Computer-Assisted Analysis of the Anderson-Hájek Ontological Controversy

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1. Introduction

We present an exemplary study in *Computational Metaphysics*, which is an emerging, interdisciplinary field aiming at the rigorous formalisation and deep logical assessment of rational, philosophical arguments on the computer. The particular focus here is on the ontological argument for the existence of God. While related work [9, 7] has concentrated on Anselm's simpler original version of the ontological argument and formalized it in classical predicate logic, we here focus on modern variants of Kurt Gödel's seminal contribution [12] requiring higher-order modal logic.

In preceding experiments [1, 4, 6], we have already demonstrated that the technology we employ is well suited for the task. In fact, computers may even contribute philosophically relevant new knowledge. For example, the theorem prover Leo-II [2] detected an inconsistency in Gödel's original script [12] of the argument, which was undetected by philosophers before. This inconsistency renders pointless Gödel's original version of the argument. Nevertheless, as our preceding experiments also confirmed, Scott's variant [13] from Fig. 1 is immune to this issue.[1] Hence, in the remainder of this article the term *Gödel's ontological argument* actually refers to Scott's variant, whose logical validity has been formally verified with our technology (for higher-order modal logic KB (or stronger logics) with constant domain or varying domain

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[1] Gödel avoids the first conjunct $\varphi(x)$ in the definition of *essence (D2)*, which Scott added for cosmetic reason. However, as Leo-II detected, one can prove from Gödel's definition that the empty property becomes an essence of every individual, which, together with theorem T1, axiom A5 and the defintion of *necessary existence (D3)* causes the inconsistency. For more details see Benzmüller and Woltzenlogel Paleo [1].

A1 Either a property or its negation is positive, but not both:

$$\forall\varphi[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$$

A2 A property necessarily implied by a positive property is positive:

$$\forall\varphi\forall\psi[(P(\varphi) \wedge \Box\forall x[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

T1 Positive properties are possibly exemplified:

$$\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$$

D1 A *God-like* being possesses all positive properties:

$$G(x) \equiv \forall\varphi[P(\varphi) \rightarrow \varphi(x)]$$

A3 The property of being God-like is positive:

$$P(G)$$

C Possibly, a God-like being exists:

$$\Diamond\exists xG(x)$$

A4 Positive properties are necessarily positive:

$$\forall\varphi[P(\varphi) \rightarrow \Box\ P(\varphi)]$$

D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties:

$$\varphi\ ess\ x \equiv \varphi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\varphi(y) \rightarrow \psi(y)))$$

T2 Being God-like is an essence of any God-like being:

$$\forall x[G(x) \rightarrow G\ ess\ x]$$

D3 *Necessary existence* of an individual is the necessary exemplification of all its essences:

$$\mathrm{NE}(x) \equiv \forall\varphi[\varphi\ ess\ x \rightarrow \Box\exists y\varphi(y)]$$

A5 Necessary existence is a positive property:

$$P(\mathrm{NE})$$

L1 If a God-like being exists, then necessarily a God-like being exists:

$$\exists xG(x) \rightarrow \Box\exists yG(y)$$

L2 If possibly a God-like being exists, then necessarily a God-like being exists:

$$\Diamond\exists xG(x) \rightarrow \Box\exists yG(y)$$

T3 Necessarily, a God-like being exists:

$$\Box\exists xG(x)$$

FIGURE 1. Scott's version of Gödel's ontological argument

semantics). Here we assume some familiarity with these related experiments and papers.

The particular motivation and context for the work presented in the given article is as follows: It is well known that the axioms in Gödel's ontological proof (cf. Fig. 1) entail what is called *modal collapse* [25, 14]. This means that the formula $\varphi \rightarrow \Box\varphi$, abbreviated as MC, holds for any formula

φ and not just for $\exists x. God(x)$ as intended. Hence, anything that is the case is necessarily the case, actual truth implies necessary truth, or, in other words, there are no contingent truths. In fact, one may even go further and interpret MC as a result against free will. The fact that MC follows from Gödel's axioms, which has also been confirmed with higher-order automated theorem provers in our previous experiments [4, 6], has led to strong criticism of the argument and stimulated attempts to remedy the problem. Hájek [22, 19] proposed the use of cautious restrictions of comprehension principles, and Fitting [16] took greater care of the semantics of higher-order quantifiers in the presence of modalities. Others, such as C.A. Anderson [23], Hájek [17] and Bjørdal [20], proposed emendations of Gödel's axioms and definitions. They require neither comprehension restrictions nor more complex semantics and are therefore technically simpler to analyze with computer support. Therefore, we concentrate on the latter emendations of Gödel's argument and leave the former ones for future work. Again, familiarity with the papers by C.A. Anderson [23], Hájek [17] and Bjørdal [20] is assumed, since we only briefly assess various claims made there.

Our formalizations employ the embedding of higher-order modal logic (HOML) in classical higher-order logic (HOL) as introduced in previous work [4, 5] as an enabling technology to settle a long-standing controversy between Hájek and Anderson regarding the redundancy of some axioms in Anderson's emendation. A core issue in the controversy was whether Anderson's emendation should be interpreted with constant or varying domains and how the provability of lemmas, theorems and even axioms was affected by this choice. In *constant domain semantics* (*possibilist notion of quantification*), the individual domains are the same in all possible worlds. In *varying domain semantics* (*actualist notion of quantification*), the domains may vary from world to world. The latter notion is technically encoded in our approach with the help of an existence relation expressing which individuals actually exist in each world. In other words, actualist quantification is formalized as possibilist quantification guarded by the existence relation. This technical solution enables us to flexibly switch between these different notions of quantification and to even mix them. When we mention varying domain semantics (actualist quantification) below, we in fact mean varying domain semantics for the domain of individuals only, while for the other types we still assume the standard constant domain semantics (possibilist quantification). This is, to our best knowledge, in line with the intention of the authors (it would of course be easily possible in our approach to study other combinations).

In our experiments we have utilized the proof assistant Isabelle/HOL [18] together with the external automated higher-order provers Leo-II [2] and Satallax [8], called through Sledgehammer. Sledgehammer analyzes the proofs generated by these provers and tells which axioms or previously proven lemmas are needed to reconstruct the proof inside Isabelle using its *metis* method [11], which relies on a reimplementation of the resolution-based first-order prover Metis [15]. To find counter-examples, we used Nitpick [10], also through Isabelle.

Our main results, which we have presented at the 1st World Congress on Logic and Religion [3], are presented in more detail below. The corresponding formalizations, containing all the details, are available online: see the subdirectories `Anderson`, `Hajek` and `Bjordal` at `github.com/FormalTheology/GoedelGod/blob/master/Formalizations/Isabelle/`. [2]

2. Results of the Experiments

For *all emendations and variants* discussed here, the axioms and definitions have been shown to be consistent and not to entail modal collapse.

Next, we investigate some controversially debated claims regarding the redundancy, superfluousness and independence of particular axioms in the investigated emendations and variants. Several novel findings are reported.

The following definitions of redundancy, superfluousness and independence are used throughout this paper:

Def. 1. *An axiom A is* redundant *w.r.t. a set of axioms S iff S entails A.*

Def. 2. *An axiom A is* superfluous *w.r.t. axiom set S iff* $S \setminus \{A\}$ *entails T3.*

Def. 3. *An axiom A is* independent *of a set of axioms S iff there are models of S where A is true and other models of S where A if false.*

For both constant domain semantics (possibilist quantification) and varying domain semantics (actualist quantification), the following results hold for *Anderson's Emendation* (cf. Fig. 2): T1, C and T3′ can be quickly automated (in logics **K**, **K** and **KB**, respectively); the axioms A4 and A5′ are proven redundant (the former in logic **K4B** and the latter already in **K**); a trivial countermodel (with two worlds and two individuals) for MC is generated by Nitpick (for all mentioned logics); all axioms and definitions are shown to be mutually consistent.

The redundancy of A4 and A5 is particularly controversial. Magari [24] claimed that A4 and A5 are superfluous, arguing that T3 is true in all models of the other axioms and definitions by Gödel. Hájek [22, pp. 5–6] investigated this further, and claimed that Magari's claim is not valid, but is nevertheless true under additional silent assumptions by Magari. Moreover, Hájek [22, p. 2] cites his earlier work[3] [19], where he claims (in Theorem 5.3) that for Anderson's emended theory [23], A4 and A5 are not only superfluous, but also redundant. Anderson and Gettings [21, footnote 1 in p. 1] rebutted Hájek's claim, arguing that the redundancy of A4 and A5 holds only under constant domain semantics, while Anderson's emended theory ought to be taken under Cocchiarella's semantics [26] (a varying domain semantics). Our results show

[2] Ideally, Logica Universalis should provide permanent storage of such additional material.
[3] Although [19] precedes [22] in writing, it was published only 5 years later, in German.

A:A1 If a property is positive, its negation is not positive:

$$\forall\varphi[P(\varphi) \to \neg P(\neg\varphi)]$$

A2 A property necessarily implied by a positive property is positive:

$$\forall\varphi\forall\psi[(P(\varphi) \land \Box\forall x[\varphi(x) \to \psi(x)]) \to P(\psi)]$$

T1 Positive properties are possibly exemplified:

$$\forall\varphi[P(\varphi) \to \Diamond\exists x\varphi(x)]$$

A:D1 A *God-like* being necessarily possesses those and only those properties that are positive:

$$G_A(x) \equiv \forall\varphi[P(\varphi) \leftrightarrow \Box\varphi(x)]$$

A3′ The property of being God-like is positive:

$$P(G_A)$$

C Possibly, a God-like being exists:

$$\Diamond\exists x G(x)$$

A4 Positive properties are necessarily positive:

$$\forall\varphi[P(\varphi) \to \Box\ P(\varphi)]$$

A:D2 An *essence* of an individual is a property that necessarily implies those and only those properties that the individual has necessarily:

$$\varphi\ ess_A\ x \equiv \forall\psi[\Box\psi(x) \leftrightarrow \Box\forall y(\varphi(y) \to \psi(y))]$$

T2′ Being God-like is an essence of any God-like being:

$$\forall x[G_A(x) \to G_A\ ess_A\ x]$$

D3′ *Necessary existence* of an individual is the necessary exemplification of all its essences:

$$\mathrm{NE}_A(x) \equiv \forall\varphi[\varphi\ ess_A\ x \to \Box\exists y\varphi(y)]$$

A5′ Necessary existence is a positive property:

$$P(\mathrm{NE}_A)$$

L1′ If a God-like being exists, then necessarily a God-like being exists:

$$\exists x G_A(x) \to \Box\exists y G_A(y)$$

L2′ If possibly a God-like being exists, then necessarily a God-like being exists:

$$\Diamond\exists x G_A(x) \to \Box\exists y G_A(y)$$

T3′ Necessarily, a God-like being exists:

$$\Box\exists x G_A(x)$$

FIGURE 2. Anderson's Emendation

that Hájek was originally right, under both constant and varying domain semantics (for the domain of individuals).

Nevertheless, Hájek [17, p. 7] acknowledges Anderson's rebuttal, and apparently accepts it, as evidenced by his use[4] of A4 and A5′, as well as

[4] A4 and A5 are used by Hájek [17, p. 11] in, respectively, Lemma 4 and Theorem 4.

H:A12 The negation of a property necessarily implied by a positive property is not positive:

$$\forall\varphi\forall\psi[(P(\varphi) \wedge \Box\forall x[\varphi(x) \rightarrow \psi(x)]) \rightarrow \neg P(\neg\psi)]$$

H:D1 A *God-like* being necessarily possesses those and only those properties that are necessarily implied by a positive property:

$$G_H(x) \equiv \forall\varphi[\Box\varphi(x) \leftrightarrow \exists\psi[P(\psi) \wedge \Box\forall x[\psi(x) \rightarrow \varphi(x))]]$$

A3′ The property of being God-like is positive:

$$P(G_H)$$

A4 Positive properties are necessarily positive:

$$\forall\varphi[P(\varphi) \rightarrow \Box\ P(\varphi)]$$

A:D2 An *essence* of an individual is a property that necessarily implies those and only those properties that the individual has necessarily:

$$\varphi\ ess_A\ x \equiv \forall\psi[\Box\psi(x) \leftrightarrow \Box\forall y(\varphi(y) \rightarrow \psi(y))]$$

D3′ *Necessary existence* of an individual is the necessary exemplification of all its essences:

$$\mathrm{NE}_A(x) \equiv \forall\varphi[\varphi\ ess_A\ x \rightarrow \Box\exists y[\varphi(y)]]$$

A5′ Necessary existence is a positive property:

$$P(\mathrm{NE}_A)$$

L3 (1) The negation of a positive property is not positive:

$$\forall\varphi[P(\varphi) \rightarrow \neg P(\neg\varphi)]$$

(2) Positive properties are possibly exemplified:

$$\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$$

(3) If a God-like being exists, then necessarily a God-like being exists:

$$\forall x[G_H(x) \rightarrow \Box G_H(x)))]$$

(4) All positive properties are necessarily implied by the property of being God-like:

$$\forall\varphi[P(\varphi) \rightarrow \Box\forall x[G_H(x) \rightarrow \varphi(x)]]$$

L4 Being God-like is an essence of any God-like being:

$$\forall x[G_H(x) \rightarrow G_H\ ess\ x]$$

T3′ Necessarily, a God-like being exists:

$$\Box\exists x G_H(x)$$

FIGURE 3. Hájek's First Emendation $\mathcal{AOE}'$

varying domain semantics, in his new emendation (named $\mathcal{AOE}'$ [17, sec. 4], cf. Fig. 3), which replaces Anderson's A:A1 and A2 by a simpler axiom H:A12. Surprisingly, the computer-assisted formalization of $\mathcal{AOE}'$ shows that A4 and A5′ are still superfluous. Moreover, A4 and A5′ are independent of the other axioms and definitions. Therefore, A4 and A5′ are not redundant, despite their superfluousness.

H:A12 The negation of a property necessarily implied by a positive property is not positive:

$$\forall\varphi\forall\psi[(P(\varphi) \wedge \Box\forall x[\varphi(x) \to \psi(x)]) \to \neg P(\neg\psi)]$$

A:D1 A *God-like* being necessarily possesses those and only those properties that are positive:

$$G_A(x) \equiv \forall\varphi[P(\varphi) \leftrightarrow \Box\varphi(x)]$$

H:A3 The property of being God-like and existing actually is positive:

$$P(G_A \wedge E)$$

T3′ Necessarily, a God-like being exists:

$$\Box\exists x G_A(x)$$

FIGURE 4. Hájek's Second Emendation $\mathcal{AOE}'_0$

Although Hájek did not notice the superfluousness of A4 and A5′ in his $\mathcal{AOE}'$, he did describe yet another emendation (his $\mathcal{AOE}'_0$, cf. Fig. 4) where A4 and A5′ are superfluous (though no claim is made w.r.t. to their redundancy), if A3′ is replaced by a stronger axiom (H:A3) additionally stating that the property of actual existence is positive when it comes to God-like beings [17, sec. 5]. Formalization of $\mathcal{AOE}'_0$ shows that A4 is not only superflous, but also redundant. In addition, A5′ becomes superfluous and independent. Surprisingly, a countermodel for the weaker A3′ was succesfully generated. This is somewhat unsatisfactory (for theistic goals), because it shows that $\mathcal{AOE}'_0$ does not entail the positiveness of being God-like.

Nevertheless, $\mathcal{AOE}'_0$ is explicitly regarded by Hájek [17, p. 12] as just an intermediary step towards a more natural theory, based on a more sophisticated notion of positiveness. That is his final emendation ($\mathcal{AOE}''$, cf. Fig. 5), which restores A3′ and does use A4 and A5′, albeit in a modified form (i.e. H:A4 and H:A5).

The formalization of $\mathcal{AOE}''$ shows that both H:A4 and H:A5 are superfluous as well as independent. For the old A5′, no conclusive results were achieved.

Additionally, Anderson [23, footnote 14] (cf. Fig. 6) remarks that only the quantifiers in T3′ and in A:D2 need to be interpreted as actualistic quantifiers, while others may be taken as possibilistic quantifiers. Our computer-assisted study of this mixed variant shows that A4 is still redundant in logic **K4B**, but A5′ becomes independent (hence not redundant). Unfortunately, a countermodel for T3 can then be found.

The controversy over the superfluousness of A4 and A5 indicates a trend to reduce the ontological argument to its bare essentials. In this regard, already C.A. Anderson [23, p. 7] indicates that, by taking a notion of *defective* as primitive and defining the notion of *positive* upon it, axioms A:A1, A2 and A4 become derivable.

H:A12 The negation of a property necessarily implied by a positive property is not positive:

$$\forall\varphi\forall\psi[(P(\varphi) \wedge \Box\forall x[\varphi(x) \to \psi(x)]) \to \neg P(\neg\psi)]$$

D4 A property is positive$^{\#}$ iff it is necessarily implied by a positive property:

$$P^{\#}(\varphi) \equiv \exists\psi[P(\psi) \wedge \Box\forall x[\psi(x) \to \varphi(x)]]$$

H:D1 A *God-like* being necessarily possesses those and only those properties that are positive$^{\#}$:

$$G_H(x) \equiv \forall\varphi[P^{\#}(\varphi) \leftrightarrow \Box\varphi(x)]$$

A3′ The property of being God-like is positive:

$$P(G_H)$$

H:A4 Positive$^{\#}$ properties are necessarily positive$^{\#}$:

$$\forall\varphi[P^{\#}(\varphi) \to \Box\, P^{\#}(\varphi)]$$

A:D2 An *essence* of an individual is a property that necessarily implies those and only those properties that the individual has necessarily:

$$\varphi\ ess_A\ x \equiv \forall\psi[\Box\psi(x) \leftrightarrow \Box\forall y(\varphi(y) \to \psi(y))]$$

D3′ *Necessary existence* of an individual is the necessary exemplification of all its essences:

$$\mathrm{NE}_A(x) \equiv \forall\varphi[\varphi\ ess_A\ x \to \Box\exists y[\varphi(y)]]$$

H:A5 Necessary existence is a positive$^{\#}$ property:

$$P^{\#}(\mathrm{NE}_A)$$

FIGURE 5. Hájek's Third Emendation $\mathcal{AOE}''$

D (Defective) is taken as primitive and P_{AS} (Positive) is defined.

AS:D1 A property is positive iff its absence necessarily renders an individual defective and it is possible that an individual has the property without being defective.

$$P_{AS}(\varphi) \equiv \Box(\forall x(\neg\varphi(x) \to D(x)) \wedge (\neg\Box\forall x(\varphi(x) \to D(x))))$$

A:A1′ If a property is positive, its negation is not positive:

$$\forall\varphi[P_{AS}(\varphi) \to \neg P_{AS}(\neg\varphi)]$$

A2′ A property necessarily implied by a positive property is positive:

$$\forall\varphi\forall\psi[(P_{AS}(\varphi) \wedge \Box\forall x[\varphi(x) \to \psi(x)]) \to P_{AS}(\psi)]$$

A4′ Positive properties are necessarily positive:

$$\forall\varphi[P_{AS}(\varphi) \to \Box P_{AS}(\varphi)]$$

FIGURE 6. Anderson's Simplification

These claims have been confirmed by the automated theorem provers (in logic **K4B**). Within the same trend, the alternative proposed by Bjørdal [20] (cf. Fig. 7) achieves a high level of minimality.

G_B (God-like) is taken as primitive and P_B (Positive) is defined.

B:D1 A property is positive iff it is necessarily possessed by every God-like being.

$$P_B(\phi) \equiv \Box\forall x(G_B(x) \rightarrow \phi(x))$$

B:L1 B:D1 is logically equivalent in **S4** with the union of D1′ and axioms A2′, A3′ and A4′.

$$\text{B:D1} \leftrightarrow \text{D1}' \wedge \text{A2}' \wedge \text{A3}' \wedge \text{A4}'$$

B:D2 a *maximal composite* of an individual's positive properties is a positive property possessed by the individual and necessarily implying every positive property possessed by the individual.

$$\text{MCP}(\phi, x) \equiv (\phi(x) \wedge P_B(\phi)) \wedge \forall\psi((\psi(x) \wedge P_B(\psi)) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$$

B:D3 *Necessary existence* of an individual is the necessary exemplification of all its maximal composites.

$$\text{NE}_B(x) \equiv \forall\phi(\text{MCP}(\phi, x) \rightarrow \Box\exists y\phi(y))$$

A:A1′ If a property is positive, its negation is not positive:

$$\forall\varphi[P_B(\varphi) \rightarrow \neg P_B(\neg\varphi)]$$

A5′ Necessary existence is a positive property.

$$P_B(\text{NE}_B)$$

T3′ Necessarily, a God-like being exists:

$$\Box\exists x G_B(x)$$

Figure 7. Bjørdal's Alternative

He takes the property of being God-like as a primitive and defines (B:D1) the positive properties as those properties necessarily possessed by every God-like being. He then briefly indicates (B:L1) that B:D1 is logically equivalent, under modal logic **S4**, to the conjunction of D1′, A2′, A3′ and A4′. This has been confirmed in the computer-assisted formalization: A2′ and A3′ can be quickly automatically derived in logic **K**. A4′ can be proved in logic **KT** (i.e. assuming reflexivity of the accessibility relation). For constant domain semantics, proving D1′ is possible in logic **K4**, whereas for varying domain semantics, a countermodel can be found even in logic **S5**. Conversely, the proof that B:D1 is entailed by D1′, A2′, A3′ and A4′ is possible already in logic **K**. The provers also show that theorem T3′ follows from B:D1, B:D2, B:D3, A:A1′ and A5′ already in logic **KB**. Bjørdal's last paragraph briefly mentions Hájek's ideas about the superfluousness of A5′ and claims that it is possible, with (unclear) additional modifications of the definitions, to eliminate A5′ from his theory as well. Without any additional modification, the automated reasoners show that A5′ is independent, but actually not superfluous. All these results, with the exception of the aforementioned countermodel for D1′, hold for both constant and varying domain semantics.

Investigated Variant	D1′	A:A1′	A2′	A3′	A4	A4′	H:A4	A5	A5′	H:A5	T3	T3′	MC
Scott (const)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Scott (var)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Anderson (const)	-	-	-	-	R(K4B)	-	-	-	R	-	-	P	CS
Anderson (var)	-	-	-	-	R(K4B)	-	-	-	R	-	-	P	CS
Anderson (mix)	-	-	-	-	R(K4B)	-	-	-	I	-	-	CS	CS
Hájek AOE′ (var)	-	-	CS		S/I	-	-	-	S/I	-	-	P(KB)	CS
Hájek AOE′_0 (var)	-	-	-	CS	R	-	-	-	S/U	-	-	P(KB)	CS
Hájek AOE″ (var)	-	-	-	-	-	-	S/I	-	-	S/I	-	P(KB)	CS
Anderson (simp) (var)	-	R	R	-	-	R(K4B)	-	-	-	-	-	-	-
Bjørdal (const)	R(K4)	-	R	R	-	R(KT)	-	-	N/I	-	-	P(KB)	CS
Bjørdal (var)	CS	-	R	R	-	R(KT)	-	-	N/I	-	-	P(KB)	CS

FIGURE 8. Summary of Results.

Figure 8 summarizes the results obtained by the automated reasoning tools. The following abbreviations are used: S/I = superfluous and independent; R = superfluous and redundant; S/U = superfluous and unknown whether redundant or independent; N/I = non-superfluous and independent; P = provable; CS = counter-satisfiable. The weakest logic required to show redundancy or provability is indicated in parentheses (**K** is the default). Cells highlighted in red contain results that differ from what had been claimed by either Magari, Anderson or Hájek. Cells highlighted in yellow contain results that are surprising, albeit not contradicting any claims. Cells highlighted in green contain results where the tools were able to obtain the same results as humans, but using weaker modal logics.

3. Conclusion

Using our approach of semantically embedding higher-order modal logics in classical higher-order logic, the formalization and (partly) automated analysis of several variants of Gödel's ontological argument has been surprisingly straightforward. The higher-order provers we employed not only confirmed many claimed results, but also exposed a few mistakes and novel insights in a long-standing controversy. We believe the technology employed in this work is ready to be fruitfully adopted in larger scale by philosophers. Moreover, our approach is by no means restricted to metaphysics or philosophy, and further work could explore similar applications in other areas, including, for example, law, politics and ethics. In fact, we claim that our approach, at least to some degree, realizes Leibniz's dream of a characteristica universalis and a calculus ratiocinator.

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