On Logic Embeddings and Gödel's God

Christoph Benzmüller[1]

(jww Bruno Woltzenlogel Paleo)

5th of September 2014

WADT 2014

[1]Supported by DFG grant BE 2501/9-1

This talk: two related topics

Embeddings of expressive logics in classical higher-order logic (HOL)

(own research since about 2008)

Application in Philosophy: study of Gödel's ontological argument

(jww with Bruno since 2013)

This talk: outline

Gödel's ontological argument — Introduction

Embeddings of expressive logics in HOL / Automation

Gödel's ontological argument — Results

Vision of Leibniz (1646–1716): *Calculemus!*

If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other . . . : Let us calculate.

(Translation by Russell)

Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo . . . dicere: calculemus. (Leibniz, 1684)

Required:
characteristica universalis and **calculus ratiocinator**

Our Contribution: Towards a Computational Metaphysics

Ontological argument for the existence of God

We focused on Gödel's modern version in higher-order modal logic

Automation with provers for higher-order classical logic (HOL)

- confirmation of known results
- detection of some novel results
- systematic variation of the logic settings
- exploited HOL as a universal metalogic via logic embeddings (characteristica universalis?)

... Anselm v. C. Gaunilo ... Th. Aquinas ...... Descartes Spinoza Leibniz ... Hume Kant ... Hegel ... Frege ... Hartshorne Malcolm Lewis Plantinga Gödel ...

Anselm's notion of God (Proslogion, 1078):

"God is that, than which nothing greater can be conceived."

Gödel's notion of God:

"A God-like being possesses all 'positive' properties."

To show by logical reasoning:

"God exists."

$\exists x G(x)$

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"God is that, than which nothing greater can be conceived."

Gödel's notion of God:

"A God-like being possesses all 'positive' properties."

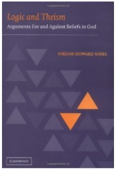
To show by logical reasoning:

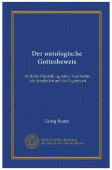
"Necessarily God exists."

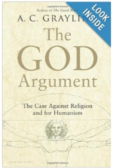
$$\Box \exists x G(x)$$

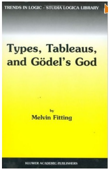
Different Interests in Ontological Arguments:

- Philosophical: Boundaries of Metaphysics & Epistemology
  - We specify a metaphysical concept (God),
  - but we want to draw a conclusion for the real world.
- Theistic: Successful argument could convince atheists?
- Ours: Can computers (theorem provers) be used …
  - …to formalize the definitions, axioms and theorems?
  - …to verify the arguments step-by-step?
  - …to fully automate (sub-)arguments?


Logic and Theism
Arguments For and Against Beliefs in God
JORDAN HOWARD SOBEL


Der ontologische Gottesbeweis
Georg Runze


A. C. GRAYLING
The GOD Argument
The Case Against Religion and for Humanism
LOOK INSIDE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Types, Tableaus, and Gödel's God
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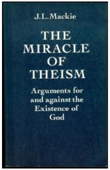
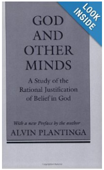
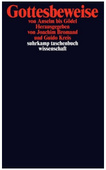
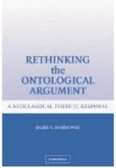
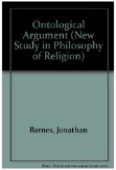

J.L.Mackie
THE MIRACLE OF THEISM
Arguments for and against the Existence of God


LOOK INSIDE
GOD AND OTHER MINDS
A Study of the Rational Justification of Belief in God
With a new Preface by the author
ALVIN PLANTINGA


Gottesbeweise
von Anselm bis Gödel
Herausgegeben von Joachim Bromand und Guido Kreis
suhrkamp taschenbuch wissenschaft

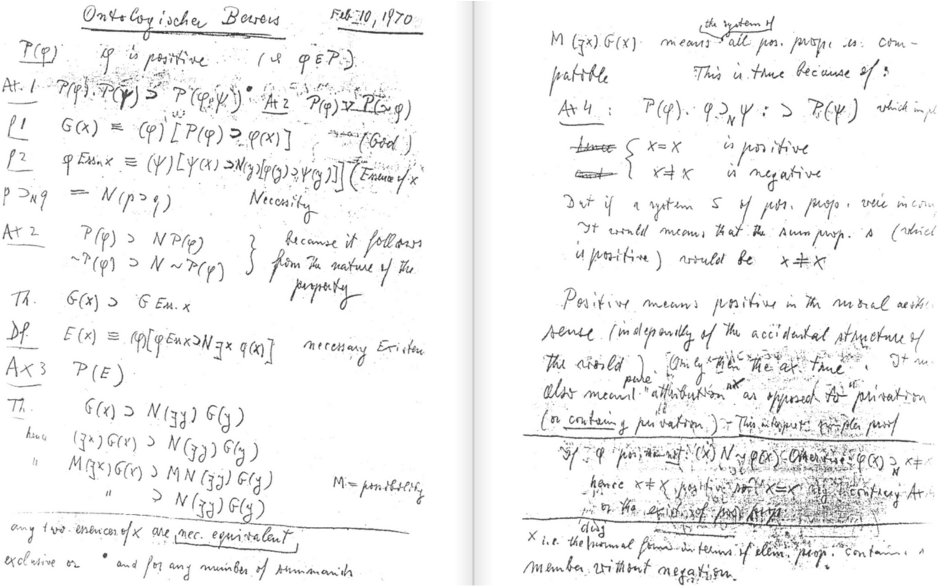

RETHINKING the ONTOLOGICAL ARGUMENT


Ontological Argument (New Study in Philosophy of Religion)
Barnes, Jonathan


The Ontological Argument from Descartes to Hegel


WOLFGANG RÖD
DER GOTT DER REINEN VERNUNFT
C.H.BECK

Gödel's Manuscript: 1930's, 1941, 1946-1955, 1970



Scott's Version of Gödel's Axioms, Definitions and Theorems

Axiom A1 Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

Axiom A2 A property necessarily implied by a positive property is positive:
$$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

Thm. T1 Positive properties are possibly exemplified: $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

Def. D1 A *God-like* being possesses all positive properties: $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$

Axiom A3 The property of being God-like is positive: $P(G)$

Cor. C Possibly, God exists: $\Diamond\exists xG(x)$

Axiom A4 Positive properties are necessarily positive: $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$

Def. D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi \; ess \; x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Thm. T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \rightarrow G \; ess \; x]$

Def. D3 *Necessary existence* of an individual is the necessary exemplification of all its essences: $E(x) \leftrightarrow \forall\phi[\phi \; ess \; x \rightarrow \Box\exists y\phi(y)]$

Axiom A5 Necessary existence is a positive property: $P(E)$

Thm. T3 Necessarily, God exists: $\Box\exists xG(x)$

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Difference to Gödel (who omits this conjunct)

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Modal operators are used

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second-order quantifiers

Proof Overview

D1: $G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$

D2: $\varphi \; ess \; x \equiv \varphi(x) \wedge \forall\psi_{\blacksquare}.(\psi(x) \rightarrow \Box\forall x_{\blacksquare}.(\varphi(x) \rightarrow \psi(x)))$

D3: $NE(x) \equiv \forall\varphi_{\blacksquare}.[\varphi \; ess \; x \rightarrow \Box\exists y.\varphi(y)]$

A3: $P(G)$

A2: $\forall\varphi_{\blacksquare}.\forall\psi_{\blacksquare}.[(P(\varphi) \wedge \Box\forall x_{\blacksquare}.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

A1a: $\forall\varphi_{\blacksquare}.[P(\neg\varphi) \rightarrow \neg P(\varphi)]$

T1: $\forall\varphi_{\blacksquare}.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]$

C: $\Diamond\exists z.G(z)$

A1b: $\forall\varphi_{\blacksquare}.[\neg P(\varphi) \rightarrow P(\neg\varphi)]$

A4: $\forall\varphi_{\blacksquare}.[P(\varphi) \rightarrow \Box \; P(\varphi)]$

T2: $\forall y_{\blacksquare}.[G(y) \rightarrow G \; ess \; y]$

A5: $P(NE)$

L1: $\exists z.G(z) \rightarrow \Box\exists x.G(x)$

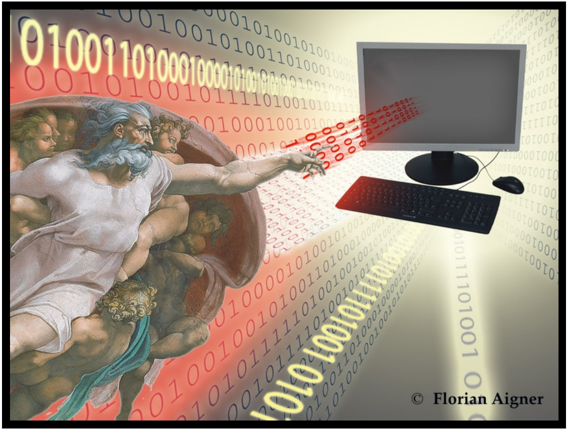
$\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)$

S5: $\forall\xi_{\blacksquare}.[\Diamond\Box\xi \rightarrow \Box\xi]$

L2: $\Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)$

C: $\Diamond\exists z.G(z)$ **L2:** $\Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)$

T3: $\Box\exists x.G(x)$



How to automate Higher-Order Modal Logic?

Challenge: No provers for *Higher-order Modal Logic* (HOML)

Our solution: **Embedding in *Higher-order Classical Logic* (HOL)**

Then use existing HOL theorem provers for reasoning in HOML

[BenzmüllerPaulson, Logica Universalis, 2013]

Previous empirical findings:

Embedding of *First-order Modal Logic* in HOL works well

[BenzmüllerOttenRaths, ECAI, 2012]

[Benzmüller, LPAR, 2013]

Simple Types $\alpha ::= o \mid \iota \mid \mu \mid \alpha_1 \rightarrow \alpha_2$

HOL $s,t ::= C_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta}\, t_\alpha)_\beta \mid$

$(\neg_{o\rightarrow o}\, s_o)_o \mid (s_o \vee_{o\rightarrow o\rightarrow o} t_o)_o \mid (\forall_{(\alpha\rightarrow o)\rightarrow o}\, s_{\alpha\rightarrow o})_o$

(note: binder notation $\forall x_\alpha t_o$ as syntactic sugar for $\forall_{(\alpha\rightarrow o)\rightarrow o}\lambda x_\alpha t_o$)

HOL with Henkin semantics is (meanwhile) well understood

Origin [Church,JSymbLog,1940]

Henkin semantics [Henkin,JSymb.Log,1950]

[Andrews, JSymbLog,1971,1972]

Extens./Intens. [BenzmüllerEtAl,JSymbLog,2004]

[Muskens,JSymbLog,2007]

Sound and complete provers do exists

interactive: Isabelle/HOL, HOL4, Hol Light, Coq/HOL, PVS, . . .

automated: TPS, LEO-II, Satallax, Nitpick, Isabelle/HOL, . . .

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HOML $\varphi, \psi \quad ::= \quad \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x_\gamma\, \varphi \mid \exists x_\gamma\, \varphi$

- Kripke style semantics (possible world semantics)

$M, g, s \models \neg\varphi$ iff not $M, g, s \models \varphi$

$M, g, s \models \varphi \wedge \psi$ iff $M, g, s \models \varphi$ and $M, g, s \models \psi$

$\ldots$

$M, g, s \models \Box\varphi$ iff $M, g, u \models \varphi$ for all u with $r(s, u)$

$\ldots$

$M, g, s \models \forall x_\gamma\, \varphi$ iff $M, [d/x]g, s \models \varphi$ for all $d \in D_\gamma$

$\ldots$

[BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

[Muskens, HandbookOfModalLogic, 2006]

HOML $\varphi, \psi ::= \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x_\gamma\, \varphi \mid \exists x_\gamma\, \varphi$

HOL $s, t ::= C \mid x \mid \lambda x s \mid s\, t \mid \neg s \mid s \vee t \mid \forall x\, t$

HOML in HOL: HOML formulas φ are mapped to HOL predicates $\varphi_{\mu\rightarrow o}$

$$\begin{aligned}
\neg &= \lambda\varphi_{\mu\rightarrow o}\lambda w_\mu \neg\varphi w \\
\wedge &= \lambda\varphi_{\mu\rightarrow o}\lambda\psi_{\mu\rightarrow o}\lambda w_\mu(\varphi w \wedge \psi w) \\
\rightarrow &= \lambda\varphi_{\mu\rightarrow o}\lambda\psi_{\mu\rightarrow o}\lambda w_\mu(\neg\varphi w \vee \psi w) \\
\forall &= \lambda h_{\gamma\rightarrow(\mu\rightarrow o)}\lambda w_\mu \forall d_\gamma\, hdw \\
\exists &= \lambda h_{\gamma\rightarrow(\mu\rightarrow o)}\lambda w_\mu \exists d_\gamma\, hdw \\
\Box &= \lambda\varphi_{\mu\rightarrow o}\lambda w_\mu \forall u_\mu\, (\neg rwu \vee \varphi u) \\
\Diamond &= \lambda\varphi_{\mu\rightarrow o}\lambda w_\mu \exists u_\mu\, (rwu \wedge \varphi u) \\
\text{valid} &= \lambda\varphi_{\mu\rightarrow o}\forall w_\mu . \varphi w
\end{aligned}$$

Ax (polymorphic over γ)

The equations in Ax are given as axioms to the HOL provers!

HOML $\varphi, \psi ::= \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x_\gamma\, \varphi \mid \exists x_\gamma\, \varphi$

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$\wedge$	$=$	$\lambda\varphi_{\mu\rightarrow o}\lambda\psi_{\mu\rightarrow o}\lambda w_\mu(\varphi w \wedge \psi w)$
$\rightarrow$	$=$	$\lambda\varphi_{\mu\rightarrow o}\lambda\psi_{\mu\rightarrow o}\lambda w_\mu(\neg\varphi w \vee \psi w)$
$\forall$	$=$	$\lambda h_{\gamma\rightarrow(\mu\rightarrow o)}\lambda w_\mu \forall d_\gamma\, h d w$
$\exists$	$=$	$\lambda h_{\gamma\rightarrow(\mu\rightarrow o)}\lambda w_\mu \exists d_\gamma\, h d w$
$\Box$	$=$	$\lambda\varphi_{\mu\rightarrow o}\lambda w_\mu \forall u_\mu\, (\neg r w u \vee \varphi u)$
$\Diamond$	$=$	$\lambda\varphi_{\mu\rightarrow o}\lambda w_\mu \exists u_\mu\, (r w u \wedge \varphi u)$
valid	$=$	$\lambda\varphi_{\mu\rightarrow o}\forall w_\mu\text{.}\, \varphi w$

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HOL $s, t \quad ::= \quad C \mid x \mid \lambda x s \mid s\, t \mid \neg s \mid s \vee t \mid \forall x\, t$

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\begin{array}{lcl}
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\text{valid} & = & \lambda\varphi_{\mu\rightarrow o}\forall w_\mu.\, \varphi w
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The equations in Ax are given as axioms to the HOL provers!

Embedding HOML in HOL

HOML $\quad \varphi, \psi \quad ::= \quad \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x_\gamma\, \varphi \mid \exists x_\gamma\, \varphi$

HOL $\quad s, t \quad ::= \quad C \mid x \mid \lambda x s \mid s\, t \mid \neg s \mid s \vee t \mid \forall x\, t$

HOML in HOL: HOML formulas φ are mapped to HOL predicates $\varphi_{\mu\rightarrow o}$

$$
\begin{array}{lcl}
\neg & = & \lambda\varphi_{\mu\rightarrow o}\lambda w_\mu \neg\varphi w \\
\wedge & = & \lambda\varphi_{\mu\rightarrow o}\lambda\psi_{\mu\rightarrow o}\lambda w_\mu(\varphi w \wedge \psi w) \\
\rightarrow & = & \lambda\varphi_{\mu\rightarrow o}\lambda\psi_{\mu\rightarrow o}\lambda w_\mu(\neg\varphi w \vee \psi w) \\
\forall & = & \lambda h_{\gamma\rightarrow(\mu\rightarrow o)}\lambda w_\mu \forall d_\gamma\, h d w \\
\exists & = & \lambda h_{\gamma\rightarrow(\mu\rightarrow o)}\lambda w_\mu \exists d_\gamma\, h d w \\
 & & \\
\forall\varphi_{\mu\rightarrow o}\forall w_\mu[\ (\Box\varphi)w & \equiv & \forall u_\mu\, (\neg r w u \vee \varphi u)\ \ ] \\
\Diamond & = & \lambda\varphi_{\mu\rightarrow o}\lambda w_\mu \exists u_\mu\, (r w u \wedge \varphi u) \\
 & & \\
\text{valid} & = & \lambda\varphi_{\mu\rightarrow o}\forall w_\mu.\, \varphi w
\end{array}
\qquad \text{Ax}
$$

The equations in Ax are given as axioms to the HOL provers!

Example

HOML formula	$\Diamond \exists x G(x)$
HOML formula in HOL	valid $(\Diamond \exists x G(x))_{\mu \to o}$
expansion, $\beta\eta$-conversion	$\forall w_\mu (\Diamond \exists x G(x))_{\mu \to o}\, w$
expansion, $\beta\eta$-conversion	$\forall w_\mu \exists u_\mu (rwu \wedge (\exists x G(x))_{\mu \to o} u)$
expansion, $\beta\eta$-conversion	$\forall w_\mu \exists u_\mu (rwu \wedge \exists x Gxu)$

Expansion: user or prover may flexibly choose expansion depth

What are we doing?

In order to prove that φ is valid in HOML,
–> we instead prove that valid $\varphi_{\mu \to o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

For the experts: soundness and completeness wrt Henkin semantics

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Propositional Quantification [Fitting, J.Symb.Log., 2002]

...

$M, g, s \models \forall p\, \varphi$ iff $M, [v/p]g, s \models \varphi$ for all $v \in P$

Embedding in HOL

...

$\forall = \lambda h_{(\mu\to o)\to(\mu\to o)} . \lambda s_{\mu} . \forall v_{(\mu\to o)}\, hvs$

Modal logic axioms

valid $\forall\varphi(\Box\varphi \supset \Diamond\varphi)$

Semantical Condition

$\forall x \exists y (rxy)$

Bridge rules

valid $\forall\varphi(\Box_r\varphi \supset \Box_s\varphi)$

Semantical Condition

$\forall x \forall y (rxy \supset sxy)$

We get a wide range of modal logics and combinations for free!

[BenzmüllerPaulson, LogicaUniversalis, 2013]

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Embeddings in HOL — Theoretical Results

Soundness and Completeness

$\models^{L} \varphi$ iff $\mathsf{Ax} \models^{HOL}_{\mathsf{Henkin}} \mathit{valid}\, \varphi_{\mu\to o}$

Logic L:

- Higher-order Modal Logics [BenzmüllerWoltezenlogelPaleo, ECAI, 2014]
- First-order Multimodal Logics [BenzmüllerPaulson, LogicaUniversalis, 2013]
- Propositional Multimodal Logics [BenzmüllerPaulson, Log.J.IGPL, 2010]
- Quantified Conditional Logics [Benzmüller, IJCAI, 2013]
- Propositional Conditional Logics [BenzmüllerEtAl., AMAI, 2012]
- Intuitionistic Logics [BenzmüllerPaulson, Log.J.IGPL, 2010]
- Access Control Logics [Benzmüller, IFIP SEC, 2009]
- Logic Combinations [Benzmüller, AMAI, 2011]
- . . . more is on the way . . .

Embeddings in HOL — Theoretical Results

Soundness and Completeness (and Cut-elimination)

$\models^{L} \varphi$ iff $\mathsf{Ax} \models^{HOL}_{\mathsf{Henkin}} \mathit{valid}\ \varphi_{\mu\to o}$ (iff $Ax \vdash^{\mathsf{seq}^{\mathsf{HOL}}}_{\mathsf{cut\text{-}free}} \mathit{valid}\ \varphi_{\mu\to o}$)

Logic L:

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- . . . more is on the way . . .

Cut-free Calculi for HOL: History

- Takeuti (1953): defined GLC (generalized logical calculus) by extending Gentzen's LK; conjectured cut-elimination for GLC
- Schütte (1960): simplified verion GLC; gave a semantic characterization Takeuti's conjecture.
- Tait (1966): proved Schütte's conjecture.
- Takahashi (1967), Prawitz (1968): proved higher-order versions of the conjecture.
- Girard (1971): Takeuti's conjecture as a consequence of strong normalization for System F.
- Andrews (1971): Completeness of resolution in elementary type theory with abstract consistency technique.
- Takeuti (1975): Henkin complete cut-free sequent calculus with extensionality.
- Brown (2004), Benzmüller et al. (2004, 2009), and Brown and Smolka (2010): Various complete cut-free calculi with/without extensionality, use of abstract consistency technique

One-sided sequent calculus $\mathcal{G}_{\beta\mathfrak{f}\mathfrak{b}}$ [BenzmüllerBrownKohlhase, LMCS, 2009]
(Δ: finite sets of β-normal closed formulas, $\Delta * \mathbf{A}$ stands for $\Delta \cup \{\mathbf{A}\}$,
$cwff_\alpha$: set of closed terms of type α, $\doteq$ abbreviates Leibniz equality):

Base Rules

$$\frac{\mathbf{A} \text{ atomic (\& } \beta\text{-normal)}}{\Delta * \mathbf{A} * \neg\mathbf{A}}\,\mathcal{G}(init) \qquad \frac{\Delta * \mathbf{A}}{\Delta * \neg\neg\mathbf{A}}\,\mathcal{G}(\neg) \qquad \frac{\Delta * \neg\mathbf{A} \quad \Delta * \neg\mathbf{B}}{\Delta * \neg(\mathbf{A} \vee \mathbf{B})}\,\mathcal{G}(\vee_-)$$

$$\frac{\Delta * \mathbf{A} * \mathbf{B}}{\Delta * (\mathbf{A} \vee \mathbf{B})}\,\mathcal{G}(\vee_+) \qquad \frac{\Delta * \neg\,(\mathbf{A}\mathbf{C})\!\downarrow_\beta \quad \mathbf{C} \in cwff_\alpha}{\Delta * \neg\Pi^\alpha\mathbf{A}}\,\mathcal{G}(\Pi^{\mathbf{C}}_-) \qquad \frac{\Delta * (\mathbf{A}c)\!\downarrow_\beta \quad c_\alpha\, new}{\Delta * \Pi^\alpha\mathbf{A}}\,\mathcal{G}(\Pi^c_+)$$

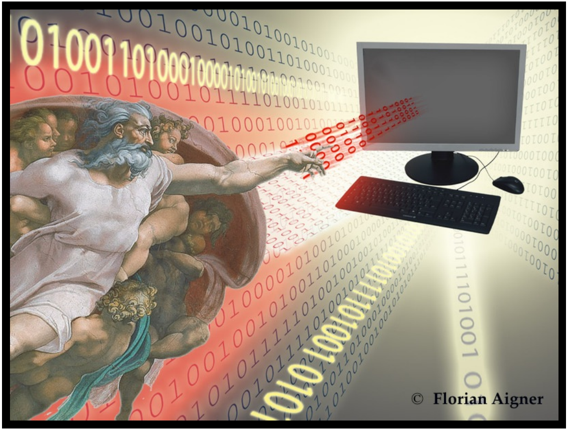
Full Extensionality

$$\frac{\Delta * (\forall X_\alpha.\mathbf{A}X \doteq^\beta \mathbf{B}X)\!\downarrow_\beta}{\Delta * (\mathbf{A} \doteq^{\alpha\to\beta} \mathbf{B})}\,\mathcal{G}(\mathfrak{f}) \qquad \frac{\Delta * \neg\mathbf{A} * \mathbf{B} \quad \Delta * \neg\mathbf{B} * \mathbf{A}}{\Delta * (\mathbf{A} \doteq^o \mathbf{B})}\,\mathcal{G}(\mathfrak{b})$$

Initial. and Decomp. of Leibniz Equality

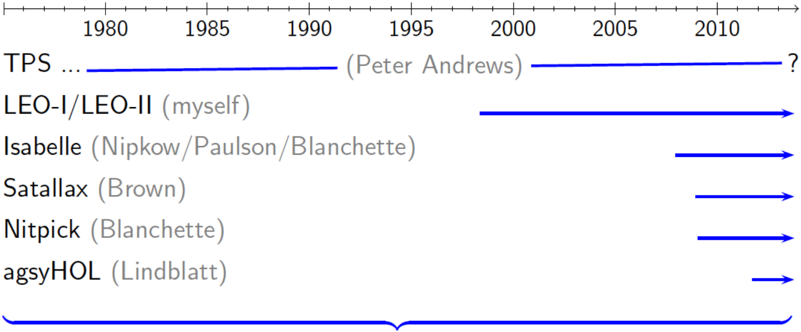
$$\frac{\Delta * (\mathbf{A} \doteq^o \mathbf{B}) \quad \mathbf{A},\mathbf{B} \text{ atomic}}{\Delta * \neg\mathbf{A} * \mathbf{B}}\,\mathcal{G}(Init^{\doteq})$$

$$\frac{\Delta * (\mathbf{A}^1 \doteq^{\alpha_1} \mathbf{B}^1) \;\cdots\; \Delta * (\mathbf{A}^n \doteq^{\alpha_n} \mathbf{B}^n) \quad n \geq 1, \beta \in \{o, \iota\}, h_{\overline{\alpha^n}\to\beta} \in \Sigma}{\Delta * (h\overline{\mathbf{A}^n} \doteq^\beta h\overline{\mathbf{B}^n})}\,\mathcal{G}(d)$$



Automated Proof Search and Consistency Check

Automated Theorem Provers and Model Finders for HOL



- all accept TPTP THF Syntax [SutcliffeBenzmüller, J.Form.Reas, 2009]
- can be called remotely via SystemOnTPTP at Miami
- they significantly gained in strength over the last years
- they can be bundled into a combined prover HOL-P

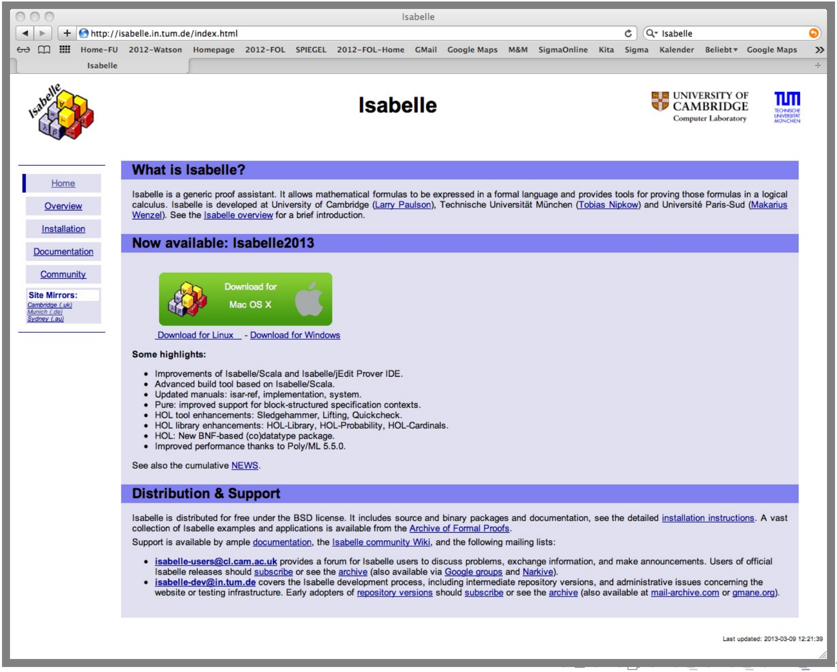
Exploit HOL with Henkin semantics as metalogic
Automate other logics (& combinations) via semantic embeddings
— HOL-P becomes a Universal Reasoner —

```
Terminal — bash — 125×32
MacBook-Chris %
MacBook-Chris %
MacBook-Chris % ./call_tptp.sh T3.p

Asking various HOL-ATPs in Miami remotely (thanks to Geoff Sutcliffe)

MacBook-Chris % agsyHOL---1.0 : T3.p ++++++ RESULT: SOT_7L4x_Y - agsyHOL---1.0 says Unknown - CPU = 0.00 WC = 0.02
LEO-II---1.6.0 : T3.p ++++++ RESULT: SOT_E4SCha - LEO-II---1.6.0 says Theorem - CPU = 0.03 WC = 0.09
Satallax---2.7 : T3.p ++++++ RESULT: SOT_kVZ1cB - Satallax---2.7 says Theorem - CPU = 0.00 WC = 0.14
Isabelle---2013 : T3.p ++++++ RESULT: SOT_xa0gEp - Isabelle---2013 says Theorem - CPU = 14.06 WC = 17.73 SolvedBy = auto
TPS---3.120601S1b : T3.p ++++++ RESULT: SOT_R0Egsg - TPS---3.120601S1b says Unknown - CPU = 33.56 WC = 41.57
Nitrox---2013 : T3.p ++++++ RESULT: SOT_WGY1Tx - Nitrox---2013 says Unknown - CPU = 75.55 WC = 49.24

MacBook-Chris %
MacBook-Chris % ./call_tptp.sh Consistency.p

Asking various HOL-ATPs in Miami remotely (thanks to Geoff Sutcliffe)

MacBook-Chris % agsyHOL---1.0 : Consistency.p ++++++ RESULT: SOT_ZtY_7o - agsyHOL---1.0 says Unknown - CPU = 0.00 WC = 0.00
Nitrox---2013 : Consistency.p ++++++ RESULT: SOT_HUz10C - Nitrox---2013 says Satisfiable - CPU = 6.56 WC = 8.50
TPS---3.120601S1b : Consistency.p ++++++ RESULT: SOT_fpJxTM - TPS---3.120601S1b says Unknown - CPU = 43.00 WC = 49.42
Isabelle---2013 : Consistency.p ++++++ RESULT: SOT_6Tpp9i - Isabelle---2013 says Unknown - CPU = 69.96 WC = 72.62
LEO-II---1.6.0 : Consistency.p ++++++ RESULT: SOT_dY10sj - LEO-II---1.6.0 says Timeout - CPU = 90 WC = 89.86
Satallax---2.7 : Consistency.p ++++++ RESULT: SOT_Q9WSLf - Satallax---2.7 says Timeout - CPU = 90 WC = 90.50

MacBook-Chris %
```

Provers are called remotely in Miami — no local installation needed!

Download our experiments from `https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/THF`

Automation and Verification in Isabelle/HOL
Interactive Verification in Coq



Isabelle

http://isabelle.in.tum.de/index.html

Home-FU 2012-Watson Homepage 2012-FOL SPIEGEL 2012-FOL-Home GMail Google Maps M&M SigmaOnline Kita Sigma Kalender Beliebt Google Maps

Isabelle

UNIVERSITY OF CAMBRIDGE Computer Laboratory

TUM TECHNISCHE UNIVERSITÄT MÜNCHEN

- Home
- Overview
- Installation
- Documentation
- Community

Site Mirrors:
Cambridge (.uk)
Munich (.de)
Sydney (.au)

What is Isabelle?

Isabelle is a generic proof assistant. It allows mathematical formulas to be expressed in a formal language and provides tools for proving those formulas in a logical calculus. Isabelle is developed at University of Cambridge (Larry Paulson), Technische Universität München (Tobias Nipkow) and Université Paris-Sud (Makarius Wenzel). See the Isabelle overview for a brief introduction.

Now available: Isabelle2013

Download for Mac OS X

Download for Linux - Download for Windows

Some highlights:

- Improvements of Isabelle/Scala and Isabelle/jEdit Prover IDE.
- Advanced build tool based on Isabelle/Scala.
- Updated manuals: isar-ref, implementation, system.
- Pure: improved support for block-structured specification contexts.
- HOL tool enhancements: Sledgehammer, Lifting, Quickcheck.
- HOL library enhancements: HOL-Library, HOL-Probability, HOL-Cardinals.
- HOL: New BNF-based (co)datatype package.
- Improved performance thanks to Poly/ML 5.5.0.

See also the cumulative NEWS.

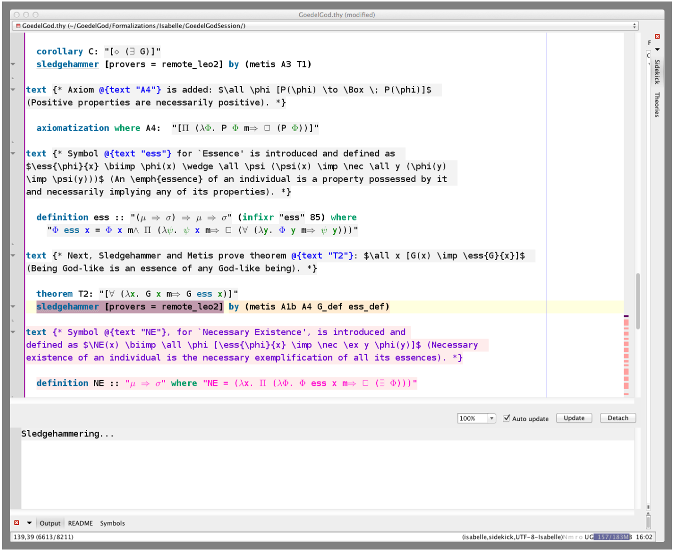
Distribution & Support

Isabelle is distributed for free under the BSD license. It includes source and binary packages and documentation, see the detailed installation instructions. A vast collection of Isabelle examples and applications is available from the Archive of Formal Proofs.

Support is available by ample documentation, the Isabelle community Wiki, and the following mailing lists:

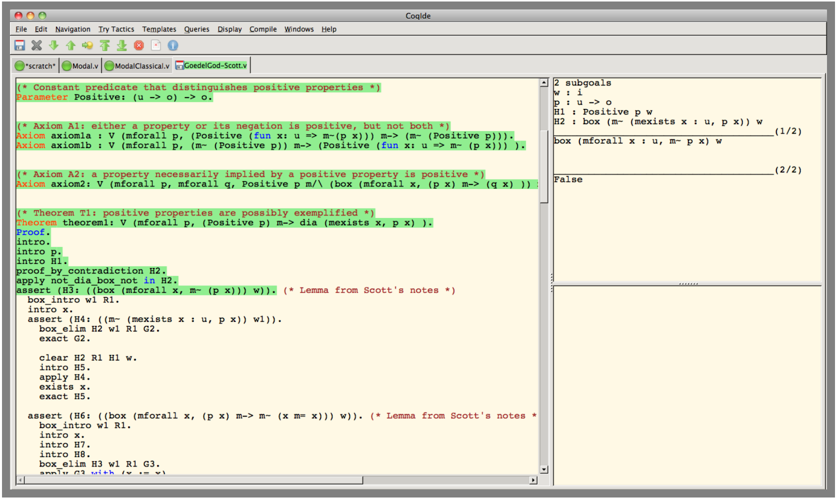
- **isabelle-users@cl.cam.ac.uk** provides a forum for Isabelle users to discuss problems, exchange information, and make announcements. Users of official Isabelle releases should subscribe or see the archive (also available via Google groups and Narkive).
- **isabelle-dev@in.tum.de** covers the Isabelle development process, including intermediate repository versions, and administrative issues concerning the website or testing infrastructure. Early adopters of repository versions should subscribe or see the archive (also available at mail-archive.com or gmane.org).

Last updated: 2013-03-09 12:21:39

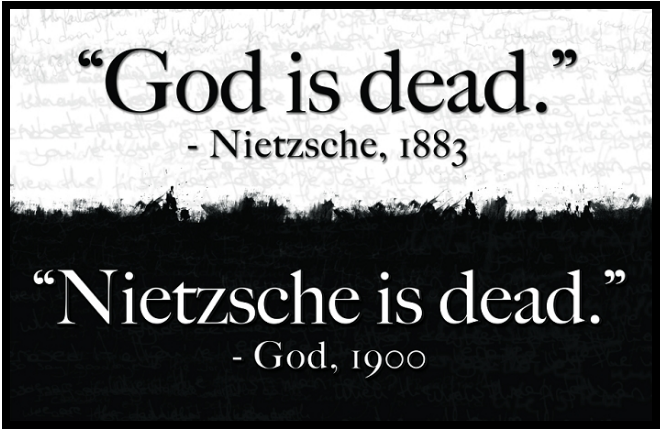


See verifiable Isabelle/HOL document (Archive of Formal Proofs) at:

`http://afp.sourceforge.net/entries/GoedelGod.shtml`



See verifiable Coq document at: `https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/Coq`



Main Findings [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

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	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$						
A2	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\wedge}\, \dot{\Box}\dot{\forall}X_\mu.\, (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$						
T1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Diamond}\dot{\exists}X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot{\Diamond}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Box}p\phi]$						
D2	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \,\dot{\wedge}\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \dot{\supset} \psi Y))$						
T2	$[\dot{\forall}X_\mu.\, g_{\mu\to\sigma}X \dot{\supset} (\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathbf{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, (\mathbf{ess}\, \phi X \dot{\supset} \dot{\Box}\dot{\exists}Y_\mu.\, \phi Y)$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathbf{NE}_{\mu\to\sigma}]$						
T3	$[\dot{\Box}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \dot{\supset} \dot{\Box}s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}X_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (\dot{\neg}(p_{(\mu\to\sigma)\to\sigma}\phi) \dot{\supset} \dot{\neg}(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot{\forall}X_\mu.\, \dot{\forall}Y_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (g_{\mu\to\sigma}Y \dot{\supset} X \dot{=} Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \dot{\supset} \psi Y))$						
CO'	$\emptyset$ (no goal, check for consistency)	A1(⊃), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

Main Findings [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\dot\forall\psi_{\mu\to\sigma}.(p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu.(\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\dot\forall\phi_{\mu\to\sigma}.p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}.p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\lambda X_\mu.\phi X \dot\wedge \dot\forall\psi_{\mu\to\sigma}.(\psi X \dot\supset \dot\Box\dot\forall Y_\mu.(\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu.g_{\mu\to\sigma}X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\dot\forall\phi_{\mu\to\sigma}.(\mathrm{ess}\,\phi X \dot\supset \dot\Box\dot\exists Y_\mu.\phi Y)$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.g_{\mu\to\sigma}X]$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall\phi_{\mu\to\sigma}.\dot\forall X_\mu.(g_{\mu\to\sigma}X \dot\supset (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \dot\supset \dot\neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu.\dot\forall Y_\mu.(g_{\mu\to\sigma}X \dot\supset (g_{\mu\to\sigma}Y \dot\supset X \dot= Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\lambda X_\mu.\dot\forall\psi_{\mu\to\sigma}.(\psi X \dot\supset \dot\Box\dot\forall Y_\mu.(\phi Y \dot\supset \psi Y))$						
CO'	$\emptyset$ (no goal, check for consistency)	A1(⊃), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall \phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu. \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\dot\forall\psi_{\mu\to\sigma}.(p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu.(\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\dot\forall\phi_{\mu\to\sigma}.p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu. g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}.p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$ess_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\lambda X_\mu.\phi X \dot\wedge \dot\forall\psi_{\mu\to\sigma}.(\psi X \dot\supset \dot\Box\dot\forall Y_\mu.(\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu. g_{\mu\to\sigma}X \dot\supset (ess_{(\mu\to\sigma)\to\mu\to\sigma} g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$NE_{\mu\to\sigma} = \lambda X_\mu.\dot\forall\phi_{\mu\to\sigma}.($e…						
A5	$[p_{(\mu\to\sigma)\to\sigma} NE_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu. g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$						
FG	$[\dot\forall\phi_{\mu\to\sigma}.\dot\forall X_\mu.(g_{\mu\to\sigma}X \dot\supset ($…						
MT	$[\dot\forall X_\mu.\dot\forall Y_\mu.(g_{\mu\to\sigma}X \dot\supset (g_\mu$…						
CO	∅ (no goal, check for cons…						
D2'	$ess_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\lambda$…						
CO'	∅ (no goal, check for cons…						

Automating Scott's proof script

T1: "Positive properties are possibly exemplified" proved by LEO-II and Satallax

- **in logic: K**
- **from axioms:**
  - **A1 and A2**
- **for domain conditions:**
  - **constant domains**

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu.\, (\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \dot\wedge \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\, (\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (\mathrm{e}$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$						
FG	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma}X \dot\supset ($						
MT	$[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma}X \dot\supset (g_\mu$						
CO	∅ (no goal, check for cons						
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$						
CO'	∅ (no goal, check for cons						

Automating Scott's proof script

T1: "Positive properties are possibly exemplified" proved by LEO-II and Satallax

- **in logic: K**
- **from axioms:**
  - **A1 and A2**
  - **A1(⊃) and A2**
- **for domain conditions:**
  - **constant domains**

Main Findings [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\, \phi \dot\wedge \dot\Box\dot\forall X_\mu.\, (\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma} X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \dot\wedge \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\, (\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma} X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (e$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma} X]$						
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$						
FG	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma} X \dot\supset ($						
MT	$[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma} X \dot\supset (g_\mu$						
CO	∅ (no goal, check for cons						
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$						
CO'	∅ (no goal, check for cons						

Automating Scott's proof script

T1: "Positive properties are possibly exemplified" proved by LEO-II and Satallax

- in logic: K
- from axioms:
  - A1 and A2
  - A1(⊃) and A2
- for domain conditions:
  - constant domains
  - varying domains (individuals)

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall \phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu. \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\dot\forall\psi_{\mu\to\sigma}.(p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu.(\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\dot\forall\phi_{\mu\to\sigma}.p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	[illegible]						
C	$[\dot\Diamond\dot\exists X_\mu. g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}.p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$ess_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\lambda X_\mu.\phi X \dot\wedge \dot\forall\psi_{\mu\to\sigma}.(\psi X \dot\supset \dot\Box\dot\forall Y_\mu.(\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu. g_{\mu\to\sigma}X \dot\supset (ess_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$NE_{\mu\to\sigma} = \lambda X_\mu.\dot\forall\phi_{\mu\to\sigma}.(e$						
A5	$[p_{(\mu\to\sigma)\to\sigma} NE_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu. g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$						
FG	$[\dot\forall\phi_{\mu\to\sigma}.\dot\forall X_\mu.(g_{\mu\to\sigma}X \dot\supset ($						
MT	$[\dot\forall X_\mu.\dot\forall Y_\mu.(g_{\mu\to\sigma}X \dot\supset (g_\mu$						
CO	∅ (no goal, check for cons						
D2'	$ess_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\lambda$						
CO'	∅ (no goal, check for cons						

Automating Scott's proof script

C: "Possibly, God exists" proved by LEO-II and Satallax

- **in logic: K**
- **from assumptions:**
  - **T1, D1, A3**
  - **A1, A2, D1, A3**
- **for domain conditions:**
  - **constant domains**
  - **varying domains (individuals)**

Main Findings [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \,\dot\equiv\, \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\wedge\, \dot\Box\dot\forall X_\mu.\, (\phi X \,\dot\supset\, \psi X)) \,\dot\supset\, p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \dot\Diamond\dot\exists X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \dot\Box p\phi]$						
D2	$\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \,\dot\wedge\, \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \,\dot\supset\, \dot\Box\dot\forall Y_\mu.\, (\phi Y \,\dot\supset\, \psi Y))$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \,\dot\supset\, (\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\text{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (e$ …						
A5	$[p_{(\mu\to\sigma)\to\sigma}\text{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \,\dot\supset\, \dot\Box s_\sigma]$						
FG	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma}X \,\dot\supset\,$ (…						
MT	$[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma}X \,\dot\supset\, (g_\mu$ …						
CO	∅ (no goal, check for cons …						
D2'	$\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$ …						
CO'	∅ (no goal, check for cons …						

Automating Scott's proof script

T2: "Being God-like is an ess. of any God-like being" proved by LEO-II and Satallax

- in logic: K
- from assumptions:
  - A1, D1, A4, D2
  - A1, A2, D1, A3, A4, D2
- for domain conditions:
  - constant domains
  - varying domains (individuals)

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$						
A2	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\wedge}\, \dot{\Box}\dot{\forall}X_\mu.\, (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$						
T1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Diamond}\dot{\exists}X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot{\Diamond}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Box}p\phi]$						
D2	$\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \,\dot{\wedge}\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \dot{\supset} \psi Y))$						
T2	$[\dot{\forall}X_\mu.\, g_{\mu\to\sigma}X \dot{\supset} (\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, gX)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$NE_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, (e$						
A5	$[p_{(\mu\to\sigma)\to\sigma} NE_{\mu\to\sigma}]$						
T3	$[\dot{\Box}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \dot{\supset} \dot{\Box}s_\sigma]$						
FG	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}X_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} ($						
MT	$[\dot{\forall}X_\mu.\, \dot{\forall}Y_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (g_\mu$						
CO	∅ (no goal, check for cons						
D2'	$\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$						
CO'	∅ (no goal, check for cons						

Automating Scott's proof script

T3: "Necessarily, God exists"
proved by LEO-II and Satallax

- **in logic: KB**
- **from assumptions:**
  - **D1, C, T2, D3, A5**
- **for domain conditions:**
  - **constant domains**
  - **varying domains (individuals)**

For logic K we got a countermodel by Nitpick

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall \phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu. \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall \phi_{\mu\to\sigma}. \dot\forall \psi_{\mu\to\sigma}. (p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu.(\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall \phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu. \dot\forall \phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu. g_{\mu\to\sigma} X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall \phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}. \lambda X_\mu. \phi X \dot\wedge \dot\forall \psi_{\mu\to\sigma}.(\psi X \dot\supset \dot\Box\dot\forall Y_\mu.(\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu. g_{\mu\to\sigma} X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu. \dot\forall \phi_{\mu\to\sigma}.(\mathrm{ess}\,\phi X \dot\supset \dot\Box\dot\exists Y_\mu.\phi Y)$						
A5	$[p_{(\mu\to\sigma)\to\sigma} \mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu. g_{\mu\to\sigma} X]$						
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$						
FG	$[\dot\forall \phi_{\mu\to\sigma}. \dot\forall X_\mu.(g_{\mu\to\sigma} X \dot\supset ($						
MT	$[\dot\forall X_\mu. \dot\forall Y_\mu.(g_{\mu\to\sigma} X \dot\supset (g_\mu$						
CO	∅ (no goal, check for cons						
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}. \lambda$						
CO'	∅ (no goal, check for cons						

Automating Scott's proof script

Summary

- **proof verified and automated**
- **KB is sufficient (critisized logic S5 not needed!)**
- **proof works for constant and varying domains**
- **exact dependencies determined experimentally**
- **ATPs have found alternative proofs (shorter)**

Main Findings [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu.\, (\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu-}$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma} X]$						
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma} X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)-}$						
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (\mathrm{e}$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma} X]$						
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma} X \dot\supset (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \dot\supset \dot\neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma} X \dot\supset (g_{\mu\to\sigma} Y \dot\supset X \dot= Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\, (\phi Y \dot\supset \psi Y))$						
CO'	∅ (no goal, check for consistency)	A1(⊃), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

Consistency check: Gödel vs. Scott

- **Scott's assumptions are consistent; shown by Nitpick**
- **Gödel's assumptions are inconsistent; shown by LEO-II (new philosophical result!)**

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\wedge\, \dot\Box\dot\forall X_\mu.\, (\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu-}$ ... $\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$						
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p$ ...						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$ ...						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)}$ ...						
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (e$ ...						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma}X \dot\supset (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \dot\supset \dot\neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma}X \dot\supset (g_{\mu\to\sigma}Y \dot\supset X \dot= Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\, (\phi Y \dot\supset \psi Y))$						
CO'	$\emptyset$ (no goal, check for consistency)	A1(⊃), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

Further Results

- **Monotheism holds**
- **God is flawless**

Modal Collapse

$\forall\varphi(\varphi \supset \Box\varphi)$

- proved by LEO-II and Satallax
- for constant and varying domains

Main critique on Gödel's ontological proof:

- there are no contingent truths
- everything is determined / no free will
- why using modal logic in the first place?

	HOL encoding						
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg$ …						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to}$ …						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists$ …						
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu}$ …						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$						
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p$ …						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$ …						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)}$ …						
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (e$ …						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma}X \dot\supset (\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \dot\supset \neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma}X \dot\supset (g_{\mu\to\sigma}Y \dot\supset X \doteq Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\, (\phi Y \dot\supset \psi Y))$						
CO'	$\emptyset$ (no goal, check for consistency)	A1($\supset$), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

Avoiding the Modal Collapse: Very recent work (not yet published)

Variants of Gödel's proof that avoid the modal collapse

- [Frode Børdal, **Understanding Gödel's Ontological Argument**, 1998] (verified and automated)
- [Anthony Anderson, **Some emendations of Gödel's ontological proof**, 1990] (verified and automated)
- [Melvin Fitting, **Types, Tableaux and Gödel's God**, 2002] (ongoing)

Future work

- [André Fuhrmann, 2005]
- [Petr Hajek, 1996, 2001, 2002, 2008, 2011]
- [Szatkowski, 2011]
- ...

Conclusion

Achievements

- significant contribution towards a **Computational Metaphysics**
- **HOL** very fruitfully exploited as a **universal metalogic**
- systematic study of a **prominent philosophical argument**
- even some **novel results** were found **by HOL-ATPs**
- infrastructure can be adapted for **other logics and logic combinations**

Relevance (wrt foundations and applications)

- Theoretical Philosophy, Artificial Intelligence, Computer Science, Maths

Little related work: only for Anselm's simpler argument

- first-order ATP PROVER9 [OppenheimerZalta, 2011]
- interactive proof assistant PVS [Rushby, 2013]

Future work

- continuation of systematic study of the ontological argument
- further studies in **Computational Metaphysics**

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Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hürter

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Kurt Gödel (um das Jahr 1935): Der Mathematiker hielt seinen Gottesbeweis jahrzehntelang geheim

Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelgebilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.

Montag, 09.09.2013 – 12:03 Uhr

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Jetzt sind die letzten Zweifel ausgeräumt: Gott existiert tatsächlich. Ein Computer hat es mit kalter Logik bewiesen - das MacBook des Computerwissenschaftlers Christoph Benzmüller von der Freien Universität Berlin.

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English Site > Germany > Science > Scientists Use Computer to Mathematically Prove Gödel God Theorem

Holy Logic: Computer Scientists 'Prove' God Exists

By David Knight

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Austrian mathematician Kurt Gödel kept his proof of God's existence a secret for decades. Now two scientists say they have proven it mathematically using a computer.

Two scientists have formalized a theorem regarding the existence of God penned by mathematician Kurt Gödel. But the God angle is somewhat of a red herring -- the real step forward is the example it sets of how computers can make scientific progress simpler.

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DIE WELT

FREITAG, 18. OKTOBER 2013

Zippert zappt

Teuer erkaufte Harmonie

Mindestlohn lockt SPD in dritte Ehe mit der Union

Mittwoch soll über eine große Koalition verhandelt werden. Votum des SPD-Konvents steht aus

THEMEN

Sport
Bundestrainer Löw unterschreibt beim DFB bis 2016

Wirtschaft
Der Jumbojet wird bei Boeing zum Ladenhüter

Karriere
Stress im Job
So legen Sie sich einen dicken Panzer zu

Aus aller Welt
Mietprozess in Paris: Dürfen 15 Quadratmeter 330 Euro kosten?

Dax
Im Minus

Göttliche Mathematik

Haben Forscher mit einem Computerprogramm die Existenz Gottes bewiesen?

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- Wiener Zeitung
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SCIENCE NEWS

HOME / SCIENCE NEWS / RESEARCHERS SAY THEY USED MACBOOK TO PROVE GOEDEL'S GOD THEOREM

Researchers say they used MacBook to prove Goedel's God theorem

Oct. 23, 2013 | 8:14 PM | 1 comments

See more serious and funny news links at
https://github.com/FormalTheology/GoedelGod/tree/master/Press

```
%----Additional base type mu (for worlds)
%----(already inbuilt: $i for individuals and $o for Booleans)
thf(mu_type,type,(mu:$tType)).
%----Reserved constant r for accessibility relation
thf(r,type,(r:$i>$i>$o)).
%----Modal operators not, or, box
thf(mnot_type,type,(mnot:($i>$o)>$i>$o)).
thf(mnot,definition,(mnot = (^[A:$i>$o,W:$i]:~(A@W)))).
thf(mor_type,type,(mor:($i>$o)>($i>$o)>$i>$o)).
thf(mor,definition,(mor = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W)|(Psi@W))))).
thf(mbox_type,type,(mbox:($i>$i>$o)>($i>$o)>$i>$o)).
thf(mbox,definition,(mbox = (^[A:$i>$o,W:$i]:![V:$i]:(~(r@W@V)|(A@V))))).
%----Quantifier (constant domains) for individuals and propositions
thf(mall_ind_type,type,(mall_ind:(mu>$i>$o)>$i>$o)).
thf(mall_ind,definition,(mall_ind = (^[A:mu>$i>$o,W:$i]:![X:mu]:(A@X@W)))).
thf(mall_indset_type,type,(mall_indset:((mu>$i>$o)>$i>$o)>$i>$o)).
thf(mall_indset,definition,(
    mall_indset = (^[A:(mu>$i>$o)>$i>$o,W:$i]:![X:mu>$i>$o]:(A@X@W)))).
%----Definition of validity
thf(v_type,type,(v:($i>$o)>$o)).
thf(mvalid,definition,(v = (^[A:$i>$o]:![W:$i]:(A@W)))).
%----Properties of accessibility relations
thf(msymmetric_type,type,(msymmetric:($i>$i>$o)>$o)).
    msymmetric = (^[R:$i>$i>$o]:![S:$i,T:$i]:((R@S@T)=>(R@T@S))))).
%----Here we work with logic KB
thf(sym,axiom,(msymmetric@r)).
```

Proof Overview

T3: $\Box\exists x.G(x)$

Proof Overview

$$\frac{\textbf{C: } \Diamond \exists z. G(z)}{\textbf{T3: } \Box \exists x. G(x)}$$

Proof Overview

$$\frac{\textbf{C: } \Diamond \exists z. G(z) \qquad \textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\textbf{T3: } \Box \exists x. G(x)}$$

Proof Overview

L2: $\Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$

$$\frac{\textbf{C: } \Diamond \exists z. G(z) \qquad \textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\textbf{T3: } \Box \exists x. G(x)}$$

$$\dfrac{\dfrac{\mathbf{S5}}{\forall \xi.\,[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\dfrac{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x) \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\frac{\dfrac{\textbf{L1: } \exists z.G(z) \rightarrow \Box\exists x.G(x)}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\frac{\textbf{C: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

D1: $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

$$\dfrac{\dfrac{\textbf{L1: } \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall \xi.[\Diamond \Box \xi \rightarrow \Box \xi]}}{\textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}$$

$$\dfrac{\textbf{C: } \Diamond \exists z.G(z) \qquad \textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\textbf{T3: } \Box \exists x.G(x)}$$

$$\textbf{D1:}\ G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$$

$$\textbf{D3*:}\ NE(x) \equiv \Box\exists y.G(y)$$

$$\dfrac{\dfrac{\dfrac{P(NE)}{\textbf{L1:}\ \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C:}\ \Diamond\exists z.G(z) \qquad \textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3:}\ \Box\exists x.G(x)}$$

D1: $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

D3*: $NE(x) \equiv \Box \exists y.G(y)$ (cheating!)

$$\frac{\dfrac{\dfrac{P(NE)}{\textbf{L1: } \exists z.G(z) \rightarrow \Box \exists x.G(x)}}{\Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall \xi.[\Diamond \Box \xi \rightarrow \Box \xi]}}{\textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}$$

$$\frac{\textbf{C: } \Diamond \exists z.G(z) \qquad \textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\textbf{T3: } \Box \exists x.G(x)}$$

D1: $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

D3*: $NE(x) \equiv \Box \exists y.G(y)$

D3: $NE(x) \equiv \forall \varphi_{\blacksquare}.[\varphi \; ess \; x \rightarrow \Box \exists y.\varphi(y)]$

$$\dfrac{\dfrac{\dfrac{\textbf{T2: } \forall y_{\blacksquare}.[G(y) \rightarrow G \; ess \; y] \qquad P(NE)}{\textbf{L1: } \exists z.G(z) \rightarrow \Box \exists x.G(x)}}{\Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall \xi_{\blacksquare}.[\Diamond \Box \xi \rightarrow \Box \xi]}}{\textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}$$

$$\dfrac{\textbf{C: } \Diamond \exists z.G(z) \qquad \textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\textbf{T3: } \Box \exists x.G(x)}$$

Proof Overview

$$\textbf{D1:}\ G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$$

$$\textbf{D3*:}\ NE(x) \equiv \Box\exists y.G(y)$$

$$\textbf{D3:}\ NE(x) \equiv \forall\varphi_{\blacksquare}.[\varphi\ ess\ x \rightarrow \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\dfrac{\textbf{T2:}\ \forall y_{\blacksquare}.[G(y) \rightarrow G\ ess\ y] \qquad \dfrac{\textbf{A5}}{P(NE)}}{\textbf{L1:}\ \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi_{\blacksquare}.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C:}\ \Diamond\exists z.G(z) \qquad \textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3:}\ \Box\exists x.G(x)}$$

Proof Overview

D1: $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

D3*: $NE(x) \equiv \Box \exists y.G(y)$

D3: $NE(x) \equiv \forall \varphi_{\blacksquare}.[\varphi \; ess \; x \rightarrow \Box \exists y.\varphi(y)]$

$$\dfrac{\textbf{T2: } \forall y_{\blacksquare}.[G(y) \rightarrow G \; ess \; y] \qquad \dfrac{\textbf{A5}}{P(NE)}}{\dfrac{\textbf{L1: } \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x)}}$$

$$\dfrac{\Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x) \qquad \dfrac{\textbf{S5}}{\forall \xi_{\blacksquare}.[\Diamond \Box \xi \rightarrow \Box \xi]}}{\textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}$$

$$\dfrac{\textbf{C: } \Diamond \exists z.G(z) \qquad \textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\textbf{T3: } \Box \exists x.G(x)}$$

D1: $G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$

D2: $\varphi\ ess\ x \equiv \varphi(x) \wedge \forall\psi.(\psi(x) \rightarrow \Box\forall x.(\varphi(x) \rightarrow \psi(x)))$

D3*: $NE(x) \equiv \Box\exists y.G(y)$

D3: $NE(x) \equiv \forall\varphi.[\varphi\ ess\ x \rightarrow \Box\exists y.\varphi(y)]$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\forall\varphi.[\neg P(\varphi) \rightarrow P(\neg\varphi)]}^{\textbf{A1b}} \quad \overline{\forall\varphi.[P(\varphi) \rightarrow \Box P(\varphi)]}^{\textbf{A4}}}{\textbf{T2: } \forall y.[G(y) \rightarrow G\ ess\ y]} \quad \overline{P(NE)}^{\textbf{A5}}}{\textbf{L1: } \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \quad \overline{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}^{\textbf{S5}}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

D1: $G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$

D2: $\varphi\ ess\ x \equiv \varphi(x) \wedge \forall\psi_{\blacksquare}.(\psi(x) \rightarrow \Box\forall x_{\blacksquare}.(\varphi(x) \rightarrow \psi(x)))$

D3*: $NE(x) \equiv \Box\exists y.G(y)$ **D3:** $NE(x) \equiv \forall\varphi_{\blacksquare}.[\varphi\ ess\ x \rightarrow \Box\exists y.\varphi(y)]$

C: $\Diamond\exists z.G(z)$

A1b: $\forall\varphi_{\blacksquare}.[\neg P(\varphi) \rightarrow P(\neg\varphi)]$

A4: $\forall\varphi_{\blacksquare}.[P(\varphi) \rightarrow \Box P(\varphi)]$

From A1b and A4: **T2:** $\forall y_{\blacksquare}.[G(y) \rightarrow G\ ess\ y]$

A5: $P(NE)$

From T2 and A5: **L1:** $\exists z.G(z) \rightarrow \Box\exists x.G(x)$

From L1: $\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)$

S5: $\forall\xi_{\blacksquare}.[\Diamond\Box\xi \rightarrow \Box\xi]$

From these: **L2:** $\Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)$

C: $\Diamond\exists z.G(z)$ **L2:** $\Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)$

T3: $\Box\exists x.G(x)$

D1: $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

D2: $\varphi \; ess \; x \equiv \varphi(x) \wedge \forall \psi.(\psi(x) \rightarrow \Box \forall x.(\varphi(x) \rightarrow \psi(x)))$

D3*: $NE(x) \equiv \Box \exists y.G(y)$ **D3:** $NE(x) \equiv \forall \varphi.[\varphi \; ess \; x \rightarrow \Box \exists y.\varphi(y)]$

$$\frac{P(G)}{\textbf{C: } \Diamond \exists z.G(z)}$$

$$\frac{\frac{\frac{\textbf{A1b}}{\forall \varphi.[\neg P(\varphi) \rightarrow P(\neg \varphi)]} \quad \frac{\textbf{A4}}{\forall \varphi.[P(\varphi) \rightarrow \Box P(\varphi)]}}{\textbf{T2: } \forall y.[G(y) \rightarrow G \; ess \; y]} \quad \frac{\textbf{A5}}{P(NE)}}{\textbf{L1: } \exists z.G(z) \rightarrow \Box \exists x.G(x)}$$

$$\frac{\frac{\textbf{L1: } \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x)} \quad \frac{\textbf{S5}}{\forall \xi.[\Diamond \Box \xi \rightarrow \Box \xi]}}{\textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}$$

$$\frac{\textbf{C: } \Diamond \exists z.G(z) \qquad \textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\textbf{T3: } \Box \exists x.G(x)}$$

Proof Overview

$$\textbf{D1: } G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$$

$$\textbf{D2: } \varphi \; ess \; x \equiv \varphi(x) \wedge \forall\psi_{\blacksquare}.(\psi(x) \rightarrow \Box\forall x_{\blacksquare}.(\varphi(x) \rightarrow \psi(x)))$$

$$\textbf{D3*: } NE(x) \equiv \Box\exists y.G(y) \qquad \textbf{D3: } NE(x) \equiv \forall\varphi_{\blacksquare}.[\varphi \; ess \; x \rightarrow \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\textbf{A3}}{P(G)}}{\textbf{C: } \Diamond\exists z.G(z)}$$

$$\dfrac{\dfrac{\textbf{A1b}}{\forall\varphi_{\blacksquare}.[\neg P(\varphi) \rightarrow P(\neg\varphi)]} \quad \dfrac{\textbf{A4}}{\forall\varphi_{\blacksquare}.[P(\varphi) \rightarrow \Box P(\varphi)]}}{\textbf{T2: } \forall y_{\blacksquare}.[G(y) \rightarrow G \; ess \; y]} \quad \dfrac{\textbf{A5}}{P(NE)}$$

$$\dfrac{\textbf{T2} \quad P(NE)}{\textbf{L1: } \exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{L1: } \exists z.G(z) \rightarrow \Box\exists x.G(x)}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)}$$

$$\dfrac{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x) \quad \dfrac{\textbf{S5}}{\forall\xi_{\blacksquare}.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

D1: $G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$

D2: $\varphi\ ess\ x \equiv \varphi(x) \wedge \forall\psi_{\blacksquare}.(\psi(x) \rightarrow \Box\forall x_{\blacksquare}.(\varphi(x) \rightarrow \psi(x)))$

D3*: $NE(x) \equiv \Box\exists y.G(y)$ **D3:** $NE(x) \equiv \forall\varphi_{\blacksquare}.[\varphi\ ess\ x \rightarrow \Box\exists y.\varphi(y)]$

$$\dfrac{\dfrac{\textbf{A3}}{P(G)} \qquad \textbf{T1: } \forall\varphi_{\blacksquare}.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]}{\textbf{C: } \Diamond\exists z.G(z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{A1b}}{\forall\varphi_{\blacksquare}.[\neg P(\varphi) \rightarrow P(\neg\varphi)]} \qquad \dfrac{\textbf{A4}}{\forall\varphi_{\blacksquare}.[P(\varphi) \rightarrow \Box P(\varphi)]}}{\textbf{T2: } \forall y_{\blacksquare}.[G(y) \rightarrow G\ ess\ y]} \qquad \dfrac{\textbf{A5}}{P(NE)}}{\textbf{L1: } \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi_{\blacksquare}.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1:}\ G(x) \equiv \forall \varphi.[P(\varphi) \to \varphi(x)]$$

$$\textbf{D2:}\ \varphi\ ess\ x \equiv \varphi(x) \wedge \forall \psi_{\blacksquare}.(\psi(x) \to \Box \forall x_{\blacksquare}.(\varphi(x) \to \psi(x)))$$

$$\textbf{D3*:}\ NE(x) \equiv \Box \exists y.G(y) \qquad \textbf{D3:}\ NE(x) \equiv \forall \varphi_{\blacksquare}.[\varphi\ ess\ x \to \Box \exists y.\varphi(y)]$$

$$\dfrac{\dfrac{}{P(G)}\textbf{A3} \qquad \dfrac{\dfrac{}{\forall \varphi_{\blacksquare}.\forall \psi_{\blacksquare}.[(P(\varphi) \wedge \Box \forall x_{\blacksquare}.[\varphi(x) \to \psi(x)]) \to P(\psi)]}\textbf{A2} \qquad \dfrac{}{\forall \varphi_{\blacksquare}.[P(\neg \varphi) \to \neg P(\varphi)]}\textbf{A1a}}{\textbf{T1:}\ \forall \varphi_{\blacksquare}.[P(\varphi) \to \Diamond \exists x.\varphi(x)]}}{\textbf{C:}\ \Diamond \exists z.G(z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall \varphi_{\blacksquare}.[\neg P(\varphi) \to P(\neg \varphi)]}\textbf{A1b} \qquad \dfrac{}{\forall \varphi_{\blacksquare}.[P(\varphi) \to \Box\, P(\varphi)]}\textbf{A4}}{\textbf{T2:}\ \forall y_{\blacksquare}.[G(y) \to G\ ess\ y]} \qquad \dfrac{}{P(NE)}\textbf{A5}}{\textbf{L1:}\ \exists z.G(z) \to \Box \exists x.G(x)}}{\Diamond \exists z.G(z) \to \Diamond \Box \exists x.G(x)} \qquad \dfrac{}{\forall \xi_{\blacksquare}.[\Diamond \Box \xi \to \Box \xi]}\textbf{S5}}{\textbf{L2:}\ \Diamond \exists z.G(z) \to \Box \exists x.G(x)}$$

$$\dfrac{\textbf{C:}\ \Diamond \exists z.G(z) \qquad \textbf{L2:}\ \Diamond \exists z.G(z) \to \Box \exists x.G(x)}{\textbf{T3:}\ \Box \exists x.G(x)}$$

$$\textbf{D1:}\ G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$$

$$\textbf{D2:}\ \varphi\ ess\ x \equiv \varphi(x) \wedge \forall \psi_{\blacksquare}.(\psi(x) \rightarrow \Box \forall x_{\blacksquare}.(\varphi(x) \rightarrow \psi(x)))$$

$$\textbf{D3:}\ NE(x) \equiv \forall \varphi_{\blacksquare}.[\varphi\ ess\ x \rightarrow \Box \exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\textbf{A3}}{P(G)} \qquad \dfrac{\dfrac{\textbf{A2}}{\forall \varphi_{\blacksquare}.\forall \psi_{\blacksquare}.[(P(\varphi) \wedge \Box \forall x_{\blacksquare}.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]} \qquad \dfrac{\textbf{A1a}}{\forall \varphi_{\blacksquare}.[P(\neg\varphi) \rightarrow \neg P(\varphi)]}}{\textbf{T1:}\ \forall \varphi_{\blacksquare}.[P(\varphi) \rightarrow \Diamond \exists x.\varphi(x)]}}{\textbf{C:}\ \Diamond \exists z.G(z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{A1b}}{\forall \varphi_{\blacksquare}.[\neg P(\varphi) \rightarrow P(\neg\varphi)]} \qquad \dfrac{\textbf{A4}}{\forall \varphi_{\blacksquare}.[P(\varphi) \rightarrow \Box\, P(\varphi)]}}{\textbf{T2:}\ \forall y_{\blacksquare}.[G(y) \rightarrow G\ ess\ y]} \qquad \dfrac{\textbf{A5}}{P(NE)}}{\textbf{L1:}\ \exists z.G(z) \rightarrow \Box \exists x.G(x)}}{\Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall \xi_{\blacksquare}.[\Diamond \Box \xi \rightarrow \Box \xi]}}{\textbf{L2:}\ \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}$$

$$\dfrac{\textbf{C:}\ \Diamond \exists z.G(z) \qquad \textbf{L2:}\ \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\textbf{T3:}\ \Box \exists x.G(x)}$$

Algebraic/Axiomatic Specification Viewpoint

HOL e.g. assume an algebraic / categorical characterization

HOML example formula: $(\Box\Diamond(\Box\varphi \supset \Box\Diamond\varphi)) \supset (\Box\varphi \supset \Box\Diamond\varphi)$

HOL e.g. assume an algebraic / categorical characterization

HOML example formula: $(\Box\Diamond(\Box\varphi \supset \Box\Diamond\varphi)) \supset (\Box\varphi \supset \Box\Diamond\varphi)$

HOML in HOL: $\textit{valid}\ (\Box\Diamond(\Box\varphi \supset \Box\Diamond\varphi)) \supset (\Box\varphi \supset \Box\Diamond\varphi)$

Signature

$$\varphi, \psi : \mu \to o$$

$$\Box, \Diamond : (\mu \to o) \to (\mu \to o)$$

$$\supset\ : (\mu \to o) \to (\mu \to o) \to (\mu \to o)$$

$$\text{valid} : (\mu \to o) \to o$$

$$r : \mu \to \mu \to o$$

HOL e.g. assume an algebraic / categorical characterization

HOML example formula: $(\Box\Diamond(\Box\varphi \supset \Box\Diamond\varphi)) \supset (\Box\varphi \supset \Box\Diamond\varphi)$

HOML in HOL: $\mathit{valid}\ (\Box\Diamond(\Box\varphi \supset \Box\Diamond\varphi)) \supset (\Box\varphi \supset \Box\Diamond\varphi)$

Signature

$$\varphi, \psi : \mu \to o$$
$$\Box, \Diamond : (\mu \to o) \to (\mu \to o)$$
$$\supset : (\mu \to o) \to (\mu \to o) \to (\mu \to o)$$
$$\text{valid} : (\mu \to o) \to o$$
$$r : \mu \to \mu \to o$$

Axiomatization (relative to HOL)

$$\forall s\,(\mathit{valid}\ s) \equiv \forall w\,(sw)$$
$$\forall s \forall t \forall w\,((s \supset t)w) \equiv ((sw) \Rightarrow (tw))$$
$$\forall s \forall w\,((\Diamond s)w) \equiv \exists v\,(rwv) \wedge (sv)$$
$$\forall s \forall w\,((\Box s)w) \equiv \forall v\,(rwv) \Rightarrow (sv)$$

HOL e.g. assume an algebraic / categorical characterization

HOML example formula: $(\Box\Diamond(\Box\varphi \supset \Box\Diamond\varphi)) \supset (\Box\varphi \supset \Box\Diamond\varphi)$

HOML in HOL: $\mathit{valid}\ (\Box\Diamond(\Box\varphi \supset \Box\Diamond\varphi)) \supset (\Box\varphi \supset \Box\Diamond\varphi)$

Signature

$$\varphi, \psi : \mu \to o$$
$$\Box, \Diamond : (\mu \to o) \to (\mu \to o)$$
$$\supset\ : (\mu \to o) \to (\mu \to o) \to (\mu \to o)$$
$$\text{valid} : (\mu \to o) \to o$$
$$r : \mu \to \mu \to o$$

Axiomatization (relative to HOL)

$$\forall s\,(\mathit{valid}\ s) \equiv \forall w\,(sw)$$
$$\forall s \forall t \forall w\,((s \supset t)w) \equiv ((sw) \Rightarrow (tw))$$
$$\forall s \forall w\,((\Diamond s)w) \equiv \exists v\,(rwv) \wedge (sv)$$
$$\Box = \lambda s \lambda w \forall u\,(\neg rwu \vee su)$$