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A Calculus of Regions Respecting Both Measure and Topology

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Abstract

Say that space is 'gunky' if every part of space has a proper part. Traditional theories of gunk, dating back to the work of Whitehead in the early part of last century, modeled space in the Boolean algebra of regular closed (or regular open) subsets of Euclidean space. More recently a complaint was brought against that tradition in Arntzenius (2008) and Russell (2008): Lebesgue measure is not even finitely additive over the algebra, and there is no countably additive measure on the algebra. Arntzenius advocated modeling gunk in measure algebras instead—in particular, in the algebra of Borel subsets of Euclidean space, modulo sets of Lebesgue measure zero. But while this algebra carries a natural, countably additive measure, it has some unattractive topological features. In this paper, we show how to construct a model of gunk that has both nice rudimentary measure-theoretic and topological properties. We then show that in modeling gunk in this way we can distinguish between finite dimensions, and that nothing in lost in terms of our ability to identify points as *locations* in space.

Keywords Regions · Gunk · Topology · Point-free space · Mereology · Mereotopology

1 Introduction

Space as we typically represent it in mathematics and physics is made up of points—indivisible, zero-dimensional parts of space. But our experience of the world seems

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to put us in contact only with regions of space that are *extended*. Philosophers have at various times doubted whether there *are* any point-sized regions of space, and relatedly, whether we do well to take points as primitives in a theory of space.

According to an alternative view of space, every part of space has a proper part. Mereologies that satisfy this condition are sometimes called 'gunky.' The term was coined by Lewis [20], and has since been taken up by those interested in a kind of infinite divisibility of space. If space is gunky, then no region of space is a point-sized part. For points are indivisible, spatial atoms; they have no proper parts.

The possibility of gunk raises several immediate questions of both a philosophical and mathematical character. If space is gunky, what are the right primitives to adopt in giving a theory of space? What axioms in these primitives best characterize space? And what models of gunky space are most realistic? These questions are most closely associated with the work of de Laguna and Whitehead in the early part of last century.[1] Since that time, a lot of new and interesting work has been done on these questions, in the areas of logic, philosophy, and computer science.[2] Modern incarnations of the Whiteheadian project typically take as primitive the two-place mereological relation of parthood and a topological predicate, 'contact.' (By topology, of course, we do not mean ordinary topology as given by open sets of *points*. Rather we mean, very loosely, something like the way in which parts of space are 'glued together.') Intuitively, two regions are in contact if they either overlap, or are adjacent to one another. In addition to contact, some have adopted as primitive the unary predicate 'limited'; intuitively, a region is limited if it is bounded on every side.

Region-based theories of space are interpreted in non-degenerate Boolean algebras together with some additional relations interpreting the primitive topological notions. The basic idea is that regions of space correspond to the non-zero elements of the algebra, and the partial order $\leq$ on the algebra interprets the mereological relation of parthood. Thus $a \leq b$ if a is a part of b, and $a < b$ if a is a proper part of b. In these models, the whole of space is itself a region, corresponding to the top element, 1, of the Boolean algebra. (The bottom element, 0, does not correspond to any region of space, but is retained for mathematical convenience). Likewise, complements in the Boolean algebra correspond to mereological differences between the whole of space and a given region of space.

It will be useful to give a name to the mereological condition stated above:

Mereological Gunk Every region of space has a proper subregion.

Boolean algebras that satisfy Mereological Gunk are atomless: for every non-zero element a of the algebra, there is a non-zero element b with $b < a$. Satisfaction of Mereological Gunk is a basic requirement of any model of the alternative picture of space sketched above, on which there are no spatial atoms. But someone interested in

[1] See de Laguna [9] and Whitehead [32].

[2] The literature here is extensive. For some examples in logic and computer science see, e.g., Tarski [28], Grzegorczyk [16], De Vries [10], Clarke [7], Randell [21], Gerla [14], Roeper [22], Cohn and Hazarika [18], Dimov and Vakarelov [11], Balbiani et al. [3], Vakarelov [29], among many others. For some examples in philosophy, see, e.g., Skyrms [27], Cohn and Varzi [8], Arntzenius and Hawthorne [2], Arntzenius [1] and Russell [23]. Two very recent monographs are Gruszczyński [17] and Shapiro and Hellman [25].

the gunky picture of space may want a further kind of infinite divisibility. Thinking topologically now, consider the requirement that for every region a there is a subregion that sits *well-inside* it: that is not in contact with any region disjoint from a (where regions x and y are *disjoint* if they do not share a common part). This condition appears in many formal region-based theories of space, where it sometimes goes under the name of 'extensionality.'[3] We borrow from Russell [23] in giving it the following name.[4]

Topological Gunk Every region a of space contains a region that is not in contact with any region disjoint from a.

One particularly important class of models of region-based theories of space arise from the regular closed subsets of an ordinary ('pointy') topological space. These sets form a complete Boolean algebra in which operations are not set-theoretic, as we'll see below. Moreover, we can equip the algebra with a standard contact relation by saying that two regular closed sets are in contact if their intersection is non-empty. The contact algebra of regular closed subsets of finite-dimensional Euclidean space, $\mathbb{R}^n$, is atomless, and hence a model of Mereological Gunk. It is also a model of Topological Gunk. (This follows from the fact that Euclidean space is regular, so every non-empty open set contains the closure of a non-empty open set.) Nevertheless, in Arntzenius [1] and Russell [23] that algebra came under fire as a model of gunk. Simply put, the problem is that our ordinary notion of size—Lebesgue measure—is not even finitely additive over the algebra, and no non-zero measure is countably additive over the algebra.

For Arntzenius, this was a fundamental problem. For, in addition to whatever topological structure we give to gunky space, there must, he thought, be a workable and realistic notion of size for regions of space. As a result, Arntzenius turned to a different model of gunk altogether: the Lebesgue measure algebra, or algebra of Borel subsets of $\mathbb{R}^n$ modulo sets of (Lebesgue) measure zero. (Recall that the algebra of Borel subsets of a topology X is the smallest collection of subsets of X containing all the open sets and closed under complements and countable unions. Lebesgue measure is the standard measure on Euclidean space; the measure of any interval in $\mathbb{R}$ is its length, the measure of any disc in $\mathbb{R}^2$ its area. While Lebesgue measure is not defined on every subset of Euclidean space, all Borel sets are Lebesgue-measurable.)

Like the algebra of regular closed subsets of $\mathbb{R}^n$, the Lebesgue measure algebra is atomless and complete. Moreover, Lebesgue measure is countably additive over the Borel subsets of $\mathbb{R}^n$. This allows us to define a countably additive measure on the Lebesgue measure algebra in a natural way (namely, by assigning to each equivalence class in the algebra the Lebesgue measure of any of its representatives). Then the

[3]See, for example, De Vries [10], Stell [26], Vakarelov [29], Vakarelov et al. [30, 31], among many others.

[4]Russell's statement of the condition is slightly different. According to him, a region x is a *boundary* of a region y if every part of x is in contact with both y and some region disjoint from y; x is *open* if no part of x is a boundary of x; and finally, space is *topologically gunky* if every region is open. This definition of topological gunk is equivalent to the definition given above, assuming standard axioms for a contact relation on a Boolean algebra: $(A_1) - (A_5)$ below.

algebra satisfies a third kind of infinite divisibility condition, this time cashed out in terms of the notion of size:

Measure-theoretic Gunk Every region of space has a subregion that is arbitrarily small.

But what about topology? As Arntzenius [1] shows, we can define a very natural contact relation on the Lebesgue measure algebra. However, when we do so the resulting contact algebra does not satisfy Topological Gunk: there are regions of space that do not contain any region well-inside them. We can sum things up by saying that the standard model of gunk (the algebra of regular closed subsets of Euclidean space) has nice topological structure, but poor measure-theoretic structure, whereas Arntzenius' model of gunk has nice measure-theoretic structure, but poor topological structure.

It would be nice to have everything we want all at once: a realistic model of gunky space with nice mereological, topological, and measure-theoretic features. Unfortunately, an impossibility result due to Russell [23] constrains any such hopes. Given some very basic assumptions about the nature of space and the parthood relation, the thesis Topological Gunk is in conflict with the existence of a countably subadditive measure on regions of space that satisfies Measure-theoretic Gunk.[5]

These problems occasion, we think, a look 'downward'—a turn, in other words, to smaller algebras that sit inside either the algebra of regular closed subsets of Euclidean space, or the Lebesgue measure algebra. In this paper we explore a model of gunk that, in fact, sits inside both. It is atomless, of course, and hence a model of Mereological Gunk. From the algebra of regular closed subsets of Euclidean space it inherits nice topological features; and from the Lebesgue measure algebra, it inherits nice rudimentary measure-theoretic features. More specifically, the algebra satisfies Topological Gunk, Measure-theoretic Gunk, and Lebesgue measure is finitely (although not countably) additive over the algebra. Moreover, we can show that this way of modeling space admits a distinction between different finite dimensions. And when it comes to identifying points as *locations* in space (not regions), we lose nothing in passing from the algebra of regular closed subsets of Euclidean space to the smaller subalgebra.

In return for these advantages, we sacrifice certain infinitary operations. As we said above, Lebesgue measure is not *countably* additive over the algebra. Moreover, the algebra is not complete.

To what extent are these sacrifices problematic? That will in general depend on one's reasons for looking to gunky models of space to begin with. The attitude

[5] More precisely, Russell [23] shows that the following five theses are inconsistent:

1. Space has a transitive and reflexive parthood relation;
2. Space has a topology with a countable basis;
3. Space is topologically gunky;
4. Space has a non-trivial countably subadditive measure;
5. Every region has an arbitrarily small subregion.

where a *countable basis* for a space is a collection B of open regions such that every open region of space is a mereological sum of elements in B. (The definition of 'open' is given in n. 4.)

adopted here is one of exploration. We put forward this model of gunk not in the spirit of adopting it as the one, true representation of space as it really is—or space as it would be if space were gunky. Rather, this is *a* model, among many, that captures some of what we want in a representation of infinitely divisible space. The existence of this particular model shows that the calculus of regions presented below is compatible with some of the topological and measure-theoretic features of space that we may be antecedently interested in.

Classical mereologists interested in the principle of Unrestricted Composition, according to which *every* collection of things has a mereological sum, will perhaps be tempted to conclude that this model violates that principle. That would be too quick. It's true that the algebra we present as a model of gunk is not closed under arbitrary suprema. But if a collection of regions fails to have a supremum in the algebra of regions, that does not mean that there is a collection of things that fails to have a mereological sum. It means only that there is no *region* which is its mereological sum. Philosophers who accept the principle of Unrestricted Composition need not commit themselves to the further claim that the mereological sum of any collection of regions is itself a region. For the mereological sum may be an entity of some kind—even a spatial entity—that is not a region. (Consider, for example, the view of points adopted in Roeper [22]. Points, as Roeper sees it, are locations in space, and hence presumably spatial entities of one kind or another. But they are not regions of space. On such a view, not every spatial entity need be a region.) This will, of course, not satisfy those metaphysicians who do buy into the idea that the mereological sum of any collection of regions must itself be a region. Our point is simply that this is a further principle—one that goes beyond the principle of Unrestricted Composition defended in places like, e.g., Lewis [19] and Bricker [6].

The paper is organized as follows. In Section 2, we recall Roeper's axioms for region-based theories of space using the primitives 'contact' and 'limited.' In Section 3, we discuss standard models of those axioms based on regular closed sets, as well as Arntzenius's alternative model of gunky space based on the Lebesgue measure algebra. In Section 4, we introduce a new model that sits inside both the algebra of regular closed subsets of Euclidean space and the Lebesgue measure algebra, and in Section 5, we show that this model satisfies all ten of Roeper's axioms. In Section 6, we prove that in modeling gunk this way, we can distinguish between different finite dimensions. In Section 7, we show that in passing to the new model of gunk, nothing is lost in terms of our ability to identify points as locations in space. And finally in Section 8, we briefly consider a related, 'junky' model of space.

2 Roeper's Axioms

We take as our starting point an axiomatization of region-based theories of space given by Roeper [22], using the primitives 'parthood,' 'contact,' and 'limited.' As Roeper shows, this axiomatization is characteristic of locally compact, Hausdorff spaces, in the following sense. Every complete algebra that satisfies the axioms is isomorphic to the algebra of regular closed subsets of a locally compact, Hausdorff space, and indeed, there is a one-to-one correspondence between complete algebras satisfying the axioms (up to isomorphism) and locally compact Hausdorff spaces (up to homeomorphism). We note, however, that not every algebra that satisfies Roeper's axioms is *atomless*; therefore, not every such algebra is a model of Mereological Gunk. Because of our interest in gunk, the specific models of Roeper's axioms we consider below are all mereologically gunky. (Moreover, additional first-order axioms could easily be added to this axiomatization, which would preclude all but atomless algebraic models.)

The first five axioms characterize the relation of contact between regions, and are standard not just in Roeper's axiomatization but in axiomatizations of region-based theories of space that use the primitive 'contact' more generally. (We use '$a \bowtie b$' to mean a is in contact with b.)

(A_1) If $a \bowtie b$, then $b \bowtie a$;
(A_2) If $a \neq 0$, then $a \bowtie a$;
(A_3) $0 \not\bowtie a$;
(A_4) If $a \bowtie b$ and $b \leq c$, then $a \bowtie c$;
(A_5) If $a \bowtie (b \vee c)$ then $a \bowtie b$ or $a \bowtie c$;

The next three axioms characterize the predicate 'limited.' Together they ensure that in any algebra in which these axioms are modeled, the collection of limited regions forms an ideal.

(A_6) 0 is limited;
(A_7) If a is limited and $b \leq a$, then b is limited;
(A_8) If a and b are both limited, then $a \vee b$ is limited;

Finally, the last two axioms govern the interaction between the predicates 'limited' and 'contact':

(A_9) If $a \bowtie b$, then there is a limited $b' \leq b$ such that $a \bowtie b'$;
(A_{10}) If a is limited, $b \neq 0$, and $a \not\bowtie -b$, then there is a non-zero limited c such that $a \not\bowtie -c$ and $c \not\bowtie -b$.

The intuitive idea behind (A_9), as Roeper explains it, is that 'a region that is connected with another region is limited where it is so connected.' (A_{10}) on the other hand expresses a kind of infinite divisibility of space that is different from the simple mereological gunky thesis that every region has a proper part. It says that if a is limited, and is well-inside the (non-zero) region b, then there is a limited (non-zero) region c 'in between' them (in the sense that a is well-inside c and c is well-inside b). In particular, letting $a = 0$, axiom (A_{10}) requires that every (non-zero) region has a limited (non-zero) region that sits well-inside it. This entails what we called above Topological Gunk.

Following Roeper, we will say that a *region-based topology* is a triple,

$$\langle \Omega, \bowtie, \Delta \rangle$$

where Ω is a non-degenerate Boolean algebra, and $\bowtie$ and Ω are, respectively, binary and unary relations on Ω that satisfy axioms (A_1) − (A_{10}).

3 Two Traditions

3.1 Standard Models

One very important class of region-based topologies arises from the regular closed subsets of a topological space. Let X be a topological space. We denote the interior and closure of a set $S \subseteq X$ by $\mathrm{Int}(S)$ and $\mathrm{Cl}(S)$, respectively. Then $S \subseteq X$ is *regular closed* if $S = \mathrm{Cl}\ \mathrm{Int}(S)$ and S is *regular open* if $S = \mathrm{Int}\ \mathrm{Cl}(S)$.

We omit the simple proof of the next lemma.

Lemma 1 *If F is a closed set and O is an open set, then*

1. $\mathrm{Cl}(\mathrm{Int}(F)) \subseteq F$*;*
2. $\mathrm{Int}(\mathrm{Cl}(O)) \supseteq O$*;*
3. $\mathrm{Int}(F)$ *is a regular open set;*
4. $\mathrm{Cl}(O)$ *is a regular closed set.*

It is well-known that the regular closed subsets of X form a complete Boolean algebra, $\mathrm{RC}(X)$, with operations $-, \vee, \wedge$ and infinitary joins and meets $\bigvee, \bigwedge$ defined as follows:[6]

$$\begin{aligned} A \vee B &= A \cup B \\ A \wedge B &= \mathrm{Cl}\ \mathrm{Int}(A \cap B) \\ -A &= \mathrm{Cl}(X \setminus A) \\ \bigvee\{A_i \mid i \in I\} &= \mathrm{Cl}\left(\bigcup\{A_i \mid i \in I\}\right) \\ \bigwedge\{A_i \mid i \in I\} &= \mathrm{Cl}\ \mathrm{Int}\left(\bigcap\{A_i \mid i \in I\}\right) \end{aligned}$$

Note that the meet is not simply set-theoretic intersection, because in general the intersection of two regular closed sets is not regular closed (the same holds for infinitary operations in the algebra). The partial order on the algebra is simply inclusion:

$$A \leq B \text{ iff } A \subseteq B$$

The relations ‘contact’ and ‘limited’ can be defined in $\mathrm{RC}(X)$ in a standard way. For any two elements $A, B \in \mathrm{RC}(X)$, put

$$A \bowtie_X B \text{ if and only if } A \cap B \neq \emptyset$$

[6]The regular open subsets of X also form a Boolean algebra, $\mathrm{RO}(X)$. Indeed, $\mathrm{RO}(X)$ is isomorphic to $\mathrm{RC}(X)$ by the mapping that takes every regular open set to its closure. Following the recent literature on region-based theories of space, we work with $\mathrm{RC}(X)$ instead of $\mathrm{RO}(X)$, but nothing significant rides on this: everything that follows could have been done in the algebra $\mathrm{RO}(X)$, with the necessary modifications.

Note that we mean here set-theoretic intersection, and *not* the Boolean meet! Thus two regions in the algebra RC(X) are in contact if they have a point in common. We put

$$A \in \Delta_X \text{ if and only if } A \text{ is compact}$$

Roeper [22] shows that if X is a locally compact, T_2 space, then $\langle \mathrm{RC}(X), \bowtie_X, \Delta_x \rangle$ is a region-based topology—i.e., satisfies axioms $(A_1) - (A_{10})$. In particular, $(A_1) - (A_8)$ are satisfied in *any* topological space X; (A_9) is satisfied if X is locally compact; and (A_{10}) is satisfied if X is both locally compact and T_2.[7]

Let us now turn to the difficulty mentioned above with modeling space in regular closed algebras, as pointed out in Arntzenius [1], and further studied in Russell [23]. The ordinary notion of 'size' in n-dimensional Euclidean space is given by Lesbesgue measure, which is defined over all measurable sets. This includes all Borel sets, and in particular the regular closed sets. The problem is that Lebesgue measure is not even finitely additive over the Boolean algebra of regular closed sets, RC($\mathbb{R}^n$).

There are many ways to see this. Here we use a 'fat' Cantor set in $\mathbb{R}$; the example can be generalized to $\mathbb{R}^n$ for any finite dimension n. To construct the fat Cantor set, begin with the closed unit interval, $I = [0, 1]$. At stage $n = 1$, remove the open middle interval of length 1/4 from I. We are left with two closed intervals, $[0, 3/8]$ and $[5/8, 1]$. At stage $n = 2$, remove the open middle intervals of length $(1/4)^2$ from each of these remaining intervals. We are left with four closed intervals. In general, at stage $n = k$, remove the open middle interval of length $(1/4)^k$ from each of the remaining closed intervals of the previous stage. Let U_k denote the union of intervals removed at stage k, and let

$$K = [0, 1] - \bigcup_{k \geq 1} U_k$$

We refer to K as the *fat Cantor set*. By adding up the measures of the U_k's, it is not difficult to see that the Lebesgue measure of $\bigcup_{k \geq 1} U_k$ is $1/2$, and therefore also the measure of K is 1/2:

$$\mu\left(\bigcup_{k \geq 1} U_k\right) = \sum_{k \geq 1} 2^{k-1} \left(\frac{1}{4}\right)^k = 1/2$$

It is easily verified that the fat Cantor set, K, has empty interior. Therefore while K is closed, it is not regular closed.

Remark 2 The fat Cantor set is topologically identical to the ordinary Cantor set, which is constructed by removing open middle thirds of remaining intervals at each stage of construction. The difference between the two is that the fat Cantor set has non-zero measure, whereas the ordinary Cantor set has measure zero.

Proposition 3 *Lebesgue measure is not finitely additive over* RC($\mathbb{R}$).

[7] See Roeper [22], Theorems 5.2, 5.3, 5.4, and Theorem 5.5.

Proof Consider the union of all open intervals removed at odd stages of construction of the fat Cantor set, and the union of all open intervals removed at even stages of construction:

$$O = \bigcup_{k \text{ odd}} U_k$$
$$E = \bigcup_{k \text{ even}} U_k$$

Readers can convince themselves that $\mathrm{Cl}(O) = O \cup K$ and $\mathrm{Cl}(E) = E \cup K$. By Lemma 1, $\mathrm{Cl}(O)$ and $\mathrm{Cl}(E)$ are regular closed sets. Moreover, they are disjoint in the algebra $\mathrm{RC}(\mathbb{R})$, since:

$$\mathrm{Cl}(O) \wedge \mathrm{Cl}(E) = \mathrm{Cl}\,\mathrm{Int}(\mathrm{Cl}(O) \cap \mathrm{Cl}(E)) = \mathrm{Cl}\,\mathrm{Int}(K) = \emptyset$$

Lebesgue measure, μ, is not finitely additive over $\mathrm{RC}(\mathbb{R})$, because

$$\mu(\mathrm{Cl}(O) \vee \mathrm{Cl}(E)) \neq \mu(\mathrm{Cl}(O)) + \mu(\mathrm{Cl}(E))$$

Indeed, the left-hand side is just $\mu([0, 1])$, which is 1. But the right-hand side is $\mu(O) + \mu(K) + \mu(E) + \mu(K)$, which is 3/2. □

The example shows that Lebesgue measure is not finitely additive over $\mathrm{RC}(\mathbb{R})$. Of course, Lebesgue measure is finitely (indeed, countably) additive over the algebra of Borel subsets of $\mathbb{R}$, and over the bigger algebra of measurable subsets. The difficulty comes from the fact that in $\mathrm{RC}(\mathbb{R})$, the Boolean meet is not simply set-theoretic intersection; thus (some) regular closed sets that are not disjoint as sets are nevertheless disjoint in $\mathrm{RC}(\mathbb{R})$, and can lead to a failure of finite additivity.[8]

3.2 The Lebesgue Measure Contact Algebra

The results of the previous section pose an impediment to modeling space in the algebra of regular closed subsets of $\mathbb{R}^n$. In response to those difficulties, Arntzenius [1] proposed to instead model gunk in the Lebesgue measure algebra.

Let $\mathrm{Borel}(\mathbb{R}^n)$ denote the σ-algebra of Borel subsets of $\mathbb{R}^n$, and let N denote the ideal of (Lebesgue) measure zero Borel subsets of $\mathbb{R}^n$. The *Lebesgue measure algebra* is the quotient:

$$B_n = \mathrm{Borel}(\mathbb{R}^n) \setminus N$$

Elements of B_n are equivalence classes of Borel subsets of $\mathbb{R}^n$. Two sets A and B belong to the same equivalence class if their symmetric difference $A \triangle B$ belongs to N. We denote by $|A|$ the equivalence class containing the set A. Operations in the quotient algebra are defined in the usual way in terms of representative sets:

$$|A| \wedge |B| = |A \cap B|$$
$$|A| \vee |B| = |A \cup B|$$
$$-|A| = |\mathbb{R}^n \setminus A|$$

[8] In addition, Birkhoff [5] showed that there is no non-zero measure on $\mathrm{RC}(\mathbb{R}^n)$ that is countably additive. See Birkhoff [5].

We can define a natural measure m on B_n by putting:

$$m(|A|) = \mu(A)$$

where μ denotes Lebesgue measure. The measure m is well-defined, because if $|A| = |B|$, then A and B differ by a set of measure zero, so $\mu(A) = \mu(B)$.

Proposition 4 *m is countably additive over B_n.*

Proof Define the projection map f from Borel($\mathbb{R}^n$) to B_n by putting: $f(A) = |A|$. Then f is a σ-homomorphism. In particular,

$$\bigvee |A_k| = \left| \bigcup_{k \in N} A_k \right|$$

Let $\{|A_k| \mid k \in \mathbb{N}\}$ be a collection of pairwise disjoint elements of B_n, and let $C_k = A_k - \bigcup_{i<k} A_i$. Note that the C_k's are pairwise disjoint Borel sets. Moreover, $A_i \cap A_j$ has measure zero for $i \neq j$, so $A_k - C_k = A_k \cap \bigcup_{i<k} A_i$ has measure zero. Therefore, $|C_k| = |A_k|$ for each $k \in \mathbb{N}$. It follows that:

$$\begin{aligned} m\left(\bigvee_{k \in \mathbb{N}} |A_k|\right) &= m\left(\bigvee_{k \in \mathbb{N}} |C_k|\right) \\ &= m\left(|\bigcup_{k \in \mathbb{N}} C_k|\right) \\ &= \mu\left(\bigcup_{k \in \mathbb{N}} C_k\right) \\ &= \sum_{k \in \mathbb{N}} \mu(C_k) \\ &= \sum_{k \in \mathbb{N}} m(|C_k|) \\ &= \sum_{k \in \mathbb{N}} m(|A_k|) \end{aligned}$$

□

Consider now the primitive topological notions ‘contact’ and ‘limited’ that appear in Roeper’s axioms. Arntzenius [1] shows that we can interpret these notions in the algebra B_n as follows. For any two elements $a = |A|$ and $b = |B|$ in B_n, say that a and b are in *contact* ($a \bowtie b$) just in case there is a point $x \in \mathbb{R}^n$ such that for every open set O containing x,

$$\mu(A \cap O) > 0 \text{ and } \mu(B \cap O) > 0$$

The reader can check that μ is well-defined (i.e., independent of the choice of representatives A and B). Say that a is *limited* just in case for some compact Borel set B, $a \leq |B|$. We denote the set of limited regions of B_n by Δ.

These definitions are very natural, but as Arntzenius [1] himself notes, the algebra $\langle B_n, \bowtie, \Delta \rangle$ satisfies only the first nine of Roeper's axioms. To see that it does not satisfy (A_{10}), let K denote, as before, the fat Cantor set. By (A_{10}) there is a non-zero region c such that c is well-inside $|K|$ (i.e., c is not in contact with the complement of $|K|$). And this cannot be the case, as the following proposition shows. (Similar examples can be constructed to show that (A_{10}) fails in B_n for any $n \in \mathbb{N}$.)

Proposition 5 *Every non-zero element of B_1 is in contact with $|\mathbb{R} \setminus K|$.*

Proof $\mathbb{R} \setminus K$ is an open dense set, since K is closed and has empty interior. Therefore, for any open set O, $O \cap (\mathbb{R} \setminus K)$ is a non-empty open set, and hence has non-zero Lebesgue measure. Let $\mathcal{B}$ be a countable open basis for $\mathbb{R}$ (for example, the set of all rational intervals centered at rational points). Suppose that $|A|$ is not in contact with $|\mathbb{R} \setminus K|$. Then for each $x \in \mathbb{R}$ there exists a basis set B_x containing x such that $\mu(A \cap B_x) = 0$. Let $S = \{B_x \mid x \in X\}$. Then S is a countable open cover of $\mathbb{R}$. So

$$\mu(A) = \mu\left(\bigcup_{U \in S} (A \cap U)\right) \leq \sum_{U \in S} \mu(A \cap U) = 0$$

Thus $|A| = 0$. We have shown that only the zero element of B_1 is not in contact with $|\mathbb{R} \setminus K|$. □

The failure of Axiom (A_{10}) is important. It shows that the algebra B_n does not satisfy Topological Gunk. While each region of space has a proper part, some regions do not have proper parts that sits 'well-inside' them.

4 Looking 'Downward'

This leaves us in a somewhat unsatisfactory position. The algebra $\mathrm{RC}(\mathbb{R}^n)$ is a poor model of gunk because the natural measure on the algebra is not even finitely additive. But likewise, for anyone interested in models of space that satisfy Topological Gunk, the algebra B_n is a poor model of gunk because, while the natural measure on B_n is countably additive, the algebra contains unintuitive regions of space like the fat Cantor region, $|K|$, which contain no parts that are well-inside them.

The problems encountered in Section 3 motivate a search for smaller algebras that are well-behaved in terms of *both* topology and measure—ones perhaps sitting inside the regular closed algebra, $\mathrm{RC}(\mathbb{R}^n)$, or the Lebesgue measure algebra, B_n. Let us begin the search for such algebras by looking again at $\mathrm{RC}(\mathbb{R}^n)$.

4.1 Regular Closed Sets with 'Null' Boundary

Recall that in ordinary (pointy) topology, the *boundary* of a set $A \subseteq \mathbb{R}^n$ is the set:

$$\partial(A) = \text{Cl}(A) - \text{Int}(A)$$

The following simple lemma about boundaries, which we state without proof, will be useful in what follows.

Lemma 6 *Let X be a topological space, and let $S, T \subseteq X$. Then,*

1. $\partial(S) = \partial(X \setminus S)$;
2. $\partial(S \cup T) \subseteq \partial(S) \cup \partial(T)$.
3. $\partial(S \cap T) \subseteq \partial(S) \cup \partial(T)$.
4. *If S is closed,* $\partial(\text{Cl Int}(S)) \subseteq \partial(S)$;

Some elements of $\text{RC}(\mathbb{R}^n)$ have boundaries with non-zero Lebesgue measure. Recall the sets O and E from the proof of Proposition 3. The boundary of both $\text{Cl}(O)$ and $\text{Cl}(E)$ is the fat Cantor set K, which has measure 1/2. These regions lead, as we saw, to a failure of finite additivity. It is natural to wonder what happens if we simply remove all sets whose boundaries have non-zero measure from $\text{RC}(\mathbb{R}^n)$. Consider the collection of regular closed subsets of $\mathbb{R}^n$ with boundaries of Lebesgue-measure zero (henceforth, 'null boundaries'). These sets form a subalgebra of $\text{RC}(\mathbb{R}^n)$; indeed, they are closed under finite unions and complements by Lemma 6, parts 1. and 2., and hence closed under Boolean operations in the algebra by Lemma 6, part. 4. We will denote this subalgebra by $\text{RCN}(\mathbb{R}^n)$, where 'N' is inserted to signify that boundaries are null. Since $\text{RCN}(\mathbb{R}^n)$ is a subalgebra of $\text{RC}(\mathbb{R}^n)$, we can interpret 'contact' and 'limited' in $\text{RCN}(\mathbb{R}^n)$ by simply restricting the interpretation of those relations in $\text{RC}(\mathbb{R}^n)$ to the subalgebra. Thus two elements of $\text{RCN}(\mathbb{R}^n)$ are in contact if they have non-empty intersection, and an element of $\text{RCN}(\mathbb{R}^n)$ is limited if it is compact.

Lemma 7 *Lebesgue measure is finitely additive over* $\text{RCN}(\mathbb{R}^n)$.

Proof Suppose that $A, B \in \text{RCN}(\mathbb{R}^n)$, and $A \wedge B = \emptyset$. Then $\text{Cl Int}(A \cap B) = \emptyset$, so $\text{Int}(A \cap B) = \emptyset$. Thus $A \cap B = \partial(A \cap B)$. Moreover, $\partial(A \cap B)$ has measure zero, since $\partial(A \cap B) \subseteq \partial(A) \cup \partial(B)$. Thus $\mu(A \cap B) = 0$. We have: $\mu(A \vee B) = \mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) = \mu(A) + \mu(B)$. □

Recall that a subalgebra B of A is *dense* in A if for every non-zero $a \in A$, there exists a non-zero $b \in B$ such that $b \leq a$.

Proposition 8 $\text{RCN}(\mathbb{R}^n)$ *is:*

1. *atomless;*
2. *dense in* $\text{RC}(\mathbb{R}^n)$;
3. *satisfies Topological Gunk;*
4. *satisfies Measure-theoretic Gunk (with Lebesgue measure).*

Proof Let A be a non-zero element of RC($\mathbb{R}$). Then Int$(A) \neq \emptyset$, so there exists an open ball $B \subseteq A$. Moreover, there exists a closed ball B' concentric with B but with strictly smaller radius. This shows both that RCN($\mathbb{R}^n$) is dense in RC($\mathbb{R}^n$), and that RCN($\mathbb{R}^n$) is atomless. Since B' is disjoint from the complement of A in RCN($\mathbb{R}^n$)", RCN($\mathbb{R}^n$) satisfies Topological Gunk. Since B' can be chosen with arbitrarily small radius, RCN($\mathbb{R}^n$) satisfies Measure-theoretic Gunk. □

The algebra RCN($\mathbb{R}^n$) is an interesting model of gunk. It is atomless, sits densely inside the standard model RC($\mathbb{R}^n$), and Lebesgue measure is finitely additive over it. Moreover, it satisfies Topological Gunk and Measure-theoretic Gunk.

We close this section by noting that RCN($\mathbb{R}^n$) is not complete, and Lebesgue measure is not *countably* additive over the algebra. Thus in return for the advantages of taking RCN($\mathbb{R}^n$) as a model of gunk, we give up certain infinitary operations. In the proofs below we appeal freely to the following proposition, whose proof can be found in, e.g., Givant and Halmos [15], Ch. 25.

Proposition 9 *If A is a dense subalgebra of B, then*

1. *Every element in B is a least uppper bound of some set of elements in A;*
2. *If a set S of elements in A has a least upper bound b in A, then b is also the least upper bound of S in B (i.e. A is a regular subalgebra of B).*

Proposition 10 RCN($\mathbb{R}^n$) *is not complete.*

Proof This follows immediately from Proposition 9. For let A be an element of RC($\mathbb{R}^n$) that is not an element of RCN($\mathbb{R}^n$)—for example, Cl(O), where O is defined in the proof of Proposition 3. Since RCN($\mathbb{R}^n$) is dense in RC($\mathbb{R}^n$), A is the least upper bound of some set S of elements in RCN($\mathbb{R}^n$). If the latter is complete, then by Proposition 9, the least upper bound of S in RC($\mathbb{R}^n$) would be equal to the least upper bound of S in RCN($\mathbb{R}^n$), and therefore A would belong to RCN($\mathbb{R}^n$), contradicting our assumption. Thus RCN($\mathbb{R}^n$) is not complete. □

Proposition 11 *Lebesgue measure is not countably additive over* RCN($\mathbb{R}^n$).

Proof Let O be the complement of the fat Cantor set K. Then O can be written as a disjoint union of open intervals $\bigcup_n O_n$. Let $C_n = Cl(O_n)$ and note that the C_n's are pairwise disjoint elements of RCN($\mathbb{R}$). Moreover, Cl($\bigcup_n C_n$) = [0, 1]. Therefore the least upper bound of $\{C_n \mid n \in \mathbb{N}\}$ in the (complete) algebra RC($\mathbb{R}$) is [0, 1]. But $[0, 1] \in$ RCN($\mathbb{R}$). So [0, 1] is also the least upper bound of $\{C_n \mid n \in \mathbb{N}\}$ in the algebra RCN($\mathbb{R}$). However, $\sum_n \mu(C_n) = \sum_n \mu(O_n) = 1/2$. So:

$$\mu\left(\bigvee_n C_n\right) = 1 \neq 1/2 = \sum_n \mu(C_n)$$

This shows that Lebesgue measure is not countably additive over RCN($\mathbb{R}$). Similar examples can be constructed in RCN($\mathbb{R}^n$) for any $n \in \mathbb{N}$. □

4.2 Clopens in the Lebesgue Measure Algebra

In the previous subsection, we presented an interesting model of gunk that sits inside the standard model, $\mathrm{RC}(\mathbb{R}^n)$. But what about looking to subalgebras of Arntzenius's model B_n instead? Perhaps here we can also find a nice model of gunk that has both nice rudimentary topological and measure-theoretic features.

The following simple idea presents itself. We can distinguish between 'open' and 'closed' elements in the algebra B_n as follows. (Scott [24] showed this in the context of giving a new semantics for the modal logic $S4$; we recall the construction briefly here.) An element $a \in B_n$ is *open* if $a = |A|$ for some open set $A \subseteq \mathbb{R}^n$. An element $a \in B_n$ is *closed* if it is the Boolean complement of an open element. Not every element of B_n is open; consider, for example, the fat Cantor region $|K|$. Its complement in the algebra is open, so it is closed. And it is not open: K differs from every open set by a set of non-zero measure.[9] So we have a non-trivial distinction between open and closed elements of B_n.

Say that an element a of B_n is *clopen* if it is both open and closed. (Thus a is clopen if it contains both an open set and a closed set.) The collection of clopen elements in B_n is clearly closed under Boolean operations $-$, $\vee$, $\wedge$, and thus forms a subalgebra of B_n. We will denote this algebra by $\mathrm{CLOP}(B_n)$.

Lemma 12 $\mathrm{CLOP}(B_n)$ *is atomless.*

Proof Let a be a non-zero element in $\mathrm{CLOP}(B_n)$. Then $a = |A|$ for some non-empty open set A. Pick an open ball $B \subseteq A$, and let B' be an open ball properly contained in B. Then $|B'|$ is clopen, $|B'| < |B| \leq a$, and $|B'| \neq 0$. □

Lemma 13 *Lebesgue measure is finitely additive over the algebra* $\mathrm{CLOP}(B_n)$.

Proof Immediate from the fact that $\mathrm{CLOP}(B_n)$ is a subalgebra of B_n and Lebesgue measure is finitely (indeed, countably) additive over B_n. □

The subalgebra $\mathrm{CLOP}(B_n)$ of B_n is also an interesting model of gunk. It is atomless, and hence satisfies Mereological Gunk. It also satisfies Measure-theoretic Gunk, for we can let B' in the proof of Lemma 12 be an arbitrarily small open ball. Moreover, as we saw, Lebesgue measure is finitely additive over the algebra. Finally, $\mathrm{CLOP}(B_n)$ does not contain the problematic region $|K| \in B_n$, since $|K|$ is closed but not open.

Of course, in choosing only the clopen elements of B_n we have, as it were, erased the natural topology on B_n. But since $\mathrm{CLOP}(B_n)$ is a subalgebra of B_n, we can now re-introduce topology via the relations 'contact' and 'limited,' by simply restricting

[9] See the proof of Proposition 5.

those relations in B_n to the clopen elements. When we do this, as we'll see below, the algebra satisfies Topological Gunk: every region has a region that sits well-inside it.

4.3 Same Algebra, Different Clothing

Up to now, we have encountered two interesting candidates for modeling gunk: the algebra $\mathrm{RCN}(\mathbb{R}^n)$, which sits densely inside the standard model of gunk, $\mathrm{RC}(\mathbb{R})$, and the algebra $\mathrm{CLOP}(B_n)$, which sits inside Arntzenius' alternative model, B_n. What is the relationship between the two?

In this section, we show that they are in fact one and the same: as Boolean algebras with the relations 'contact' and 'limited', $\mathrm{RCN}(\mathbb{R}^n)$ and $\mathrm{CLOP}(B_n)$ are isomorphic. So $\mathrm{RCN}(\mathbb{R}^n)$ sits inside both the standard model $\mathrm{RC}(\mathbb{R}^n)$ and Arntzenius' model, B_n, inheriting nice topological properties from the former, and nice measure-theoretic properties from the latter.

If $R_1 = \langle \Omega_1, \bowtie_1, \Delta_1 \rangle$ and $R_2 = \langle \Omega_2, \bowtie_2, \Delta_2 \rangle$ are region-based topologies, then a mapping $h : \Omega_1 \to \Omega_2$ is a *Boolean isomorphism* if it is a bijection that preserves all Boolean operations. We say that h is an *isomorphism of region-based topologies* if h is a Boolean isomorphism that also preserves the relations 'contact' and 'limited':

$a \bowtie_1 b$ iff $h(a) \bowtie_2 h(b)$.
$a \in \Delta_1$ iff $h(a) \in \Delta_2$.

If there is an isomorphism of region-based topologies from R_1 to R_2, we say that R_1 and R_2 are *isomorphic*.

Note that if A is a regular closed set with null boundary, then $|A|$ is clopen in B_n. Indeed, A is closed, so $|A|$ is closed. Moreover, A has null boundary, so A is equal in measure to $\mathrm{Int}(A)$. It follows that $|\mathrm{Int}(A)| = |A|$, so $|A|$ is also open, hence clopen. Therefore we can define a mapping $h : \mathrm{RCN}(\mathbb{R}^n) \to \mathrm{CLOP}(B_n)$ by putting:

$$h(A) = |A|$$

Proposition 14 *h is an isomorphism of region-based topologies.*

Proof The verification that h preserves Boolean operations is routine and is left to the reader.

- *h is injective.*
  Suppose $F, G \in \mathrm{RCN}(\mathbb{R})$, and $F \neq G$. WLOG there exists $x \in F \setminus G$. Since $x \notin G$ and G is closed, x is not a point of closure for G. Thus there exists an open set O such that $x \in O$ and $O \cap G = \emptyset$. But since $x \in F$ and F is regular closed, x is a point of closure of $\mathrm{Int}(F)$. So $\mathrm{Int}(F) \cap O$ is a non-empty open set, and hence has non-zero measure. Moreover, $\mathrm{Int}(F) \cap O \subseteq F \setminus G$. So $h(F) = |F| \neq |G| = h(G)$.
- *h is surjective.*
  Let $a \in \mathrm{CLOP}(B_n)$, let F be a closed set in a, and let O be an open set in a. Then $O \subseteq F$. (Suppose not. Then $O \setminus F$ is a non-empty open set, and hence has non-zero measure. But then $|O| \neq |F|$.) Therefore $O \subseteq \mathrm{Cl}(O) \subseteq F$. Since

$|O| = |F|$, also $|O| = |\operatorname{Cl}(O)|$. But then $\operatorname{Cl}(O)$ has null boundary. By Lemma 1, $\operatorname{Cl}(O)$ is a regular closed set. Therefore $\operatorname{Cl}(O) \in \mathrm{RCN}(\mathbb{R}^n)$ and $h(\operatorname{Cl}(O)) = |\operatorname{Cl}(O)| = a$.

– *h preserves 'limited'.*

  Suppose that $A \in \mathrm{RCN}(\mathbb{R}^n)$ is limited. Then A is compact. So $h(A) = |A|$ is limited in $\mathrm{CLOP}(B_n)$. Conversely, suppose that $h(A)$ is limited in $\mathrm{CLOP}(B_n)$. Then $|A| \leq |B|$ for some compact set B. Since B is a compact subset of a Hausdorff space, B is closed. We show that $A \subseteq B$. Suppose not. Then $A \setminus B$ is non-empty. Let $x \in A \setminus B$. Since $x \in \mathbb{R}^n \setminus B$ and $x \in A = \operatorname{Cl}\operatorname{Int}(A)$, we know $\operatorname{Int}(A) \cap (\mathbb{R}^n \setminus B)$ is a non-empty open set, and hence has non-zero Lebesgue measure. But $\operatorname{Int}(A) \cap (\mathbb{R}^n \setminus B) \subseteq A \setminus B$, contradicting the fact that $|A| \leq |B|$. We conclude that $A \subseteq B$. Thus A is a closed subset of the compact set B, and hence is compact. Equivalently, A is limited in $\mathrm{RCN}(\mathbb{R}^n)$.

– *h preserves 'contact'.*

  Suppose that $A \bowtie B$ in $\mathrm{RCN}(\mathbb{R}^n)$. Then $A \cap B \neq \emptyset$. Let $x \in A \cap B$. Since A and B are regular closed sets, x is a point of closure of $\operatorname{Int}(A)$ and of $\operatorname{Int}(B)$. Let O be an open set with $x \in O$. Then $O \cap \operatorname{Int}(A)$ and $O \cap \operatorname{Int}(B)$ are non-empty open sets, hence sets of non-zero Lebesgue measure. So $\mu(O \cap A) > 0$ and $\mu(O \cap B) > 0$. This shows that $h(A) = |A| \bowtie |B| = h(B)$.

  Conversely, suppose that $h(A) \bowtie h(B)$ in $\mathrm{CLOP}(B_n)$. Then there is a point $x \in \mathbb{R}^n$ such that for any open set O with $x \in O$, $\mu(A \cap O) > 0$ and $\mu(B \cap O) > 0$. In particular, $A \cap O \neq \emptyset$, and $B \cap O \neq \emptyset$. So x is a point of closure of A and of B. But A and B are closed sets. So $x \in A \cap B$, and $A \bowtie B$. □

5 The Axioms Satisfied

We have passed from the bigger algebra $\mathrm{RC}(\mathbb{R}^n)$ to the subalgebra $\mathrm{RCN}(\mathbb{R}^n)$, and likewise from the bigger algebra B_n to the subalgebra $\mathrm{CLOP}(B_n)$. But we don't yet know whether these smaller algebras are region-based topologies—i.e., whether they satisfy all ten of Roeper's axioms. This section is devoted to answering that question in the affirmative.

The first thing to note is that there is no difficulty with the axioms $(A_1) - (A_8)$. Satisfaction of those axioms in $\mathrm{RCN}(\mathbb{R}^n)$ is immediate from the fact that they are satisfied in $\mathrm{RC}(\mathbb{R})$, and $\mathrm{RCN}(\mathbb{R}^n)$ is a subalgebra of $\mathrm{RC}(\mathbb{R}^n)$. So we can focus entirely on axioms (A_9) and (A_{10}):

(A_9) If $a \bowtie b$, then there is a limited $b' \leq b$ such that $a \bowtie b'$;

(A_{10}) If a is limited, $b \neq 0$, and $a \not\bowtie -b$, then there a non-zero limited c such that $a \not\bowtie -c$ and $c \not\bowtie -b$.

The following lemma is proved in Roeper [22].

Lemma 15 *Let X be a topology. If $S_1 \subseteq X$ is open, and $S_2 \subseteq X$, then $S_1 \cap \operatorname{Cl}(S_2) \subseteq \operatorname{Cl}(S_1 \cap S_2)$.*

Proposition 16 $\mathrm{RCN}(\mathbb{R}^n)$ *satisfies* (A_9).

Proof We must show that if $A, B \in \mathrm{RCN}(\mathbb{R}^n)$ and $A \cap B \neq \emptyset$, then there is a *compact* set $B' \in \mathrm{RCN}(\mathbb{R}^n)$ such that $B' \subseteq B$ and $A \cap B' \neq \emptyset$. Let $x \in A \cap B$. Pick a closed ball $D \subseteq \mathbb{R}^n$ such that $x \in \mathrm{Int}(D)$. Note that $D \in \mathrm{RCN}(\mathbb{R}^n)$ and D is compact. Let $B' = D \wedge B = \mathrm{Cl}\ \mathrm{Int}(D \cap B)$. Then $B' \in \mathrm{RCN}(\mathbb{R}^n)$. Moreover, $B' \subseteq B$, and since D is compact and B' is a closed subset of D, B' is compact.

We want to show that $A \cap B' \neq \emptyset$. We have:

$$\begin{aligned} x &\in \mathrm{Int}(D) \cap B \\ &= \mathrm{Int}(D) \cap \mathrm{Cl}(\mathrm{Int}(B)) && \text{since } B \text{ is regular closed} \\ &\subseteq \mathrm{Cl}(\mathrm{Int}(D) \cap \mathrm{Int}(B)) && \text{by Lemma 15} \\ &= \mathrm{Cl}(\mathrm{Int}(D \cap B)) \\ &= B' \end{aligned}$$

So $x \in A \cap B'$, and $A \cap B' \neq \emptyset$. □

Note that for $X, Y \in \mathrm{RCN}(\mathbb{R}^n)$, $X \not\lhd -Y$ is equivalent to the condition that $X \cap \mathrm{Cl}(\mathbb{R}^n \setminus Y) = \emptyset$, or simply $X \subseteq \mathrm{Int}(Y)$. We use this repeatedly in the proof of the next proposition.

Proposition 17 $\mathrm{RCN}(\mathbb{R}^n)$ *satisfies* (A_{10}).

Proof We need to show that for any $A, B \in \mathrm{RCN}(\mathbb{R}^n)$, if A is compact, $B \neq \emptyset$, and $A \subseteq \mathrm{Int}(B)$, then there exists a non-empty, compact set $C \in \mathrm{RCN}(\mathbb{R}^n)$ such that $A \subseteq \mathrm{Int}(C)$ and $C \subseteq \mathrm{Int}(B)$. First, if $A = \emptyset$, then since B is a non-empty regular closed set, $\mathrm{Int}(B) \neq \emptyset$. Let C be a closed ball contained in $\mathrm{Int}(B)$. Then C satisfies the desideratum. Second, if $A \neq \emptyset$, then since $A \subseteq \mathrm{Int}(B)$, A is disjoint from $\mathrm{Cl}(\mathbb{R}^n \setminus B)$. Since $\mathbb{R}^n$ is a normal topology and A is closed, there exist disjoint open sets O and U such that $A \subseteq O$ and $\mathrm{Cl}(\mathbb{R}^n \setminus B) \subseteq U$. Note that the set $\mathcal{B}$ of all open balls centered at rational points (points whose coordinates are rational numbers) with rational radii forms a countable open basis for $\mathbb{R}^n$, and for each $V \in \mathcal{B}$, $\mathrm{Cl}(V) \in \mathrm{RCN}(\mathbb{R}^n)$. We can write O as a union $\bigcup_{n \in \mathbb{N}} B_n$ of elements in $\mathcal{B}$. Since A is compact and $\{B_n \mid n \in \mathbb{N}\}$ is an open cover of A, there is a finite subcover $\{B_{n_1}, \ldots, B_{n_k}\}$ of A. Let $C = \mathrm{Cl}(\bigcup_{i \leq k} B_{n_i}) = \bigcup_{i \leq k} \mathrm{Cl}(B_{n_i})$. Then C is non-empty, since A is non-empty; $C \in \mathrm{RCN}(\mathbb{R}^n)$; and C is compact because it is the finite union of compact sets. Moreover, $A \subseteq \bigcup_{i \leq k} B_{n_i} \subseteq C$, so $A \subseteq \mathrm{Int}(C)$. Finally, $C \subseteq \mathrm{Cl}(O) \subseteq \mathbb{R}^n \setminus U$, and since $\mathrm{Cl}(\mathbb{R}^n \setminus B) \subseteq U$, we have $\mathbb{R}^n \setminus U \subseteq \mathrm{Int}(B)$. Thus $C \subseteq \mathrm{Int}(B)$. □

Theorem 18 $\mathrm{RCN}(\mathbb{R}^n)$*, hence also* $\mathrm{CLOP}(B_n)$*, is a region-based topology.*

Proof Immediate from Propositions 16 and 17, and Proposition 14. □

Remark 19 Note that in the proof of Proposition 17, we did not make use of the fact that A and B were elements of $\mathrm{RCN}(\mathbb{R}^n)$; in fact, the proof goes through under the weaker assumption that A and B are simply regular closed sets. It follows that if $A, B \in \mathrm{RC}(\mathbb{R}^n)$, A is limited, $B \neq 0$, and $A \not\bowtie -B$, then there exists a non-zero $C \in \mathrm{RCN}(\mathbb{R}^n)$ such that $A \not\bowtie -C$ and $C \not\bowtie -B$. We will use this fact repeatedly in Section 7.

6 Dimension

We showed in the previous section that $\mathrm{RCN}(\mathbb{R}^n)$ is a region-based topology. The next question to consider is whether we can distinguish between finite dimensions when modeling gunk in algebras of this kind. In particular, we would like to show that for $n \neq m$, the region-based topologies $\mathrm{RCN}(\mathbb{R}^n)$ and $\mathrm{RCN}(\mathbb{R}^m)$ are not isomorphic.

Our strategy will be to use the fact that the bigger region-based topologies $\mathrm{RC}(\mathbb{R}^n)$ and $\mathrm{RC}(\mathbb{R}^m)$ are not isomorphic for $n \neq m$, to show that the smaller region-based topologies $\mathrm{RCN}(\mathbb{R}^n)$ and $\mathrm{RCN}(\mathbb{R}^m)$ cannot be isomorphic. In particular, we show that any isomorphism between these smaller region-based topologies gives rise to an isomorphism between the bigger region-based topologies, so there can be no such isomorphism.

Recall that a Boolean algebra B is the *completion* of a Boolean algebra A if A is a dense subalgebra of B and every set of elements in A has a least upper bound in B. In fact, up to (Boolean) isomorphism, there is only one completion of a given Boolean algebra (see, e.g., Givant and Halmos [15], p. 218, Theorem 24), so we can refer without misunderstanding to *the* completion of a Boolean algebra. The following proposition is well-known.[10]

Proposition 20 *If B_1 is the completion of A_1, B_2 is the completion of A_2, and $f : A_1 \to A_2$ is a Boolean isomorphism, then there is a Boolean isomorphism from B_1 to B_2 that extends f.*

Proposition 21 $\mathrm{RC}(\mathbb{R}^n)$ *is the completion of* $\mathrm{RCN}(\mathbb{R}^n)$.

Proof We saw above that $\mathrm{RCN}(\mathbb{R}^n)$ is a subalgebra of $\mathrm{RC}(\mathbb{R}^n)$, $\mathrm{RC}(\mathbb{R}^n)$ is complete, and $\mathrm{RCN}(\mathbb{R}^n)$ is dense in $\mathrm{RC}(\mathbb{R}^n)$. Thus $\mathrm{RC}(\mathbb{R}^n)$ is the completion of $\mathrm{RCN}(\mathbb{R}^n)$. □

Lemma 22 *If $A \in \mathrm{RC}(\mathbb{R}^n)$ and A is limited, then there exists $A' \in \mathrm{RCN}(\mathbb{R}^n)$ such that $A' \geq A$ and A' is limited.*

[10]See, e.g., Givant and Halmos [15], p. 217–218. If A is isomorphic to A' via a mapping f, B is a completion of A and C is a completion of A', then the same argument given in the proof of Theorem 23 on p. 217 shows that there is an embedding g of B into C that extends f. So $g(B)$ is a complete extension of A' that is a subalgebra of C. Since C is the completion of A', by Corollary 1 on p. 218, $g(B)$ is identical to C. So B is isomorphic to C via an embedding that extends $f : A \to A'$.

Proof If A is limited, then A is a compact subset of $\mathbb{R}^n$ and is therefore contained in a closed ball, A'. Clearly A' is a compact element of $\mathrm{RCN}(\mathbb{R}^n)$. □

Lemma 23 *If $A, B \in \mathrm{RC}(\mathbb{R}^n)$ and $A \bowtie B$, then there exist $A', B' \in \mathrm{RCN}(\mathbb{R}^n)$ such that $A' \leq A$, $B' \leq B$, and $A' \bowtie B'$.*

Proof Suppose $A, B \in \mathrm{RC}(\mathbb{R}^n)$ and $A \bowtie B$. Then there exists $x \in A \cap B$. Since A is regular closed and $x \in A$, x is a point of closure of $\mathrm{Int}(A)$. For each $n \in \mathbb{N}$, let B_n be an open ball centered at x with radius $1/n$. Then $B_n \cap Int(A)$ is a non-empty open set, so let C_n be a closed ball contained in $B_n \cap \mathrm{Int}(A)$. Let

$$A' = \mathrm{Cl}\left(\bigcup_{n\in\mathbb{N}} C_n\right)$$

We claim that: (1) $A' \subseteq A$; (2) $x \in A'$; (3) $A' \in \mathrm{RCN}(\mathbb{R}^n)$.

For (1), note that $\bigcup_{n\in\mathbb{N}} C_n \subseteq \mathrm{Int}(A)$, so $A' = \mathrm{Cl}(\bigcup_{n\in\mathbb{N}} C_n) \subseteq \mathrm{Cl}(\mathrm{Int}(A)) = A$. For (2), note that if O is an open set containing x, then $B_n \subseteq O$ for some $n \in \mathbb{N}$, and $C_n \subseteq B_n$. So x is a point of closure of $\bigcup_{n\in\mathbb{N}} C_n$. Equivalently, $x \in A'$. Finally, for (3) note that A' is a regular closed set since each of the C_n's are regular closed. It remains to show that A' has null boundary. To this end, we claim that:

$$A' = \bigcup_{n\in\mathbb{N}} C_n \cup \{x\} \tag{1}$$

To see this, suppose that $y \in A'$ and $y \neq x$. Then there is some $k \in \mathbb{N}$ such that $y \notin \mathrm{Cl}(B_k)$. But then since $\bigcup_{n\geq k} C_n \subseteq B_k$, also $y \notin \mathrm{Cl}(\bigcup_{n\geq k} C_n)$. Nevertheless,

$$y \in \mathrm{Cl}\left(\bigcup_{n\in\mathbb{N}} C_n\right) = \bigcup_{n<k} C_n \cup \mathrm{Cl}\left(\bigcup_{n\geq k} C_n\right)$$

Therefore $y \in C_n$ for some $n < k$. This proves (1). It follows that:

$$\begin{aligned}
\partial(A') &= A' \setminus \mathrm{Int}(A') \\
&= \left(\bigcup_{n\in\mathbb{N}} C_n \cup \{x\}\right) \setminus \mathrm{Int}(A') && \text{by (1)} \\
&\subseteq \{x\} \cup \bigcup_{n\in\mathbb{N}} (C_n \setminus \mathrm{Int}(C_n)) \\
&= \{x\} \cup \bigcup_{n\in\mathbb{N}} \partial(C_n)
\end{aligned}$$

and RHS has measure zero, since $\partial(C_n)$ has measure zero for each $n \in \mathbb{N}$. Thus $A' \in \mathrm{RCN}(\mathbb{R}^n)$.

We have shown that there exists $A' \in \mathrm{RCN}(\mathbb{R}^n)$ such that $A' \leq A$ and $x \in A'$. By symmetry, there exists also $B' \in \mathrm{RCN}(\mathbb{R}^n)$ such that $B' \leq B$ and $x \in B'$. Therefore $A' \bowtie B'$ in $\mathrm{RCN}(\mathbb{R}^n)$.[11] □

Proposition 24 *Region-based topologies* $\mathrm{RC}(\mathbb{R}^n)$ *and* $\mathrm{RC}(\mathbb{R}^m)$ *are not isomorphic, for* $n \neq m$.

Proof Roeper [22] shows that if X_1 and X_2 are locally compact, T_2 topologies, then X_1 is homeomorphic to X_2 if and only if the region-based topologies $\mathrm{RC}(X_1)$ and $\mathrm{RC}(X_2)$ are isomorphic. (See Roeper [22], Theorem 5.8, p. 278.) It thus follows from the fact that $\mathbb{R}^n$ and $\mathbb{R}^m$ are locally compact, T_2 topologies, and $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for $n \neq m$, that the region-based topologies $\mathrm{RC}(\mathbb{R}^n)$ and $\mathrm{RC}(\mathbb{R}^m)$ are not isomorphic. □

Theorem 25 *Region-based topologies* $\mathrm{RCN}(\mathbb{R}^n)$ *and* $\mathrm{RCN}(\mathbb{R}^m)$ *are not isomorphic, for* $n \neq m$.

Proof Suppose (toward contradiction) that $f : \mathrm{RCN}(\mathbb{R}^n) \to \mathrm{RCN}(\mathbb{R}^m)$ is an isomorphism of region-based topologies. Since $\mathrm{RC}(\mathbb{R}^n)$ is the completion of $\mathrm{RCN}(\mathbb{R}^n)$ and $\mathrm{RC}(\mathbb{R}^m)$ is the completion of $\mathrm{RCN}(\mathbb{R}^m)$, by Proposition 20 we can extend f to a Boolean isomorphism g from $\mathrm{RC}(\mathbb{R}^n)$ to $\mathrm{RC}(\mathbb{R}^m)$. We need to show that g is an isomorphism of region-based topologies—i.e., that g preserves the relations 'contact' and 'limited.'

– *g preserves 'contact'*.
  If $A, B \in \mathrm{RC}(\mathbb{R}^n)$ and $A \bowtie B$, then by Lemma 23, there exist $A', B' \in \mathrm{RCN}(\mathbb{R}^n)$ with $A' \leq A$, $B' \leq B$, and $A' \bowtie B'$. Since $f : \mathrm{RCN}(\mathbb{R}^n) \to \mathrm{RCN}(\mathbb{R}^m)$ is an isomorphism of region-based topologies, $f(A') \bowtie f(B')$. Since g is an extension of f, $g(A') \bowtie g(B')$. But g is a Boolean isomorphism, hence preserves order. So $g(A') \leq g(A)$ and $g(B') \leq g(B)$. Therefore, $g(A) \bowtie g(B)$. We have shown that if $A \bowtie B$ in $\mathrm{RC}(\mathbb{R}^n)$, then $g(A) \bowtie g(B)$ in $\mathrm{RC}(\mathbb{R}^m)$. The converse is proved by noting that g^{-1} is an extension of f^{-1} to a Boolean isomorphism from $\mathrm{RC}(\mathbb{R}^m)$ to $\mathrm{RC}(\mathbb{R}^n)$, and using the same argument.
– *g preserves 'limited'*.
  If $A \in \mathrm{RC}(\mathbb{R}^n)$ is limited, then by Lemma 22, there exists $A' \in \mathrm{RCN}(\mathbb{R}^n)$ such that $A \leq A'$ and A' is limited. Since $f : \mathrm{RCN}(\mathbb{R}^n) \to \mathrm{RCN}(\mathbb{R}^m)$ is an isomorphism of region-based topologies, $f(A')$ is limited. Since g is an extension of f, $g(A')$ is limited. But g is a Boolean isomorphism, hence preserves order. So $g(A) \leq g(A')$. Therefore, $g(A)$ is limited. We have shown that if A is limited in $\mathrm{RC}(\mathbb{R}^n)$, then $g(A)$ is limited in $\mathrm{RC}(\mathbb{R}^m)$.
  The converse is proved by noting that g^{-1} is an extension of f^{-1} to a Boolean isomorphism from $\mathrm{RC}(\mathbb{R}^m)$ to $\mathrm{RC}(\mathbb{R}^n)$, and using the same argument.

[11] We thank an anonymous referee for a helpful suggestion that allowed us to simplify this proof.

We have shown that $g : \mathrm{RC}(\mathbb{R}^n) \to \mathrm{RC}(\mathbb{R}^m)$ is an isomorphism of region-based topologies, contradicting Proposition 24. We conclude that the region-based topologies $\mathrm{RCN}(\mathbb{R}^n)$ and $\mathrm{RCN}(\mathbb{R}^m)$ are not isomorphic. □

7 Points

Region-based theories of space take as primitive the notion of an extended region. However, this does not mean that points must disappear altogether from such theories of space. With Roeper [22], we can think of points as *locations* in space. Although points are not themselves regions, Roeper shows that we can recover points from region-based topologies via certain infinitary constructions.

The basic idea is familiar from the Stone representation theorem. Starting from a region-based topology, $R = \langle \Omega, \bowtie, \triangle \rangle$, we can consider the set of ultrafilters in Ω. A first, naive thought is to identify points with ultrafilters—much as points in the Stone space of a Boolean algebra B are the ultrafilters of B. However, this won't work for two reasons.

First, there may be ultrafilters in Ω that do not contain any limited region. It is difficult to think of such ultrafilters as identifying any precise location in space. Second, distinct ultrafilters may be 'co-located.' Consider, for example, the region-based topology $\mathrm{RC}(\mathbb{R})$, and let $a \in \mathbb{R}$. The collection of regions $A \in \mathrm{RC}(\mathbb{R})$ such that $[a, y] \subseteq A$ for some $y > a$ form a proper filter F in $\mathrm{RC}(\mathbb{R})$. By the Ultrafilter Lemma, F can be extended to an ultrafilter F'. But also, the set of regions $A \in \mathrm{RC}(\mathbb{R})$ with $[y, a] \subseteq A$ for some $y < a$ form a proper filter G in $\mathrm{RC}(\mathbb{R})$. By the Ultrafilter Lemma, G can be extended to an ultrafilter G'. Both F' and G' intuitively identify the same point, a. But they are distinct ultrafilters.

Instead of identifying points with ultrafilters, the idea in Roeper [22] is to identify points with equivalence classes of 'limited' ultrafilters, or ultrafilters that contain some limited region. Let $R = \langle \Omega, \bowtie, \triangle \rangle$ be a region-based topology, let ∇ and ∇' be ultrafilters on the Boolean algebra Ω, and let $\alpha \in \Omega$. We say that α is *connected* to ∇ ($\alpha \bowtie \nabla$) if $\alpha \bowtie \beta$ for each $\beta \in \nabla$. We say that ∇ is *connected* to ∇' ($\nabla \bowtie \nabla'$) if $\alpha \bowtie \beta$ for all $\alpha \in \nabla$ and $\beta \in \nabla'$. We say that ∇ is *limited* if there exists a limited $\alpha \in \nabla$.

The following lemma is proved in Roeper [22].[12]

Lemma 26 *The relation of connection on the set of limited ultrafilters in Ω is an equivalence relation.*

If ∇ is a limited ultrafilter, we denote by $[\nabla]_{\bowtie}$ the equivalence class containing ∇. Following Roeper, we denote the set of points associated with $R = \langle \Omega, \bowtie, \triangle \rangle$ by P_R:

$$P_R = \{[\nabla]_{\bowtie} \mid \nabla \text{ a limited ultrafilter in } R\}$$

[12] See Roeper [22], Theorem 2.2.

Note that the Boolean structure of R determines what ultrafilters exist in R, whereas the contact relation determines the equivalence classes. Thus what points are associated with a given region-based topology depends, as Roeper points out, on both the mereological and 'topological' structure of the region-based topology.

Once we have points, we can define a topology on the set of points in a canonical way. Recall that the topology of a space is fully given by specifying a closed (or open) basis. A collection $\mathbf{B}$ of subsets of a set X forms a closed basis for some topology on X if (1) $X \in \mathbf{B}$; (2) if $A, B \in \mathbf{B}$, and $x \notin A \cup B$, then there exists $C \in \mathbf{B}$ such that $A \cup B \subseteq C$ and $x \notin C$. The closed subsets of the topology given by $\mathbf{B}$ are arbitrary intersections of elements in $\mathbf{B}$.

Following Roeper, we associate to each region $\alpha \in R$ a set of points $C(\alpha)$ in P_R as follows.

$$C(\alpha) = \{[\nabla]_{\bowtie} \mid \alpha \in \bigcup[\nabla]_{\bowtie}\}$$

That is, $[\nabla]_{\bowtie} \in C(\alpha)$ if and only if $\alpha \in \nabla'$ for some $\nabla' \in [\nabla]_{\bowtie}$. It is not difficult to see that

$$\mathbf{B} = \{C(\alpha) \mid \alpha \in \Omega\}$$

is a closed basis for a topology on P_R.[13] Let C_R be the family of closed subsets of P_R. Then we denote by $\mathcal{P}_R$ the topological space $\langle P_R, C_R\rangle$. Thus to each region-based topology $R = \langle \Omega, \bowtie, \triangle\rangle$, we associate a pointy topological space $\mathcal{P}_R$. Roeper [22] shows that the pointy topology $\mathcal{P}_R$ is a locally compact, T_2 space. Indeed, he shows that there is a one-to-one correspondence between complete region-based topologies (up to isomorphism) and locally compact, T_2 spaces (up to homeomorphism).[14,15]

In modeling gunk above, we passed from $\mathrm{RC}(\mathbb{R}^n)$ to the subalgebra $\mathrm{RCN}(\mathbb{R}^n)$. Both of these are region-based topologies, and so both have an associated pointy topological space defined as above. It is natural to wonder about the relationship between these associated pointy topological spaces. Does the smaller algebra $\mathrm{RCN}(\mathbb{R}^n)$ give rise to a different set of points, a different topology on the same set of points, or the very same topology?

This section is devoted to answering that question. We show that the region-based topologies $\mathrm{RC}(\mathbb{R}^n)$ and $\mathrm{RCN}(\mathbb{R}^n)$ give rise to homeomorphic pointy topological spaces. Thus nothing is lost in terms of the identification of points (or locations) in space when we pass from the bigger, standard model, $\mathrm{RC}(\mathbb{R}^n)$, to the smaller one, $\mathrm{RCN}(\mathbb{R}^n)$.

In the remainder of this section, let R_1 and R_2 denote, respectively, the region-based topologies $\mathrm{RC}(\mathbb{R}^n)$ and $\mathrm{RCN}(\mathbb{R}^n)$, and let $\bowtie_1$ and $\bowtie_2$ denote the contact relations in R_1 and R_2.

Lemma 27 *If ∇ is a limited ultrafilter in R_1, then $\nabla \cap R_2$ is a limited ultrafilter in R_2.*

[13] See Roeper [22], Theorem 4.2.

[14] A region-based topology $R = \langle \Omega, \bowtie, \triangle\rangle$ is *complete* if Ω is a complete Boolean algebra.

[15] See Roeper [22], MAIN THEOREM, p. 279.

Proof Clearly $\nabla \cap R_2$ is an ultrafilter in R_2. Since ∇ is limited, there exists a limited $A \in \nabla$. By Lemma 22, there exists a limited $A' \in R_2$ with $A \leq A'$. Since ∇ is an ultrafilter (hence closed upward), $A' \in \nabla \cap R_2$. So $\nabla \cap R_2$ is limited. □

Let f be the map from the set of limited ultrafilters in R_1 to the set of limited ultrafilters in R_2, defined by:

$$f(\nabla) = \nabla \cap R_2$$

Lemma 28 *If ∇ and ∇' are limited ultrafilters in R_1, $\nabla \bowtie_1 \nabla'$ if and only if $f(\nabla) \bowtie_2 f(\nabla')$.*

Proof The left-to-right direction is obvious. For the reverse direction, suppose that $\nabla \not\bowtie_1 \nabla'$. Then there exist $A \in \nabla$ and $B \in \nabla'$ with $A \not\bowtie_1 B$. Since ∇ and ∇' are limited, we can assume WLOG that both A and B are limited. (Indeed, since ∇ and ∇' are limited, there exists a limited $C \in \nabla$ and a limited $D \in \nabla'$. Then $A' = A \wedge C \in \nabla$ and $B' = B \wedge D \in \nabla'$. Moreover, A' and B' are limited and $A' \not\bowtie_1 B'$.)

Note that since $A \in \nabla$, $A \neq 0$. Therefore $B \neq 1$ (else $A \bowtie_1 B$), so $-B \neq 0$. By Remark 19, there exists $C \in R_2$ such that $A \not\bowtie_1 -C$ and $C \not\bowtie_1 B$. So $A \leq C$. Now note that $C \neq 1$ (else $C \bowtie_1 B$), so $-C \neq 0$. By a similar argument, there exists $D \in R_2$ such that $B \not\bowtie_1 -D$ and $D \not\bowtie_1 C$; and since $D, C \in R_2$, we can write $D \not\bowtie_2 C$. So $B \leq D$. Clearly $C \in \nabla \cap R_2 = f(\nabla)$ and $D \in \nabla' \cap R_2 = f(\nabla')$. Finally, since $D \not\bowtie_2 C$, we have $f(\nabla) \not\bowtie_2 f(\nabla')$. □

Since points are equivalence classes of limited ultrafilters under the equivalence relation $\bowtie$, we can define a mapping $h : \mathcal{P}_{R_1} \to \mathcal{P}_{R_2}$ by putting,

$$h([\nabla]_{\bowtie_1}) = [f(\nabla)]_{\bowtie_2}$$

The definition of h is correct (i.e., independent of the choice of $\nabla \in [\nabla]_{\bowtie_1}$) by Lemma 28. We need to show that h is a homeomorphism, and hence that the region-based topologies R_1 and R_2 give rise to the 'same' pointy space.

Lemma 29 *h is a bijection.*

Proof To see that h is injective, note that if $h([\nabla]_{\bowtie_1}) = h([\nabla']_{\bowtie_1})$, then $f(\nabla) \bowtie_2 f(\nabla')$. By Lemma 28, $\nabla \bowtie_1 \nabla'$, and therefore $[\nabla]_{\bowtie_1} = [\nabla']_{\bowtie_1}$.

To see that h is surjective, note that for any Boolean algebras A and B, if A is a subalgebra of B then every ultrafilter F in A can be written as $G \cap A$ for some ultrafilter G in B. Indeed, F generates a filter in the algebra B and by the Ultrafilter Lemma, this filter can be extended to an ultrafilter G in B. Then $F = G \cap A$. Therefore the map f from the set of limited ultrafilters in R_1 to the set of limited ultrafilters in R_2 is surjective. It follows immediately that h is surjective. □

It remains to show that h preserves closed sets. In the remainder of this section, let C_1 be the mapping of members of R_1 to closed basis sets in $\mathcal{P}_{R_1}$, and C_2 be the mapping of members of R_2 to closed basis sets in $\mathcal{P}_{R_2}$ defined above. That is, for any $A \in R_i$, $i = 1, 2$,

$$C_i(A) = \{[\nabla]_{\bowtie_i} \in \mathcal{P}_{R_i} \mid A \in \bigcup[\nabla]_{\bowtie_i}\}$$

We begin by recalling the following useful lemma proved in Roeper [22].[16]

Lemma 30 *Let $A \in \Omega$. If ∇ is a limited ultrafilter in Ω and $A \bowtie \nabla$, then there exists a limited ultrafilter ∇' such that $A \in \nabla'$ and $\nabla' \bowtie \nabla$.*

Proposition 31 *For any $A \in R_1$, $h(C_1(A)) = \bigcap\{C_2(B) \mid B \in R_2$ and $A \leq B\}$*

Proof Suppose $x \in h(C_1(A))$. Then $x = h([\nabla]_{\bowtie_1})$ for some $[\nabla]_{\bowtie_1} \in C_1(A)$. Then $A \in \nabla'$ for some $\nabla' \in [\nabla]_{\bowtie_1}$. Suppose $B \in R_2$ and $A \leq B$. Then $B \in \nabla'$, since ∇' is an ultrafilter. So $B \in \nabla' \cap R_2 = f(\nabla')$ and $f(\nabla') \bowtie_2 f(\nabla)$. Thus $B \in \bigcup[f(\nabla)]_{\bowtie_2}$ and $x = [f(\nabla)]_{\bowtie_2} \in C_2(B)$. This shows that $h(C_1(A)) \subseteq \bigcap\{C_2(B) \mid B \in R_2$ and $A \leq B\}$.

For the reverse inclusion, suppose that $x \in \mathcal{P}_{R_2}$ and $x \notin h(C_1(A))$. Since h is surjective, $x = h([\nabla]_{\bowtie_1})$ for some limited ultrafilter ∇ in R_1. Then $[\nabla]_{\bowtie_1} \notin C_1(A)$, so $A \notin \bigcup[\nabla]_{\bowtie_1}$. Thus $A \notin \nabla'$ for all ∇' such that $\nabla' \bowtie_1 \nabla$. By Lemma 30, $A \not\bowtie_1 \nabla$. So there exists $C \in \nabla$ such that $C \not\bowtie_1 A$, and of course C is non-zero, since ∇ is an ultrafilter. Since ∇ is limited, we can assume WLOG that C is limited (else pick a limited $D \in \nabla$ and let $C' = C \wedge D$. Then C' is limited and $A \not\bowtie_1 C'$). Now $A \neq 1$ (else $A \bowtie_1 C$). So $-A \neq 0$. By Remark 19, there exists a non-zero $D \in R_2$ such that $C \not\bowtie_1 -D$ and $D \not\bowtie_1 A$. Therefore $A \leq -D$. We claim that $-D \notin \bigcup[f(\nabla)]_{\bowtie_2}$. (Suppose not. Then $-D \in \Gamma$ for some limited ultrafilter Γ in R_2 with $\Gamma \bowtie_2 f(\nabla)$. By surjectivity of f, $\Gamma = f(\nabla'')$ for some limited ultrafilter ∇'' in R_1. So $f(\nabla'') \bowtie_2 f(\nabla)$. By Lemma 28, $\nabla'' \bowtie_1 \nabla$. Clearly $-D \in \nabla''$, contradicting the fact that $-D \not\bowtie_1 C$.) So now we have: $x = h([\nabla]_{\bowtie_1}) = [f(\nabla)]_{\bowtie_2} \notin C_2(-D)$. We have shown that if $x \notin h(C_1(A))$, then $x \notin \bigcap\{C_2(B) \mid B \in R_2$ and $A \leq B\}$. □

Corollary 32 *For any $A \in R_2$, $h(C_1(A)) = C_2(A)$.*

Proof Note that if $A, B \in R_2$ and $A \leq B$, then $C_2(A) \subseteq C_2(B)$. Indeed, if $[\nabla]_{\bowtie_2} \in C_2(A)$, then $A \in \nabla'$ for some ∇' with $\nabla' \bowtie_2 \nabla$. Since ∇' is an ultrafilter, $B \in \nabla'$. So $B \in \bigcup[\nabla]_{\bowtie_2}$, and $[\nabla]_{\bowtie_2} \in C_2(B)$. Therefore, for any $A \in R_2$ we have: $\bigcap\{C_2(B) \mid B \in R_2$ and $A \leq B\} = C_2(A)$. The result now follows from Proposition 31. □

Proposition 33 *A set F is closed in $\mathcal{P}_{R_1}$ iff $h(F)$ is closed in $\mathcal{P}_{R_2}$.*

Proof Let $\mathbf{B_1}$ be the closed basis $\{C_1(A) \mid A \in R_1\}$ for $\mathcal{P}_{R_1}$, and let $\mathbf{B_2}$ be the closed basis $\{C_2(A) \mid A \in R_2\}$ for $\mathcal{P}_{R_2}$. It is sufficient to show that

[16] See Roeper [22], Lemma 2.7.

1. If $F \in \mathbf{B_1}$, then $h(F)$ is closed;
2. If $G \in \mathbf{B_2}$ then $h^{-1}(G)$ is closed.

For 1., note that if $F \in \mathbf{B_1}$, then $F = C_1(A)$ for some $A \in R_1$. By Proposition 31, $h(F) = \bigcap\{C_2(B) \mid B \in R_2 \text{ and } A \leq B\}$, and RHS is and intersection of closed sets, hence closed. For 2., note that if $G \in \mathbf{B_2}$, then $G = C_2(A)$ for some $A \in R_2$. By Corollary 32, $h^{-1}(G) = C_1(A)$, and RHS is closed. □

Proposition 34 *$\mathcal{P}_{R_1}$ is homeomorphic to $\mathcal{P}_{R_2}$.*

Proof Immediate from Lemmas 29 and 33. □

We showed that the pointy topology associated with RCN($\mathbb{R}^n$) is homeomoprhic to the pointy topology associated with RC($\mathbb{R}^n$). In fact, the topology associated with RC($\mathbb{R}^n$) is (up to homeomorphism) just $\mathbb{R}^n$.[17] Thus, both region-based topologies RCN($\mathbb{R}^n$) and RC($\mathbb{R}^n$) give rise to ordinary, finite-dimensional Euclidean space, $\mathbb{R}^n$.

8 Conclusion

The impossibility result proved in Russell [23] shows that there is no model of gunk that satisfies all of the features we might antecedently be interested in. Nevertheless, we showed that there is a nice model of gunk, RCN($\mathbb{R}^n$), that sits densely inside the standard model RC($\mathbb{R}^n$); this model satisfies all of Roeper's axioms (A_1) - (A_{10}), and thus is a region-based topology; it is atomless; and finally, Lebesgue measure is finitely (although not countably) additive over it. Moreover, RCN($\mathbb{R}^n$) is isomorphic to an interesting region-based topology that sits inside of Arntzenius's alternative model of gunk: namely, the algebra of clopens in B_n. In modeling space in RCN($\mathbb{R}^n$), we can distinguish between finite dimensions of any size: the region-based topologies RCN($\mathbb{R}^n$) and RCN($\mathbb{R}^m$) are not isomorphic for $n \neq m$. And in terms of points, we can recover from RCN($\mathbb{R}^n$) the pointy topology $\mathbb{R}^n$ by identifying points with equivalence classes of limited ultrafilters in RCN($\mathbb{R}^n$).

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[17] See Roeper [22], Theorem 5.7, p. 278.

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