Multimodal and Intuitionistic Logics in Simple Type Theory*

C. Benzmüller and L. C. Paulson

Articulate Software, Angwin, CA, U.S.
Computer Laboratory, University of Cambridge, Cambridge CB3 0FD, UK.

Abstract. We study straightforward embeddings of propositional normal multimodal logic and propositional intuitionistic logic in simple type theory. The correctness of these embeddings is easily shown. We give examples to demonstrate that these embeddings provide an effective framework for computational investigations of various non-classical logics. We report some experiments using the higher-order automated theorem prover LEO-II.

1 Introduction

There are two well investigated approaches to automate reasoning in modal logics: the direct approach and the translational approach. The direct approach [9, 10, 18, 28] develops specific calculi and tools for the task; the translational approach [29, 30] transforms modal logic formulas into first-order logic and applies standard first-order tools. Embeddings of modal logics into higher-order logic, however, have not yet been widely studied, although multimodal logic can be regarded as a natural fragment of simple type theory. Gallin [19] appears to mention the idea first. He presents an embedding of modal logic into a 2-sorted type theory. This idea is picked up by Gamut [20] and a related embedding has recently been studied by Hardt and Smolka [22]. Carpenter [16] proposes to use lifted connectives, an idea that also underlies the embeddings presented by Merz [27], Brown [15], Harrison [23, Chap. 20], and Kaminski and Smolka [25].

In this article we pick up and extend the embedding of multimodal logics in simple type theory as proposed by Brown [15]. The starting point is a characterization of multimodal logic formulas as particular λ-terms in simple type theory. A distinctive characteristic of the encoding is that the definiens of the $\Box_R$ operator λ-abstracts over the accessibility relation R. We prove this encoding sound and complete. Moreover, we illustrate that this encoding supports the formulation of meta properties of encoded multimodal logics such as the correspondence between certain axioms and properties of the accessibility relation R. We show that some of these meta properties can even be efficiently automated

* We thank Chad Brown for inspiring this work with his talk at Loria, Nancy in 2005. The research was supported by the Engineering and Physical Sciences Research Council [grant number EP/D070511/1] and by the German Research Foundation (DFG) [grant number BE 2501/6-1].

within our higher-order theorem prover LEO-II [14] via cooperation with the first-order automated theorem prover E [31]. We also discuss some challenges to higher-order reasoning implied by this application direction.

New in this article with regard to previous (non-reviewed) work [13, 14] are the soundness and completeness proofs for the embedding. So far this has only been proven for the monomodal logics S4 and T [11]. In the second part of this article we then combine our results with Gödel's interpretation [21] of propositional intuitionistic logic in modal logic $S4$ to obtain a sound and complete embedding of propositional intuitionistic logic in simple type theory.

2 Preliminaries

We assume familiarity with the syntax and semantics of multimodal logics and simple type theory and only briefly review the most important notions.

The multimodal logic language $\mathcal{MM}$ is defined by

$$s, t ::= p \mid \neg\, s \mid s \vee t \mid \Box_r\, s$$

where p denotes atomic primitives and r denotes accessibility relations (distinct from p). Other logical connectives can be defined from the chosen ones in the usual way.

A *Kripke frame* for $\mathcal{MM}$ is a pair $\langle W, (R_r)_{r\in S}\rangle$, where W is a non-empty set (called *possible worlds*), $S = \{1, \ldots, n\}$ is an index set and the R_r are binary relations on W (called *accessibility relations*). A *Kripke model* for $\mathcal{MM}$ is a triple $\langle W, (R_r)_{r\in S}, \models\rangle$, where $\langle W, (R_r)_{r\in S}\rangle$ is a Kripke frame, and $\models$ is a satisfaction relation between nodes of W and formulas of $\mathcal{MM}$ satisfying $w \models \neg\, s$ if and only if $w \not\models s$, $w \models s \vee t$ if and only if $w \models s$ or $w \models t$, $w \models \Box_r\, s$ if and only if for all u with $R_r(w, u)$ holds $u \models s$. The satisfaction relation $\models$ is uniquely determined by its value on the atomic primitives p. A formula s is *valid in a Kripke model* $\langle W, (R_r)_{r\in S}, \models\rangle$, if $w \models s$ for all $w \in W$. Also s is *valid in a Kripke frame* $\langle W, (R_r)_{r\in S}\rangle$ if it is valid in $\langle W, (R_r)_{r\in S}, \models\rangle$ for all possible $\models$. If s is valid for all possible Kripke frames $\langle W, (R_r)_{r\in S}\rangle$ then s is called *valid* and we write $\models^K s$.

Classical higher-order logic or *simple type theory* $\mathcal{STT}$ [4, 17] is a formalism built on top of the simply typed λ-calculus. The set $\mathcal{T}$ of simple types is usually freely generated from a set of basic types $\{o, \iota\}$ (where o denotes the type of Booleans) using the right-associative function type constructor $\rightarrow$.

The simple type theory language $\mathcal{STT}$ is defined by ($\alpha, \beta, o \in \mathcal{T}$):

$$\begin{aligned} s, t ::= {} & p_\alpha \mid X_\alpha \mid (\lambda X_\alpha . s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta}\, t_\alpha)_\beta \mid (\neg_{o\rightarrow o}\, s_o)_o \mid \\ & (s_o \vee_{o\rightarrow o\rightarrow o} t_o)_o \mid (\Pi_{(\alpha\rightarrow o)\rightarrow o}\, s_{\alpha\rightarrow o})_o \end{aligned}$$

Here p_α denotes typed constants and X_α typed variables (distinct from p_α). Complex typed terms are constructed via abstraction and application. Our logical connectives of choice are $\neg_{o\rightarrow o}$, $\vee_{o\rightarrow o\rightarrow o}$ and $\Pi_{(\alpha\rightarrow o)\rightarrow o}$ (for each type α). From these connectives, other logical connectives can be defined in the usual way. We often use binder notation $\forall X_\alpha . s$ for $(\Pi_{(\alpha\to o)\to o}(\lambda X_\alpha . s_o))$. We denote *substitution* of a term A_α for a variable X_α in a term B_β by $[A/X]B$. Since we consider α-conversion implicitly, we assume the bound variables of B avoid variable capture. Two common relations on terms are given by β-reduction and η-reduction. A β-redex has the form $(\lambda X . s)t$ and β-reduces to $[t/X]s$. An η-redex has the form $(\lambda X . sX)$ where the variable X is not free in s; it η-reduces to s. We write $s =_\beta t$ to mean s can be converted to t by a series of β-reductions and expansions. Similarly, $s =_{\beta\eta} t$ means s can be converted to t using both β and η. For each $s \in \mathcal{STT}$ there is a unique *β-normal form* and a unique *$\beta\eta$-normal form*.

The semantics of $\mathcal{STT}$ is well understood and thoroughly documented in the literature [1, 2, 12, 24]; our summary below is adapted from Andrews [5].

A *frame* is a collection $\{D_\alpha\}_{\alpha\in\mathcal{T}}$ of nonempty domains (sets) D_α, such that $D_o = \{T, F\}$ (where T represents truth and F represents falsehood). The $D_{\alpha\to\beta}$ are collections of functions mapping D_α into D_β. The members of D_ι are called *individuals*. An *interpretation* is a tuple $\langle\{D_\alpha\}_{\alpha\in\mathcal{T}}, I\rangle$ where function I maps each typed constant c_α to an appropriate element of D_α, which is called the *denotation* of c_α (the logical symbols $\neg$, $\vee$ and Π are always given the standard denotations). A *variable assignment* ϕ maps variables X_α to elements in D_α. An interpretation $\langle\{D_\alpha\}_{\alpha\in\mathcal{T}}, I\rangle$ is a *Henkin model* (equivalently, a *general model*) if and only if there is a binary function $\mathcal{V}$ such that $\mathcal{V}_\phi\, s_\alpha \in D_\alpha$ for each variable assignment ϕ and term $s_\alpha \in L$, and the following conditions are satisfied for all ϕ and all $s, t \in L$: (a) $\mathcal{V}_\phi X_\alpha = \phi X_\alpha$, (b) $\mathcal{V}_\phi\, p_\alpha = I p_\alpha$, (c) $\mathcal{V}_\phi(s_{\alpha\to\beta}\, t_\alpha) = (\mathcal{V}_\phi\, s_{\alpha\to\beta})(\mathcal{V}_\phi t_\alpha)$, and (d) $\mathcal{V}_\phi(\lambda X_\alpha . s_\beta)$ is that function from D_α into D_β whose value for each argument $z \in D_\alpha$ is $\mathcal{V}_{[z/X_\alpha],\phi} s_\beta$, where $[z/X_\alpha], \phi$ is that variable assignment such that $([z/X_\alpha], \phi)X_\alpha = z$ and $([z/X_\alpha], \phi)Y_\beta = \phi Y_\beta$ if $Y_\beta \neq X_\alpha$. Since $I\neg$, $I\vee$, and $I\Pi$ always denote the standard truth functions, we have $\mathcal{V}_\phi\,(\neg s) = T$ iff $\mathcal{V}_\phi\, s = F$, $\mathcal{V}_\phi\,(s \vee t) = T$ iff $\mathcal{V}_\phi\, s = T$ or $\mathcal{V}_\phi\, t = T$, and $\mathcal{V}_\phi\,(\forall X_\alpha . s_o) = \mathcal{V}_\phi\,(\Pi^\alpha(\lambda X_\alpha . s_o)) = T$ iff for all $z \in D_\alpha$ we have $\mathcal{V}_{[z/X_\alpha],\phi}\, s_o = T$. Moreover, we have $\mathcal{V}_\phi\, s = \mathcal{V}_\phi\, t$ whenever $s =_{\beta\eta} t$; in order to emphasize this correspondence we sometimes write $\mathcal{V}_\phi\, s =_{\beta\eta} \mathcal{V}_\phi\, t$.

If an interpretation $\langle\{D_\alpha\}_{\alpha\in\mathcal{T}}, I\rangle$ is a Henkin model, then the function $\mathcal{V}_\phi$ is uniquely determined. An interpretation $\langle\{D_\alpha\}_{\alpha\in\mathcal{T}}, I\rangle$ is a *standard model* if and only if for all α and β, $D_{\alpha\to\beta}$ is the set of all functions from D_α into D_β. Each standard model is also a Henkin model.

We say that formula $A \in L$ is *valid in a model* $\langle\{D_\alpha\}_{\alpha\in\mathcal{T}}, I\rangle$ if and only if $\mathcal{V}_\phi A = T$ for every variable assignment ϕ. A model for a set of formulas H is a model in which each formula of H is valid.

A formula A is Henkin-valid (resp., standard-valid) if and only if A is valid in every Henkin (resp., standard) model. Clearly each formula which is Henkin-valid is also standard-valid, but the converse of this statement is false. We write $\models^{\mathcal{STT}} A$ if A is Henkin-valid and we write $\Gamma \models^{\mathcal{STT}} A$ if A is valid in all Henkin models in which all formulas of Γ are valid.

3 Propositional Normal Multimodal Logics in Simple Type Theory

Simple type theory is an expressive logic and it is thus no surprise that modal logic can be encoded in several ways in it. Harrison [23], for instance, presents a 'deep embedding' of modal logics by formalizing standard Kripke semantics and a 'shallow embedding' of the temporal logic LTL. The latter encoding more naturally exploits the expressiveness of higher-order logic. Harrison's shallow embedding is an instance of the encoding due to Brown [15]. Here we adapt and further extend Brown's suggestion and show that this approach is well suited for reasoning within and about modal logics.

The idea of the encoding is simple: Choose a base type — we choose ι — to denote the set of all possible worlds. Certain formulas of type $\iota \to o$ then correspond to multimodal logic expressions. The multimodal connectives $\neg$, $\vee$, and $\Box_r$ become λ-terms of types $(\iota \to o) \to (\iota \to o)$, $(\iota \to o) \to (\iota \to o) \to (\iota \to o)$, and $(\iota \to \iota \to o) \to (\iota \to o) \to (\iota \to o)$ respectively. Note that $\neg$ forms the complement of a set of worlds, while $\vee$ forms the union of two such sets. Our encoding actually only exploits the first-order fragment of simple type theory enhanced with lambda-notation. Some examples below additionally employ quantification over relations.

Definition 1 (Propositional Multimodal Logic $\mathcal{MM}^{\mathcal{STT}}$). *Let $\mathcal{MM}$ be a propositional multimodal logic with atomic primitives $p^1, \ldots, p^m$ $(m \geq 1)$ and box-operators $\Box_{r^1}, \ldots, \Box_{r^n}$ $(n \geq 1)$ for accessibility relations $r^1, \ldots, r^n$.*

We define the set $\mathcal{MM}^{\mathcal{STT}}$ of corresponding propositional multimodal logic propositions in $\mathcal{STT}$ as follows.

1. *For the atomic primitives $p^1, \ldots, p^m$ we introduce corresponding predicate constants $p^1_{\iota\to o}, \ldots, p^m_{\iota\to o}$ and for the accessibility relations $r^1, \ldots, r^n$ we provide corresponding relation constants $r^1_{\iota\to\iota\to o}, \ldots, r^n_{\iota\to\iota\to o}$.*
2. *We introduce the logical connectives of $\mathcal{MM}^{\mathcal{STT}}$ as abbreviations for the following λ-terms:*

$$\begin{aligned} \neg_{(\iota\to o)\to(\iota\to o)} &= \lambda A_{\iota\to o}.\lambda X_\iota.\neg A\,X \\ \vee_{(\iota\to o)\to(\iota\to o)\to(\iota\to o)} &= \lambda A_{\iota\to o}.\lambda B_{\iota\to o}.\lambda X_\iota.A\,X \vee B\,X \\ \Box_{(\iota\to\iota\to o)\to(\iota\to o)\to(\iota\to o)} &= \lambda R_{\iota\to\iota\to o}.\lambda A_{\iota\to o}.\lambda X_\iota.\forall Y_\iota.\neg R\,X\,Y \vee A\,Y \end{aligned}$$

3. *We define the set of $\mathcal{MM}^{\mathcal{STT}}$-propositions as the smallest set of simply typed λ-terms for which the following hold:*
   - *The predicate constants $p^1_{\iota\to o}, \ldots, p^m_{\iota\to o}$ define the atomic $\mathcal{MM}^{\mathcal{STT}}$-propositions.*
   - *If ϕ and ψ are $\mathcal{MM}^{\mathcal{STT}}$-propositions, then so are $\neg\,\phi$, $\phi \vee \psi$ and $(\Box\, r^i_{\iota\to\iota\to o})\,\phi$, where $\neg$, $\vee$, and $\Box$ are defined as above and where $r^i_{\iota\to\iota\to o}$ for $1 \leq i \leq n$ is a relation constant. (In the following we write $\Box_{r^i_{\iota\to\iota\to o}}$ instead of $(\Box\, r^i_{\iota\to\iota\to o})$.)*
4. *The propositional multimodal logic operators $\supset$, $\Leftrightarrow$, $\Diamond_r$, etc. can be defined in terms of $\neg$, $\vee$ and $\Box_r$ in the usual way.*

Note that the encoding of the modal operator $\Box_r$ depends explicitly on an accessibility relation r of type $\iota \to \iota \to o$ given as its first argument. Hence, we basically introduce a generic framework for modeling multimodal logics. This idea is where Brown [15] differs from the LTL encoding of Harrison. The latter chooses the interpreted type *num* of numerals and then uses the predefined relation $\leq$ over numerals as a fixed accessibility relation in the definitions of $\Box$ and $\Diamond$.

By making the dependency of $\Box_r$ and $\Diamond_r$ on the accessibility relation R explicit, we can formalize and automatically prove some properties of multimodal logics in simple type theory, as we will illustrate later in Example 4.

Example 1. Given a multimodal logic $\mathcal{MM}$ with an atomic proposition a and box operators $\Box_r$ and $\Box_s$. The $\mathcal{MM}$ proposition[1] $\Box_s (\Box_r a \supset \Box_r a)$ is translated into the corresponding $\mathcal{MM}^{\mathcal{STT}}$ term $\Box_s (\Box_r a \supset \Box_r a)$ for constant symbols $s_{\iota \to \iota \to o}, r_{\iota \to \iota \to o}$ and $a_{\iota \to o}$. By unfolding the abbreviations and by $\beta\eta$-reduction we obtain

$$\lambda X_\iota . \forall Y_\iota . \neg s\, X\, Y \vee (\neg(\forall Z_\iota . \neg r\, Y\, Z \vee a\, Z) \vee (\forall Z_\iota . \neg r\, Y\, Z \vee a\, Z))$$

of type $\iota \to o$.

Next, we define validity of modal logic expressions $A_{\iota \to o} \in \mathcal{MM}^{\mathcal{STT}}$: the formula A is valid iff for all possible worlds W_ι we have $W \in A$, that is, iff $A\,W$ holds.

Definition 2 (Validity). *Validity is modeled as an abbreviation for the following simply typed λ-term:*

$$\texttt{valid} := \lambda A_{\iota \to o} . \forall W_\iota . A\, W$$

Note that we could define validity also as $\texttt{valid} := \Pi_{(\iota \to o) \to o}$.

Example 2 (Ex. 1 contd.). The validity statement for the multimodal logic formula $\Box_s (\Box_r a \supset \Box_r a)$ is transformed into the $\mathcal{MM}^{\mathcal{STT}}$ formula $(\texttt{valid}\ (\Box_s (\Box_r a \supset \Box_r a)))$. By unfolding the abbreviations and by $\beta\eta$-reduction we obtain

$$\forall W_\iota . \forall Y_\iota . \neg s\, W\, Y \vee (\neg(\forall Z_\iota . \neg r\, Y\, Z \vee a\, Z) \vee (\forall Z_\iota . \neg r\, Y\, Z \vee a\, Z)).$$

It is easy to verify that this is a tautology in $\mathcal{STT}$.

3.1 Soundness and Completeness

In our soundness proof we exploit the following mapping of Kripke frames into Henkin models.

[1] We assume that $\Box_r$ binds more strongly than the propositional connectives, and hence, $\Box_r a \supset \Box_r a$ stands for $(\Box_r a) \supset (\Box_r a)$.

Definition 3 (Henkin model M^K for Kripke model K). *Let $p^1, \ldots, p^m$ be the atomic primitives occuring in modal language $\mathcal{MM}$. Furthermore, let $\Box_{r_1}, \ldots, \Box_{r_n}$ be the box operators for accessibility relations $r_1, \ldots, r_n$ in $\mathcal{MM}$. Note that the p^j $(1 \leq j \leq m)$ are mapped to predicate constants $p^j_{\iota \to o}$ and the r^i $(1 \leq i \leq n)$ to relation constants $r^i_{\iota \to \iota \to o}$. These are the only constant symbols provided in $\mathcal{MM}^{\mathcal{STT}}$.*

Now, given a Kripke model $K = \langle W, (R_r)_{r \in S}, \models \rangle$, the corresponding Henkin model $M^K = \langle \{D_\alpha\}_{\alpha \in \mathcal{T}}, I \rangle$ is defined as follows. We choose the set of individuals D_ι as the set of possible worlds W and we choose the $D_{\alpha \to \beta}$ as the set of all functions from D_α to D_β. Furthermore, for $1 \leq j \leq m$ we choose $Ip^j_{\iota \to o} \in D_{\iota \to o}$ such that $(Ip^j_{\iota \to o})(w) = T$ for all worlds $w \in D_\iota$ with $w \models p^j$ in Kripke model K and $(Ip^j_{\iota \to o})(w) = F$ otherwise. Similarly, we choose $Ir^i_{\iota \to \iota \to o} \in D_{\iota \to \iota \to o}$ such that $(Ir^i_{\iota \to \iota \to o})(w, w') = T$ if $R_{r^i}(w, w')$ in Kripke model K and $(Ir^i_{\iota \to \iota \to o})(w, w') = F$ otherwise. It is easy to check that $M^K = \langle \{D_\alpha\}_{\alpha \in \mathcal{T}}, I \rangle$ is a Henkin model. In fact it is a standard model since the function spaces are full.

Lemma 1. *Let $\mathcal{MM}$ be a multimodal language and $\mathcal{MM}^{\mathcal{STT}}$ its corresponding logic in $\mathcal{STT}$. Let $q \in \mathcal{MM}$ be arbitrary and let $q_{\iota \to o}$ be the corresponding term in $\mathcal{MM}^{\mathcal{STT}}$. Furthermore, let $K = \langle W, (R_r)_{r \in S}, \models \rangle$ be a Kripke model for $\mathcal{MM}$ and let $M^K = \langle \{D_\alpha\}_{\alpha \in \mathcal{T}}, I \rangle$ be the corresponding Henkin model for K. For all worlds $w \in W$ and variable assignments ϕ we have $w \models q$ in K if and only if $\mathcal{V}_{[w/X_\iota],\phi}\,(q_{\iota \to o} X_\iota) = T$ in M^K for all variables X_ι.*

Proof. The proof is by induction on the structure of $q \in \mathcal{MM}$. Let $q = p$ for some atomic primitive $p \in \mathcal{MM}$. By construction of M^K, we have $\mathcal{V}_{[w/X_\iota],\phi}\,(p_{\iota \to o}\, X_\iota) = (I\, p_{\iota \to o})(w) = T$ if and only if $w \models p$. Let $q = \neg\, s$ for $s \in \mathcal{MM}$. We have $w \models \neg s$ if and only if $w \not\models s$, which is equivalent by induction to $\mathcal{V}_{[w/X_i],\phi}\,(s_{\iota \to o}\, X_\iota) = F$ and hence to $\mathcal{V}_{[w/X_\iota],\phi}\, \neg(s_{\iota \to o}\, X_\iota) =_{\beta\eta} \mathcal{V}_{[w/X_\iota],\phi}\,((\neg\, s_{\iota \to o})\, X_\iota) = T$. Let $q = (s \vee t)$ for $s, t \in \mathcal{MM}$. We have $w \models (s \vee t)$ if and only if $w \models s$ or $w \models t$. The latter condition is equivalent by induction to $\mathcal{V}_{[w/X_\iota],\phi}\,(s_{\iota \to o}\; X_\iota) = T$ or $\mathcal{V}_{[w/X_\iota],\phi}\,(t_{\iota \to o}\; X_\iota) = T$ and therefore to $\mathcal{V}_{[w/X_\iota],\phi}\,(s_{\iota \to o}\; X_\iota) \vee (t_{\iota \to o}\, X_\iota) =_{\beta\eta} \mathcal{V}_{[w/X_\iota],\phi}\,((s_{\iota \to o}\; \vee\, t_{\iota \to o})\; X_\iota) = T$. Let $q = \Box_r\, s$ for $s \in \mathcal{MM}$. We have $w \models \Box_r\, s$ if and only if for all u with $R_r(w, u)$ we have $u \models s$. The latter condition is equivalent by induction to this one: for all u with $R_r(w, u)$ we have $\mathcal{V}_{[u/V_\iota],\phi}\,(s_{\iota \to o}\, V_\iota) = T$. That is equivalent to $\mathcal{V}_{[u/V_\iota],[w/X_\iota],\phi}\,(\neg(r_{\iota \to \iota \to o}\, X_\iota\, V_\iota) \vee (s_{\iota \to o}\, V_\iota)) = T$ and thus to $\mathcal{V}_{[w/X_\iota],\phi}\,(\forall Y_\iota.\; (\neg(r_{\iota \to \iota \to o}\, X_\iota\, Y_\iota) \vee (s_{\iota \to o}\, Y_\iota))) =_{\beta\eta} \mathcal{V}_{[w/X_\iota],\phi}\,((\Box_{r_{\iota \to \iota \to o}}\, s_{\iota \to o})\, X_\iota) = T$.

We exploit this result to prove the soundness of our embedding of propositional multimodal logics into $\mathcal{STT}$.

Theorem 1 (Soundness of Embedding $\mathcal{MM}^{\mathcal{STT}}$). *Suppose that $\mathcal{MM}$ is a multimodal language and $\mathcal{MM}^{\mathcal{STT}}$ its corresponding logic in $\mathcal{STT}$. Let $s \in \mathcal{MM}$ be a multimodal logic proposition and let $s_{\iota \to o}$ be the corresponding term in $\mathcal{MM}^{\mathcal{STT}}$. If $\models^{\mathcal{STT}} (\mathtt{valid}\, s_{\iota \to o})$ then $\models s$.*

Proof. The proof is by contraposition. For this, assume $\not\models s$, that is, there is a Kripke model $K = \langle W, (R_r)_{r \in S}, \models \rangle$ with $w \not\models s$ for some $w \in W$. By Lemma

1, for arbitrary ϕ we have $\mathcal{V}_{[w/W_\iota],\phi}\,(s_{\iota\to o}\,W_\iota) = F$ in Henkin model M^K for K. Thus, $\mathcal{V}_\phi\,(\forall W_\iota \centerdot (s_{\iota\to o}\,W)) =_{\beta\eta} \mathcal{V}_\phi\,(\mathtt{valid}\,s_{\iota\to o}) = F$. Hence, $\not\models^{\mathcal{STT}}\,(\mathtt{valid}\,s_{\iota\to o})$.

In order to prove completeness, we reverse our mapping from Henkin models to Kripke models.

Definition 4 (Kripke Model K^M for Henkin model M). *Let $p^1, \ldots, p^m$ be the atomic primitives occuring in the modal language $\mathcal{MM}$. Furthermore, let $\Box_{r_1}, \ldots, \Box_{r_n}$ be the box operators for accessibility relations $r_1, \ldots, r_n$ in $\mathcal{MM}$. Remember that the p^j $(1 \leq j \leq m)$ are mapped to predicate constants $p^j_{\iota\to o}$ and the r^i $(1 \leq i \leq n)$ to relation constants $r^i_{\iota\to\iota\to o}$. Except for these, there are no other constant symbols given in language $\mathcal{MM}^{\mathcal{STT}}$.*

Now, let Henkin model $M = \langle \{D_\alpha\}_{\alpha\in\mathcal{T}}, I\rangle$ be given. The Kripke model $K^M = \langle W, (R_r)_{r\in S}, \models\rangle$ for M is defined as follows: We choose the set of worlds W as the set of individuals D_ι. Moreover, we choose $\models$ such that $w \models p^j$ in K^M if $(Ip^j_{\iota\to o})(w) = T$ in M and $w \not\models p^i$ otherwise. Similarly, we choose R_{r^i} such that $w\,R_{r^i}\,w'$ in K^M if $(Ir^i_{\iota\to\iota\to o})(w,w') = T$ in M and $\neg(w\,R_{r^i}\,w')$ otherwise. It is easy to check that K^M is a Kripke model.

Lemma 2. *Let $K^M = \langle W, (R_r)_{r\in S}, \models\rangle$ be a Kripke model for Henkin model $M = \langle \{D_\alpha\}_{\alpha\in\mathcal{T}}, I\rangle$. Furthermore, let $q \in \mathcal{MM}$ be a multimodal logic proposition and let $q_{\iota\to o}$ be the corresponding term in $\mathcal{MM}^{\mathcal{STT}}$. For all worlds $w \in W$ and variable assignments ϕ, we have $w \models q$ if and only if $\mathcal{V}_{[w/X_\iota],\phi}\,(q_{\iota\to o}\,X_\iota) = T$ for all variables X_ι.*

Proof. The proof is by induction on the structure of $q_{\iota\to o} \in \mathcal{MM}^{\mathcal{STT}}$. Let $q = p_{\iota\to o}$ for some predicate constant $p_{\iota\to o} \in \mathcal{MM}^{\mathcal{STT}}$. By definition, $p_{\iota\to o}$ corresponds to atomic primitive $p \in \mathcal{MM}$. By construction of Kripke model K^M we have $w \models p$ if and only if $\mathcal{V}_{[w/X_\iota],\phi}\,(p_{\iota\to o}\,X_\iota) = I(p_{\iota\to o})(w) = T$. Let $q = \neg\, s_{\iota\to o}$ for $s_{\iota\to o} \in \mathcal{MM}^{\mathcal{STT}}$. Since K^M is a Kripke model, $w \models \neg\, s$ if and only if $w \not\models s$. By induction, $w \not\models s$ if and only if $\mathcal{V}_{[w/X_\iota],\phi}\,(s_{\iota\to o}\,X_\iota) = F$. Since M is a Henkin model this is equivalent to $\mathcal{V}_{[w/X_\iota],\phi}\,(\neg(s_{\iota\to o}\,X_\iota)) =_{\beta\eta} \mathcal{V}_{[w/X_\iota],\phi}\,((\neg\, s_{\iota\to o})\,X_\iota) = T$. Let $q = s_{\iota\to o} \vee t_{\iota\to o}$ for $s_{\iota\to o}, t_{\iota\to o} \in \mathcal{MM}^{\mathcal{STT}}$. K^M is a Kripke model, so $w \models s \vee t$ is equivalent to $w \models s$ or $w \models t$. By induction, $w \models s$ or $w \models t$ is equivalent to $\mathcal{V}_{[w/X_\iota],\phi}\,(s_{\iota\to o}\,X_\iota) = T$ or $\mathcal{V}_{[w/X_\iota],\phi}\,(t_{\iota\to o}\,X_\iota) = T$, and, since M is a Henkin model, to $\mathcal{V}_{[w/X_\iota],\phi}\,((s_{\iota\to o}\,X_\iota) \vee (s_{\iota\to o}\,X_\iota)) =_{\beta\eta} \mathcal{V}_{[w/X_\iota],\phi}\,((\,s_{\iota\to o}\,\vee\,t_{\iota\to o})\,X_\iota) =_{\beta\eta} T$. Let $q = \Box_{r_{\iota\to\iota\to o}}\,s_{\iota\to o}$ for $s_{\iota\to o} \in \mathcal{MM}^{\mathcal{STT}}$ and accessibility relation constant $r_{\iota\to\iota\to o}$. $r_{\iota\to\iota\to o}$ corresponds to r resp. R_r, that is, for all u we have $w\,R_r\,u$ if and only if $(I\,r_{\iota\to\iota\to o})(w,u) = T$. K^M is a Kripke model, so $w \models \Box_r\,s$ if and only if for all u with $R_r(w,u)$ we have $u \models s$. By induction and the above correspondence, this is equivalent to the following: for all u with $(I\,r_{\iota\to\iota\to o})(w,u) = T$ we have $\mathcal{V}_{[u/V_\iota],\phi}\,(s_{\iota\to o}\,V_\iota) = T$. This is equivalent to the statement for all u we have $\mathcal{V}_{[u/V_\iota],[w/X_\iota],\phi}\,(\neg(r_{\iota\to\iota\to o}\,X_\iota\,V_\iota) \vee (s_{\iota\to o}\,V_\iota)) = T$, and hence to $\mathcal{V}_{[w/X_\iota],\phi}\,(\forall Y_\iota \centerdot (\neg(r_{\iota\to\iota\to o}\,X_\iota\,Y_\iota) \vee (s_{\iota\to o}\,Y_\iota))) =_{\beta\eta} \mathcal{V}_{[w/X_\iota],\phi}\,((\Box_{r_{\iota\to\iota\to o}}\,s_{\iota\to o})\,X_\iota) = T$.

Theorem 2 (Completeness of Embedding $\mathcal{MM}^{\mathcal{STT}}$). *Let $s \in \mathcal{MM}$ be a monomodal logic proposition and let $s_{\iota\to o}$ be the corresponding term in $\mathcal{MM}^{\mathcal{STT}}$. If $\models s$ then $\models^{\mathcal{STT}}\,(\mathtt{valid}\,s_{\iota\to o})$.*

Proof. The proof is by contraposition. Assume $\not\models^{\mathcal{STT}}$ $(\texttt{valid}\, s_{\iota\to o})$, that is, for a Henkin model $M = \langle \{D_\alpha\}_{\alpha\in\mathcal{T}}, I\rangle$ and a variable assignment ϕ we have $\mathcal{V}_\phi\,(\texttt{valid}\, s_{\iota\to o}) = F$ in M. This implies that there is some $w \in D_i$ such that $\mathcal{V}_{[w/W_\iota],\phi}\,(s_{\iota\to o}\, W_\iota) = F$ in M. By Lemma 2 we know that $w \not\models s$ in Kripke model $K^M = \langle W, (R_r)_{r\in S}, \models\rangle$ for M. Hence, $\not\models s$.

3.2 Reasoning in and about Propositional Normal Multimodal Logics

A prominent monomodal logic is logic $S4$. For modeling $S4$ in our framework we consider a single $\Box_r$-operator, and therefore one single accessibility relation r. In $S4$ the accessibility relation r is is required to be reflexive and transitive. These semantic properties of r correspond to the well known axioms $\Box_r\, a \supset a$ and $\Box_r\, a \supset \Box_r\, \Box_r\, a$. Any proof problem $t \in \mathcal{MM}$ for modal logic $S4$ can thus be translated using our embedding into the following proof problem t' in $\mathcal{STT}$:

$$t' := (\forall A_{\iota\to o}\centerdot \texttt{valid}\, \Box_r\, A \supset A) \wedge (\forall A_{\iota\to o}\centerdot \texttt{valid}\, \Box_r\, A \supset \Box_r\, \Box_r\, A) \supset \texttt{valid}\, t_{\iota\to o}$$

Similarly, we can model other normal multimodal and monomodal logics in simple type theory. Hence, we can exploit off-the-shelf automated higher-order theorem provers such as our LEO-II [14] as generic reasoners for reasoning in these logics. For simple problems, the performance results of LEO-II are encouraging [13]. For instance, Example 2 can be proven automatically in less than 0.1 seconds on a standard notebook computer and similar performance results can be achieved with the prover TPS [6, 7].

We can even use higher-order theorem provers to investigate meta-theoretic properties of various modal logics automatically. This issue has been studied in our previous work [13]; here we give two examples.

Example 3. The equivalence between axioms $\Box_r\, a \supset a$ and $\Box_r\, a \supset \Box_r\, \Box_r\, a$ and the reflexivity and transitivity properties of the accessibility relation r is encoded as the following proof problem in $\mathcal{STT}$:

$$\begin{aligned}\forall R_{\iota\to\iota\to o}\centerdot\ & ((\forall A_{\iota\to o}\centerdot \texttt{valid}\ \Box_R\, A \supset A) \\ & \quad \wedge(\forall A_{\iota\to o}\centerdot \texttt{valid}\ \Box_R\, A \supset \Box_R\, \Box_R\, A)) \\ & \Leftrightarrow (\texttt{refl}\ R \wedge \texttt{trans}\ R)\end{aligned}$$

where **refl** and **trans** are abbreviations for the terms $\lambda R_{\iota\to\iota\to o}\centerdot \forall X_\iota\centerdot R\, X\, X$ and $\lambda R_{\iota\to\iota\to o}\centerdot \forall X_\iota\centerdot \forall Y_\iota\centerdot \forall Z_\iota\centerdot R\, X\, Y \wedge R\, Y\, Z \Rightarrow R\, X\, Z$ resp. LEO-II can prove this well known modal logic meta-level problem in less than 0.3 seconds.

As example Example 3 confirms, we can translate a proof problem $t \in \mathcal{MM}$ for modal logic $S4$ alternatively into a problem t'' in $\mathcal{STT}$ of the following form:

$$t'' := (\texttt{refl}\, r) \wedge (\texttt{trans}\, r) \supset \texttt{valid}\, t_{\iota\to o}$$

Example 4. We can exploit our higher-order framework to study questions such as, *Is the axiom $\Box_r\, a \supset a$ valid in basic modal logic K for arbitrary accessibility relations r?*

This question is encoded as $\forall R. \forall A.\, \texttt{valid}\ \Box_R\, A \supset A$. As expected, LEO-II fails to prove this problem. But we may also formalize the question,

Is there a relation r such that for all modal propositions a, axiom $\Box_r\, a \supset a$ is valid in K?

This is encoded as $\exists R. \forall A.\, \texttt{valid}\ \Box_R\, A \supset A$. LEO-II can solve this problem in 3.0 seconds. A clever instantiation for relation R is actually needed to solve this problem. (R obviously needs to be reflexive.) This instantiation cannot be synthesized in LEO-II by higher-order pre-unification. In fact, LEO-II needs to guess an appropriate instantiation by applying primitive substitutions (also called set instantiations) [3] and what LEO-II essentially proposes based on primitive substitutions is to consider the universal relation as a candidate relation (a predecessor version of LEO-II suggested the equality relation [13]).

It is no surprise that LEO-II can also prove in 0.2 seconds the correspondence of the reflexivity axiom and the reflexivity property of the accessibility relation.

Example 4 illustrates that our embedding is generally suited to support the computational exploration of multimodal logics and their properties within a uniform framework. However, non-trivial challenges are raised; for effectively answering questions as illustrated in Example 4 within a higher-order automated theorem prover, further progress is required for handling primitive substitutions and set instantiations. So far, primitive substitutions blindly guess some logical structure for free predicate or set variables in a clause that cannot be synthesized otherwise, and they introduce new free variables in order to delay some further decisions. The instantiation of the new and the remaining free variables is ideally supported by higher-order pre-unification. Generally, however, the primitive substitution process has to be iterated which leads to very challenging search space for clause sets containing many free variables.

4 Propositional Intuitionistic Logic in Simple Type Theory

In this section we combine Gödels interpretation of propositional intuitionistic logic in propositional modal logic $S4$ [21] with our results from the previous section in order to provide a sound and complete embedding of propositional intuitionistic logic into simple type theory.

Gödel studies the propositional intuitionistic logic $\mathcal{IPL}$ defined by

$$s, t ::= p \mid \dot{\neg}\, s \mid s \mathbin{\dot{\supset}} t \mid s \mathbin{\dot{\vee}} t \mid s \mathbin{\dot{\wedge}} t$$

He introduces a mapping from $\mathcal{IPL}$ into propositional modal logic $S4$ which maps $\dot{\neg}\, s$ to $\neg\, \Box_r\, s$, $s \mathbin{\dot{\supset}} t$ to $\Box_r\, s \supset \Box_r\, t$, $s \mathbin{\dot{\vee}} t$ to $\Box_r\, s \vee \Box_r\, t$, and $s \mathbin{\dot{\wedge}} t$ to $s \wedge t$. (Alternative mappings have been proposed and studied in the literature which we could employ here equally as well.) By simply combining Gödel's mapping with our mapping from before we obtain the following embedding of $\mathcal{IPL}$ in simple type theory.

Definition 5 (Propositional Intuitionistic Logic $\mathcal{IPL}^{\mathcal{STT}}$). *Let $\mathcal{IPL}$ be a propositional intuitionistic logic with atomic primitives $p^1, \ldots, p^m$ $(m \geq 1)$.*

We define the set $\mathcal{IPL}^{\mathcal{STT}}$ of corresponding propositional intuitionistic logic propositions in $\mathcal{STT}$ as follows.

1. *For the atomic primitives $p^1, \ldots, p^m$ we introduce corresponding predicate constants $\mathrm{p}^1_{\iota\to o}, \ldots, \mathrm{p}^m_{\iota\to o}$. Moreover, we provide the single accessibility relation constant $r_{\iota\to\iota\to o}$.*
2. *Corresponding to Gödel's mapping we introduce the logical connectives of $\mathcal{IPL}^{\mathcal{STT}}$ as abbreviations for the following λ-terms (we omit the types here):*

$$\begin{aligned}
\dot{\neg} &= \lambda A. \lambda X. \forall Y. \neg r\, X\, Y \vee A\, Y \\
\dot{\supset} &= \lambda A. \lambda B. \lambda X. \neg(\forall Y. \neg r\, X\, Y \vee A\, Y) \vee (\forall Y. \neg r\, X\, Y \vee B\, Y) \\
\dot{\vee} &= \lambda A. \lambda B. \lambda X. (\forall Y. \neg r\, X\, Y \vee A\, Y) \vee (\forall Y. \neg r\, X\, Y \vee B\, Y) \\
\dot{\wedge} &= \lambda A. \lambda B. \lambda X. \neg(\neg A\, X \vee \neg B\, X)
\end{aligned}$$

3. *We define the set of $\mathcal{IPL}^{\mathcal{STT}}$-propositions as the smallest set of simply typed λ-terms for which the following hold:*
   - *The predicate constants $\mathrm{p}^1_{\iota\to o}, \ldots, \mathrm{p}^m_{\iota\to o}$ define the atomic $\mathcal{MM}^{\mathcal{STT}}$-propositions.*
   - *If ϕ and ψ are $\mathcal{IPL}^{\mathcal{STT}}$-propositions, then so are $\dot{\neg}\, \phi$, $\phi \,\dot{\supset}\, \psi$, $\phi \,\dot{\vee}\, \psi$, and $\phi \,\dot{\wedge}\, \psi$.*

The notion of validity we adopt is the same as given before in Definition 2. However, since Gödel connects $\mathcal{IPL}$ with modal logic $S4$, we transform each proof problem $t \in \mathcal{IPL}$ into a corresponding proof problem t' in $\mathcal{STT}$ of the following form

$$t' := (\forall A_{\iota\to o}. \mathtt{valid}\, \Box_r\, A \supset A) \wedge (\forall A_{\iota\to o}. \mathtt{valid}\, \Box_r\, A \supset \Box_r\, \Box_r\, A) \supset \mathtt{valid}\, t_{\iota\to o}$$

where $t_{\iota\to o}$ is the $\mathcal{IPL}^{\mathcal{STT}}$ term for t according to Definition 5. Alternatively we may translate t into t'':

$$t'' := (\mathtt{refl}\, r) \wedge (\mathtt{trans}\, r) \supset \mathtt{valid}\, t_{\iota\to o}$$

Combining soundness [21] and completeness [26] of Gödel's embedding with our soundness and completeness theorems 1 and 2 we obtain the following corollary:

Corollary 1 (Soundness and Completeness of Embedding $\mathcal{IPL}^{\mathcal{STT}}$). *Let $t \in \mathcal{IPL}$ and let $t' \in \mathcal{STT}$ as constructed above. t is valid in propositional intuitionistic logic if and only if t' is valid in $\mathcal{STT}$.*

5 Conclusion

In this paper we have explored an interesting and promising research direction: the embedding of propositional normal multimodal logic and propositional intuitionistic logic in simple type theory. We argue that simple type theory can thus provide a fruitful, uniform basis for modeling and exploring different non-classical logics. Our results provide a theoretical foundation for reasoning not only within but also *about* these logics in simple type theory by employing off-the-shelf higher-order proof assistants and higher-order automated theorem provers. Preliminary experiments with our approach have been promising [13]. A small corpus of related example problems has meanwhile been entered into the new TPTP library for automated higher-order theorem proving [32] in order to stimulate further experiments with our approach and to foster the improvement of existing higher-order automated theorem provers for the task.

Future work includes the study of further embeddings of non-classical logics into simple type theory. For example, access control logics, which are important in the field of computer security, can be embedded analogously [11]. Ongoing and future work also includes extending our embeddings to first-order and higher-order multimodal logics (or intuitionistic logics).

Acknowledgements: We thank the anonymous reviewers of this paper for their comments.

References

1. Peter B. Andrews. General models and extensionality. *Journal of Symbolic Logic*, 37:395–397, 1972.
2. Peter B. Andrews. General models, descriptions, and choice in type theory. *Journal of Symbolic Logic*, 37:385–394, 1972.
3. Peter B. Andrews. On connections and higher-order logic. *Journal of Automated Reasoning*, 5:257–291, 1989.
4. Peter B. Andrews. *An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof.* Kluwer Academic Publishers, second edition, 2002.
5. Peter B. Andrews. Church's type theory. In E. N. Zalta, editor, *The Stanford Encyclopedia of Philosophy.* Center for the Study of Language and Information, Stanford University, 2008. http://plato.stanford.edu/archives/fall2008/entries/type-theory-church/.
6. Peter B. Andrews, Matthew Bishop, Sunil Issar, Dan Nesmith, Frank Pfenning, and Hongwei Xi. TPS: A theorem proving system for classical type theory. *Journal of Automated Reasoning*, 16:321–353, 1996.
7. Peter B. Andrews and Chad E. Brown. Tps: A hybrid automatic-interactive system for developing proofs. *Journal of Applied Logic*, 4(4):367–395, 2006.
8. Alessandro Armando, Peter Baumgartner, and Gilles Dowek, editors. *Automated Reasoning, 4th International Joint Conference, IJCAR 2008, Sydney, Australia, August 12-15, 2008, Proceedings*, volume 5195 of *LNCS*. Springer, 2008.

9. Philippe Balbiani, Luis Fariñas del Cerro, and Andreas Herzig. Declarative semantics for modal logic programs. In *Fifth Generation Computer Systems 1988, Proceedings of the International Conference on Fifth Generation Computer Systems 1988, Tokyo, Japan, November 28-December 2, 1988*, pages 507–514. OHMSHA Ltd. Tokyo and Springer-Verlag, 1988.
10. Matteo Baldoni, Laura Giordano, and Alberto Martelli. A framework for a modal logic programming. In *Joint International Conference and Symposium on Logic Programming*, pages 52–66, 1996.
11. Christoph Benzmüller. Automating access control logic in simple type theory with LEO-II. In Dimitris Gritzalis and Javier López, editors, *Emerging Challenges for Security, Privacy and Trust, 24th IFIP TC 11 International Information Security Conference, SEC 2009, Pafos, Cyprus, May 18-20, 2009. Proceedings*, volume 297 of *IFIP*, pages 387–398. Springer, 2009.
12. Christoph Benzmüller, Chad E. Brown, and Michael Kohlhase. Higher order semantics and extensionality. *Journal of Symbolic Logic*, 69:1027–1088, 2004.
13. Christoph Benzmüller and Lawrence Paulson. *Festschrift in Honor of Peter B. Andrews on His 70th Birthday*, chapter Exploring Properties of Normal Multimodal Logics in Simple Type Theory with LEO-II. Studies in Logic, Mathematical Logic and Foundations. College Publications, 2008. ISBN 978-1-904987-70-3.
14. Christoph Benzmüller, Frank Theiss, Larry Paulson, and Arnaud Fietzke. LEO-II - a cooperative automatic theorem prover for higher-order logic. In Armando et al. [8], pages 162–170.
15. Chad E. Brown. Encoding hybrid logic in higher-order logic. Unpublished slides from an invited talk presented at Loria Nancy, France, April 2005. `http://mathgate.info/cebrown/papers/hybrid-hol.pdf`.
16. Bob Carpenter. *Type-logical semantics*. MIT Press, Cambridge, MA, USA, 1998.
17. Alonzo Church. A Formulation of the Simple Theory of Types. *Journal of Symbolic Logic*, 5:56–68, 1940.
18. Luis Fariñas del Cerro. Molog: A system that extends PROLOG with modal logic. *New Generation Comput.*, 4(1):35–50, 1986.
19. Daniel Gallin. *Intensional and Higher-Order Modal Logic*, volume 19 of *North-Holland Mathematics Studies*. North-Holland, Amsterdam, 1975.
20. L. T. F. Gamut. *Logic, Language, and Meaning. Volume II. Intensional Logic and Logical Grammar*, volume 2. The University of Chicago Press, 1991.
21. Kurt Gödel. Eine interpretation des intuitionistischen aussagenkalküls. *Ergebnisse eines Mathematischen Kolloquiums*, 8:39–40, 1933. Also published in Gödel [**?**], 296–302.
22. Moritz Hardt and Gert Smolka. Higher-order syntax and saturation algorithms for hybrid logic. *Electronic Notes in Theoretical Computer Science*, 174(6):15–27, 2007.
23. John Harrison. *HOL Light Tutorial (for version 2.20).* Intel JF1-13, September 2006. `http://www.cl.cam.ac.uk/~jrh13/hol-light/tutorial\_220.pdf`.
24. Leon Henkin. Completeness in the theory of types. *Journal of Symbolic Logic*, 15:81–91, 1950.
25. Mark Kaminski and Gert Smolka. Terminating tableaux for hybrid logic with the difference modality and converse. In Armando et al. [8], pages 210–225.
26. John Charles Chenoweth McKinsey and Alfred Tarski. Some theorems about the sentential calculi of lewis and heyting. *Journal of Symbolic Logic*, 13:1–15, 1948.
27. Stephan Merz. Yet another encoding of TLA in Isabelle. Available on the Internet: `http://www.loria.fr/~merz/projects/isabelle-tla/doc/design.ps.gz`, 1999.

28. Linh Anh Nguyen. A fixpoint semantics and an SLD-resolution calculus for modal logic programs. *Fundamenta Informaticae*, 55(1):63–100, 2003.
29. Andreas Nonnengart. How to use modalities and sorts in Prolog. In Craig MacNish, David Pearce, and Luís Moniz Pereira, editors, *Logics in Artificial Intelligence, European Workshop, JELIA '94*, volume 838 of *Lecture Notes in Computer Science*, pages 365–378. Springer, 1994.
30. Hans Jürgen Ohlbach. A resolution calculus for modal logics. In Ewing L. Lusk and Ross A. Overbeek, editors, *9th International Conference on Automated Deduction, Argonne, Illinois, USA, May 23-26, 1988, Proceedings*, volume 310 of *Lecture Notes in Computer Science*, pages 500–516. Springer, 1988.
31. Stephan Schulz. E – A Brainiac Theorem Prover. *Journal of AI Communications*, 15(2/3):111–126, 2002.
32. Geoff Sutcliffe, Christoph Benzmüller, Chad Brown, and Frank Theiss. Progress in the development of automated theorem proving for higher-order logic. In Renate Schmidt, editor, *Automated Deduction - CADE-22, 22nd International Conference on Automated Deduction, Montreal, Canada, August 2-7, 2009. Proceedings*, volume 5663 of *LNCS*, pages 116–130. Springer, 2009.