{\alpha \rightarrow \alpha}^{\sim}$ be given. $v^{\sim}(\mathcal{E}^{\sim}(\Pi^{\alpha}) @ \sim \mathbf{P}) \equiv \mathbf{T}$, iff $v(\mathcal{E}(\Pi^{\alpha}) @ \mathbf{P}^*) \equiv \mathbf{T}$, iff (by ∂^{Π}) $v(\mathbf{P}^* @ \mathbf{a}) \equiv \mathbf{T}$ for each $\mathbf{a} \in \overline{\mathcal{D}_{\alpha}^{\sim}}$, iff $v^{\sim}(\mathbf{P} @ \sim \mathbf{A}) \equiv \mathbf{T}$ for each $\mathbf{A} \in \mathcal{D}_{\alpha}^{\sim}$. This last equivalence is true since for every $\mathbf{a} \in \overline{\mathcal{D}_{\alpha}^{\sim}}$, we can use $\llbracket \mathbf{a} \rrbracket_{\sim} \in \mathcal{D}_{\alpha}^{\sim}$ with $\mathbf{a} \sim \llbracket \mathbf{a} \rrbracket_{\sim}^*$ to determine $v(\mathbf{P}^* @ \mathbf{a}) \equiv \mathbf{T}$, and for every $\mathbf{A} \in \mathcal{D}_{\alpha}^{\sim}$, we can use $\mathbf{A}^* \in \overline{\mathcal{D}_{\alpha}^{\sim}}$ to determine $v^{\sim}(\mathbf{P} @ \sim \mathbf{A}) \equiv \mathbf{T}$.

So, we have the desired Σ -model \mathcal{M} . Now, we check the other properties.

1. Suppose \mathcal{P} satisfies $\partial^{\mathbf{q}}$. We must show \mathcal{M} satisfies property \mathbf{q} . Let α be a type and $\mathbf{q}^{\alpha} \in \overline{\mathcal{D}_{\alpha \rightarrow \alpha \rightarrow \alpha}^{\sim}}$ be the element guaranteed to exist by $\partial^{\mathbf{q}}$. We will show $\llbracket \mathbf{q}^{\alpha} \rrbracket_{\sim} \in \mathcal{D}_{\alpha \rightarrow \alpha \rightarrow \alpha}^{\sim}$ is the required witness for \mathbf{q} . Let $\mathbf{A}, \mathbf{B} \in \mathcal{D}_{\alpha}^{\sim}$ be given. We have $v^{\sim}(\llbracket \mathbf{q}^{\alpha} \rrbracket_{\sim} @ \sim \mathbf{A} @ \sim \mathbf{B}) \equiv \mathbf{T}$, iff $v(\mathbf{q}^{\alpha} @ \mathbf{A}^* @ \mathbf{B}^*) \equiv \mathbf{T}$, iff (by $\partial^{\mathbf{q}}$) $\mathbf{A}^* \sim \mathbf{B}^*$, iff $\mathbf{A} \equiv \mathbf{B}$. So, if \mathcal{P} satisfies $\partial^{\mathbf{q}}$, then \mathcal{M} satisfies \mathbf{q} .
2. Suppose \mathcal{P} satisfies ∂^{η} . To check that the model \mathcal{M} satisfies property η , let $\mathbf{A} \in \text{wff}_{\alpha}(\Sigma)$ and an assignment φ for \mathcal{M} be given. By ∂^{η} , we have $\mathcal{E}_{\varphi^*}(\mathbf{A}) \sim \mathcal{E}_{\varphi^*}(\mathbf{A} \downarrow_{\beta\eta})$ and so $\mathcal{E}_{\varphi}^{\sim}(\mathbf{A}) \equiv \mathcal{E}_{\varphi}^{\sim}(\mathbf{A} \downarrow_{\beta\eta})$.
3. Suppose \mathcal{P} satisfies ∂^{ξ} . To check property ξ , suppose $\mathbf{M}, \mathbf{N} \in \text{wff}_{\beta}$, φ is an assignment into $(\mathcal{D}^{\sim}, @^{\sim})$, X_{α} is a variable and $\mathcal{E}_{\varphi, [\mathbf{A}/X]}^{\sim}(\mathbf{M}) \equiv \mathcal{E}_{\varphi, [\mathbf{A}/X]}^{\sim}(\mathbf{N})$ for every $\mathbf{A} \in \mathcal{D}_{\alpha}^{\sim}$. This implies $\mathcal{E}_{\varphi^*, [\mathbf{a}/X]}(\mathbf{M}) \sim \mathcal{E}_{\varphi^*, [\mathbf{a}/X]}(\mathbf{N})$ for every $\mathbf{a} \in \overline{\mathcal{D}_{\alpha}^{\sim}}$. Since $\varphi^* \sim \varphi^*$ we conclude $\mathcal{E}_{\varphi^*}(\lambda X. \mathbf{M}) \sim \mathcal{E}_{\varphi^*}(\lambda X. \mathbf{N})$ from ∂^{ξ} . That is, $\mathcal{E}_{\varphi}^{\sim}(\lambda X. \mathbf{M}) \equiv \mathcal{E}_{\varphi}^{\sim}(\lambda X. \mathbf{N})$.
4. Suppose \mathcal{P} satisfies $\partial^{\mathbf{f}}$. It easily follows that \mathcal{M} satisfies property \mathbf{f} . Let $\mathbf{G}, \mathbf{H} \in \mathcal{D}_{\alpha \rightarrow \beta}^{\sim}$ be such that for every $\mathbf{A} \in \mathcal{D}_{\alpha}^{\sim}$ we have $\mathbf{G} @ \sim \mathbf{A} \equiv \mathbf{H} @ \sim \mathbf{A}$. This implies $\mathbf{G}^* @ \mathbf{a} \sim \mathbf{H}^* @ \mathbf{a}$ for every $\mathbf{a} \in \overline{\mathcal{D}_{\alpha}^{\sim}}$. If we take any $\mathbf{a}, \mathbf{b} \in \mathcal{D}_{\alpha}$ with

$a \sim b$, then we have $G^* @ a \sim H^* @ a \sim H^* @ b$ by $\partial^@$ since $H^* \in \overline{\mathcal{D}_{\alpha \rightarrow \beta}^\sim}$. By ∂^f , we have $G^* \sim H^*$. Hence $G \equiv H$, as desired.

5. Finally, if \mathcal{P} satisfies ∂^b , then \mathcal{D}_o^\sim has only two elements and so \mathcal{M} satisfies property **b**. \dashv

7.2. Possible Values Structures. We now develop in the abstract the structures which will be used to construct many models. The elements of our semantical domains \mathcal{D}_α will consist of tuples $\langle \mathbf{T}, t \rangle$ associating terms (\mathbf{T}) with possible values (t). The semantic techniques used by Takahashi (cf. [14]) and Prawitz (cf. [11]) to prove cut-elimination for “relational” formulations of simple type theory (without extensionality) were extended by Andrews (cf. [1]) for Church-style formulations (without extensionality). The idea is to define \mathcal{D}_o to determine the truth value of some sentence and delay determining the truth value of others. We have adopted the phrase “possible values” from Prawitz (cf. [11]). A similar abstract theory of possible values structures is also developed in [6].

DEFINITION 7.6 (Possible Values Structure). A *possible values structure* [with η] $\mathcal{F} \equiv (\mathcal{D}, @)$ is an applicative structure $(\mathcal{D}, @)$ where

1. For each type α , $a \in \mathcal{D}_\alpha$ implies $a \equiv \langle \mathbf{A}, a \rangle$ for some β -normal [$\beta\eta$ -normal] term $\mathbf{A} \in \text{cwff}_\alpha(\Sigma)$ and some value a .
2. At each base type α , for every $\mathbf{A} \in \text{cwff}_\alpha(\Sigma)$ in β -normal form [$\beta\eta$ -normal form], there exists an a with $\langle \mathbf{A}, a \rangle \in \mathcal{D}_\alpha$.
3. For each function type $\alpha \rightarrow \beta$, $\langle \mathbf{G}, g \rangle \in \mathcal{D}_{\alpha \rightarrow \beta}$ iff $\mathbf{G} \in \text{cwff}_{\alpha \rightarrow \beta}(\Sigma)$ is β -normal [$\beta\eta$ -normal], $g: \mathcal{D}_\alpha \rightarrow \mathcal{D}_\beta$ and for every $\langle \mathbf{A}, a \rangle \in \mathcal{D}_\alpha$ the first component of $g(\langle \mathbf{A}, a \rangle)$ is the β -normal form [$\beta\eta$ -normal form] of \mathbf{GA} .
4. For each $\langle \mathbf{G}, g \rangle \in \mathcal{D}_{\alpha \rightarrow \beta}$ and $\langle \mathbf{A}, a \rangle \in \mathcal{D}_\alpha$, $\langle \mathbf{G}, g \rangle @ \langle \mathbf{A}, a \rangle \equiv g(\langle \mathbf{A}, a \rangle)$.

We use the notation \mathbf{A}_\star to mean the β -normal form in the β case and $\beta\eta$ -normal form in the $\beta\eta$ case. We also use the term “normal” ambiguously to consider both cases.

DEFINITION 7.7 (Possible Value). Let $\mathcal{F} \equiv (\mathcal{D}, @)$ be a possible values structure [with η]. We will call a a *possible value* for $\mathbf{A} \in \text{cwff}_\alpha(\Sigma)$ if $\langle \mathbf{A}_\star, a \rangle \in \mathcal{D}_\alpha$.

LEMMA 7.8 (Possible Values Existence). Let \mathcal{F} be a possible values structure [with η]. For each closed term $\mathbf{A} \in \text{cwff}_\alpha(\Sigma)$, there is a possible value p for \mathbf{A} in \mathcal{F} .

PROOF. The proof is by induction on the type α (following [1]). If α is a base type, we know there is a possible value for \mathbf{A} by condition (2) of Definition 7.6. Suppose α is $\beta \rightarrow \gamma$. By the induction hypothesis, there are possible values $p^{\mathbf{AB}}$ for \mathbf{AB} for each $\langle \mathbf{B}, b \rangle \in \mathcal{D}_\beta$. Using the axiom of choice (at the meta-level), there is a function $p: \mathcal{D}_\beta \rightarrow \mathcal{D}_\gamma$ such that $p(\langle \mathbf{B}, b \rangle) \equiv \langle (\mathbf{AB})_\star, p^{\mathbf{AB}} \rangle$. This p is a possible value for \mathbf{A} . \dashv

Since we are interested in interpreting a term $\mathbf{A} \in \text{cwff}_\alpha(\Sigma)$ as a pair of the form $\langle \mathbf{A}_\star, a \rangle$ we let $\mathcal{D}_\alpha^{\mathbf{A}}$ denote

$$\{ \langle \mathbf{A}_\star, a \rangle \mid a \text{ is a possible value for } \mathbf{A} \}.$$

REMARK 7.9. Note that $\mathcal{D}_\alpha^{\mathbf{A}} \subseteq \mathcal{D}_\alpha$ for each type α and that the syntactic condition on $\mathcal{D}_{\alpha \rightarrow \beta}$ guarantees that if $\langle \mathbf{G}, g \rangle \in \mathcal{D}_{\alpha \rightarrow \beta}$, then $g: \mathcal{D}_\alpha \rightarrow \mathcal{D}_\beta$ restricts to a mapping $g: \mathcal{D}_\alpha^{\mathbf{A}} \rightarrow \mathcal{D}_\beta^{\mathbf{GA}}$ for any $\mathbf{A} \in \text{cuff}_\alpha(\Sigma)$.

DEFINITION 7.10 (Possible Values Evaluation). A *possible values evaluation* [with η] is a Σ -evaluation $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$ where $(\mathcal{D}, @)$ is a possible values structure [with η] and $\mathcal{E}(c) \in \mathcal{D}_\alpha^c$ for each constant $c \in \Sigma_\alpha$.

Let φ be a variable assignment. Note that we can extract a substitution θ^φ from φ by taking $\theta^\varphi(x) := \mathbf{A}$ whenever $\varphi(x) \equiv \langle \mathbf{A}, a \rangle$. Also, for each variable x_α , $\theta^\varphi(x)$ is a closed normal term of type α .¹

We can always extend an interpretation of constants in a possible values structure to obtain a $(\xi$ -functional) possible values evaluation.

THEOREM 7.11. Let $\mathcal{F} := (\mathcal{D}, @)$ be a possible values structure [with η] and $\mathcal{I}: \Sigma \rightarrow \mathcal{D}$ be a typed function such that $\mathcal{I}(c) \in \mathcal{D}_\alpha^c$ for every $c \in \Sigma_\alpha$. There is an evaluation function \mathcal{E} so that $(\mathcal{D}, @, \mathcal{E})$ is a ξ -functional possible values evaluation [with η] such that $\mathcal{E}|_\Sigma \equiv \mathcal{I}$. Also, $\mathcal{E}_\varphi(\mathbf{A}) \in \mathcal{D}_\alpha^{\theta^\varphi(\mathbf{A})}$ for each $\mathbf{A} \in \text{wff}_\alpha(\Sigma)$, assignment φ . Furthermore, if \mathcal{F} is a possible values structure with η , then $\mathcal{E}_\varphi(\mathbf{A}) \equiv \mathcal{E}_\varphi(\mathbf{A}|_{\beta\eta})$ for every $\mathbf{A} \in \text{wff}_\alpha(\Sigma)$ and assignment φ .

PROOF. We extend \mathcal{I} to an evaluation function \mathcal{E} by induction on terms. At each stage, we ensure that the first component of $\mathcal{E}_\varphi(\mathbf{A}_\alpha)$ is $(\theta^\varphi(\mathbf{A}))_\downarrow$, i.e., $\mathcal{E}_\varphi(\mathbf{A}) \in \mathcal{D}_\alpha^{\theta^\varphi(\mathbf{A})}$.

- For each $c_\alpha \in \Sigma$, we must let $\mathcal{E}_\varphi(c) := \mathcal{I}(c) \in \mathcal{D}_\alpha^c$.
- For variables x_α , let $\mathcal{E}_\varphi(x) := \varphi(x) \in \mathcal{D}_\alpha^{\theta^\varphi(x)}$. This ensures that \mathcal{E} satisfies the first condition to be an evaluation function.
- For application, we have $\mathcal{E}_\varphi(\mathbf{G}) \equiv \langle \theta^\varphi(\mathbf{G})_\downarrow, g \rangle$ where $g: \mathcal{D}_\alpha^{\theta^\varphi(\mathbf{A})} \rightarrow \mathcal{D}_\beta^{\theta^\varphi(\mathbf{G})\theta^\varphi(\mathbf{A})}$ and $\mathcal{E}_\varphi(\mathbf{A}) \equiv \langle \theta^\varphi(\mathbf{A})_\downarrow, a \rangle$. Let $\mathcal{E}_\varphi(\mathbf{GA}) := \mathcal{E}_\varphi(\mathbf{G})@_\varphi \mathcal{E}_\varphi(\mathbf{A})$. This definition ensures the second condition for \mathcal{E} to be an evaluation function. Also, $\mathcal{E}_\varphi(\mathbf{GA}) \equiv g(\langle \mathbf{A}, a \rangle) \in \mathcal{D}_\beta^{\theta^\varphi(\mathbf{GA})}$ guarantees the first component of $\mathcal{E}_\varphi(\mathbf{GA})$ is $(\theta^\varphi(\mathbf{GA}))_\downarrow$.
- For abstraction, suppose we have $\mathcal{E}_{\varphi, [\langle \mathbf{A}, a \rangle / X]}(\mathbf{B}) \in \mathcal{D}_\beta^{\theta^\varphi([\mathbf{A}/X]\mathbf{B})}$ for each $\langle \mathbf{A}, a \rangle \in \mathcal{D}_\alpha$. This defines a function g from \mathcal{D}_α to \mathcal{D}_β which properly restricts as $g: \mathcal{D}_\alpha^{\theta^\varphi(\mathbf{A})} \rightarrow \mathcal{D}_\beta^{\theta^\varphi([\mathbf{A}/X]\mathbf{B})}$. Let $\mathcal{E}_\varphi(\lambda X_\alpha. \mathbf{B}) := \langle (\theta^\varphi(\lambda X_\alpha. \mathbf{B}))_\downarrow, g \rangle \in \mathcal{D}_{\alpha \rightarrow \beta}^{\theta^\varphi(\lambda X. \mathbf{B})}$. Note that we have chosen the first component to be $(\theta^\varphi(\lambda X_\alpha. \mathbf{B}))_\downarrow$, thus maintaining this invariant.

To complete the verification that \mathcal{E} is an evaluation function, we must check two more conditions.

- Suppose φ and ψ coincide on $\text{free}(\mathbf{A})$. An easy induction using the definition of \mathcal{E} shows $\mathcal{E}_\varphi(\mathbf{A}) \equiv \mathcal{E}_\psi(\mathbf{A})$.
- To show \mathcal{E} respects β -reduction, we show \mathcal{E} respects a single reduction, then use induction on the number of reductions.

First, we show $\mathcal{E}_{\varphi, [\mathcal{E}_\varphi(\mathbf{B})/X]}(\mathbf{A}) \equiv \mathcal{E}_\varphi([\mathbf{B}/X]\mathbf{A})$ by induction on \mathbf{A} . If \mathbf{A} is X , then $\mathcal{E}_{\varphi, [\mathcal{E}_\varphi(\mathbf{B})/X]}(X) \equiv \mathcal{E}_\varphi(\mathbf{B}) \equiv \mathcal{E}_\varphi([\mathbf{B}/X]X)$. If \mathbf{A} is a constant or

¹Technically, θ^φ is an infinite substitution. However, in any particular case we can consider $\theta^\varphi(\mathbf{A})$ to be $\theta(\mathbf{A})$ where θ is θ^φ restricted to the finite set $\text{free}(\mathbf{A})$.

any variable other than X , then $\mathcal{E}_{\varphi, [\mathcal{E}_{\varphi}(\mathbf{B})/X]}(\mathbf{A}) \equiv \mathcal{E}_{\varphi}(\mathbf{A}) \equiv \mathcal{E}_{\varphi}([\mathbf{B}/X]\mathbf{A})$.
 If \mathbf{A} is an application \mathbf{FC} , then the induction hypothesis implies

$$\begin{aligned} \mathcal{E}_{\varphi, [\mathcal{E}_{\varphi}(\mathbf{B})/X]}(\mathbf{FC}) &\equiv \mathcal{E}_{\varphi, [\mathcal{E}_{\varphi}(\mathbf{B})/X]}(\mathbf{F}) @ \mathcal{E}_{\varphi, [\mathcal{E}_{\varphi}(\mathbf{B})/X]}(\mathbf{C}) \\ &\equiv \mathcal{E}_{\varphi}([\mathbf{B}/X]\mathbf{F}) @ \mathcal{E}_{\varphi}([\mathbf{B}/X]\mathbf{C}) \\ &\equiv \mathcal{E}_{\varphi}([\mathbf{B}/X]\mathbf{FC}) \end{aligned}$$

If \mathbf{A} is an abstraction $\lambda Y_{\beta}. \mathbf{C}_{\gamma}$, then we check that the first and second components are equal. The first component of $\mathcal{E}_{\varphi, [\mathcal{E}_{\varphi}(\mathbf{B})/X]}(\mathbf{A})$ is $(\theta^{\varphi}, [\theta^{\varphi}(\mathbf{B})]_{\ast} / X])(\mathbf{A})_{\ast}$. This simplifies to $\theta^{\varphi}([\mathbf{B}/X]\mathbf{A})_{\ast}$, which is the first component of $\mathcal{E}_{\varphi}([\mathbf{B}/X]\mathbf{A})$. So, we know the first components are equal. The second component of $\mathcal{E}_{\varphi, [\mathcal{E}_{\varphi}(\mathbf{B})/X]}(\mathbf{A})$ is the function $g: \mathcal{D}_{\beta} \longrightarrow \mathcal{D}_{\gamma}$ such that $g(\mathbf{b}) \equiv \mathcal{E}_{\varphi, [\mathcal{E}_{\varphi}(\mathbf{B})/X], [\mathbf{b}/Y]}(\mathbf{C})$ for every $\mathbf{b} \in \mathcal{D}_{\beta}$. The second component of $\mathcal{E}_{\varphi}([\mathbf{B}/X]\mathbf{A})$ is the function $h: \mathcal{D}_{\beta} \longrightarrow \mathcal{D}_{\gamma}$ such that $h(\mathbf{b}) \equiv \mathcal{E}_{\varphi, [\mathbf{b}/Y]}([\mathbf{B}/X]\mathbf{C})$ for every $\mathbf{b} \in \mathcal{D}_{\beta}$. The induction hypothesis implies $g(\mathbf{b}) \equiv h(\mathbf{b})$ for every $\mathbf{b} \in \mathcal{D}_{\beta}$. That is, $g \equiv h$. Hence, the second components are also equal and we are done.

Now, using the definition of \mathcal{E} on applications and abstractions, we have

$$\mathcal{E}_{\varphi}((\lambda X. \mathbf{A})\mathbf{B}) \equiv \mathcal{E}_{\varphi}(\lambda X. \mathbf{A}) @ \mathcal{E}_{\varphi}(\mathbf{B}) \equiv \mathcal{E}_{\varphi, [\mathcal{E}_{\varphi}(\mathbf{B})/X]}(\mathbf{A}) \equiv \mathcal{E}_{\varphi}([\mathbf{B}/X]\mathbf{A})$$

Next, if \mathbf{C} β -reduces to \mathbf{D} in a single step, then induction on the position of the redex in \mathbf{C} shows $\mathcal{E}_{\varphi}(\mathbf{C}) \equiv \mathcal{E}_{\varphi}(\mathbf{D})$. Finally, induction on the number of β -reduction steps shows $\mathcal{E}_{\varphi}(\mathbf{C}) \equiv \mathcal{E}_{\varphi}(\mathbf{C} \downarrow_{\beta})$.

Therefore, \mathcal{E} is an evaluation function.

Next we show $(\mathcal{D}, @, \mathcal{E})$ is ξ -functional. Suppose $\mathbf{M}, \mathbf{N} \in wff_{\beta}(\Sigma)$ and for every $\mathbf{a} \in \mathcal{D}_{\alpha}$ we have $\mathcal{E}_{\varphi, [\mathbf{a}/X]}(\mathbf{M}) \equiv \mathcal{E}_{\varphi, [\mathbf{a}/X]}(\mathbf{N})$. We must show $\mathcal{E}_{\varphi}(\lambda X. \mathbf{M}) \equiv \mathcal{E}_{\varphi}(\lambda X. \mathbf{N})$. First, we know $\mathcal{E}_{\varphi}(\lambda X. \mathbf{M})$ is of the form $\langle (\theta^{\varphi}(\lambda X. \mathbf{M}))_{\ast}, g \rangle$ and $\mathcal{E}_{\varphi}(\lambda X. \mathbf{N})$ is of the form $\langle (\theta^{\varphi}(\lambda X. \mathbf{N}))_{\ast}, h \rangle$. For every $\mathbf{a} \in \mathcal{D}_{\alpha}$, we have

$$\begin{aligned} g(\mathbf{a}) &\equiv \mathcal{E}_{\varphi, [\mathbf{a}/X]}((\lambda X. \mathbf{M})X) \equiv \mathcal{E}_{\varphi, [\mathbf{a}/X]}(\mathbf{M}) \equiv \mathcal{E}_{\varphi, [\mathbf{a}/X]}(\mathbf{N}) \\ &\equiv \mathcal{E}_{\varphi, [\mathbf{a}/X]}((\lambda X. \mathbf{N})X) \equiv h(\mathbf{a}) \end{aligned}$$

using the properties of the evaluation function \mathcal{E} . Thus $g \equiv h$. It only remains to show the first components are equal. Choose a parameter w_{α} which does not occur in either \mathbf{M} or \mathbf{N} and let \mathbf{w} be $\mathcal{E}(w)$. The first component of $\mathcal{E}_{\varphi}([w/X]\mathbf{M})$ is $(\theta^{\varphi}([w/X]\mathbf{M}))_{\ast}$ and the first component of $\mathcal{E}_{\varphi}([w/X]\mathbf{N})$ is $(\theta^{\varphi}([w/X]\mathbf{N}))_{\ast}$. By the Substitution-Value Lemma (cf. Lemma 3.20 in [4]),

$$\mathcal{E}_{\varphi}([w/X]\mathbf{M}) \equiv \mathcal{E}_{\varphi, [\mathbf{w}/X]}(\mathbf{M}) \equiv \mathcal{E}_{\varphi, [\mathbf{w}/X]}(\mathbf{N}) \equiv \mathcal{E}_{\varphi}([w/X]\mathbf{N}).$$

Hence $(\theta^{\varphi}([w/X]\mathbf{M}))_{\ast} \equiv (\theta^{\varphi}([w/X]\mathbf{N}))_{\ast}$. Let \mathbf{M}' be $(\theta^{\varphi}, [X/X])(\mathbf{M})_{\ast}$ and \mathbf{N}' be $(\theta^{\varphi}, [X/X])(\mathbf{N})_{\ast}$. Note that $(\lambda X. \mathbf{M}')_{\ast}$ is $(\theta^{\varphi}(\lambda X. \mathbf{M}))_{\ast}$ (the first component of $\mathcal{E}_{\varphi}(\lambda X. \mathbf{M})$) and $(\lambda X. \mathbf{N}')_{\ast}$ is $(\theta^{\varphi}(\lambda X. \mathbf{N}))_{\ast}$ (the first component of $\mathcal{E}_{\varphi}(\lambda X. \mathbf{N})$). Hence it is enough to show $\mathbf{M}' \equiv \mathbf{N}'$. We know $[w/X]\mathbf{M}'$ and $[w/X]\mathbf{N}'$ are β -normal (or $\beta\eta$ -normal) since \mathbf{M}' and \mathbf{N}' are normal and w is a parameter. Thus

$$[w/X]\mathbf{M}' \equiv (\theta^{\varphi}([w/X]\mathbf{M}))_{\ast} \equiv (\theta^{\varphi}([w/X]\mathbf{N}))_{\ast} \equiv [w/X]\mathbf{N}'.$$

An easy induction on terms shows that for any terms $\mathbf{A}, \mathbf{B} \in wff_{\beta}(\Sigma)$ and parameter w_{α} , if $[w/X]\mathbf{A} \equiv [w/X]\mathbf{B}$ and w does not occur in either \mathbf{A} or \mathbf{B} , then $\mathbf{A} \equiv \mathbf{B}$. We conclude $\mathbf{M}' \equiv \mathbf{N}'$ and so $\mathcal{E}_{\varphi}(\lambda X. \mathbf{M}) \equiv \mathcal{E}_{\varphi}(\lambda X. \mathbf{N})$. Therefore, $(\mathcal{D}, @, \mathcal{E})$ is ξ -functional.

In case \mathcal{F} is an possible values structure with η , we can show $\mathcal{E}_\varphi(\lambda X_{\beta\bullet}\mathbf{A}_{\alpha\rightarrow\beta}X) \equiv \mathcal{E}_\varphi(\mathbf{A})$ if $X \notin \text{free}(\mathbf{A})$. The first components are equal using the invariant and the fact that $\theta^\varphi(\lambda X_{\beta\bullet}\mathbf{A}X) \downarrow_{\beta\eta} \equiv \theta^\varphi(\mathbf{A}) \downarrow_{\beta\eta}$. Let $g: \mathcal{D}_\alpha \rightarrow \mathcal{D}_\beta$ be the second component of $\mathcal{E}_\varphi(\lambda X_{\beta\bullet}\mathbf{A}X)$ and $h: \mathcal{D}_\alpha \rightarrow \mathcal{D}_\beta$ be the second component of $\mathcal{E}_\varphi(\mathbf{A})$. By definition of \mathcal{E}_φ , we know

$$g(\mathbf{a}) \equiv \mathcal{E}_{\varphi, [\mathbf{a}/X]}(\mathbf{A}X) \equiv \mathcal{E}_{\varphi, [\mathbf{a}/X]}(\mathbf{A}) @ \mathcal{E}_{\varphi, [\mathbf{a}/X]}(X) \equiv \mathcal{E}_\varphi(\mathbf{A}) @ \mathbf{a}$$

By definition of $@$, we know $\mathcal{E}_\varphi(\mathbf{A}) @ \mathbf{a} \equiv h(\mathbf{a})$. So, $g \equiv h$.

Now, just as in the β -reduction case, we can show that $\mathcal{E}_\varphi(\mathbf{C}) \equiv \mathcal{E}_\varphi(\mathbf{D})$ whenever \mathbf{C} η -reduces to \mathbf{D} in one step by induction on the position of the η -redex in \mathbf{C} . Then we have $\mathcal{E}_\varphi(\mathbf{A}) \equiv \mathcal{E}_\varphi(\mathbf{A} \downarrow_{\beta\eta})$ by induction on the number of $\beta\eta$ -reductions. \dashv

§8. The Strong Model Existence Theorem. Suppose Γ_Σ is an acceptable abstract consistency class in \mathfrak{Acc}_* and $\Phi \in \Gamma_\Sigma$. Our goal in this section is to find a model in \mathfrak{M}_* with $\mathcal{M} \models \Phi$. This will be used to prove completeness of the sequent calculi \mathcal{G}_* and the Saturated Extension Theorem.

THEOREM 8.1 (Model Existence for \mathfrak{Acc}_*). *Let $*$ $\in \mathfrak{D}$, Γ_Σ be an acceptable \mathfrak{Acc}_* , and let $\Phi \in \Gamma_\Sigma$ be a sufficiently Σ -pure set of Σ -sentences. There exists a model $\mathcal{M} \in \mathfrak{M}_*$, such that $\mathcal{M} \models \Phi$.*

PROOF. Suppose we have a sufficiently Σ -pure $\Phi \in \Gamma_\Sigma$ where Γ_Σ is an acceptable abstract consistency class in \mathfrak{Acc}_* . Let Γ'_Σ be the acceptable compactification of Γ_Σ given by Theorem 4.3. Let \mathcal{H} be an acceptable Hintikka set in Γ'_Σ such that $\Phi \subseteq \mathcal{H}$ guaranteed to exist by Theorem 6.3. We now construct a model \mathcal{M} in \mathfrak{M}_* such that $\mathcal{M} \models \mathcal{H}$, hence $\mathcal{M} \models \Phi$.

$\mathfrak{Acc}_\beta, \mathfrak{Acc}_{\beta\eta}, \mathfrak{Acc}_{\beta\xi}, \mathfrak{Acc}_{\beta\mathfrak{f}}$: These cases follow from the model existence theorem 8.6 proven in the next subsection using (essentially) Andrews' v -complex construction.

$\mathfrak{Acc}_{\beta\mathfrak{fb}}$: This case follows from the model existence theorem 8.12 which we develop in Subsection 8.2. Andrews' v -complex construction gives possible values structures which do not satisfy property \mathfrak{b} . Using compatibility we can construct a possible values structure and an appropriate per. The quotient by the per will give the desired model. \dashv

Consequently, we can conclude completeness of \mathcal{G}_* for \mathfrak{M}_* where $*$ $\in \mathfrak{D}$. We also conclude $\mathcal{G}(\text{cut})$ is admissible in \mathcal{G}_* .

COROLLARY 8.2. *Let $*$ $\in \mathfrak{D}$. The sequent calculus \mathcal{G}_* is complete for \mathfrak{M}_* . Furthermore, $\mathcal{G}(\text{cut})$ is admissible in \mathcal{G}_* .*

PROOF. Suppose $\Vdash_{\mathcal{G}_*} \Delta$ does not hold. By Lemmas 3.5 and 5.2, $\neg\Delta \in \Gamma_\Sigma^{\mathcal{G}_*}$. By Theorem 5.4, $\Gamma_\Sigma^{\mathcal{G}_*}$ is acceptable in \mathfrak{Acc}_* . By Theorem 8.1, there is a model $\mathcal{M} \in \mathfrak{M}_*$ such that $\mathcal{M} \models \neg\Delta$ (since $\neg\Delta$ is finite, hence sufficiently pure). Thus Δ is not valid in \mathfrak{M}_* . Therefore, \mathcal{G}_* is complete.

To show $\mathcal{G}(\text{cut})$ is admissible, assume $\Vdash_{\mathcal{G}_*} \Delta * \mathbf{C}$ and $\Vdash_{\mathcal{G}_*} \Delta * \neg\mathbf{C}$ hold but $\Vdash_{\mathcal{G}_*} \Delta$ does not. By soundness (cf. Theorem 5.3), $\Delta * \mathbf{C}$ and $\Delta * \neg\mathbf{C}$ are valid

for \mathfrak{M}_* . By completeness, there is a model $\mathcal{M} \in \mathfrak{M}_*$ such that $\mathcal{M} \models \neg\Delta$. Either $\mathcal{M} \models \mathbf{C}$ (contradicting validity of $\Delta * \neg\mathbf{C}$) or $\mathcal{M} \models \neg\mathbf{C}$ (contradicting validity of $\Delta * \mathbf{C}$). Thus $\mathcal{G}(\text{cut})$ is admissible in \mathcal{G}_* . \dashv

8.1. Model Existence for β , $\beta\eta$, $\beta\xi$ and $\beta\mathfrak{f}$. We will now use the machinery developed so far to prove our first model existence theorems. Note that in three of these case, we need not be concerned with acceptability conditions, since for β , $\beta\eta$ and $\beta\xi$ all Hintikka sets are acceptable (see Definition 6.1).

DEFINITION 8.3 (\mathcal{H} -Possible Booleans). Let $\mathcal{H} \subseteq \text{cwff}_o(\Sigma)$ and $\mathbf{A} \in \text{cwff}_o(\Sigma)$. We define the set $\mathcal{B}_{\mathcal{H}}^{\mathbf{A}}$ of \mathcal{H} -possible Booleans for \mathbf{A} by

$$\mathcal{B}_{\mathcal{H}}^{\mathbf{A}} := \begin{cases} \{\mathbf{T}, \mathbf{F}\} & \text{if } \mathbf{A} \notin \mathcal{H} \text{ and } \neg\mathbf{A} \notin \mathcal{H} \\ \{\mathbf{T}\} & \text{if } \mathbf{A} \in \mathcal{H} \text{ and } \neg\mathbf{A} \notin \mathcal{H} \\ \{\mathbf{F}\} & \text{if } \mathbf{A} \notin \mathcal{H} \text{ and } \neg\mathbf{A} \in \mathcal{H} \\ \emptyset & \text{if } \mathbf{A} \in \mathcal{H} \text{ and } \neg\mathbf{A} \in \mathcal{H} \end{cases}$$

DEFINITION 8.4 (Andrews Structure). Let $\mathcal{H} \subseteq \text{cwff}_o(\Sigma)$ and let \diamond be some fixed arbitrary value. We define a structure $(\mathcal{D}, @)$ we will call the *Andrews Structure* [with η] $\mathcal{F}^{\mathcal{H}} := (\mathcal{D}, @)$ [$\mathcal{F}_{\eta}^{\mathcal{H}} := (\mathcal{D}, @)$] for \mathcal{H} as follows:

- Let \mathcal{D}_o be the set of pairs $\langle \mathbf{A}_o, p \rangle$ where $\mathbf{A} \in \text{cwff}_o(\Sigma)$ is β -normal [$\beta\eta$ -normal] and $p \in \mathcal{B}_{\mathcal{H}}^{\mathbf{A}}$.
- Let \mathcal{D}_i be the set of pairs $\langle \mathbf{A}_i, \diamond \rangle$ where $\mathbf{A} \in \text{cwff}_i(\Sigma)$ is β -normal [$\beta\eta$ -normal].
- Let $\mathcal{D}_{\alpha \rightarrow \beta}$ be the set of pairs $\langle \mathbf{G}, g \rangle$ where $\mathbf{G} \in \text{cwff}_{\alpha \rightarrow \beta}(\Sigma)$ is β -normal [$\beta\eta$ -normal] and $g: \mathcal{D}_{\alpha} \rightarrow \mathcal{D}_{\beta}$ such that for every $\langle \mathbf{A}, a \rangle \in \mathcal{D}_{\alpha}$, $g(\langle \mathbf{A}, a \rangle) \equiv \langle \mathbf{B}, b \rangle$ implies \mathbf{B} is the β -normal form [$\beta\eta$ -normal form] of \mathbf{GA} .

We define $@$ by $\langle \mathbf{G}, g \rangle @ \langle \mathbf{A}, a \rangle \equiv g(\langle \mathbf{A}, a \rangle)$, where \mathbf{B} is the β -normal form [$\beta\eta$ -normal form] of \mathbf{GA} for each $\langle \mathbf{G}, g \rangle \in \mathcal{D}_{\alpha \rightarrow \beta}$ and $\langle \mathbf{A}, a \rangle \in \mathcal{D}_{\alpha}$.

Assuming \mathcal{H} satisfies $\vec{\nabla}_c$, we can show the Andrews Structure is a possible values structure and the Andrews Structure with η is a possible values structure with η .

LEMMA 8.5. *Let $\mathcal{H} \subseteq \text{cwff}_o(\Sigma)$ be a set of sentences. If \mathcal{H} satisfies $\vec{\nabla}_c$, then the Andrews Structure [with η] is a possible values structure [with η].*

PROOF. For any set \mathcal{H} , every condition in Definition 7.6 except (2) holds for $\mathcal{F}^{\mathcal{H}}$ and $\mathcal{F}_{\eta}^{\mathcal{H}}$ directly by the definitions of $\mathcal{F}^{\mathcal{H}}$ and $\mathcal{F}_{\eta}^{\mathcal{H}}$. We only check condition (2). Let \mathcal{D}_{α} denote the domain of type α . For any β -normal ($\beta\eta$ -normal if $\mathcal{F}_{\eta}^{\mathcal{H}}$) $\mathbf{A} \in \text{cwff}_i(\Sigma)$, we know $\langle \mathbf{A}, \diamond \rangle$ is in \mathcal{D}_i . Suppose $\mathbf{A} \in \text{cwff}_o(\Sigma)$ is β -normal [$\beta\eta$ -normal]. By $\vec{\nabla}_c$, we cannot have $\mathbf{A}, \neg\mathbf{A} \in \mathcal{H}$, so $\mathbf{p} \in \mathcal{B}_{\mathcal{H}}^{\mathbf{A}}$ for \mathbf{p} either \mathbf{T} or \mathbf{F} . Hence $\langle \mathbf{A}, \mathbf{p} \rangle$ is in \mathcal{D}_o . \dashv

We can use the Andrews structure to construct models for Hintikka sets in $\mathfrak{H}\text{int}_{\beta}$ and $\mathfrak{H}\text{int}_{\beta\xi}$. We can use the possible values structure with η to construct models for Hintikka sets in $\mathfrak{H}\text{int}_{\beta\eta}$. If we have a Hintikka set $\mathcal{H} \in \mathfrak{H}\text{int}_{\beta\mathfrak{f}}$ which satisfies $\vec{\nabla}_{\eta}$, then we can also use the possible values structure with η to construct a model for \mathcal{H} . Hence we are in a position to prove a model existence theorem for acceptable Hintikka sets in these four classes. (Note that the assumption of

acceptability of \mathcal{H} is only useful if $*$ is $\beta\mathfrak{f}$, in which case we assume \mathcal{H} satisfies $\vec{\nabla}_\eta$.)

THEOREM 8.6 (Model Existence $(\beta, \beta\eta, \beta\xi, \beta\mathfrak{f})$).

Let $*$ $\in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}\}$ and \mathcal{H} be an acceptable Hintikka set in $\mathfrak{H}\text{int}_*$. There is a model \mathcal{M} in \mathfrak{M}_* such that $\mathcal{M} \models \mathcal{H}$.

PROOF. By the Impredicativity Gap (cf. Theorem 6.5), we know \mathcal{H} is saturated or Leibniz-free. If \mathcal{H} is saturated, we already constructed a term model modulo Leibniz in Theorem 6.33 in [4]. Hence we assume \mathcal{H} is Leibniz-free. (This will be important when we check the model satisfies property \mathfrak{q} .)

We directly follow Andrews' v -complex construction [1]. If $*$ $\in \{\beta, \beta\xi\}$, then let $\mathcal{F} := (\mathcal{D}, @)$ be the Andrews Structure for \mathcal{H} (a possible values structure by Lemma 8.5). Otherwise, let $\mathcal{F} := (\mathcal{D}, @)$ be the Andrews Structure with η for \mathcal{H} (a possible values structure with η by Lemma 8.5).

By Lemma 7.8, every closed term has a possible value. For each parameter $c_\alpha \in \Sigma$, choose p^c to be some possible value for c , so $\langle c, p^c \rangle \in \mathcal{D}_\alpha$. For the logical constants \neg , \vee and Π^α , we must take particular values.

We define values for the logical constants as follows:

- \neg : Let $p^\neg : \mathcal{D}_o \rightarrow \mathcal{D}_o$ be defined by $p^\neg(\langle \mathbf{A}, a \rangle) := \langle \neg \mathbf{A}, b \rangle$ where b is T if a is F and b is F if a is T. The $\vec{\nabla}_\neg$ and $\vec{\nabla}_c$ properties of \mathcal{H} guarantees this is well-defined. So, p^\neg is a possible value for \neg .
- \vee : For each $\langle \mathbf{A}, \mathbf{F} \rangle \in \mathcal{D}_o$, let $p_{\langle \mathbf{A}, \mathbf{F} \rangle}^\vee : \mathcal{D}_o \rightarrow \mathcal{D}_o$ be the function defined by $p_{\langle \mathbf{A}, \mathbf{F} \rangle}^\vee(\langle \mathbf{B}, b \rangle) := \langle \mathbf{A} \vee \mathbf{B}, b \rangle$. For each $\langle \mathbf{A}, \mathbf{T} \rangle \in \mathcal{D}_o$, let $p_{\langle \mathbf{A}, \mathbf{T} \rangle}^\vee : \mathcal{D}_o \rightarrow \mathcal{D}_o$ be the function defined by $p_{\langle \mathbf{A}, \mathbf{T} \rangle}^\vee(\langle \mathbf{B}, b \rangle) := \langle \mathbf{A} \vee \mathbf{B}, \mathbf{T} \rangle$. The properties $\vec{\nabla}_\vee$, $\vec{\nabla}_\wedge$ and $\vec{\nabla}_c$ of \mathcal{H} guarantees these are well-defined and $\langle \vee \mathbf{A}, p_{\langle \mathbf{A}, a \rangle}^\vee \rangle \in \mathcal{D}_{o \rightarrow o}$. Now, let $p^\vee : \mathcal{D}_o \rightarrow \mathcal{D}_{o \rightarrow o}$ be the function defined by $p^\vee(\langle \mathbf{A}, a \rangle) := \langle \vee \mathbf{A}, p_{\langle \mathbf{A}, a \rangle}^\vee \rangle$. Clearly, p^\vee is a possible value for \vee .
- Π^α : Let $p^{\Pi^\alpha} : \mathcal{D}_{\alpha \rightarrow o} \rightarrow \mathcal{D}_o$ be the function defined by $p^{\Pi^\alpha}(\mathbf{F}, f) := \langle \Pi^\alpha \mathbf{F}, p \rangle$ where $p \equiv \mathbf{T}$ if for every $\langle \mathbf{A}, a \rangle \in \mathcal{D}_\alpha$, the second component of $f(\langle \mathbf{A}, a \rangle)$ is T, and $p \equiv \mathbf{F}$ otherwise. This is well-defined by $\vec{\nabla}_\forall$, $\vec{\nabla}_\exists$ and $\vec{\nabla}_c$, and p^{Π^α} is a possible value for Π^α .

Let \mathcal{E} be the evaluation function extending \mathcal{I} guaranteed to exist by Theorem 7.11 so that $(\mathcal{D}, @, \mathcal{E})$ is a ξ -functional possible values evaluation [with η].

To make this a Σ -model, we must define a valuation $v : \mathcal{D}_o \rightarrow \{\mathbf{T}, \mathbf{F}\}$. We take the obvious choice $v(\langle \mathbf{A}, p \rangle) := p$. Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, v)$. To check \mathcal{M} is a Σ -model, we must check the requirements for v .

- \neg : $v(\mathcal{E}(\neg)@a) \equiv \mathbf{T}$, iff $v(a) \equiv \mathbf{F}$ by the definition of p^\neg .
- \vee : $v(\mathcal{E}(\vee)@a@b) \equiv \mathbf{T}$, iff $v(a) \equiv \mathbf{T}$ or $v(b) \equiv \mathbf{T}$ by the definition of p^\vee .
- Π : $v(\mathcal{E}(\Pi^\alpha)@f) \equiv \mathbf{T}$, iff $v(f@a) \equiv \mathbf{T}$ for each $a \in \mathcal{D}_\alpha$ by the definition of p^{Π^α} .

We verify $\mathcal{M} \models \mathcal{H}$. Suppose $\mathbf{A} \in \mathcal{H}$ and let \mathbf{B} be \mathbf{A}_* . Note that $\mathcal{E}(\mathbf{A}) \equiv \langle \mathbf{B}, \mathbf{p} \rangle \in \mathcal{D}_o$ for some $\mathbf{p} \in \mathcal{B}_\mathcal{H}^\mathbf{B}$. Since $\mathbf{A} \in \mathcal{H}$, we have $\mathbf{B} \in \mathcal{H}$ by $\vec{\nabla}_\beta$ (if $*$ $\in \{\beta, \beta\xi\}$) or $\vec{\nabla}_\eta$ (if $*$ $\in \{\beta\eta, \beta\mathfrak{f}\}$).² Thus $\mathcal{B}_\mathcal{H}^\mathbf{B} \equiv \{\mathbf{T}\}$, $\mathbf{p} \equiv \mathbf{T}$ and so $\mathcal{M} \models \mathbf{A}$.

²This is where the construction would fail if $\mathcal{H} \in \mathfrak{H}\text{int}_{\beta\mathfrak{f}}$ did not satisfy $\vec{\nabla}_\eta$.

If $*$ $\in \{\beta\eta, \beta f\}$, then Theorem 7.11 guarantees \mathcal{M} satisfies property η . We also know \mathcal{M} satisfies property ξ since $(\mathcal{D}, @, \mathcal{E})$ is ξ -functional by Theorem 7.11.³ If $*$ is βf , then \mathcal{M} satisfies property f since \mathcal{M} satisfies properties η and ξ (cf. Theorem 6.31 in [4]).⁴

In general, we can use Theorem 3.62 in [4] to obtain a model of \mathcal{H} satisfying property q , though this would not preserve properties ξ or f (cf. Remark 3.57 in [4]). Instead, we use the assumption that \mathcal{H} is Leibniz-free to show the possible values model \mathcal{M} *already* satisfies property q . To see this, for each $\langle \mathbf{A}, a \rangle \in \mathcal{D}_\alpha$, let $s_{\langle \mathbf{A}, a \rangle} : \mathcal{D}_\alpha \rightarrow \mathcal{D}_o$ be defined by $s_{\langle \mathbf{A}, a \rangle}(\langle \mathbf{A}, a \rangle) := \langle (\mathbf{A} \doteq \mathbf{A}) \downarrow_*, \mathbf{T} \rangle$ and $s_{\langle \mathbf{A}, a \rangle}(\langle \mathbf{B}, b \rangle) := \langle (\mathbf{A} \doteq \mathbf{B}) \downarrow_*, \mathbf{F} \rangle$ for $\langle \mathbf{B}, b \rangle \neq \langle \mathbf{A}, a \rangle$. This is well-defined since we never have $\neg(\mathbf{A} \doteq \mathbf{A}) \downarrow_* \in \mathcal{H}$, and $(\mathbf{A} \doteq \mathbf{B}) \downarrow_* \notin \mathcal{H}$ since \mathcal{H} is Leibniz-free. Then, define $q^\alpha := \langle \doteq^\alpha, \mathbf{l} \rangle$ where $l(\langle \mathbf{A}, a \rangle) := \langle (\lambda X. \mathbf{A} \doteq x) \downarrow_*, s_{\langle \mathbf{A}, a \rangle} \rangle$. This witnesses \mathcal{M} satisfies property q .

Therefore, we know $\mathcal{M} \in \mathfrak{M}_*$ as desired. \dashv

8.2. Model Existence for $\beta f b$. We now turn to the cases where a Hintikka set \mathcal{H} is acceptable in $\mathfrak{H}int_{\beta f b}$ and we wish to obtain a model of \mathcal{H} satisfying properties f and b . We construct such models by putting a per evaluation over a possible values structure similar to the Andrews Structure.

DEFINITION 8.7 (\mathcal{H} -Compatibility Structure). Let \mathcal{H} be a Hintikka set. We define a structure $(\mathcal{D}, @)$ we will call the \mathcal{H} -Compatibility Structure $\mathcal{F}_c^\mathcal{H} := (\mathcal{D}, @)$ and a binary relation \sim on \mathcal{D} by induction on types as follows:

- Let \mathcal{D}_o be the set of pairs $\langle \mathbf{A}_o, p \rangle$ where $\mathbf{A}_o \in cwf\!f_o(\Sigma)$ is β -normal and $p \in \mathcal{B}_\mathcal{H}^\mathbf{A}$. For each $\langle \mathbf{A}, p \rangle, \langle \mathbf{B}, q \rangle \in \mathcal{D}_o$, $\langle \mathbf{A}_o, p \rangle \sim \langle \mathbf{B}_o, q \rangle$ holds if $p \equiv q$.
- Let \mathcal{D}_l be the set of pairs $\langle \mathbf{A}_l, S \rangle$ where $\mathbf{A}_l \in cwf\!f_l(\Sigma)$ is β -normal, $\mathbf{A}_l \in S$ and S is an \mathcal{H} -compatible subset of $cwf\!f_l(\Sigma)$ (cf. Definition 6.7). For each $\langle \mathbf{A}, S \rangle, \langle \mathbf{B}, S' \rangle \in \mathcal{D}_l$, $\langle \mathbf{A}_o, S \rangle \sim \langle \mathbf{B}_o, S' \rangle$ holds if $S \equiv S'$.
- Let $\mathcal{D}_{\alpha \rightarrow \beta}$ be the set of pairs $\langle \mathbf{G}, g \rangle$ where $\mathbf{G} \in cwf\!f_{\alpha \rightarrow \beta}(\Sigma)$ is β -normal and $g : \mathcal{D}_\alpha \rightarrow \mathcal{D}_\beta$ such that for every $\langle \mathbf{A}, a \rangle \in \mathcal{D}_\alpha$, $g(\langle \mathbf{A}, a \rangle) \equiv \langle \mathbf{B}, b \rangle$ implies \mathbf{B} is the β -normal form of \mathbf{GA} . For each $\langle \mathbf{G}_{\alpha \rightarrow \beta}, g \rangle, \langle \mathbf{H}_{\alpha \rightarrow \beta}, h \rangle \in \mathcal{D}_{\alpha \rightarrow \beta}$, $\langle \mathbf{G}_{\beta \rightarrow \gamma}, g \rangle \sim \langle \mathbf{H}_{\beta \rightarrow \gamma}, h \rangle$ holds if $g(\mathbf{a}) \sim h(\mathbf{b})$ for every $\mathbf{a}, \mathbf{b} \in \mathcal{D}_\beta$ with $\mathbf{a} \sim \mathbf{b}$.

We define $@$ by

$$\langle \mathbf{G}, g \rangle @ \langle \mathbf{A}, a \rangle \equiv g(\langle \mathbf{A}, a \rangle)$$

for each $\langle \mathbf{G}, g \rangle \in \mathcal{D}_{\alpha \rightarrow \beta}$ and $\langle \mathbf{A}, a \rangle \in \mathcal{D}_\alpha$.

As in Definition 7.1, at each type α we let

$$\overline{\mathcal{D}}_\alpha^\sim := \{ \langle \mathbf{A}, a \rangle \in \mathcal{D}_\alpha \mid \langle \mathbf{A}, a \rangle \sim \langle \mathbf{A}, a \rangle \}$$

Combining this with the notation restricting the first components, we let $\overline{\mathcal{D}}_\alpha^\mathbf{A}$ be $\mathcal{D}_\alpha^\mathbf{A} \cap \overline{\mathcal{D}}_\alpha^\sim$.

LEMMA 8.8 (Compatibility Structures). *If \mathcal{H} is an acceptable Hintikka set in $\mathfrak{H}int_{\beta f b}$, then the applicative structure $\mathcal{F}_c^\mathcal{H}$ is a possible values structure.*

³We do not need to assume \mathcal{H} satisfies $\vec{\nabla}_\xi$ in this case. The condition $\vec{\nabla}_\xi$ was needed in the case where \mathcal{H} is saturated.

⁴Again we do not need to assume \mathcal{H} satisfies $\vec{\nabla}_f$ in this case. The condition $\vec{\nabla}_f$ is used in the saturated case.

PROOF. Every condition in Definition 7.6 except (2) is trivial to check, so we only check (2). Let \mathcal{D}_α denote the domain of $\mathcal{F}_c^\mathcal{H}$ of type α . By Lemma 6.10, $\mathbf{A} \parallel \mathbf{A}$ for every closed term \mathbf{A} . In particular, for every β -normal $\mathbf{A} \in \text{cwff}_\iota(\Sigma)$ we know $\mathbf{A} \parallel \mathbf{A}$ and so $\langle \mathbf{A}, \{\mathbf{A}\} \rangle \in \mathcal{D}_\iota$. By $\vec{\nabla}_c$, $\mathcal{B}_\mathcal{H}^\mathbf{A}$ is nonempty. Hence there is some $\mathbf{p} \in \mathcal{B}_\mathcal{H}^\mathbf{A}$ such that $\langle \mathbf{A}, \mathbf{p} \rangle \in \mathcal{D}_o$. \dashv

Since every $\mathbf{A} \in \text{cwff}_\alpha(\Sigma)$ has a possible value by Lemma 7.8 in $\mathcal{F}_c^\mathcal{H}$, we choose a particular one $r^\mathbf{A}$ for each \mathbf{A} to act as a default value when necessary.

The binary relation \sim on $\mathcal{F}_c^\mathcal{H}$ is a typed per.

LEMMA 8.9. *Let \mathcal{H} be a Hintikka set and $\mathcal{F}_c^\mathcal{H} \equiv (\mathcal{D}, @)$ be the \mathcal{H} -compatibility structure with binary relation \sim (cf. Definition 8.7). The relation \sim is a typed per on \mathcal{D} .*

PROOF. One can show \sim is symmetric and transitive on each \mathcal{D}_α by induction on types. The only interesting aspect of the proof is that symmetry of \sim_β is used to show transitivity of $\sim_{\beta \rightarrow \gamma}$. \dashv

LEMMA 8.10. *Let \mathcal{H} be an acceptable Hintikka set in $\mathfrak{Sint}_{\beta\text{fb}}$ and $\mathcal{F}_c^\mathcal{H} \equiv (\mathcal{D}, @)$ be the \mathcal{H} -compatibility structure. For each type α , we have*

1. *If $\langle \mathbf{A}, a \rangle \sim \langle \mathbf{B}, b \rangle$ in \mathcal{D}_α , then $\mathbf{A} \parallel \mathbf{B}$.*
2. *If $S \subseteq \text{cwff}_\alpha(\Sigma)$ is \mathcal{H} -compatible, then there is a family of possible values $(p^\mathbf{A})_{\mathbf{A} \in S}$ where $p^\mathbf{A}$ is a possible value for $\mathbf{A} \in S$ in $\mathcal{F}_c^\mathcal{H}$ and $\langle \mathbf{A} \downarrow_\beta, p^\mathbf{A} \rangle \sim \langle \mathbf{B} \downarrow_\beta, p^\mathbf{B} \rangle$ for each $\mathbf{A}, \mathbf{B} \in S$.*

PROOF. These two statements are proven by a mutual induction over the type α . We consider three cases for the first statement.

- o:* Assume $\langle \mathbf{A}, a \rangle \sim \langle \mathbf{B}, b \rangle$ and so $a \equiv b \in \{\mathbf{T}, \mathbf{F}\}$. If \mathbf{A} and \mathbf{B} were \mathcal{H} -incompatible, then, without loss of generality, we can assume $\mathbf{A}, \neg \mathbf{B} \in \mathcal{H}$. Since $\mathbf{A} \in \mathcal{H}$, a cannot be \mathbf{F} , and so $a \equiv \mathbf{T}$. Since $\neg \mathbf{B} \in \mathcal{H}$, b cannot be \mathbf{T} , and so $b \equiv \mathbf{F}$. Since $\mathbf{T} \neq \mathbf{F}$, we have a contradiction.
- ι:* $\langle \mathbf{A}, A \rangle \sim \langle \mathbf{B}, B \rangle$ implies $A \equiv B$. So, $\mathbf{A} \parallel \mathbf{B}$ as members of the \mathcal{H} -compatible set A .
- β → γ:* Suppose $\langle \mathbf{G}, g \rangle \sim \langle \mathbf{H}, h \rangle$. Let $\mathbf{A} \parallel \mathbf{B}$ in $\text{cwff}_\beta(\Sigma)$ be given. Applying the induction hypothesis for part 2 at type β to the set $S := \{\mathbf{A}, \mathbf{B}\}$, we obtain $p^\mathbf{A}$ and $p^\mathbf{B}$ with $\langle \mathbf{A} \downarrow_\beta, p^\mathbf{A} \rangle \sim \langle \mathbf{B} \downarrow_\beta, p^\mathbf{B} \rangle$. So, $g(\langle \mathbf{A} \downarrow_\beta, p^\mathbf{A} \rangle) \sim h(\langle \mathbf{B} \downarrow_\beta, p^\mathbf{B} \rangle)$. The first components of $g(\langle \mathbf{A} \downarrow_\beta, p^\mathbf{A} \rangle)$ and $h(\langle \mathbf{B} \downarrow_\beta, p^\mathbf{B} \rangle)$ are $(\mathbf{GA}) \downarrow_\beta$ and $(\mathbf{HB}) \downarrow_\beta$, resp. Applying the induction hypothesis for part 1 to these terms at type γ , we have $(\mathbf{GA}) \downarrow_\beta \parallel (\mathbf{HB}) \downarrow_\beta$. By Lemma 6.8, $(\mathbf{GA}) \parallel (\mathbf{HB})$. Generalizing over \mathbf{A} and \mathbf{B} , we have $\mathbf{G} \parallel \mathbf{H}$.

We now consider three cases for the second statement.

- o:* We must either be able to let $p^\mathbf{A} := \mathbf{T}$ for every $\mathbf{A} \in S$ or let $p^\mathbf{A} := \mathbf{F}$ for every $\mathbf{A} \in S$. If neither is the case, then by the definition of \mathcal{D}_o there must be $\mathbf{A}, \mathbf{B} \in S$ with $\mathbf{A} \downarrow_\beta, \neg \mathbf{B} \downarrow_\beta \in \mathcal{H}$. But this contradicts \mathcal{H} -compatibility of S and Lemma 6.8.
- ι:* Let For each $\mathbf{A} \in S$, let $p^\mathbf{A} := S^\beta$ where S^β is $\{\mathbf{A} \downarrow_\beta \mid \mathbf{A} \in S\}$. \mathcal{H} -compatibility of S^β follows from Lemma 6.8. By definition of \mathcal{D}_ι and \sim , $\langle \mathbf{A} \downarrow_\beta, S^\beta \rangle \in \mathcal{D}_\iota$ and $\langle \mathbf{A} \downarrow_\beta, S^\beta \rangle \sim \langle \mathbf{B} \downarrow_\beta, S^\beta \rangle$ for all $\mathbf{A}, \mathbf{B} \in S$.

$\beta \rightarrow \gamma$: Suppose we are given the set $S \subseteq \text{cuff}_{\beta \rightarrow \gamma}(\Sigma)$.

For $\langle \mathbf{B}, b \rangle \in \mathcal{D}_\beta \setminus \overline{\mathcal{D}_\beta}$ and $\mathbf{G} \in S$, we let $p^{\mathbf{GB}}$ be the default possible value $r^{\mathbf{GB}}$ for \mathbf{GB} .

For each $\langle \mathbf{B}, b \rangle \in \overline{\mathcal{D}_\beta}$, we choose a particular representative $\langle \mathbf{B}^\sim, b^\sim \rangle$ in the equivalence class of $\langle \mathbf{B}, b \rangle$ with respect to \sim . For a particular $\langle \mathbf{B}^\sim, b^\sim \rangle$, let

$$\mathcal{B} := \{ \langle \mathbf{B}, b \rangle \mid \langle \mathbf{B}, b \rangle \sim \langle \mathbf{B}^\sim, b^\sim \rangle \}$$

and let

$$\mathcal{G}_\mathcal{B} := \{ \mathbf{GB} \mid \mathbf{G} \in S, \langle \mathbf{B}, b \rangle \in \mathcal{B} \text{ for some } b \}$$

For each $\langle \mathbf{B}, b \rangle, \langle \mathbf{C}, c \rangle \in \mathcal{B}$, Applying the induction hypothesis for part 1 to $\langle \mathbf{B}, b \rangle \sim \langle \mathbf{C}, c \rangle$ at type β , we have $\mathbf{B} \parallel \mathbf{C}$. By the definition of \parallel at function types, the set $\mathcal{G}_\mathcal{B}$ is \mathcal{H} -compatible since S is \mathcal{H} -compatible

By applying the induction hypothesis for part 2 to $\mathcal{G}_\mathcal{B}$ at type γ we obtain related possible values $p^{\mathbf{GB}}$ for each $\mathbf{GB} \in \mathcal{G}_\mathcal{B}$. This defines $p^{\mathbf{GB}}$ for each $\mathbf{G} \in S$ and $\mathbf{B} \in \overline{\mathcal{D}_\beta}$.

Now, for each $\mathbf{G} \in S$, we can use the axiom of choice (at the meta-level) to define a function $p^{\mathbf{G}}: \mathcal{D}_\beta \rightarrow \mathcal{D}_\gamma$ such that

$$p^{\mathbf{G}}(\langle \mathbf{B}, b \rangle) \equiv \langle (\mathbf{GB}) \downarrow_\beta, p^{\mathbf{GB}} \rangle$$

This $p^{\mathbf{G}}$ does map into \mathcal{D}_γ since each $p^{\mathbf{GB}}$ is a possible value for \mathbf{GB} . Note that the choices of $p^{\mathbf{GB}}$ imply the functions $p^{\mathbf{G}}$ are related as

$$p^{\mathbf{G}}(\langle \mathbf{B}, b \rangle) \equiv \langle (\mathbf{GB}) \downarrow_\beta, p^{\mathbf{GB}} \rangle \sim \langle (\mathbf{HC}) \downarrow_\beta, p^{\mathbf{HC}} \rangle \equiv p^{\mathbf{H}}(\langle \mathbf{C}, c \rangle)$$

whenever $\langle \mathbf{B}, b \rangle \sim \langle \mathbf{C}, c \rangle$ for each $\mathbf{G}, \mathbf{H} \in S$. So, $\langle \mathbf{G}, p^{\mathbf{G}} \rangle \sim \langle \mathbf{H}, p^{\mathbf{H}} \rangle$ for each $\mathbf{G}, \mathbf{H} \in S$.

⊢

THEOREM 8.11. *Let \mathcal{H} is an acceptable Hintikka set in $\mathfrak{H}\text{int}_{\beta\mathfrak{fb}}$. Every closed term $\mathbf{A} \in \text{cuff}_\alpha(\Sigma)$ has a possible value a with $\langle \mathbf{A} \downarrow_\beta, a \rangle \in \overline{\mathcal{D}_\alpha}$. That is, $(\overline{\mathcal{D}_\alpha})^{\mathbf{A}}$ is nonempty.*

PROOF. This follows simply by applying Lemma 8.10(2) to the singleton set $\{\mathbf{A}\}$ since $\mathbf{A} \parallel \mathbf{A}$ by Lemma 6.10. ⊢

In particular, we now know \sim is a nonempty per on each domain \mathcal{D}_α . We can now use this construction to prove the following theorem.

THEOREM 8.12 (Model Existence ($\beta\mathfrak{fb}$)). *Let \mathcal{H} is an acceptable Hintikka set in $\mathfrak{H}\text{int}_{\beta\mathfrak{fb}}$. There exists a model \mathcal{M} in $\mathfrak{M}_{\beta\mathfrak{fb}}$ such that $\mathcal{M} \models \mathcal{H}$.*

PROOF. First, if \mathcal{H} is not Leibniz-free, then it is saturated by Theorem 6.5. In this case, we are done by Theorem 6.33 in [4]. So, we may assume \mathcal{H} is Leibniz-free.

Let $\mathcal{F} := \mathcal{F}_c^\mathcal{H} \equiv (\mathcal{D}, @)$ be the \mathcal{H} -Compatibility Structure. This is a possible values structure by Lemma 8.8. To apply Theorem 7.11, we must interpret the constants in Σ . For parameters $c_\alpha \in \Sigma$, then choose any possible value p^c with $\langle c, p^c \rangle \in \overline{\mathcal{D}_\alpha}$ for c . Such a possible value exists by Theorem 8.11. Let $\mathcal{I}(c) := \langle c, p^c \rangle$.

To interpret the logical constants, we must check we can interpret \neg , \vee , and each Π^α for each α in the intended way. So, we define the appropriate function and then check that this is an appropriate possible value.

We define a function $v: \mathcal{D}_o \rightarrow \{\mathbf{T}, \mathbf{F}\}$ by $v(\langle \mathbf{A}, \mathbf{p} \rangle) := \mathbf{p}$. Later, this will serve as the valuation in our per evaluation.

Let $n: \mathcal{D}_o \rightarrow \mathcal{D}_o$ be the function taking $\langle \mathbf{A}, a \rangle$ to $\langle \neg \mathbf{A}, b \rangle$ where $b \equiv \mathbf{F}$ if $a \equiv \mathbf{T}$ and $b \equiv \mathbf{T}$ if $a \equiv \mathbf{F}$. It is easy to check using $\vec{\nabla}_\neg$ and $\vec{\nabla}_c$ that $\langle \neg, n \rangle \in \mathcal{D}_{o \rightarrow o}$. It is also easy to check that $\langle \neg, n \rangle \in \overline{\mathcal{D}_{o \rightarrow o}^\sim}$ using Lemma 6.10 and the definition of \sim . Let $\mathcal{I}(\neg) := \langle \neg, n \rangle$.

For each $\langle \mathbf{A}, a \rangle \in \mathcal{D}_o$, let $d_{\langle \mathbf{A}, a \rangle}: \mathcal{D}_o \rightarrow \mathcal{D}_o$ be defined by $d_{\langle \mathbf{A}, a \rangle}(\langle \mathbf{B}, b \rangle) := \langle \mathbf{A} \vee \mathbf{B}, c \rangle$ where $c \equiv \mathbf{T}$ if $a \equiv \mathbf{T}$ or $b \equiv \mathbf{T}$, and $c \equiv \mathbf{F}$ otherwise. Using $\vec{\nabla}_\vee$, $\vec{\nabla}_\vee$ and $\vec{\nabla}_c$, Lemma 6.10 and the definition of \sim , we can easily check that each $\langle \mathbf{A}, d_{\langle \mathbf{A}, a \rangle} \rangle \in \overline{\mathcal{D}_{o \rightarrow o}^\sim}$. Furthermore, we can use these properties to show $\langle \mathbf{A}, d_{\langle \mathbf{A}, a \rangle} \rangle \sim \langle \mathbf{A}', d_{\langle \mathbf{A}', a \rangle} \rangle$ for any other $\langle \mathbf{A}', a \rangle \in \mathcal{D}_o$. That is, \sim -related values in \mathcal{D}_o give \sim -related values d_* . So, we let $d: \mathcal{D}_o \rightarrow \mathcal{D}_{o \rightarrow o}$ be defined by $d(\langle \mathbf{A}, a \rangle) := \langle \mathbf{A}, d_{\langle \mathbf{A}, a \rangle} \rangle$, and conclude $\langle \vee, d \rangle \in \overline{\mathcal{D}_{o \rightarrow o \rightarrow o}^\sim}$. Let $\mathcal{I}(\vee) := \langle \vee, d \rangle$.

The most interesting case is Π^α . Here we let $\langle \Pi^\alpha, \pi^\alpha \rangle$ where $\pi^\alpha: \mathcal{D}_{\alpha \rightarrow o} \rightarrow \mathcal{D}_o$ is defined by $\pi^\alpha(\langle \mathbf{F}, f \rangle) := \langle \Pi^\alpha \mathbf{F}, p \rangle$ where $p \equiv \mathbf{T}$ if $v(f(\mathbf{a})) \equiv \mathbf{T}$ for every $\mathbf{a} \in \overline{\mathcal{D}_\alpha^\sim}$ and $p \equiv \mathbf{F}$ otherwise. Note that we have relativized the Π^α quantifier to $\overline{\mathcal{D}_\alpha^\sim}$. In general, this will *not* give a Σ -model directly on the structure \mathcal{F} . But it will give an appropriate per evaluation on the evaluation we will build over \mathcal{F} .

We must check that $\langle \Pi^\alpha, \pi^\alpha \rangle \in \overline{\mathcal{D}_{(\alpha \rightarrow o) \rightarrow o}^\sim}$. First, $\vec{\nabla}_\forall$, $\vec{\nabla}_\exists$, $\vec{\nabla}_\beta$ and $\vec{\nabla}_c$ (along with $\mathcal{I}(w) \sim \mathcal{I}(w)$ for parameters) imply π^α is well-defined and $\langle \Pi^\alpha, \pi^\alpha \rangle \in \mathcal{D}_{(\alpha \rightarrow o) \rightarrow o}$. To check $\langle \Pi^\alpha, \pi^\alpha \rangle \sim \langle \Pi^\alpha, \pi^\alpha \rangle$, let $\langle \mathbf{F}, f \rangle \sim \langle \mathbf{G}, g \rangle$ be given. By the definition of \sim , this implies $v(f(\mathbf{a})) \sim v(g(\mathbf{a}))$ for every $\mathbf{a} \in \overline{\mathcal{D}_\alpha^\sim}$. So, $v(\pi^\alpha(\langle \mathbf{F}, f \rangle)) \equiv v(\pi^\alpha(\langle \mathbf{G}, g \rangle))$ which precisely means $\pi^\alpha(\langle \mathbf{F}, f \rangle) \sim \pi^\alpha(\langle \mathbf{G}, g \rangle)$. So, let $\mathcal{I}(\Pi^\alpha) := \langle \Pi^\alpha, \pi^\alpha \rangle$.

Now we have an interpretation function \mathcal{I} such that the first component of each $\mathcal{I}(c)$ is c for every $c \in \Sigma$. Furthermore, $\mathcal{I}(c) \sim \mathcal{I}(c)$ for every $c \in \Sigma$, so by Theorem 7.11, there is an evaluation function \mathcal{E} such that

1. $\mathcal{E}|_\Sigma \equiv \mathcal{I}$,
2. $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$ is a possible values evaluation, hence a Σ -evaluation, and
3. $\mathcal{E}_\varphi(\mathbf{A}) \in \mathcal{D}_\alpha^{\theta^\varphi(\mathbf{A})}$ for each $\mathbf{A} \in \text{wff}_\alpha(\Sigma)$.

Using $\vec{\nabla}_\beta$, this last condition implies $v(\mathcal{E}(\mathbf{A})) \equiv \mathbf{T}$ for each $\mathbf{A} \in \mathcal{H}$, since $\mathcal{B}_\mathcal{H}^\mathbf{A} \equiv \{\mathbf{T}\}$.

We now show (\mathcal{J}, \sim, v) is a Σ -per evaluation satisfying ∂^Σ , ∂^{sub} , ∂^v , ∂^{TF} , ∂^\neg , ∂^\vee , ∂^Π , ∂^q , ∂^f and ∂^b . Then we will apply Theorem 7.5. We already know \mathcal{J} is a Σ -evaluation. We know \sim is a typed per on the domains \mathcal{D}_α by Lemma 8.9. To show \sim is nonempty on \mathcal{D}_α , we can simply choose a parameter w_α and note there is some p^w such that $\langle w, p^w \rangle \in \mathcal{D}_\alpha$ and $\langle w, p^w \rangle \sim \langle w, p^w \rangle$ by Lemma 8.11.

Thus $\mathcal{P} := (\mathcal{J}, \sim, v)$ is a Σ -per evaluation. We now check \mathcal{P} has each of the desired properties.

∂^Σ : For each constant $c \in \Sigma$, $\mathcal{E}(c) \equiv \mathcal{I}(c) \sim \mathcal{I}(c) \equiv \mathcal{E}(c)$.

$\partial^@$: By the definition of \sim on function domains $\mathcal{D}_{\alpha \rightarrow \beta}$, for each $\mathbf{g}, \mathbf{h} \in \mathcal{D}_{\alpha \rightarrow \beta}$ and $\mathbf{a}, \mathbf{b} \in \mathcal{D}_\alpha$, if $\mathbf{g} \sim \mathbf{h}$ and $\mathbf{a} \sim \mathbf{b}$, then $\mathbf{g}@\mathbf{a} \sim \mathbf{h}@\mathbf{b}$.

- ∂^f : Let $g, h \in \mathcal{D}_{\alpha \rightarrow \beta}$ be given. Suppose for every $a, b \in \mathcal{D}_\alpha$ $a \sim b$ implies $g@a \sim h@b$. Again by the definition of \sim on function domains, we know $g \sim h$.
- ∂^{sub} : This follows from ∂^Σ , ∂^\oplus and ∂^f using Lemma 7.4.
- ∂^v : We have $a \sim b$ implies $v(a) \equiv v(b)$ for each $a, b \in \mathcal{D}_o$ by the definition of \sim at type o .
- ∂^{TF} : By $\vec{\nabla}_c$, $\vec{\nabla}_\neg$ and $\vec{\nabla}_\beta$, there must be a normal $\mathbf{A} \in cwoff_o(\Sigma)$ with $\neg \mathbf{A} \notin \mathcal{H}$. Hence $\langle \mathbf{A}, T \rangle, \langle \neg \mathbf{A}, F \rangle \in \mathcal{D}_o$. Let $p := \langle \mathbf{A}, T \rangle$ and $q := \langle \neg \mathbf{A}, F \rangle$. Thus $v(p) \equiv T$ and $v(q) \equiv F$ as desired.
- ∂^\neg : This follows from the definition of $n: \mathcal{D}_o \rightarrow \mathcal{D}_o$ above.
- ∂^\vee : This follows from the definition of d above.
- ∂^Π : This follows from the definition of π^α (which was relativized to the smaller domains $\overline{\mathcal{D}_\alpha}$) above.
- ∂^q : We will use the fact that \mathcal{H} is Leibniz-free. Let a type α be given. We must find an element $q^\alpha \in \mathcal{D}_{\alpha \rightarrow \alpha \rightarrow o}$ such that for all $a, b \in \overline{\mathcal{D}_\alpha}$, $v(q^\alpha@a@b) \equiv T$ iff $a \sim b$. For each $a \equiv \langle \mathbf{A}, a \rangle \in \mathcal{D}_\alpha$, let $s_a: \mathcal{D}_\alpha \rightarrow \mathcal{D}_o$ be defined by $s_a(b) := \langle (\mathbf{A} \dot{=}^\alpha \mathbf{B}) \downarrow_\beta, T \rangle$ for $b \equiv \langle \mathbf{B}, b \rangle$ if $b \sim a$ in \mathcal{D}_α , and $s_a(b) := \langle (\mathbf{A} \dot{=}^\alpha \mathbf{B}) \downarrow_\beta, F \rangle$ otherwise. To show this is well-defined, we must check $s_a(b)$ really is in \mathcal{D}_o . If $a \sim b$ and $s_a(b) \equiv \langle (\mathbf{A} \dot{=}^\alpha \mathbf{B}) \downarrow_\beta, T \rangle \notin \mathcal{D}_o$, then we must have $\neg (\mathbf{A} \dot{=}^\alpha \mathbf{B}) \downarrow_\beta \in \mathcal{H}$. This contradicts $a \sim b$ by Lemma 8.10(1). If $a \not\sim b$ and $s_a(b) \equiv \langle (\mathbf{A} \dot{=}^\alpha \mathbf{B}) \downarrow_\beta, F \rangle \notin \mathcal{D}_o$, then we must have $(\mathbf{A} \dot{=}^\alpha \mathbf{B}) \downarrow_\beta \in \mathcal{H}$, contradicting the assumption that \mathcal{H} is Leibniz-free. Note that this includes the case where $a \not\sim a$. So we now have $\langle (\lambda X_{\alpha*} \mathbf{A} \dot{=}^\alpha x) \downarrow_\beta, s_a \rangle \in \mathcal{D}_{\alpha \rightarrow o}$.
- Now, define $l: \mathcal{D}_\alpha \rightarrow \mathcal{D}_{\alpha \rightarrow o}$ by $l(a) := \langle (\lambda X_{\alpha*} \mathbf{A} \dot{=}^\alpha x) \downarrow_\beta, s_a \rangle$ for any $a \equiv \langle \mathbf{A}, a \rangle \in \mathcal{D}_\alpha$. Since each $s_a \in \mathcal{D}_{\alpha \rightarrow o}$, this is well-defined. Let $q^\alpha := \langle \dot{=}^\alpha, l \rangle$. We must check $q^\alpha \sim q^\alpha$. Let $a \equiv \langle \mathbf{A}, a \rangle \in \mathcal{D}_\alpha$ and $b \equiv \langle \mathbf{B}, b \rangle \in \mathcal{D}_\alpha$ with $a \sim b$ be given. We must check $l(a) \sim l(b)$. That is, we must check that s_a and s_b sends \sim -related values to \sim -related results. Let $c \equiv \langle \mathbf{C}, c \rangle \in \mathcal{D}_\alpha$ and $d \equiv \langle \mathbf{D}, d \rangle \in \mathcal{D}_\alpha$ with $c \sim d$ be given. $v(s_a(c)) \equiv T$ iff $a \sim c$ iff $b \sim d$ iff $v(s_b(d)) \equiv T$. So, $s_a(c) \sim s_b(d)$.
- ∂^b : We clearly have only two \sim -equivalence classes in \mathcal{D}_o since there are only two possibilities for the second component of each $a \in \mathcal{D}_o$.
- We can now apply Theorem 7.5 to obtain a model $\mathcal{M} \equiv (\mathcal{D}^\sim, @^\sim, \mathcal{E}^\sim, v^\sim)$ in $\mathfrak{M}_{\beta\text{fb}}$ such that $v^\sim(\mathcal{E}^\sim(\mathbf{A})) \equiv v(\mathcal{E}(\mathbf{A}))$ for all $\mathbf{A} \in cwoff_o(\Sigma)$. Note that $\mathcal{M} \models \mathcal{H}$ since for each $\mathbf{A} \in \mathcal{H}$ $v^\sim(\mathcal{E}^\sim(\mathbf{A})) \equiv v(\mathcal{E}(\mathbf{A})) \equiv T$. \dashv

§9. The Saturated Extension Theorem. For each class of models \mathfrak{M}_* , we can produce saturated abstract consistency classes in \mathfrak{Acc}_* . For $* \in \mathfrak{A}$, this class will be a saturated extension of all acceptable Γ_Σ in \mathfrak{Acc}_* . Let $\Gamma_\Sigma^{\mathfrak{M}_*}$ be the class of all Φ such that there exists a model $\mathcal{M} \in \mathfrak{M}_*$ such that $\mathcal{M} \models \Phi$. We prove below that each $\Gamma_\Sigma^{\mathfrak{M}_*}$ is in \mathfrak{Acc}_* .

DEFINITION 9.1 ($\Gamma_\Sigma^{\mathfrak{M}_*}$). For each $* \in \mathfrak{A}$, we define $\Gamma_\Sigma^{\mathfrak{M}_*}$ to consist of the class of all $\Phi \in cwoff(\Sigma)$ such that there exists a model $\mathcal{M} \in \mathfrak{M}_*$ with $\mathcal{M} \models \Phi$.

THEOREM 9.2. *For each $*$ in \mathfrak{A} , $\Gamma_{\Sigma}^{\mathfrak{M}_*}$ is in \mathfrak{Acc}_* and saturated.*

PROOF. Let $\Phi \in \Gamma_{\Sigma}^{\mathfrak{M}_*}$ be given and let $\mathcal{M} \equiv (\mathcal{D}, @, \mathcal{E}, v)$ be a Σ -model in \mathfrak{M}_* such that $\mathcal{M} \models \Phi$. Let $\mathbf{A}, \mathbf{B} \in \text{cwff}_o(\Sigma)$, $\mathbf{F}, \mathbf{G} \in \text{cwff}_{\alpha \rightarrow \beta}(\Sigma)$ and $\mathbf{P} \in \text{cwff}_{\alpha \rightarrow o}(\Sigma)$

∇_c : Since \mathcal{M} cannot model both \mathbf{A} and $\neg \mathbf{A}$, we cannot have $\mathbf{A}, \neg \mathbf{A} \in \Phi$.

$\nabla_{\neg}, \nabla_{\beta}, \nabla_v, \nabla_{\wedge}, \nabla_{\forall}$: If $\neg \neg \mathbf{A} \in \Phi$, then $\mathcal{M} \models \Phi * \mathbf{A}$. So, $\Phi * \mathbf{A} \in \Gamma_{\Sigma}^{\mathfrak{M}_*}$. The other properties follow by the same argument.

∇_{\exists} : Suppose $\mathcal{M} \models \neg(\Pi \mathbf{P})$. Then there is some $\mathbf{a} \in \mathcal{D}_{\alpha}$ with $\mathcal{M} \models_{\varphi} \neg(\mathbf{P}X_{\alpha})$ where $\varphi(X) \equiv \mathbf{a}$. Let $w \in \Sigma_{\alpha}$ be any constant that does not occur in Φ (note that the definition of ∇_{\exists} does not require that there be such a constant). By a standard parameter change argument we see that there is a model $\mathcal{M}' \equiv (\mathcal{D}, @, \mathcal{E}', v) \in \mathfrak{M}_*$ such that $\mathcal{M}' \models \Phi$ (since w does not occur in any sentence in Φ) and $\mathcal{E}'(w) \equiv \mathbf{a}$. So, $\mathcal{M}' \models \neg(\mathbf{P}w)$.

∇_{η} : We need only check this in case $*$ in $\{\beta\eta, \beta\eta\mathbf{b}\}$. In this case, η directly implies $\mathcal{E}(\mathbf{A}) \equiv \mathcal{E}(\mathbf{A} \downarrow_{\beta\eta})$ for any $\mathbf{A} \in \text{cwff}_o(\Sigma)$. As a result, $\mathcal{M} \models \mathbf{A}$ iff $\mathcal{M} \models \mathbf{A} \downarrow_{\beta\eta}$. Suppose $\Phi \in \Gamma_{\Sigma}^{\mathfrak{M}_*}$, $\mathbf{A} \equiv_{\beta\eta} \mathbf{B}$ and $\mathbf{A} \in \Phi$. So, there is a model $\mathcal{M} \in \mathfrak{M}_*$ with $\mathcal{M} \models \Phi$. We have $\mathcal{M} \models \mathbf{A}$, and so $\mathcal{M} \models \mathbf{A} \downarrow_{\beta\eta}$ and $\mathcal{M} \models \mathbf{B}$. So, $\Phi * \mathbf{B} \in \Gamma_{\Sigma}^{\mathfrak{M}_*}$.

$\nabla_{\dot{f}}$: We must check this if $*$ in $\{\beta\dot{f}, \beta\dot{f}\mathbf{b}\}$. Suppose $\mathcal{M} \models \neg(\mathbf{F} \dot{\equiv}^{\alpha \rightarrow \beta} \mathbf{G})$. We know $\mathcal{E}(\mathbf{F})$ and $\mathcal{E}(\mathbf{G})$ are distinct elements of $\mathcal{D}_{\alpha \rightarrow \beta}$ (cf. Lemma 4.2(1) in [4]). By property \dot{f} , there must be an element $\mathbf{a} \in \mathcal{D}_{\alpha}$ with $\mathcal{E}(\mathbf{F})@ \mathbf{a} \neq \mathcal{E}(\mathbf{G})@ \mathbf{a}$. Let X_{α} be a variable of type α and φ be any assignment. Using property \mathbf{q} , we know $\mathcal{M} \models_{\varphi, [\mathbf{a}/X]} \neg((\mathbf{F}X) \dot{\equiv}^{\beta} (\mathbf{G}X))$ (cf. Lemma 4.2(2) in [4]). Let $w \in \Sigma_{\alpha}$ be any constant which does not occur in Φ . By a standard parameter change argument, there is a model $\mathcal{M}' \equiv (\mathcal{D}, @, \mathcal{E}', v) \in \mathfrak{M}_*$ such that $\mathcal{M}' \models \Phi$ (since w does not occur in any sentence in Φ). Also, $\mathcal{M} \models_{\varphi, [\mathbf{a}/X]} (\mathbf{F}X) \dot{\equiv}^{\beta} (\mathbf{G}X)$ implies $\mathcal{M}' \models \neg((\mathbf{F}w) \dot{\equiv}^{\beta} (\mathbf{G}w))$ (since w does not occur in \mathbf{F} or \mathbf{G} and $\neg((\mathbf{F}w) \dot{\equiv}^{\beta} (\mathbf{G}w))$ is closed). Thus, $\Phi * \neg(\mathbf{F}w \dot{\equiv}^{\beta} \mathbf{G}w) \in \Gamma_{\Sigma}^{\mathfrak{M}_*}$.

∇_{ξ} : We need only check this in case $*$ in $\{\beta\xi, \beta\xi\mathbf{b}\}$ and the proof is analogous to the $\nabla_{\dot{f}}$ case. Suppose $\mathcal{M} \models \neg((\lambda X. \mathbf{M}) \dot{\equiv}^{\alpha \rightarrow \beta} \lambda X. \mathbf{N})$. Then $\mathcal{E}(\lambda X. \mathbf{M}) \neq \mathcal{E}(\lambda X. \mathbf{N})$ and so $\mathcal{E}_{\varphi, [\mathbf{a}/X]}(\mathbf{M}) \neq \mathcal{E}_{\varphi, [\mathbf{a}/X]}(\mathbf{N})$ for some $\mathbf{a} \in \mathcal{D}_{\alpha}$ by property ξ . Let $w \in \Sigma_{\alpha}$ be any constant which does not occur in Φ . By changing the value of $\mathcal{E}(w)$, there is a model $\mathcal{M}' \equiv (\mathcal{D}, @, \mathcal{E}', v) \in \mathfrak{M}_*$ such that $\mathcal{M}' \models \Phi$ and $\mathcal{M}' \models \neg([w/X]\mathbf{M} \dot{\equiv}^{\beta} [w/X]\mathbf{N})$. Thus $\Phi * \neg([w/X]\mathbf{M} \dot{\equiv}^{\beta} [w/X]\mathbf{N}) \in \Gamma_{\Sigma}^{\mathfrak{M}_*}$.

$\nabla_{\mathbf{b}}$: We only must check this in case $*$ in $\{\beta\mathbf{b}, \beta\eta\mathbf{b}, \beta\xi\mathbf{b}, \beta\dot{f}\mathbf{b}\}$. In this case \mathcal{D}_o has only two elements. Without loss of generality, assume \mathcal{D}_o is $\{\mathbf{T}, \mathbf{F}\}$ and v is the identity function. If $\mathcal{M} \models \neg(\mathbf{A} \dot{\equiv}^o \mathbf{B})$, then $\mathcal{E}(\mathbf{A})$ and $\mathcal{E}(\mathbf{B})$ must be distinct elements of \mathcal{D}_o . There are only two possibilities. We could have $\mathcal{E}(\mathbf{A}) \equiv \mathbf{T}$ (so, $\mathcal{M} \models \mathbf{A}$) and $\mathcal{E}(\mathbf{B}) \equiv \mathbf{F}$ (so, $\mathcal{M} \models \neg \mathbf{B}$). In this case, $\Phi \cup \{\mathbf{A}, \neg \mathbf{B}\} \in \Gamma_{\Sigma}^{\mathfrak{M}_*}$. Otherwise, we must have $\mathcal{E}(\mathbf{A}) \equiv \mathbf{F}$ and $\mathcal{E}(\mathbf{B}) \equiv \mathbf{T}$. In this case, $\Phi \cup \{\neg \mathbf{A}, \mathbf{B}\} \in \Gamma_{\Sigma}^{\mathfrak{M}_*}$.

We must finally show that each $\Gamma_\Sigma^{\mathfrak{M}_*}$ is saturated. Let $\Phi \in \Gamma_\Sigma^{\mathfrak{M}_*}$ be given and let $\mathbf{A} \in \text{cuff}_o(\Sigma)$. Let $\mathcal{M} \in \mathfrak{M}_*$ be a model of Φ . This model either satisfies \mathbf{A} or $\neg\mathbf{A}$. So, \mathcal{M} either witnesses $\Phi * \mathbf{A} \in \Gamma_\Sigma^{\mathfrak{M}_*}$ or $\Phi * \neg\mathbf{A} \in \Gamma_\Sigma^{\mathfrak{M}_*}$. Hence, $\Gamma_\Sigma^{\mathfrak{M}_*}$ is saturated. \dashv

We now present the proof of the Saturated Extension Theorem. For each $* \in \mathfrak{D}$, one can consider the collection \mathfrak{Acc}_* of abstract consistency classes, the smaller collection of acceptable classes or the even smaller collection of saturated classes. A primary goal of this paper was to show that a saturated extension result would have proof strength at least as strong as cut-elimination. This fact follows from Theorem 3.10. For Theorem 3.10 we only needed to know every acceptable Γ_Σ in \mathfrak{Acc}_* has a saturated extension. However, in the end, we realized we could show the stronger result that there is a single saturated Γ'_Σ in \mathfrak{Acc}_* (defined from the model class \mathfrak{M}_*) which uniformly extends every acceptable Γ_Σ in \mathfrak{Acc}_* (for $* \in \mathfrak{D}$). On the other hand, one should not believe that this uniform saturated extension Γ'_Σ is maximal in the entire class \mathfrak{Acc}_* (unless $* \in \{\beta, \beta\eta, \beta\xi\}$). In particular, Examples 3.2 and 3.3 indicate how to construct abstract consistency classes in $\mathfrak{Acc}_{\beta\eta}$, $\mathfrak{Acc}_{\beta\xi}$, $\mathfrak{Acc}_{\beta\eta\beta}$, $\mathfrak{Acc}_{\beta\xi\beta}$ and $\mathfrak{Acc}_{\beta\eta\beta\xi}$ which have no saturated extension.

THEOREM 9.3 (Saturated Extension Theorem).

Let $ \in \mathfrak{D}$. There is a saturated abstract consistency class Γ'_Σ in \mathfrak{Acc}_* such that for every acceptable abstract consistency class Γ_Σ in \mathfrak{Acc}_* , Γ'_Σ is a saturated extension of Γ_Σ .*

PROOF. Let Γ_Σ be an acceptable abstract consistency class in \mathfrak{Acc}_* . We know $\Gamma_\Sigma^{\mathfrak{M}_*}$ is a saturated abstract consistency class in \mathfrak{Acc}_* by Theorem 9.2. To check $\Gamma_\Sigma^{\mathfrak{M}_*}$ is an extension of Γ_Σ , let $\Phi \in \Gamma_\Sigma$ be sufficiently Σ -pure. By the Model Existence Theorem 8.1, we have a model \mathcal{M} in \mathfrak{M}_* such that $\mathcal{M} \models \Phi$. This verifies $\Phi \in \Gamma_\Sigma^{\mathfrak{M}_*}$ and we are done. \dashv

§10. Conclusion. In [4] we have introduced and studied eight different model classes (including Henkin models) for classical type theory which generalize the notion of standard models and which allow for complete calculi. These model classes are motivated by different roles of extensionality and they adequately characterize the deductive power of existing theorem-proving calculi. In [4] we have also adapted the abstract consistency method (resp. Andrews Unifying Principle) to these model classes and proved respective model existence theorems. We have then exploited the framework to prove completeness (by syntactic means) for different natural deduction calculi introduced for each of the model classes. Due to the strong saturation condition employed in this framework it is, however, not (easily) applicable to investigate the completeness of machine-oriented calculi such as higher-order sequent, resolution or tableaux calculi.

In this paper we have therefore addressed the saturation problem and our contributions in summary are:

- We have introduced prototypical machine-oriented (cut-free) higher-order sequent calculi for five of the eight model classes as counterparts to the

human-oriented natural deduction calculi developed in [4] and we have proven their soundness.

- By studying their completeness with respect to the model classes within our framework we have illustrated that the saturation condition of [4] may be as hard to show for machine-oriented calculi as cut-elimination.
- Another interesting insight is the Leibniz Gap: Hintikka sets are either saturated or free of positive Leibniz equations. The problem in general is related to impredicativity.
- In some cases, we have developed acceptability conditions which may replace the saturation condition and which are easier to show for machine-oriented calculi. Our saturated extension theorem proves that the acceptability conditions actually still guarantee model existence.
- We have proven an unexpectedly strong formulation of the saturated extension theorem: There exists one saturated abstract consistency class that is an extension of all acceptable abstract consistency classes.
- Our model constructions employ Peter Andrews' ν -complexes [1] as a point of origin.
- We have applied our extended framework to show completeness (by syntactical means) of five prototypical sequent calculi with respect to their associated model classes.

Together with [4] we have thus achieved a framework that supports the development and proof-theoretical investigation of human-oriented as well as machine-oriented (ground) calculi for classical type theory. It remains to develop acceptability conditions and prove model existence for the three cases with Boolean extensionality but without full functional extensionality.

For non-ground machine-oriented calculi the lifting issue has to be additionally addressed and extending our framework by tools that may also support lifting arguments remains future work.

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