Cut Elimination with ξ-Functionality

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1 Introduction

In this paper we will give a complete, cut-free sequent calculus for a fragment of higher-order logic with ξ-extensionality, a weak form of functional extensionality. When one wants a calculus appropriate for automation, a common first step is to find such a complete, cut-free sequent calculus. For example, Andrews proved cut-elimination for a sequent calculus for elementary type theory [1]. Elementary type theory is a fragment of higher-order logic with no extensionality principles. In [3], we showed there is a cube of eight model classes which vary with respect to extensionality principles. For purposes of automation, we would like to have a complete, cut-free sequent calculus for each of these eight model classes. The essential ingredients for obtaining complete, cut-free sequent calculi for three of the eight model classes are already known. The sequent calculus for elementary type theory in [1] essentially provides a complete, cut-free calculus for one of the eight model classes. (Completeness of a similar calculus will follow from the results in this paper.) There are also results for two of the other model classes (the case with η and the fully extensional case) in [6, 7], though the framework is slightly different from the one here. In this paper, we prove completeness of a cut-free sequent calculus for a fourth point on the extensionality cube, elementary type theory with ξ-extensionality. To handle the case with ξ-extensionality, we will combine the techniques used in [1] with recent results on cut-simulation [4, 5].

As mentioned above, in [1] Andrews proved cut-elimination for a sequent calculus $\mathscr{G}$ for elementary type theory (Corollary 4.11 there). In particular, Andrews proved that provability in $\mathscr{G}$ is equivalent to provability in a Hilbert-style calculus $\mathscr{T}$ (where admissibility of cut is immediate). The difficult direction of this equivalence is proving that one has $\vdash_{\mathscr{T}} \mathbf{A}$ whenever one has $\vdash_{\mathscr{G}} \mathbf{A}$ (Theorem 4.10 in [1]). A key fact in Andrews' proof is his Theorem 3.5: "If Γ is an abstract consistency property and $\mathscr{S}$ is a finite set of wffs$_o$ such that $\Gamma(\mathscr{S})$, then $\mathscr{S}$ is consistent." Here, consistency is defined with respect to the calculus $\mathscr{T}$. In order to prove Theorem 3.5 from [1], Andrews proves that $\mathscr{S}$ can be used to obtain a semivaluation V such that $V\mathbf{A} = \mathtt{T}$ for all $\mathbf{A} \in \mathscr{S}$ (see Theorem 3.3 in [1]) and then proves the set $\{\mathbf{A}|V\mathbf{A} = \mathtt{T}\}$ is consistent (see Theorem 3.4 in [1]). To prove consistency of $\{\mathbf{A}|V\mathbf{A} = \mathtt{T}\}$, Andrews constructs a structure of V-complexes (following ideas in Takahashi [11] and Prawitz [10]) and proves this structure satisfies properties similar to that of a Henkin model [9].

The Andrews structure of V-complexes is not a Henkin model. In fact, it was not until much later that a notion of a nonextensional model of higher-order logic was proposed which includes V-complexes (see [3]). Given this more recent notion of a model, the Andrews structure can be viewed in a new light, and further properties of the structure can be proven.

The V-complex construction (up to the treatment of free variables) was generalized to the notion of a possible values structure in [6, 7]. We will use several results from [6, 7] to quickly conclude that the construction (starting from a Hintikka set $\mathcal{H}$ instead of a semivaluation V) yields a model. Furthermore, the model will satisfy a weak form of functional extensionality known as property ξ (cf. Definition 3.46 in [3]). If the Hintikka set is not saturated, then the model will satisfy property $\mathfrak{q}$ (cf. Definition 3.46 in [3]). Using these facts, we will prove completeness of a sequent calculus for higher-order logic with ξ-functionality relative to the model class $\mathfrak{M}_{\beta\xi}$ (cf. Definition 3.49 in [3]) and conclude cut-elimination for the sequent calculus. We will also prove completeness of a sequent calculus for elementary type theory (similar to the one given in [1]) relative to the model class $\mathfrak{M}_{\beta}$.

2 Preliminaries

We review the fundamental framework from [3] (which can be consulted for details).

As in [8], we formulate higher-order logic ($\mathcal{HOL}$) based on the simply typed λ-calculus. The set of simple types $\mathcal{T}$ is freely generated from basic types o and ι using the function type constructor $\rightarrow$.

We start with a set $\mathcal{V}$ of (typed) variables (denoted by $X_\alpha, Y, Z, X^1_\beta, X^2_\gamma \ldots$) and a signature Σ of (typed) constants (denoted by $c_\alpha, f_{\alpha\rightarrow\beta}, \ldots$). We let $\mathcal{V}_\alpha$ (Σ_α) denote the set of variables (constants) of type α. The signature Σ of constants includes the logical constants $\neg_{o\rightarrow o}$, $\vee_{o\rightarrow o\rightarrow o}$ and $\Pi^\alpha_{(\alpha\rightarrow o)\rightarrow o}$ for each type α. All other constants in Σ are called parameters. As in [3], we assume there is an infinite cardinal $\aleph_s$ such that the cardinality of Σ_α is $\aleph_s$ for each type α (cf. Remark 3.16 in [3]). The set of $\mathcal{HOL}$-formulae (or terms) are constructed from typed variables and constants using application and λ-abstraction. We let $\mathit{wff}_\alpha(\Sigma)$ be the set of all terms of type α and $\mathit{wff}(\Sigma)$ be the set of all terms.

We use vector notation to abbreviate k-fold applications and abstractions as $\mathbf{A}\overline{\mathbf{U}^k}$ and $\lambda\overline{X^k}\mathbf{.A}$, respectively. We also use Church's dot notation so that $\mathbf{.}$ stands for a (missing) left bracket whose mate is as far to the right as possible (consistent with given brackets). We use infix notation $\mathbf{A} \vee \mathbf{B}$ for $((\vee\mathbf{A})\mathbf{B})$ and binder notation $\forall X_\alpha\mathbf{.A}$ for $(\Pi^\alpha \lambda X_\alpha\mathbf{.A}_o)$. We further use $\mathbf{A} \wedge \mathbf{B}$, $\mathbf{A} \Rightarrow \mathbf{B}$, $\mathbf{A} \Leftrightarrow \mathbf{B}$ and $\exists X_\alpha\mathbf{.A}$ as shorthand for formulae defined in terms of $\neg$, $\vee$ and Π^α (cf. [3]). Finally, we let $(\mathbf{A}_\alpha \doteq^\alpha \mathbf{B}_\alpha)$ denote the Leibniz equation $\forall P_{\alpha\to o}\mathbf{.}(P\mathbf{A}) \Rightarrow (P\mathbf{B})$.

Each occurrence of a variable in a term is either bound by a λ or free. We use $free(\mathbf{A})$ to denote the set of free variables of $\mathbf{A}$ (i.e., variables with a free occurrence in $\mathbf{A}$). We consider two terms to be equal if the terms are the same up to the names of bound variables (i.e., we consider α-conversion implicitly). A term $\mathbf{A}$ is closed if $free(\mathbf{A})$ is empty. We let $cwff_\alpha(\Sigma)$ denote the set of closed terms of type α and $cwff(\Sigma)$ denote the set of all closed terms. Each term $\mathbf{A} \in wff_o(\Sigma)$ is called a proposition and each term $\mathbf{A} \in cwff_o(\Sigma)$ is called a sentence.

We denote substitution of a term $\mathbf{A}_\alpha$ for a variable X_α in a term $\mathbf{B}_\beta$ by $[\mathbf{A}/X]\mathbf{B}$. Since we consider α-conversion implicitly, we assume the bound variables of $\mathbf{B}$ avoid variable capture. Similarly, we consider simultaneous substitutions σ for the finitely many free variables in the domain $\mathbf{Dom}(\sigma)$ of σ. A substitution $\sigma, [\mathbf{A}/X]$ is the substitution such that $(\sigma, [\mathbf{A}/X])(X) \equiv \mathbf{A}$ and $(\sigma, [\mathbf{A}/X])(Y) \equiv \sigma(Y)$ for variables Y other than X. Note that here and everywhere we use $\cdot \equiv \cdot$ for syntactical identity (modulo α equivalence).

A common relation on terms is given by β-reduction. A β-redex $(\lambda X\mathbf{.A})\mathbf{B}$ β-reduces to $[\mathbf{B}/X]\mathbf{A}$. For $\mathbf{A}, \mathbf{B} \in wff_\alpha(\Sigma)$, we write $\mathbf{A}{\equiv_\beta}\mathbf{B}$ to mean $\mathbf{A}$ can be converted to $\mathbf{B}$ by a series of β-reductions and expansions. For each $\mathbf{A} \in wff(\Sigma)$ there is a unique β-normal form (denoted $\mathbf{A}{\downarrow_\beta}$; the set of all β-normal formulae is denoted by $wff(\Sigma){\downarrow_\beta}$). From this fact we know $\mathbf{A}{\equiv_\beta}\mathbf{B}$ iff $\mathbf{A}{\downarrow_\beta}$ and $\mathbf{B}{\downarrow_\beta}$ are syntactically equal ($\mathbf{A}{\downarrow_\beta} \equiv \mathbf{B}{\downarrow_\beta}$).

A model of $\mathcal{HOL}$ is given by four objects: a typed collection of nonempty sets $(\mathcal{D}_\alpha)_{\alpha\in\mathcal{T}}$, an application operator $@{:}\,\mathcal{D}_{\alpha\to\beta} \times \mathcal{D}_\alpha \longrightarrow \mathcal{D}_\beta$, an evaluation function $\mathcal{E}$ for terms and a valuation function $\upsilon{:}\,\mathcal{D}_o \longrightarrow \{\mathtt{T}, \mathtt{F}\}$. A pair $(\mathcal{D}, @)$ is called a Σ-applicative structure (cf. Definition 16). If $\mathcal{E}$ is an evaluation function for $(\mathcal{D}, @)$ (cf. Definition 3.18 in [3]), then we call the triple $(\mathcal{D}, @, \mathcal{E})$ a Σ-evaluation. If υ satisfies appropriate properties, then we call the tuple $(\mathcal{D}, @, \mathcal{E}, \upsilon)$ a Σ-model (cf. Definitions 3.40 and 3.41 in [3]).

Given an applicative structure $(\mathcal{D}, @)$, an assignment φ is a (typed) function from $\mathcal{V}$ to $\mathcal{D}$. An evaluation function $\mathcal{E}$ maps an assignment φ and a term $\mathbf{A}_\alpha \in wff_\alpha(\Sigma)$ to an element $\mathcal{E}_\varphi(\mathbf{A}) \in \mathcal{D}_\alpha$. Evaluations $\mathcal{E}$ are required to satisfy four properties (cf. Definition 3.18 in [3]):

1. $\mathcal{E}_\varphi\big|_{\mathcal{V}} \equiv \varphi$.
2. $\mathcal{E}_\varphi(\mathbf{FA}) \equiv \mathcal{E}_\varphi(\mathbf{F})@\mathcal{E}_\varphi(\mathbf{A})$ for any $\mathbf{F} \in \mathit{wff}_{\alpha\to\beta}(\Sigma)$ and $\mathbf{A} \in \mathit{wff}_\alpha(\Sigma)$ and types α and β.
3. $\mathcal{E}_\varphi(\mathbf{A}) \equiv \mathcal{E}_\psi(\mathbf{A})$ for any type α and $\mathbf{A} \in \mathit{wff}_\alpha(\Sigma)$, whenever φ and ψ coincide on $free(\mathbf{A})$.
4. $\mathcal{E}_\varphi(\mathbf{A}) \equiv \mathcal{E}_\varphi(\mathbf{A}{\downarrow_\beta})$ for all $\mathbf{A} \in \mathit{wff}_\alpha(\Sigma)$.

If $\mathbf{A}$ is closed, then we can simply write $\mathcal{E}(\mathbf{A})$ since the value $\mathcal{E}_\varphi(\mathbf{A})$ cannot depend on φ.

Given an evaluation $(\mathcal{D}, @, \mathcal{E})$, Figure 1 shows the definition of several properties a function $\upsilon: \mathcal{D}_o \longrightarrow \{\mathtt{T}, \mathtt{F}\}$ may satisfy (cf. Definition 3.40 in [3]). A valuation $\upsilon: \mathcal{D}_o \longrightarrow \{\mathtt{T}, \mathtt{F}\}$ is required to satisfy $\mathfrak{L}_\neg(\mathcal{E}(\neg))$, $\mathfrak{L}_\vee(\mathcal{E}(\vee))$ and $\mathfrak{L}_\forall^\alpha(\mathcal{E}(\Pi^\alpha))$ for every type α.

prop.	where	holds when			for all
$\mathfrak{L}_\neg(\mathsf{n})$	$\mathsf{n} \in \mathcal{D}_{o\to o}$	$\upsilon(\mathsf{n}@\mathsf{a}) \equiv \mathtt{T}$	iff	$\upsilon(\mathsf{a}) \equiv \mathtt{F}$	$\mathsf{a} \in \mathcal{D}_o$
$\mathfrak{L}_\vee(\mathsf{d})$	$\mathsf{d} \in \mathcal{D}_{o\to o\to o}$	$\upsilon(\mathsf{d}@\mathsf{a}@\mathsf{b}) \equiv \mathtt{T}$	iff	$\upsilon(\mathsf{a}) \equiv \mathtt{T}$ or $\upsilon(\mathsf{b}) \equiv \mathtt{T}$	$\mathsf{a}, \mathsf{b} \in \mathcal{D}_o$
$\mathfrak{L}_\forall^\alpha(\pi)$	$\pi \in \mathcal{D}_{(\alpha\to o)\to o}$	$\upsilon(\pi@\mathsf{f}) \equiv \mathtt{T}$	iff	$\forall \mathsf{a} \in \mathcal{D}_\alpha\ \upsilon(\mathsf{f}@\mathsf{a}) \equiv \mathtt{T}$	$\mathsf{f} \in \mathcal{D}_{\alpha\to o}$
$\mathfrak{L}_=^\alpha(\mathsf{q})$	$\mathsf{q} \in \mathcal{D}_{\alpha\to\alpha\to\mathsf{o}}$	$\upsilon(\mathsf{q}@\mathsf{a}@\mathsf{b}) \equiv \mathtt{T}$	iff	$\mathsf{a} \equiv \mathsf{b}$	$\mathsf{a}, \mathsf{b} \in \mathcal{D}_\alpha$

Figure 1. Logical Properties in Σ-Models

Given a model $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$, an assignment φ and a proposition $\mathbf{A}$ (or set of propositions Φ), we say $\mathcal{M}$ satisfies $\mathbf{A}$ (or Φ) and write $\mathcal{M} \models_\varphi \mathbf{A}$ (or $\mathcal{M} \models_\varphi \Phi$) if $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) \equiv \mathtt{T}$ (or $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) \equiv \mathtt{T}$ for each $\mathbf{A} \in \Phi$). If $\mathbf{A}$ is closed (or every member of Φ is closed), then we simply write $\mathcal{M} \models \mathbf{A}$ (or $\mathcal{M} \models \Phi$) and say $\mathcal{M}$ is a model of $\mathbf{A}$ (or Φ).

In order to define model classes which correspond to different notions of extensionality, five properties of models are defined ($\mathfrak{q}$, η, ξ, $\mathfrak{f}$, and $\mathfrak{b}$; cf. Definitions 3.46, 3.21 and 3.5 in [3]). In this paper, we will only refer to properties $\mathfrak{q}$ and ξ. Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a model. We say $\mathcal{M}$ has property

$\mathfrak{q}$ iff for all $\alpha \in \mathcal{T}$ there is some $\mathsf{q}^\alpha \in \mathcal{D}_{\alpha\to\alpha\to\mathsf{o}}$ such that $\mathfrak{L}_=^\alpha(\mathsf{q}^\alpha)$ holds.

ξ iff $(\mathcal{D}, @, \mathcal{E})$ is ξ-functional (i.e., for each $\mathbf{M}, \mathbf{N} \in \mathit{wff}_\beta(\Sigma)$, $X \in \mathcal{V}_\alpha$ and assignment φ, we have $\mathcal{E}_\varphi(\lambda X_\alpha \mathbf{.} \mathbf{M}_\beta) \equiv \mathcal{E}_\varphi(\lambda X_\alpha \mathbf{.} \mathbf{N}_\beta)$ whenever $\mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{M}) \equiv \mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{N})$ for every $\mathsf{a} \in \mathcal{D}_\alpha$).

For each $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ there is a model class $\mathfrak{M}_*$ (cf. Definition 3.49 in [3]). Here we only consider $* \in \{\beta, \beta\xi\}$: $\mathfrak{M}_\beta$ is the class of all Σ-models $\mathcal{M}$ satisfying property $\mathfrak{q}$. $\mathfrak{M}_{\beta\xi}$ is the class of all Σ-models $\mathcal{M}$ satisfying properties $\mathfrak{q}$ and ξ.

Finally, we review the model existence theorems proved in [3]. There are three stages to obtaining a model in our framework. First, we obtain an abstract consistency class Γ_Σ (usually defined as the class of irrefutable sets of sentences with respect to some calculus). Second, given a (sufficiently pure) set of sentences Φ in the abstract consistency class Γ_Σ we construct a Hintikka set $\mathcal{H}$ extending Φ. Third, we construct a model of this Hintikka set (hence a model of Φ).

We say Γ_Σ is an abstract consistency class if it is closed under subsets and satisfies properties $\nabla_c, \nabla_\neg, \nabla_\beta, \nabla_\vee, \nabla_\wedge, \nabla_\forall$ and $\nabla_\exists$ (cf. Definitions 6.1 and 6.5 in [3]). We let $\mathfrak{Acc}_\beta$ denote the collection of all abstract consistency classes. For each $* \in$ ⊞ we refine $\mathfrak{Acc}_\beta$ to a collection $\mathfrak{Acc}_*$ where the additional properties $\{\nabla_\eta, \nabla_\xi, \nabla_\mathfrak{f}, \nabla_\mathfrak{b}\}$ indicated by $*$ are required (cf. Definition 6.7 in [3]). We say an abstract consistency class Γ_Σ is saturated if ∇_{sat} holds. The only condition we will explicitly use in this paper is ∇_ξ which is defined as follows:

∇_ξ If $\neg(\lambda X_\alpha\textbf{.}\mathbf{M} \doteq^{\alpha\to\beta} \lambda X_\alpha\textbf{.}\mathbf{N}) \in \Phi$, then $\Phi * \neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N}) \in \Gamma_\Sigma$ for any parameter $w_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

In order to obtain a Hintikka set extending a set Φ, we must have parameters which will act as witnesses. For this we require sufficient purity of Φ. A set Φ of Σ-sentences is called sufficiently Σ-pure (cf. Definition 6.3 in [3]) if for each type α there is a set $\mathcal{P}_\alpha$ of parameters of type α with cardinality $\aleph_s$ (the cardinality of $\mathit{wff}_\alpha(\Sigma)$), such that no parameter in $\mathcal{P}$ occurs in a sentence in Φ. Note that since Σ is assumed to have infinite cardinality $\aleph_s$ for each type, every finite set of Σ-sentences is sufficiently Σ-pure.

A Hintikka set is a set of sentences satisfying certain properties. The following is a list of some of the properties a set $\mathcal{H}$ of sentences may satisfy (cf. Definition 6.19 in [3]):

$\vec{\nabla}_c$ $\mathbf{A} \notin \mathcal{H}$ or $\neg\mathbf{A} \notin \mathcal{H}$.

$\vec{\nabla}_\neg$ If $\neg\neg\mathbf{A} \in \mathcal{H}$, then $\mathbf{A} \in \mathcal{H}$.

$\vec{\nabla}_\beta$ If $\mathbf{A} \in \mathcal{H}$ and $\mathbf{A} \equiv_\beta \mathbf{B}$, then $\mathbf{B} \in \mathcal{H}$.

$\vec{\nabla}_\vee$ If $\mathbf{A} \vee \mathbf{B} \in \mathcal{H}$, then $\mathbf{A} \in \mathcal{H}$ or $\mathbf{B} \in \mathcal{H}$.

$\vec{\nabla}_\wedge$ If $\neg(\mathbf{A} \vee \mathbf{B}) \in \mathcal{H}$, then $\neg\mathbf{A} \in \mathcal{H}$ and $\neg\mathbf{B} \in \mathcal{H}$.

$\vec{\nabla}_\forall$ If $\Pi^\alpha\mathbf{F} \in \mathcal{H}$, then $\mathbf{FW} \in \mathcal{H}$ for each $\mathbf{W} \in \mathit{cwff}_\alpha(\Sigma)$.

$\vec{\nabla}_{\exists}$ If $\neg\Pi^{\alpha}\mathbf{F} \in \mathcal{H}$, then there is a parameter $w_{\alpha} \in \Sigma_{\alpha}$ such that $\neg(\mathbf{F}w) \in \mathcal{H}$.

$\vec{\nabla}_{\xi}$ If $\neg(\lambda X_{\alpha}.\mathbf{M} \doteq^{\alpha\to\beta} \lambda X.\mathbf{N}) \in \mathcal{H}$, then there is a parameter $w_{\alpha} \in \Sigma_{\alpha}$ such that $\neg([w/X]\mathbf{M} \doteq^{\beta} [w/X]\mathbf{N}) \in \mathcal{H}$.

$\vec{\nabla}_{sat}$ Either $\mathbf{A} \in \mathcal{H}$ or $\neg\mathbf{A} \in \mathcal{H}$.

[3] also defines properties $\vec{\nabla}_{\eta}$, $\vec{\nabla}_{\mathfrak{b}}$, and $\vec{\nabla}_{\mathfrak{f}}$, but these will not be used here. A set $\mathcal{H}$ of sentences is called a Σ-*Hintikka set* if $\vec{\nabla}_{c}$, $\vec{\nabla}_{\neg}$, $\vec{\nabla}_{\beta}$, $\vec{\nabla}_{\vee}$, $\vec{\nabla}_{\wedge}$, $\vec{\nabla}_{\forall}$ and $\vec{\nabla}_{\exists}$ hold. We define the following collections of Hintikka sets: $\mathfrak{Hint}_{\beta}$, $\mathfrak{Hint}_{\beta\eta}$, $\mathfrak{Hint}_{\beta\xi}$, $\mathfrak{Hint}_{\beta\mathfrak{f}}$, $\mathfrak{Hint}_{\beta\mathfrak{b}}$, $\mathfrak{Hint}_{\beta\eta\mathfrak{b}}$, $\mathfrak{Hint}_{\beta\xi\mathfrak{b}}$, and $\mathfrak{Hint}_{\beta\mathfrak{f}\mathfrak{b}}$, where we indicate by indices which additional properties from $\{\vec{\nabla}_{\eta}, \vec{\nabla}_{\xi}, \vec{\nabla}_{\mathfrak{f}}, \vec{\nabla}_{\mathfrak{b}}\}$ are required (cf. Definition 6.20 in [3]). We call a Hintikka set $\mathcal{H}$ *saturated* if $\vec{\nabla}_{sat}$ holds (cf. Definition 6.24 in [3]).

One of the main theorems of [3] is the Model Existence Theorem for Saturated Sets which states the following:

THEOREM 1 (Model Existence Theorem for Saturated Sets (Theorem 6.33 in [3])).
For all $* \in \boxdot$ *we have: If* $\mathcal{H}$ *is a saturated Hintikka set in* $\mathfrak{Hint}_{*}$*, then there exists a model* $\mathcal{M} \in \mathfrak{M}_{*}$ *that satisfies* $\mathcal{H}$*. Furthermore, each domain* $\mathcal{D}_{\alpha}$ *of* $\mathcal{M}$ *has cardinality at most* $\aleph_{s}$*.*

Since saturated abstract consistency classes give rise to saturated Hintikka sets, we conclude a corresponding model existence theorem for saturated abstract consistency classes.

THEOREM 2 (Theorem 6.34 in [3]). *For all* $* \in \boxdot$*, if* Γ_{Σ} *is a saturated abstract consistency class in* $\mathfrak{Acc}_{*}$ *and* $\Phi \in \Gamma_{\Sigma}$ *is a sufficiently* Σ*-pure set of sentences, then there exists a model* $\mathcal{M} \in \mathfrak{M}_{*}$ *that satisfies* Φ*. Furthermore, each domain of* $\mathcal{M}$ *has cardinality at most* $\aleph_{s}$*.*

3 Possible Values

We now review a framework developed in [6, 7] which is essentially a general version of Andrews construction using V-complexes given in [1]. There are slight differences between the construction here and that in [1]. One difference is that our domains are constructed using pairs $\langle \mathbf{A}, \mathsf{a} \rangle$ where $\mathbf{A}$ is *closed*, whereas in [1] $\mathbf{A}$ may contain free variables. This difference stems from the fact that we use parameters as existential witnesses and Andrews uses variables for this purpose in [1]. Another difference is that we start from a Hintikka set $\mathcal{H}$ instead of a semivaluation V.

Except for the different treatment of variables, the V-complex construction provides an instance of a *possible values structure for* β (cf. Definition 3) and a *possible values evaluation for* β (cf. Definition 8). The definitions in [6, 7] are for both the β and $\beta\eta$ cases. We repeat these definitions (specialized for the β case) and a few results here. We then prove that any possible values evaluation is ξ-functional (a new result).

The results in [7] are stated with respect to a signature of logical constants $\mathcal{S}$ which is distinct from the set of parameters $\mathcal{P}$. In order to apply the results from [7] we take $\mathcal{S}$ to be the set

$$\{\neg, \vee\} \cup \{\Pi^\alpha | \alpha \in \mathcal{T}\}$$

and $\mathcal{P}$ to be the typed family of sets of parameters (non-logical constants) in Σ. Note that $\Sigma \equiv (\mathcal{S} \cup \mathcal{P})$.

DEFINITION 3 (Definition 4.1.1 from [6, 7]). A *possible values structure for* β is an applicative structure $\mathcal{F} \equiv (\mathcal{D}, @)$ satisfying the following:

1. For each type $\alpha \in \mathcal{T}$, $\mathsf{a} \in \mathcal{D}_\alpha$ implies $\mathsf{a} \equiv \langle \mathbf{A}, a\rangle$ for some a and term $\mathbf{A} \in \mathit{cwff}_\alpha(\Sigma)$ such that $\mathbf{A}{\downarrow_\beta} \equiv \mathbf{A}$.
2. At each base type $\alpha \in \{o, \iota\}$, for every $\mathbf{A} \in \mathit{cwff}_\alpha(\Sigma)$, there exists some p with $\langle \mathbf{A}{\downarrow_\beta}, p\rangle \in \mathcal{D}_\alpha$.
3. For each function type $\alpha \to \beta$, $\langle \mathbf{G}, g\rangle \in \mathcal{D}_{\alpha\to\beta}$ iff $\mathbf{G} \in \mathit{cwff}_{\alpha\to\beta}(\Sigma)$, $\mathbf{G}{\downarrow_\beta} \equiv \mathbf{G}$, $g : \mathcal{D}_\alpha \longrightarrow \mathcal{D}_\beta$ and for every $\langle \mathbf{A}, a\rangle \in \mathcal{D}_\alpha$ the first component of $g(\langle \mathbf{A}, a\rangle)$ is $[\mathbf{G}\,\mathbf{A}]{\downarrow_\beta}$.
4. For each $\langle \mathbf{G}, g\rangle \in \mathcal{D}_{\alpha\to\beta}$ and $\langle \mathbf{A}, a\rangle \in \mathcal{D}_\alpha$,
$$\langle \mathbf{G}, g\rangle @ \langle \mathbf{A}, a\rangle \equiv g(\langle \mathbf{A}, a\rangle).$$

DEFINITION 4 (Definition 4.1.2 from [6, 7]). Let $\mathcal{A} \equiv (\mathcal{D}, @)$ be a possible values structure for β. We call p a *possible value* for $\mathbf{A} \in \mathit{cwff}_\alpha(\Sigma)$ if $\langle \mathbf{A}{\downarrow_\beta}, p\rangle \in \mathcal{D}_\alpha$.

The next lemma is similar to Lemma 3.4.2 in [1] which provided the idea for the proof by induction on types.

LEMMA 5 (Lemma 4.1.3 from [6, 7]). *Let* $\mathcal{F}$ *be a possible values structure for* β*. For each closed term* $\mathbf{A} \in \mathit{cwff}_\alpha(\Sigma)$*, there is a possible value* p *for* $\mathbf{A}$ *in* $\mathcal{F}$*.*

DEFINITION 6 (Definition 4.1.4 from [6, 7]). Let $\mathcal{A} \equiv (\mathcal{D}, @)$ be a possible values structure for β. We define

$$\mathcal{D}^{\mathbf{A}}_\alpha := \{\langle \mathbf{A}{\downarrow_\beta}, a\rangle \in \mathcal{D}_\alpha \mid a \text{ is a possible value for } \mathbf{A}\}.$$

for each $\mathbf{A} \in \mathit{cwff}_\alpha$.

DEFINITION 7 (Definition 4.1.5 from [6, 7]). Let $\mathcal{A} \equiv (\mathcal{D}, @)$ be a possible values structure for β and φ be an assignment into $\mathcal{A}$. For any $\mathbf{A} \in \mathit{wff}_\alpha(\Sigma)$, we define $\varphi_1(\mathbf{A})$ to be $\theta(\mathbf{A}) \in \mathit{cwff}_\alpha(\Sigma)$ where θ is the substitution with $\mathbf{Dom}(\theta) \equiv free(\mathbf{A})$ and $\varphi(x_\beta) \equiv \langle \theta(x_\beta), b\rangle \in \mathcal{D}_\beta$ for each variable $x_\beta \in free(\mathbf{A})$. We define $\varphi_1^\beta(\mathbf{A})$ to be $\varphi_1(\mathbf{A})\big\downarrow_\beta$.

DEFINITION 8 (Definition 4.1.7 from [6, 7]). We call an evaluation $\mathcal{J} \equiv (\mathcal{D}, @, \mathcal{E})$ a *possible values evaluation for* β if $(\mathcal{D}, @)$ is a possible values structure for β and $\mathcal{E}_\varphi(\mathbf{A}) \in \mathcal{D}_\alpha^{\varphi_1(\mathbf{A})}$ for every $\mathbf{A} \in \mathit{wff}_\alpha(\Sigma)$ and assignment φ.

We can always extend an appropriate interpretation of parameters and constants in a possible values structure to obtain a possible values evaluation.

THEOREM 9 (Theorem 4.1.8 from [6, 7]). *Let $\mathcal{A} \equiv (\mathcal{D}, @)$ be a possible values structure for β and $\mathcal{I} : \Sigma \longrightarrow \mathcal{D}$ be an interpretation of parameters and constants such that $\mathcal{I}(c_\alpha) \in \mathcal{D}_\alpha^c$ for every $c \in \Sigma$. There is an evaluation function $\mathcal{E}$ such that $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$ is a possible values evaluation for β, $\mathcal{E}(c_\alpha) \equiv \mathcal{I}(c_\alpha)$ for every $c_\alpha \in \Sigma$.*

We now verify the only new result of this section: possible values evaluations are ξ-functional.

PROPOSITION 10. *Every possible values evaluation for β is ξ-functional.*

Proof. Let $(\mathcal{D}, @, \mathcal{E})$ be a possible values evaluation. Let $\mathbf{M}, \mathbf{N} \in \mathit{wff}_\beta(\Sigma)$ and X_α be a variable such that $\mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{M}) \equiv \mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{N})$ for all $\mathsf{a} \in \mathcal{D}_\alpha$. We must verify $\mathcal{E}_\varphi(\lambda X_\alpha \mathbf{.M}) \equiv \mathcal{E}_\varphi(\lambda X_\alpha \mathbf{.N})$. We know $\mathcal{E}_\varphi(\lambda X_\alpha \mathbf{.M}) \equiv \langle \varphi_1^\beta(\lambda X_\alpha \mathbf{.M}), f\rangle$ and $\mathcal{E}_\varphi(\lambda X_\alpha \mathbf{.N}) \equiv \langle \varphi_1^\beta(\lambda X_\alpha \mathbf{.N}), g\rangle$ for some $f, g : \mathcal{D}_\alpha \to \mathcal{D}_\beta$. We first check that the first components are equal. Let w_α be a parameter which occurs neither in $\mathbf{M}$ nor in $\mathbf{N}$. By Lemma 5 there is some p such that $\langle w, p\rangle \in \mathcal{D}_\alpha$. By assumption, $\mathcal{E}_{\varphi,[\langle w,p\rangle/X]}(\mathbf{M}) \equiv \mathcal{E}_{\varphi,[\langle w,p\rangle/X]}(\mathbf{N})$. Since $\mathcal{E}$ is a possible values evaluation, the first component of $\mathcal{E}_{\varphi,[\langle w,p\rangle/X]}(\mathbf{M})$ is $(\varphi, [\langle w,p\rangle/X])_1^\beta(\mathbf{M})$. It is easy to see that this is the same as $[w/X]\varphi_1^\beta(\mathbf{M})$. Similarly, the first component of $\mathcal{E}_{\varphi,[\langle w,p\rangle/X]}(\mathbf{N})$ is $[w/X]\varphi_1^\beta(\mathbf{N})$. Hence $[w/X]\varphi_1^\beta(\mathbf{M}) \equiv [w/X]\varphi_1^\beta(\mathbf{N})$. Since w was chosen to be fresh, $\varphi_1^\beta(\mathbf{M}) \equiv \varphi_1^\beta(\mathbf{N})$ and so

$$\varphi_1^\beta(\lambda X \mathbf{.M}) \equiv \lambda X \mathbf{.}\varphi_1^\beta(\mathbf{M}) \equiv \lambda X \mathbf{.}\varphi_1^\beta(\mathbf{N}) \equiv \varphi_1^\beta(\lambda X \mathbf{.N}).$$

Next, we show the second components are equal. Using the properties of evaluation functions and the definition of @, we easily compute

$$f(\mathsf{a}) \equiv \mathcal{E}_\varphi(\lambda X.\mathbf{M})@\mathsf{a} \equiv \mathcal{E}_{\varphi,[\mathsf{a}/X]}((\lambda X.\mathbf{M})X) \equiv \mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{M})$$

and

$$g(\mathsf{a}) \equiv \mathcal{E}_\varphi(\lambda X.\mathbf{N})@\mathsf{a} \equiv \mathcal{E}_{\varphi,[\mathsf{a}/X]}((\lambda X.\mathbf{N})X) \equiv \mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{N})$$

for any $\mathsf{a} \in \mathcal{D}_\alpha$. Hence $f = g$ as desired. ∎

4 Model Existence Theorems Without Saturation

Model existence theorems generally say that in order to show that a set Φ of formulae has a model $\mathcal{M}$ in a given class $\mathfrak{M}$, it is sufficient to prove that Φ is a member of suitably defined abstract consistency classes Γ. Model existence theorems are usually proven in two steps: first we show that any $\Phi \in \Gamma$ can be extended to a Hintikka set $\mathcal{H} \in \Gamma$ with $\Phi \subseteq \mathcal{H}$, and then for a given Hintikka set $\mathcal{H}$ we construct a model $\mathcal{M} \in \mathfrak{M}$ that satisfies $\mathcal{H}$. The first step is already addressed by the Abstract Extension Lemma (Lemma 6.32) in [3] and it will be reused below. The second step — for the model classes $\mathfrak{M}_\beta$ and $\mathfrak{M}_{\beta\xi}$ and without assuming saturation — is a novel contribution of this paper.

When constructing models in $\mathfrak{M}_*$ of a Hintikka set $\mathcal{H}$, we must verify property $\mathfrak{q}$. For this purpose, the assumption that $\mathcal{H}$ contains no Leibniz equations is very helpful.

DEFINITION 11. Let $\mathcal{H}$ be a set of formulae. We say $\mathcal{H}$ is *Leibniz-free* if there are no terms $\mathbf{A}_\alpha, \mathbf{B}_\alpha$ such that $(\mathbf{A} \doteq^\alpha \mathbf{B}) \in \mathcal{H}$.

We can now show every Hintikka set is either saturated (in which case we have already constructed models in [3]) or Leibniz-free. Hence we will only need to construct models for Leibniz-free Hintikka sets. This result is closely related to the fact the Leibniz equations are cut-strong (see Example 14 in [4]).

THEOREM 12. *Let $\mathcal{H}$ be a Hintikka set. Either $\mathcal{H}$ is saturated or $\mathcal{H}$ is Leibniz-free.*

Proof. Suppose $\mathcal{H}$ is not Leibniz-free. Then $(\mathbf{A} \doteq^\alpha \mathbf{B}) \in \mathcal{H}$ for some $\mathbf{A}_\alpha, \mathbf{B}_\alpha$. We show $\mathcal{H}$ satisfies $\vec{\nabla}_{sat}$. Let $\mathbf{C}_o$ be a closed formula. Since $(\forall Q_{\alpha\to o}.Q\mathbf{A} \Rightarrow Q\mathbf{B}) \in \mathcal{H}$, we know $(\neg\mathbf{C} \vee \mathbf{C}) \in \mathcal{H}$ by $\vec{\nabla}_\forall$ (with the term $\lambda X_\alpha.\mathbf{C}$) and $\vec{\nabla}_\beta$. By $\vec{\nabla}_\vee$, either $\neg\mathbf{C} \in \mathcal{H}$ or $\mathbf{C} \in \mathcal{H}$. ∎

The proof of the following theorem is the main contribution in the paper. It the construction is based on Andrew's V-complexes but extends the argument by checking properties ξ (from Prop 10) and $\mathfrak{q}$ (by choosing Leibniz and using that H is Leibniz free since unsaturated).

THEOREM 13. *Let $\mathcal{H}$ be a Σ-Hintikka set which is not saturated. There is a Σ-model $\mathcal{M} \in \mathfrak{M}_{\beta\xi}$ such that $\mathcal{M} \models \mathcal{H}$.*

Proof. We first define a set $\mathcal{B}^{\mathbf{A}}_{\mathcal{H}}$ of *possible booleans* for each $\mathbf{A} \in \mathit{wff}_o(\Sigma)$:

$$\mathcal{B}^{\mathbf{A}}_{\mathcal{H}} := \begin{cases} \{\mathtt{T}\} & \text{if } \mathbf{A} \in \mathcal{H} \\ \{\mathtt{F}\} & \text{if } \neg\mathbf{A} \in \mathcal{H} \\ \{\mathtt{T}, \mathtt{F}\} & \text{otherwise.} \end{cases}$$

We define $\mathcal{D}_\alpha$ for each type $\alpha \in \mathcal{T}$ by induction:

- $\mathcal{D}_o := \{\langle \mathbf{A}_o, p\rangle | \mathbf{A} \in \mathit{wff}_o(\Sigma){\downarrow_\beta}, p \in \mathcal{B}^{\mathbf{A}}_{\mathcal{H}}\}$.
- $\mathcal{D}_\iota := \{\langle \mathbf{A}_\iota, \iota\rangle | \mathbf{A} \in \mathit{wff}_\iota(\Sigma){\downarrow_\beta}\}$.
- $\mathcal{D}_{\alpha\to\beta} := \{\langle \mathbf{F}_{\alpha\to\beta}, f\rangle | \mathbf{F} \in \mathit{wff}_{\alpha\to\beta}(\Sigma){\downarrow_\beta}, f : \mathcal{D}_\alpha \to \mathcal{D}_\beta,$

  $$\forall \langle \mathbf{A}, a\rangle \in \mathcal{D}_\alpha, \langle \mathbf{B}, b\rangle \in \mathcal{D}_\beta\, f(\langle \mathbf{A}, a\rangle) \equiv \langle \mathbf{B}, b\rangle \Rightarrow \mathbf{B} \equiv (\mathbf{F}\,\mathbf{A}){\downarrow_\beta}\}.$$

We define an application operator @ by setting $\langle \mathbf{F}, f\rangle @ \mathsf{a}$ to be $f(\mathsf{a})$ for each $\langle \mathbf{F}, f\rangle \in \mathcal{D}_{\alpha\to\beta}$ and $\mathsf{a} \in \mathcal{D}_\alpha$. It is easy to check that $(\mathcal{D}, @)$ is a possible values structure for β. Note that for all $\mathbf{A} \in \mathit{wff}_o(\Sigma)$ either $\mathbf{A} \notin \mathcal{H}$ or $\neg\mathbf{A} \notin \mathcal{H}$ (by $\vec{\nabla}_c$) and so either $\langle \mathbf{A}, \mathtt{F}\rangle \in \mathcal{D}_o$ or $\langle \mathbf{A}, \mathtt{T}\rangle \in \mathcal{D}_o$. (It is possible that both $\langle \mathbf{A}, \mathtt{F}\rangle \in \mathcal{D}_o$ and $\langle \mathbf{A}, \mathtt{T}\rangle \in \mathcal{D}_o$.)

For each parameter w_α, we can choose some p^w such that $\langle w, p^w\rangle \in \mathcal{D}_\alpha$ using Lemma 5. These values can be used to interpret parameters. To interpret logical constants, we must make appropriate choices so that the corresponding logical properties will hold.

$\neg$ Let $p^\neg : \mathcal{D}_o \to \mathcal{D}_o$ be defined by $p^\neg(\langle \mathbf{A}, a\rangle) := \langle \neg\mathbf{A}, b\rangle$ where b is $\mathtt{T}$ if a is $\mathtt{F}$ and b is $\mathtt{F}$ if a is $\mathtt{T}$. The $\vec{\nabla}_\neg$ and $\vec{\nabla}_c$ properties of $\mathcal{H}$ guarantees this is well-defined. So, $p^\neg$ is a possible value for $\neg$.

$\vee$ For each $\langle \mathbf{A}, \mathtt{F}\rangle \in \mathcal{D}_o$, let $p^\vee_{\langle \mathbf{A}, \mathtt{F}\rangle} : \mathcal{D}_o \to \mathcal{D}_o$ be the function defined by $p^\vee_{\langle \mathbf{A}, \mathtt{F}\rangle}(\langle \mathbf{B}, b\rangle) := \langle \mathbf{A} \vee \mathbf{B}, b\rangle$. For each $\langle \mathbf{A}, \mathtt{T}\rangle \in \mathcal{D}_o$, let $p^\vee_{\langle \mathbf{A}, \mathtt{T}\rangle} : \mathcal{D}_o \to \mathcal{D}_o$ be the function defined by $p^\vee_{\langle \mathbf{A}, \mathtt{T}\rangle}(\langle \mathbf{B}, b\rangle) := \langle \mathbf{A} \vee \mathbf{B}, \mathtt{T}\rangle$.

The properties $\vec{\nabla}_\vee$, $\vec{\nabla}_\wedge$ and $\vec{\nabla}_c$ of $\mathcal{H}$ guarantees these are well-defined and $\langle \vee\mathbf{A}, p^\vee_{\langle \mathbf{A}, a\rangle}\rangle \in \mathcal{D}_{o\to o}$. Now, let $p^\vee : \mathcal{D}_o \to \mathcal{D}_{o\to o}$ be defined by $p^\vee(\langle \mathbf{A}, a\rangle) := \langle \vee\mathbf{A}, p^\vee_{\langle \mathbf{A}, a\rangle}\rangle$. Clearly, $p^\vee$ is a possible value for $\vee$.

Π^α Let $p^{\Pi^\alpha} : \mathcal{D}_{\alpha\to o} \to \mathcal{D}_o$ be the function defined by $p^{\Pi^\alpha}(\mathbf{F}, f) := \langle \Pi^\alpha \mathbf{F}, p\rangle$ where $p \equiv \mathtt{T}$ if for every $\langle \mathbf{A}, a\rangle \in \mathcal{D}_\alpha$, the second component of $f(\langle \mathbf{A}, a\rangle)$ is $\mathtt{T}$, and $p \equiv \mathtt{F}$ otherwise. This is well-defined by $\vec{\nabla}_\forall$, $\vec{\nabla}_\exists$ and $\vec{\nabla}_c$, and p^{Π^α} is a possible value for Π^α.

Let $\mathcal{I}(c) := \langle c, p^c \rangle$ for each $c \in \Sigma$ and $\mathcal{E}$ be the evaluation function extending $\mathcal{I}$ guaranteed to exist by Theorem 9 so that $(\mathcal{D}, @, \mathcal{E})$ is a ξ-functional possible values evaluation.

To make this a Σ-model, we must define a valuation $\upsilon : \mathcal{D}_o \longrightarrow \{\mathtt{T}, \mathtt{F}\}$. We take the obvious choice $\upsilon(\langle \mathbf{A}, p \rangle) := p$. So let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$. To check $\mathcal{M}$ is a Σ-model, we must check the requirements for υ. Each condition is trivial:

$\neg$: $\upsilon(\mathcal{E}(\neg)@\mathsf{a}) \equiv \mathtt{T}$ iff $\upsilon(\mathsf{a}) \equiv \mathtt{F}$ by the definition of $p^{\neg}$.

$\vee$: $\upsilon(\mathcal{E}(\vee)@\mathsf{a}@\mathsf{b}) \equiv \mathtt{T}$ iff $\upsilon(\mathsf{a}) \equiv \mathtt{T}$ or $\upsilon(\mathsf{b}) \equiv \mathtt{T}$ by the definition of $p^{\vee}$.

Π: $\upsilon(\mathcal{E}(\Pi^\alpha)@\mathsf{f}) \equiv \mathtt{T}$ iff $\upsilon(\mathsf{f}@\mathsf{a}) \equiv \mathtt{T}$ for each $\mathsf{a} \in \mathcal{D}_\alpha$ by the definition of p^{Π^α}.

We verify $\mathcal{M} \models \mathcal{H}$. Suppose $\mathbf{A} \in \mathcal{H}$ and let $\mathbf{B}$ be $\mathbf{A}{\downarrow}_\beta$. Note that $\mathcal{E}(\mathbf{A}) \equiv \langle \mathbf{B}, p \rangle \in \mathcal{D}_o$ for some $p \in \mathcal{B}^{\mathbf{B}}_{\mathcal{H}}$. Since $\mathbf{A} \in \mathcal{H}$, we have $\mathbf{B} \in \mathcal{H}$ by $\vec{\nabla}_\beta$. Thus $\mathcal{B}^{\mathbf{B}}_{\mathcal{H}} \equiv \{\mathtt{T}\}$, $p \equiv \mathtt{T}$ and so $\mathcal{M} \models \mathbf{A}$.

In general, we can use Theorem 3.62 in [3] to obtain a model of $\mathcal{H}$ satisfying property $\mathfrak{q}$, though this would not preserve property ξ (cf. Remark 3.57 in [3]). Instead, we use the assumption that $\mathcal{H}$ is not saturated and hence Leibniz-free to show the possible values model $\mathcal{M}$ *already* satisfies property $\mathfrak{q}$. To see this, for each $\langle \mathbf{A}, a \rangle \in \mathcal{D}_\alpha$, let $s_{\langle \mathbf{A}, a \rangle} : \mathcal{D}_\alpha \to \mathcal{D}_o$ be defined by

$$s_{\langle \mathbf{A}, a \rangle}(\langle \mathbf{B}, b \rangle) := \begin{cases} \langle (\mathbf{A} \doteq \mathbf{A}){\downarrow}_\beta, \mathtt{T} \rangle & \text{if } \mathbf{A} = \mathbf{B} \text{ and } a = b \\ \langle (\mathbf{A} \doteq \mathbf{B}){\downarrow}_\beta, \mathtt{F} \rangle & \text{else} \end{cases}$$

This is well-defined since we never have $\neg\,(\mathbf{A} \doteq \mathbf{A}){\downarrow}_\beta \in \mathcal{H}$, and at the same time $(\mathbf{A} \doteq \mathbf{B}){\downarrow}_\beta \notin \mathcal{H}$ since $\mathcal{H}$ is Leibniz-free. Then, $\mathfrak{q}^\alpha := \langle \doteq^\alpha, \mathsf{l} \rangle$ with $l(\langle \mathbf{A}, a \rangle) := \langle (\lambda X.\mathbf{A} \doteq x){\downarrow}_\beta, s_{\langle \mathbf{A}, a \rangle} \rangle$ witnesses that $\mathcal{M}$ satisfies property $\mathfrak{q}$. Thus, $\mathcal{M} \in \mathfrak{M}_{\beta\xi}$ as desired. ■

THEOREM 14 (Model Existence for $\mathfrak{Hint}_\beta$ and $\mathfrak{Hint}_{\beta\xi}$). *For each $* \in \{\beta, \beta\xi\}$ and Σ-Hintikka set $\mathcal{H} \in \mathfrak{Hint}_*$, there is a Σ-model $\mathcal{M} \in \mathfrak{M}_*$ such that $\mathcal{M} \models \mathcal{H}$.*

Proof. If $\mathcal{H}$ is not saturated, then we can obtain such an $\mathcal{M}$ by applying Theorem 13 above. If $\mathcal{H}$ is saturated, then we can obtain such an $\mathcal{M}$ by applying the Model Existence Theorem for Saturated Sets (Theorem 1). ■

THEOREM 15 (Model Existence for $\mathfrak{Acc}_\beta$ and $\mathfrak{Acc}_{\beta\xi}$). *For each $* \in \{\beta, \beta\xi\}$, abstract consistency class $\Gamma_\Sigma \in \mathfrak{Acc}_*$ and sufficiently Σ-pure $\Phi \in \Gamma_\Sigma$, there is a Σ-model $\mathcal{M} \in \mathfrak{M}_*$ such that $\mathcal{M} \models \Phi$.*

Proof. By the Abstract Extension Lemma (Lemma 6.32 in [3]), there is a Hintikka set $\mathcal{H} \in \mathfrak{Hint}_*$ such that $\Phi \subseteq \mathcal{H}$. By Theorem 14 above there is a Σ-model $\mathcal{M} \in \mathfrak{M}_*$ such that $\mathcal{M} \models \mathcal{H}$. ■

5 A Sequent Calculus

As in [4, 5], we consider a sequent to be a finite set Δ of β-normal sentences from $cwff_o(\Sigma)$. A sequent calculus $\mathcal{G}$ provides an inductive definition for when $\Vdash_{\mathcal{G}} \Delta$ holds. We say a sequent calculus rule

$$\frac{\Delta_1 \quad \cdots \quad \Delta_n}{\Delta}$$

is *admissible* if $\Vdash_{\mathcal{G}} \Delta$ holds whenever $\Vdash_{\mathcal{G}} \Delta_i$ for all $1 \leq i \leq n$. Given a sequent Δ and a model $\mathcal{M}$, we say Δ is *valid for* $\mathcal{M}$ if $\mathcal{M} \models \mathbf{D}$ for some $\mathbf{D} \in \Delta$. For a class $\mathfrak{M}$ of models, we say Δ is *valid for* $\mathfrak{M}$ if Δ is valid for every $\mathcal{M} \in \mathfrak{M}$. As for sets in abstract consistency classes, we use the notation $\Delta * \mathbf{A}$ to denote the set $\Delta \cup \{\mathbf{A}\}$ (which is simply Δ if $\mathbf{A} \in \Delta$). We adopt the notation $\neg\Phi$ for the set $\{\neg\mathbf{A} | \mathbf{A} \in \Phi\}$ where $\Phi \subseteq cwff_o(\Sigma)$. Furthermore, we assume this use of $\neg$ binds more strongly than $\cup$ or $*$, so that $\neg\Phi \cup \Delta$ means $(\neg\Phi) \cup \Delta$ and $\neg\Phi * \mathbf{A}$ means $(\neg\Phi) * \mathbf{A}$. For any sequent calculus $\mathcal{G}$, we can define a class of sets of sentences $\Gamma_\Sigma^{\mathcal{G}}$ as in [4, 5].

DEFINITION 16 (Definition 1 from [4]/Definition 3.1 from [5]). Let $\mathcal{G}$ be a sequent calculus. We define $\Gamma_\Sigma^{\mathcal{G}}$ to be the class of all finite $\Phi \subset cwff_o(\Sigma)$ such that $\Vdash_{\mathcal{G}} \neg\, \Phi{\downarrow_\beta}$ does not hold.

Under certain conditions, $\Gamma_\Sigma^{\mathcal{G}}$ will be an abstract consistency class. The conditions are the admissibility of certain rules given in Figures 2 and 3.

LEMMA 17 (Lemma 2 from [4]/Lemma 3.2 from [5]). *Let $\mathcal{G}$ be a sequent calculus such that $\mathcal{G}(Inv^\neg)$ is admissible. For any finite sets Φ and Δ of sentences, if $\Phi \cup \neg\Delta \notin \Gamma_\Sigma^{\mathcal{G}}$, then $\Vdash_{\mathcal{G}} \neg\, \Phi{\downarrow_\beta} \cup \Delta{\downarrow_\beta}$ holds.*

THEOREM 18 (Theorem 3 from [4]/Theorem 3.3 from [5]). *Let $\mathcal{G}$ be a sequent calculus. If the rules $\mathcal{G}(Inv^\neg)$, $\mathcal{G}(\neg)$, $\mathcal{G}(weak)$, $\mathcal{G}(init)$, $\mathcal{G}(\vee_-)$, $\mathcal{G}(\vee_+)$, $\mathcal{G}(\Pi_-^{\mathbf{C}})$ and $\mathcal{G}(\Pi_+^c)$ are admissible in $\mathcal{G}$, then $\Gamma_\Sigma^{\mathcal{G}} \in \mathfrak{Acc}_\beta$.*

We also have the following result relating saturation with admissibility of cut.

THEOREM 19 (Theorem 4 from [4]/Theorem 3.4 from [5]). *Let $\mathcal{G}$ be a sequent calculus.*

1. *If $\mathcal{G}(cut)$ is admissible in $\mathcal{G}$, then $\Gamma_\Sigma^{\mathcal{G}}$ is saturated.*
2. *If $\mathcal{G}(\neg)$ and $\mathcal{G}(Inv^\neg)$ are admissible in $\mathcal{G}$ and $\Gamma_\Sigma^{\mathcal{G}}$ is saturated, then $\mathcal{G}(cut)$ is admissible in $\mathcal{G}$.*

$$\frac{\mathbf{A} \text{ atomic}}{\Delta * \mathbf{A} * \neg\mathbf{A}}\,\mathcal{G}(init) \qquad \frac{\Delta * \mathbf{A}}{\Delta * \neg\neg\mathbf{A}}\,\mathcal{G}(\neg)$$

$$\frac{\Delta * \neg\mathbf{A} \quad \Delta * \neg\mathbf{B}}{\Delta * \neg(\mathbf{A} \vee \mathbf{B})}\,\mathcal{G}(\vee_-) \qquad \frac{\Delta * \mathbf{A} * \mathbf{B}}{\Delta * (\mathbf{A} \vee \mathbf{B})}\,\mathcal{G}(\vee_+)$$

$$\frac{\Delta * \neg\,(\mathbf{AC}){\downarrow_\beta} \quad \mathbf{C} \in cwff_\alpha(\Sigma)}{\Delta * \neg\Pi^\alpha\mathbf{A}}\,\mathcal{G}(\Pi_-^{\mathbf{C}})$$

$$\frac{\Delta * (\mathbf{A}c){\downarrow_\beta} \quad c_\alpha \in \Sigma \;\text{ fresh parameter}}{\Delta * \Pi^\alpha\mathbf{A}}\,\mathcal{G}(\Pi_+^c)$$

Figure 2. Basic Sequent Calculus Rules

$$\frac{\Delta * \neg\neg\mathbf{A}}{\Delta * \mathbf{A}}\,\mathcal{G}(Inv^\neg)$$

$$\frac{\Delta}{\Delta \cup \Delta'}\,\mathcal{G}(weak) \qquad \frac{\Delta * \mathbf{C} \quad \Delta * \neg\mathbf{C}}{\Delta}\,\mathcal{G}(cut)$$

Figure 3. Inversion Rule, Weakening Rule and Cut Rule

The proofs of the previous three results are given in the appendix of [5].

We now turn our attention to the two particular sequent calculi of interest in this paper.

DEFINITION 20 (Sequent Calculi $\mathcal{G}_\beta$ and $\mathcal{G}_{\beta\xi}$). Let $\mathcal{G}_\beta$ be the sequent calculus defined by the rules in Figure 2. Let $\mathcal{G}_{\beta\xi}$ be the sequent calculus defined by the rules in Figure 2 and the $\mathcal{G}(\xi)$ rule in Figure 4

A straightforward induction on derivations proves that $\mathcal{G}_\beta$ and $\mathcal{G}_{\beta\xi}$ are sound with respect to to the model classes $\mathfrak{M}_\beta$ and $\mathfrak{M}_{\beta\xi}$, respectively. The only case which presents any difficulty is that for $\mathcal{G}(\Pi_+^c)$ which uses a fresh

$$\frac{\Delta * (\forall X_\alpha.\mathbf{M} \doteq^{\beta} \mathbf{N})}{\Delta * (\lambda X_\alpha.\mathbf{M} \doteq^{\alpha\to\beta} \lambda X_\alpha.\mathbf{N})}\ \mathcal{G}(\xi)$$

Figure 4. ξ Extensionality Rule

parameter c. We will show only this case. In this case one can modify a given model by changing the value of the parameter c in the model. This is worked out in detail in [7] and we will refer to some of the results there.

THEOREM 21. *Let* $* \in \{\beta, \beta\xi\}$ *and* Δ *be a sequent. If* $\Vdash_{\mathcal{G}_*} \Delta$*, then for all* $\mathcal{M} \in \mathfrak{M}_*$ *there is some* $\mathbf{A} \in \Delta$ *such that* $\mathcal{M} \models \mathbf{A}$*.*

Proof. This can be proven by induction on the derivation of $\Vdash_{\mathcal{G}_*} \Delta$. Suppose $\mathcal{G}(\Pi_+^c)$ is the last rule of the derivation. Then Δ is $\Delta' * \Pi^\alpha\mathbf{A}$ and $\Vdash_{\mathcal{G}_*} \Delta' * (\mathbf{A}c)\big\downarrow_\beta$ for some parameter c which occurs neither in $\mathbf{A}$ nor in any sentence in Δ'. Let $\mathcal{M} \equiv (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_*$ be given. If $\mathcal{M} \models \mathbf{B}$ for some $\mathbf{B} \in \Delta'$, then we are done. Assume there is no such $\mathbf{B} \in \Delta'$, then we must prove $\mathcal{M} \models \Pi^\alpha\mathbf{A}$, i.e. that $\upsilon(\mathcal{E}(\mathbf{A})@\mathsf{a}) \equiv \mathtt{T}$ for all $\mathsf{a} \in \mathcal{D}_\alpha$. Let $\mathsf{a} \in \mathcal{D}_\alpha$ be given. We let $\mathcal{E}^{c\mapsto\mathsf{a}}$ denote the function from Definition 3.2.16 in [7] and $\mathcal{M}^{c\mapsto\mathsf{a}}$ denote $(\mathcal{D}, @, \mathcal{E}^{c\mapsto\mathsf{a}}, \upsilon)$. We have the following:

- $\mathcal{E}^{c\mapsto\mathsf{a}}(c) \equiv \mathsf{a}$ (see Theorem 3.2.18 in [7]).
- $\mathcal{E}^{c\mapsto\mathsf{a}}(\mathbf{D}) \equiv \mathcal{E}(\mathbf{D})$ if c does not occur in $\mathbf{D}$ (see Theorem 3.2.18 in [7]).
- $\mathcal{M}^{c\mapsto\mathsf{a}} \in \mathfrak{M}_*$ (see Theorem 3.3.14 in [7]).

Applying the inductive hypothesis using $\mathcal{M}^{c\mapsto\mathsf{a}}$, we have $\mathcal{M}^{c\mapsto\mathsf{a}} \models (\mathbf{A}c)\big\downarrow_\beta$. Hence $\upsilon(\mathcal{E}^{c\mapsto\mathsf{a}}(\mathbf{A}c)) \equiv \mathtt{T}$. Using the properties above, we have $\upsilon(\mathcal{E}(\mathbf{A})@\mathsf{a}) \equiv \mathtt{T}$ as desired. ∎

We can also prove that $\mathcal{G}_\beta$ and $\mathcal{G}_{\beta\xi}$ are complete with respect to the model classes $\mathfrak{M}_\beta$ and $\mathfrak{M}_{\beta\xi}$, respectively. In order to apply the results from [4, 5], we begin by noting that certain rules are admissible.

LEMMA 22. $\mathcal{G}(weak)$ *and* $\mathcal{G}(Inv^\neg)$ *(see Figure 3) are admissible in* $\mathcal{G}_\beta$ *and* $\mathcal{G}_{\beta\xi}$*.*

Proof. Both of these follow by an induction on derivations. In the case of weakening we must also carry a parameter renaming to ensure freshness of the parameter in each application of $\mathcal{G}(\Pi_+^c)$. ∎

Using this result, we can conclude that $\Gamma_\Sigma^{\mathcal{G}_\beta}$ and $\Gamma_\Sigma^{\mathcal{G}_{\beta\xi}}$ are abstract consistency classes.

PROPOSITION 23. *$\Gamma_\Sigma^{\mathcal{G}_\beta} \in \mathfrak{Acc}_\beta$ and $\Gamma_\Sigma^{\mathcal{G}_{\beta\xi}} \in \mathfrak{Acc}_{\beta\xi}$.*

Proof. By Lemma 22 and Theorem 18. we know $\Gamma_\Sigma^{\mathcal{G}_\beta} \in \mathfrak{Acc}_\beta$ and $\Gamma_\Sigma^{\mathcal{G}_{\beta\xi}} \in \mathfrak{Acc}_\beta$. To complete the proof, we must verify ∇_ξ holds in $\Gamma_\Sigma^{\mathcal{G}_{\beta\xi}}$. Suppose $\neg(\lambda X_\alpha\mathbf{.M} \doteq^{\alpha\to\beta} \lambda X_\alpha\mathbf{.N}) \in \Phi \in \Gamma_\Sigma^{\mathcal{G}_{\beta\xi}}$ but $\Phi * \neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N}) \notin \Gamma_\Sigma^{\mathcal{G}_{\beta\xi}}$ where w_α is a parameter which does not occur in any sentence in Φ. By Lemma 17, we have $\Vdash_{\mathcal{G}_{\beta\xi}} \neg\, \Phi{\downarrow_\beta} * ([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N})\Big\downarrow_\beta$. Using the rule $\mathcal{G}(\Pi_+^w)$, we have $\Vdash_{\mathcal{G}_{\beta\xi}} \neg\, \Phi{\downarrow_\beta} * (\forall X\mathbf{.}(\mathbf{M}{\downarrow_\beta} \doteq^\beta \mathbf{N}{\downarrow_\beta})$. Using the rule $\mathcal{G}(\xi)$, we have $\Vdash_{\mathcal{G}_{\beta\xi}} \neg\, \Phi{\downarrow_\beta} * (\lambda X\mathbf{.M} \doteq^{\alpha\to\beta} \lambda X\mathbf{.N})\Big\downarrow_\beta$. Using the rule $\mathcal{G}(\neg)$, we have $\Vdash_{\mathcal{G}_{\beta\xi}} \neg\, \Phi{\downarrow_\beta} * \neg\neg\, (\lambda X\mathbf{.M} \doteq^{\alpha\to\beta} \lambda X\mathbf{.N})\Big\downarrow_\beta$. and so $\Vdash_{\mathcal{G}_{\beta\xi}} \neg\, \Phi{\downarrow_\beta}$ since $\neg(\lambda X_\alpha\mathbf{.M} \doteq^{\alpha\to\beta} \lambda X_\alpha\mathbf{.N}) \in \Phi$. This contradicts $\Phi \in \Gamma_\Sigma^{\mathcal{G}_{\beta\xi}}$. ■

We can now prove completeness.

THEOREM 24. *Let $* \in \{\beta, \beta\xi\}$ and Δ be a sequent. If for all $\mathcal{M} \in \mathfrak{M}_*$ there is some $\mathbf{A} \in \Delta$ such that $\mathcal{M} \models \mathbf{A}$, then $\Vdash_{\mathcal{G}_*} \Delta$.*

Proof. Assume Δ is a sequent such that $\not\Vdash_{\mathcal{G}_*} \Delta$. Our goal is to find a model $\mathcal{M} \in \mathfrak{M}_*$ such that $\mathcal{M} \not\models \mathbf{A}$ for all $\mathbf{A} \in \Delta$ (i.e., $\mathcal{M} \models \neg\Delta$). Since $\mathcal{G}(Inv^\neg)$ is admissible, we can apply Lemma 17 to conclude that $\neg\Delta \in \Gamma_\Sigma^{\mathcal{G}_*}$. Since $\neg\Delta$ is finite, it is sufficiently Σ-pure. Hence we obtain an $\mathcal{M} \in \mathfrak{M}_*$ such that $\mathcal{M} \models \neg\Delta$ by applying Theorem 15. ■

Consequently, cut is admissible in both calculi.

COROLLARY 25. *For each $* \in \{\beta, \beta\xi\}$, the cut rule $\mathcal{G}(cut)$ is admissible in the calculus $\mathcal{G}_*$.*

Proof. Let Δ be a sequent and $\mathbf{C}$ be a sentence such that $\Vdash_{\mathcal{G}_*} \Delta * \mathbf{C}$ and $\Vdash_{\mathcal{G}_*} \Delta * \neg\mathbf{C}$. Using Theorem 24 we can prove $\Vdash_{\mathcal{G}_*} \Delta$ by proving for every $\mathcal{M} \in \mathfrak{M}_*$ there is some $\mathbf{A} \in \Delta$ such that $\mathcal{M} \models \mathbf{A}$. Let $\mathcal{M} \in \mathfrak{M}_*$ be given. Assume $\mathcal{M} \not\models \mathbf{A}$ for all $\mathbf{A} \in \Delta$. By soundness (Theorem 21), $\mathcal{M} \models \mathbf{C}$ since $\Vdash_{\mathcal{G}_*} \Delta * \mathbf{C}$. Also, $\mathcal{M} \models \neg\mathbf{C}$ since $\Vdash_{\mathcal{G}_*} \Delta * \neg\mathbf{C}$. This is a contradiction. ■

Note that since cut is admissible, we can conclude that $\Gamma_\Sigma^{\mathcal{G}_*}$ is saturated (by Theorem 19). If we had known $\Gamma_\Sigma^{\mathcal{G}_*}$ were saturated in advance, then we could have used the model existence theorems from [3] instead of the new model existence theorems proven in this paper. However, there seems to be no easier way to prove $\Gamma_{\Sigma}^{\mathcal{G}_*}$ is saturated than to prove cut-elimination, and there seems to be no easier way to prove cut-elimination than construction of a V-complex/possible values style of model.

6 Conclusion and Further Work

In this paper, we have employed a construction based Peter Andrews' V-complexes to prove a model existence theorem for a form of higher-order logic with a weak form of functional extensionality.

In [3] we have introduced and studied eight different model classes (including Henkin models) for classical type theory which generalize the notion of standard models and which allow for complete calculi. These model classes were motivated by different roles of extensionality and they adequately characterize the deductive power of existing theorem-proving calculi. Unfortunately, the model existence theorems in [3] assume saturation, which makes them useless for proving completeness of machine-oriented calculi since saturation is equivalent to cut-elimination.

This paper addresses the saturation problem for two of the model classes. This gives a framework that supports the development and proof-theoretical investigation of human-oriented as well as machine-oriented (ground) calculi for the corresponding type theories. For non-ground machine-oriented calculi the lifting issue has to be additionally addressed and extending our framework by tools that may also support lifting arguments remains future work.

For a complete picture, the results reported here need to be extended for the remaining six model classes, and for logics that include primitive equality (see, e.g. [2]). As mentioned in the introduction, the essential ingredients for handling two of the other six model classes are in [6, 7]. The remaining four cases include the case with full functional extensionality but not Boolean extensionality and the three cases with Boolean extensionality but not full functional extensionality.

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