Some Reflections on a Computer-aided Theory Exploration Study in Category Theory (Extended Abstract)

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We present some reflections on the use of automated theorem proving and model finding technology in the context of a recent theory exploration study in category theory [1,2].

In our stepwise development of mutually equivalent axioms sets for category theory we started out with a generalised notion of monoids. More precisely, the first axiom system in our study was obtained by generalizing the standard axioms for a monoid to a partial composition operation. In subsequent development steps we simplified this initial axioms set until we reached the axioms as proposed by Scott [15] in the 1970s. We then compared this axioms set with an alternative proposal by Freyd and Scedrov [11]. In the course of this comparison we revealed a technical flaw for the axiom set of Freyd and Scedrov: either all operations, e.g. morphism composition, are total or their axiom system is inconsistent. The repair for this problem is quite straightforward and it essentially corresponds to the set of axioms proposed by Scott.

Our experiments were enabled by a semantical embedding of free logic [14] in classical higher-order logic (HOL), which we implemented in the proof assistant system Isabelle/HOL [12]. Free logic was utilised to support an adequate handling of partiality and undefinedness in the modeling of morphism composition, and the domain and codomain operators. Our experiments were substantially supported by automated reasoning technology, in particular, by the model finder Nitpick [7] and by various automated theorem provers (CVC4 [10], E [13], Leo-II [3], Satallax [8], SPASS [6], Z3 [9], etc.) integrated with Isabelle/HOL via Sledgehammer [5].

In our presentation at AITP 2018 we particularly want to reflect on the role these systems played in our experiments. This is of practical and also of epistemological relevance, since these systems, as we will evidence, can indeed substantially foster the gain of new knowledge. We will therefore highlight relevant points in our stepwise development in which these systems, in particular, the model finder Nitpick, supported the gain of intuition by providing countermodels to still slightly flawed axioms or definitions. And the theorem provers supported the detection of the constricted inconsistency, in addition to the important, albeit more traditional, role they played in confirming equivalences between different axioms sets as soon as we arrived at their correct formulations.

Despite our reassuring overall teamwork experience, which involved a domain expert (Scott), a theorem proving expert (Benzmüller) and the Isabelle/HOL framework, we also collected several critical remarks pointing to a range of improvement opportunities. Some of these improvement opportunities are of technical nature, others may include theoretical aspects. For example, Nitpick should be improved by devising and implementing better readable and eventually more domain specific representations of models and countermodels. In our experiments such conversions were in fact laboriously handled by hand by Benzmüller and the results were then communicated by email to Scott. In some cases calls of external theorem provers via Sledgehammer resulted in technical error messages, which may demotivate non-expert users, and when the theorem provers succeeded, then their proofs could most of the time not be converted into informative Isar style proofs. The constricted inconsistency result, for example, had to be reconstructed by hand to obtain an human-friendly Isar style proof (see [4] for a similar experience in a different context).

Hence, our successful experiments, in which automated reasoning tools integrated in Isabelle/HOL have demonstrated their capabilities beyond mere proof verification, still required a close interaction between three players: a domain expert, a theorem proving expert and the Isabelle/HOL proof assistant. The challenge in fact still is to get the second player completely out of the loop, without requiring the first player to adopt a nearly identical level of technical expertise in a resource-intensive, laborious manner.

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