Gödel’s Proof of God’s Existence

Christoph Benzmüller and Bruno Woltzenlogel Paleo

Square of Opposition
Vatican, May 6, 2014

A gift to Priest Edvaldo in Piracicaba, Brazil
First time mechanization and automation of
- (variants of) a modern ontological argument
- (variants of) higher-order modal logic

Work context/history:

- **Proposal:** exploit classical higher-order logic (HOL) as universal meta-logic — cf. previous talks at UNILOG
  - for object-level reasoning (in embedded non-classical logics)
  - for meta-level reasoning (about embedded non-classical logics)
- **Proof of concept:** demonstrate practical relevance of the approach by an interesting and relevant application
- **Experiments:** systematic study of Gödel’s argument
- **Relation to Square of Opposition:** should be easy to analyze variants of the Square within our approach
Challenge: No provers for *Higher-order Quantified Modal Logic* (QML)

Our solution: Embedding in *Higher-order Classical Logic* (HOL)

What we did:

A: Pen and paper: detailed natural deduction proof

B: Formalization: in classical higher-order logic (HOL)
   Automation: theorem provers LEO-II(\texttt{E}) and \texttt{Satallax}
   Consistency: model finder \texttt{Nitpick (Nitrox)}

C: Step-by-step verification: proof assistant \texttt{Coq}

D: Automation & verification: proof assistant \texttt{Isabelle}

Did we get any new results? Yes — let’s discuss this later!
Introduction

Germany
- Telepolis & Heise
- Spiegel Online
- FAZ
- Die Welt
- Berliner Morgenpost
- Hamburger Abendpost
- ...

Austria
- Die Presse
- Wiener Zeitung
- ORF
- ...

Italy
- Repubblica
- Ilsussidario
- ...

India
- DNA India
- Delhi Daily News
- India Today
- ...

US
- ABC News
- ...

International
- Spiegel International
- Yahoo Finance
- United Press Intl.
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Do you really need a MacBook to obtain the results? No

Did Apple send us some money? No
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Rich history on ontological arguments (pros and cons)

Anselm’s notion of God:
“God is that, than which nothing greater can be conceived.”

Gödel’s notion of God:
“A God-like being possesses all ‘positive’ properties.”

To show by logical reasoning:
“(Necessarily) God exists.”
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Different Interests in Ontological Arguments:

- **Philosophical:** Boundaries of Metaphysics & Epistemology
  - We talk about a metaphysical concept (God),
  - but we want to draw a conclusion for the real world.

- **Theistic:** Successful argument should convince atheists

- **Ours:** Can computers (theorem provers) be used . . .
  - . . . to formalize the definitions, axioms and theorems?
  - . . . to verify the arguments step-by-step?
  - . . . to fully automate (sub-)arguments?

Towards: ‘*Computer-assisted Theoretical Philosophy*”

(cf. Leibniz dictum — Calculemus!)

Gödel's Proof of God's Existence 8
Axiom A1  Either a property or its negation is positive, but not both: \( \forall \phi [P(\neg \phi) \equiv \neg P(\phi)] \)

Axiom A2  A property necessarily implied by a positive property is positive:

\[ \forall \phi \forall \psi [(P(\phi) \land \Box \forall x[\phi(x) \supset \psi(x)]) \supset P(\psi)] \]

Thm. T1  Positive properties are possibly exemplified:

\[ \forall \phi [P(\phi) \supset \Diamond \exists x \phi(x)] \]

Def. D1  A God-like being possesses all positive properties:

\[ G(x) \equiv \forall \phi [P(\phi) \supset \phi(x)] \]

Axiom A3  The property of being God-like is positive:

Cor. C  Possibly, God exists:

\[ \Diamond \exists x G(x) \]

Axiom A4  Positive properties are necessarily positive:

\[ \forall \phi [P(\phi) \supset \Box P(\phi)] \]

Def. D2  An essence of an individual is a property possessed by it and necessarily implying any of its properties:

\[ \phi \text{ ess. } x \equiv \phi(x) \land \forall \psi(\psi(x) \supset \Box \forall y(\phi(y) \supset \psi(y))) \]

Thm. T2  Being God-like is an essence of any God-like being:

\[ \forall x [G(x) \supset G \text{ ess. } x] \]

Def. D3  Necessary existence of an individ. is the necessary exemplification of all its essences:

\[ NE(x) \equiv \forall \phi [\phi \text{ ess. } x \supset \Box \exists y \phi(y)] \]

Axiom A5  Necessary existence is a positive property:

Thm. T3  Necessarily, God exists:

\[ \Box \exists x G(x) \]
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\[ P(G) \]

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Scott’s Version of Gödel’s Axioms, Definitions and Theorems

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• Embedding of QML in HOL and Proof Automation (myself)
• Proof Overview (Bruno)
• Experiments and Results (Bruno)
• Conclusion and Outlook (Bruno)
Embedding of QML in HOL and Proof Automation
Challenge: No provers for Higher-order Quantified Modal Logic (QML)

Our solution: Embedding in Higher-order Classical Logic (HOL)
Then use existing HOL theorem provers for reasoning in QML

Previous empirical findings:

Embedding of First-order Modal Logic in HOL works well

[BenzmüllerOttenRaths, ECAI, 2012]
[BenzmüllerRaths, LPAR, 2013]
Formalization in HOL

QML

\( \varphi, \psi \ ::= \ldots | \neg \varphi | \varphi \land \psi | \varphi \lor \psi | \Box \varphi | \Diamond \varphi | \forall x \varphi | \exists x \varphi | \forall P \varphi \)

- Kripke style semantics (possible world semantics)

HOL

\( s, t \ ::= C | x | \lambda xs | s \ t | \neg s | s \lor t | \forall x \ t \)

- meanwhile very well understood
- **Henkin semantics** vs. standard semantics
- various theorem provers do exist

interactive: Isabelle/HOL, HOL4, Hol Light, Coq/HOL, PVS, . . .
automated: TPS, LEO-II, Satallax, Nitpick, Isabelle/HOL, . . .
Formalization in HOL

\[ QML \quad \varphi, \psi ::= \ldots \mid \neg \varphi \mid \varphi \land \psi \mid \varphi \supset \psi \mid \Box \varphi \mid \Diamond \varphi \mid \forall x \varphi \mid \exists x \varphi \mid \forall P \varphi \]

\[ HOL \quad s, t ::= C \mid x \mid \lambda x s \mid st \mid \neg s \mid s \lor t \mid \forall x t \]

**QML in HOL:** QML formulas \( \varphi \) are mapped to HOL predicates \( \varphi_{t \rightarrow o} \)

\[
\begin{align*}
\neg &= \lambda \varphi_{t \rightarrow o} \lambda s_t \neg \varphi s \\
\land &= \lambda \varphi_{t \rightarrow o} \lambda \psi_{t \rightarrow o} \lambda s_t (\varphi s \land \psi s) \\
\supset &= \lambda \varphi_{t \rightarrow o} \lambda \psi_{t \rightarrow o} \lambda s_t (\neg \varphi s \lor \psi s) \\
\Box &= \lambda \varphi_{t \rightarrow o} \lambda s_t \forall u_t (\neg rsu \lor \varphi u) \\
\Diamond &= \lambda \varphi_{t \rightarrow o} \lambda s_t \exists u_t (rsu \land \varphi u) \\
\forall &= \lambda h_{(\mu \rightarrow (t \rightarrow o))} \lambda s_t \forall d_{\mu} hds \\
\exists &= \lambda h_{(\mu \rightarrow (t \rightarrow o))} \lambda s_t \exists d_{\mu} hds \\
\forall &= \lambda H_{(\mu \rightarrow (t \rightarrow o)) \rightarrow (t \rightarrow o)} \lambda s_t \forall d_{\mu} Hds \\
\text{valid} &= \lambda \varphi_{t \rightarrow o} \forall w_t \varphi w
\end{align*}
\]

The equations in Ax are given as axioms to the HOL provers!

(Remark: Note that we are here dealing with constant domain quantification)
Formalization in HOL

QML \( \phi, \psi ::= \ldots | \neg \phi | \phi \land \psi | \phi \supset \psi | \Box \phi | \Diamond \phi | \forall x \phi | \exists x \phi | \forall P \phi \)

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\end{align*}
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The equations in \( \text{Ax} \) are given as axioms to the HOL provers!

(Remark: Note that we are here dealing with constant domain quantification)
Example:

QML formula

QML formula in HOL
expansion, $\beta\eta$-conversion
expansion, $\beta\eta$-conversion
expansion, $\beta\eta$-conversion

What are we doing?

In order to prove that $\varphi$ is valid in QML,
$\rightarrow$ we instead prove that $\text{valid } \varphi_{\iota \rightarrow 0}$ can be derived from $\text{Ax}$ in HOL.

This can be done with interactive or automated HOL theorem provers.

Soundness and Completeness: wrt. Henkin semantics
Formalization in HOL

Example:

QML formula
QML formula in HOL

expansion, $\beta\eta$-conversion
expansion, $\beta\eta$-conversion
expansion, $\beta\eta$-conversion

$\forall w_i (\diamond \exists x G(x))_{t\rightarrow o} w$

$\forall w_i \exists u_i (r w u \land (\exists x G(x))_{t\rightarrow o} u)$

$\forall w_i \exists u_i (r w u \land \exists x G x u)$

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Gödel's Proof of God’s Existence
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\[ \diamond \exists x G(x) \]
\[ \forall w_i (\diamond \exists x G(x))_{i\rightarrow 0} \]
\[ \forall w_i \exists u_i (r w u \land (\exists x G(x))_{i\rightarrow 0} u) \]
\[ \forall w_i \exists u_i (r w u \land \exists x G x u) \]

What are we doing?

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Soundness and Completeness: wrt. Henkin semantics
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$\Diamond \exists x G(x)$
valid $(\Diamond \exists x G(x))_{t \rightarrow o}$
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valid \( (\diamond \exists x G(x))_{t \rightarrow o} \)
\[ \forall w (\diamond \exists x G(x))_{t \rightarrow o} w \]
\[ \forall w \exists u (r w u \land (\exists x G(x))_{t \rightarrow o} u) \]
\[ \forall w \exists u (r w u \land \exists G xu) \]

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Soundness and Completeness: wrt. Henkin semantics
Automated Theorem Provers and Model Finders for HOL


TPS ... (Peter Andrews)  
LEO-I/LEO-II (myself)  
Isabelle (Nipkow/Paulson/Blanchette)  
Satallax (Brown)  
Nitpick (Blanchette)  
agsyHOL (Lindblatt)

- all accept TPTP THF Syntax [SutcliffeBenzmüller, J.Form.Reas, 2009]
  - can be called remotely via SystemOnTPTP at Miami
  - they significantly gained in strength over the last years
  - they can be bundled into a combined prover HOL-P

Exploit HOL with Henkin semantics as metalogic
Automate other logics (& combinations) via semantic embeddings
--- HOL-P becomes a Universal Reasoner ---
Proof Overview
Experiments and Results
Gödel’s Manuscript: 1930’s, 1941, 1946-1955, 1970

On the logic of the world
Feb 10, 1970

\[ P(P) \] is positive \( \iff \forall \phi \in P \]

\[ \begin{align*}
\text{At. 1} & : P(P) \implies P(P(\phi)) \\
\text{At. 2} & : P(P) \implies P(P(\phi)) \wedge P(x) \end{align*} \]

\[ G(x) = \phi \left[ P(\phi) \implies P(x) \right] \] (Godel)

\[ \phi \text{ Em} \times \equiv \psi \left[ \psi(x) \implies \psi(y) \right] \] (Em of \( \psi \))

\[ p \implies q \iff N(p) \iff \text{Necessity} \]

\[ \begin{align*}
\text{At. 2} & : P(P) \implies N(P(\phi)) \quad \text{because it follows} \\
& \quad \text{from the nature of the property} \\
\end{align*} \]

\[ T \& G(x) \iff G \text{ Em} \times \]

\[ E(x) = \phi \left[ \phi \text{ Em} \times N \exists x \psi(x) \right] \] necessary Existence

\[ \begin{align*}
\text{At. 3} & : P(E) \\
\text{Th. 1} & : G(x) \iff N(\exists y) G(y)
\end{align*} \]

\[ \begin{align*}
G(x) & \iff G(y) \\
G(x) & \iff N(\exists y) G(y)
\end{align*} \]

\[ M(x) \iff M(\exists y) G(y) \]

P(x) \iff M(\exists y) G(y) \iff N(\exists y) G(y)

\[ \text{M = positivity} \]

\[ \text{any two chances of } x \text{ are nec. equivalent} \]

\[ \text{exclusive or for any number of chances} \]

\[ x \neq x \text{ i.e. the normal form in terms of selves, each contains a member without negation} \]
T3: $\Box \exists x. G(x)$
Proof Overview

\[ C1: \Diamond \exists z. G(z) \]

\[
\begin{array}{c}
\hline
T3: \Box \exists x. G(x)
\end{array}
\]

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Gödel's Proof of God's Existence
Proof Overview

$\textbf{C1: } \Diamond \exists z. G(z)$

$\textbf{L2: } \Diamond \exists z. G(z) \supset \Box \exists x. G(x)$

$\textbf{T3: } \Box \exists x. G(x)$
\[
\begin{align*}
\text{C1: } & \Diamond \exists z.G(z) \\
\text{L2: } & \Diamond \exists z.G(z) \supset \Box \exists x.G(x) \\
\text{T3: } & \Box \exists x.G(x)
\end{align*}
\]
\[ \text{Proof Overview} \]

\[ \text{L2:} \quad \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \]

\[ \text{C1:} \quad \Diamond \exists z. G(z) \quad \text{L2:} \quad \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \]

\[ \text{T3:} \quad \Box \exists x. G(x) \]
Proof Overview

\( \forall \xi. [\Box \xi \supset \Box \xi] \)

**L2:** \( \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \)

**C1:** \( \Diamond \exists z. G(z) \)

**L2:** \( \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \)

**T3:** \( \Box \exists x. G(x) \)
Proof Overview

\[ \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \]

**L2:** \[ \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \]

\[ \forall \xi. [\Diamond \Box \xi \supset \Box \Diamond \xi] \]

**S5**

**C1:** \[ \Diamond \exists z. G(z) \]

**L2:** \[ \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \]

**T3:** \[ \Box \exists x. G(x) \]
L1: $\exists z. G(z) \supset \Box \exists x. G(x)$

$L2: \Diamond \exists z. G(z) \supset \Box \exists x. G(x)$

C1: $\Diamond \exists z. G(z)$  L2: $\Diamond \exists z. G(z) \supset \Box \exists x. G(x)$

T3: $\Box \exists x. G(x)$

S5: $\forall \xi. [\Diamond \Box \xi \supset \Box \Box \xi]$
**D1:** $G(x) \equiv \forall \varphi. [P(\varphi) \supset \varphi(x)]$

**L1:** $\exists z. G(z) \supset \square \exists x. G(x)$

- **S5**
- $\forall \xi. [\square \xi \supset \square \xi]$

- **L2:** $\diamond \exists z. G(z) \supset \square \exists x. G(x)$

**C1:** $\diamond \exists z. G(z)$

- **L2:** $\diamond \exists z. G(z) \supset \square \exists x. G(x)$

- **T3:** $\square \exists x. G(x)$
**Proof Overview**

**D1:** $G(x) \equiv \forall \varphi. [P(\varphi) \supset \varphi(x)]$

**D3:** $E(x) \equiv \forall \varphi. [\varphi \text{ ess. } x \supset \Box \exists y. \varphi(y)]$

**T2:** $\forall y. [G(y) \supset G \text{ ess. } y]$

**L1:** $\exists z. G(z) \supset \Box \exists x. G(x)$

**S5**

**L2:** $\Box \exists z. G(z) \supset \Box \exists x. G(x)$

**C1:** $\Diamond \exists z. G(z)$

**L2:** $\Diamond \exists z. G(z) \supset \Box \exists x. G(x)$

**T3:** $\Box \exists x. G(x)$
Proof Overview

\textbf{D1:} \( G(x) \equiv \forall \varphi. [P(\varphi) \supset \varphi(x)] \)

\textbf{D3:} \( E(x) \equiv \forall \varphi. [\varphi \text{ ess. } x \supset \Box \exists y. \varphi(y)] \)

\textbf{T2:} \( \forall y. [G(y) \supset G \text{ ess. } y] \)

\textbf{L1:} \( \exists z. G(z) \supset \Box \exists x. G(x) \)

\textbf{L2:} \( \Diamond \exists z. G(z) \supset \Diamond \Box \exists x. G(x) \)

\textbf{C1:} \( \Diamond \exists z. G(z) \)

\textbf{L2:} \( \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \)

\textbf{T3:} \( \Box \exists x. G(x) \)

\textbf{A5} \( P(E) \)

\textbf{S5} \( \forall \xi. [\Diamond \Box \xi \supset \Box \xi] \)
**D1:** $G(x) \equiv \forall \varphi. [P(\varphi) \supset \varphi(x)]$

**D3:** $E(x) \equiv \forall \varphi. [\varphi \text{ ess. } x \supset \Box \exists y. \varphi(y)]$

\[\begin{align*}
\text{T2:} & \quad \forall y. [G(y) \supset G \text{ ess. } y] \\
\text{L1:} & \quad \exists z. G(z) \supset \Box \exists x. G(x) \\
\phantom{\text{T2:}} & \quad \Diamond \exists z. G(z) \supset \Diamond \Box \exists x. G(x) \\
\text{L2:} & \quad \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \\
\text{C1:} & \quad \Diamond \exists z. G(z) \\
\text{L2:} & \quad \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \\
\text{T3:} & \quad \Box \exists x. G(x)
\end{align*}\]
Proof Overview

**D1:** $G(x) \equiv \forall \varphi. [P(\varphi) \supset \varphi(x)]$

**D2:** $\varphi \text{ ess. } x \equiv \varphi(x) \land \forall \psi. (\psi(x) \supset \Box \forall x. (\varphi(x) \supset \psi(x)))$

**D3:** $E(x) \equiv \forall \varphi. [\varphi \text{ ess. } x \supset \Box \exists y. \varphi(y)]$

**A1b**

\[
\forall \varphi. [\neg P(\varphi) \supset \Box \neg \varphi] \quad \forall \varphi. [P(\varphi) \rightarrow \Box \neg P(\varphi)]
\]

**A4**

\[
\Box E(\varphi) \quad \Box \exists G(z) \supset \Box \exists x. G(x)
\]

**A5**

\[
\forall \varphi. [\neg P(\varphi) \supset P(\neg \varphi)]
\]

**T2:** $\forall y. [G(y) \supset G \text{ ess. } y]$

**L1:** $\exists z. G(z) \supset \Box \exists x. G(x)$

\[
\Rightarrow \exists z. G(z) \supset \Diamond \exists x. G(x)
\]

**L2:** $\Diamond \exists z. G(z) \supset \Box \exists x. G(x)$

**S5**

\[
\forall \xi. [\Diamond \xi \supset \Box \xi]
\]

**C1:** $\Diamond \exists z. G(z)$

**L2:** $\Diamond \exists z. G(z) \supset \Box \exists x. G(x)$
**D1:** \( G(x) \equiv \forall \varphi. [P(\varphi) \supset \varphi(x)] \)

**D2:** \( \varphi \text{ ess. } x \equiv \varphi(x) \land \forall \psi. (\psi(x) \supset \Box \forall x. (\varphi(x) \supset \psi(x))) \)

**D3:** \( E(x) \equiv \forall \varphi. [\varphi \text{ ess. } x \supset \Box \exists y. \varphi(y)] \)

**C1:** \( \Diamond \exists z. G(z) \)

\[
\begin{align*}
\text{A1b} & : \forall \varphi. [\neg P(\varphi) \supset P(\neg \varphi)] \\
\text{A4} & : \forall \varphi. [P(\varphi) \rightarrow \Box P(\varphi)] \\
\text{A5} & : P(E)
\end{align*}
\]

\[
\begin{align*}
\text{T2: } & \forall y. [G(y) \supset G \text{ ess. } y] \\
\text{L1: } & \exists z. G(z) \supset \Box \exists x. G(x) \\
\text{L2: } & \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \\
\text{S5} & : \forall \xi. [\Diamond \Box \xi \supset \Box \xi]
\end{align*}
\]

**C1:** \( \Diamond \exists z. G(z) \)

**L2:** \( \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \)

**T3:** \( \Box \exists x. G(x) \)
**D1:** \[ G(x) \equiv \forall \varphi. [P(\varphi) \supset \varphi(x)] \]

**D2:** \( \varphi \text{ ess. } x \equiv \varphi(x) \land \forall \psi. (\psi(x) \supset \Box \forall x. (\varphi(x) \supset \psi(x))) \)

**D3:** \( E(x) \equiv \forall \varphi. [\varphi \text{ ess. } x \supset \Box \exists y. \varphi(y)] \)

\[
\begin{align*}
A3 \quad & \quad P(G) \\
\hline
T1: \quad & \quad \forall \varphi. [P(\varphi) \supset \Diamond \exists x. \varphi(x)] \\
C1: \quad & \quad \Diamond \exists z. G(z) \\

A1b \quad & \quad \forall \varphi. [\neg P(\varphi) \supset P(\neg \varphi)] \\
\hline
A4 \quad & \quad \forall \varphi. [P(\varphi) \rightarrow \Box \neg P(\varphi)] \\

A5 \quad & \quad P(E) \\
T2: \quad & \quad \forall y. [G(y) \supset G \text{ ess. } y] \\

L1: \quad & \quad \exists z. G(z) \supset \Box \exists x. G(x) \\
\hline
\quad & \quad \Diamond \exists z. G(z) \supset \Diamond \Box \exists x. G(x) \\
L2: \quad & \quad \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \\
\quad & \quad \forall \xi. [\Diamond \Box \xi \supset \Box \xi] \\
S5 \quad & \quad \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \\

C1: \quad & \quad \Diamond \exists z. G(z) \\
L2: \quad & \quad \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \\
T3: \quad & \quad \Box \exists x. G(x) \\
\end{align*}
\]
Proof Overview

**D1:** \(G(x) \equiv \forall \varphi. [P(\varphi) \supset \varphi(x)]\)

**D2:** \(\varphi \text{ ess. } x \equiv \varphi(x) \land \forall \psi. (\psi(x) \supset \Box \forall x. (\varphi(x) \supset \psi(x)))\)

**D3:** \(E(x) \equiv \forall \varphi. [\varphi \text{ ess. } x \supset \Box \exists y. \varphi(y)]\)

\[
\begin{array}{c}
\textbf{A3} \quad \forall \varphi. \forall \psi. [(P(\varphi) \land \Box \forall x. (\varphi(x) \supset \psi(x))] \supset P(\psi)] \\
\textbf{A2} \quad \forall \varphi. [P(\varphi) \supset \exists x. \varphi(x)] \\
\textbf{A1a} \quad \forall \varphi. [P(\varphi) \supset \Diamond \exists x. \varphi(x)]
\end{array}
\]

\[
\begin{array}{c}
\textbf{T1:} \forall \varphi. [P(\varphi) \supset \Diamond \exists x. \varphi(x)] \\
\textbf{C1:} \Diamond \exists z. G(z)
\end{array}
\]

\[
\begin{array}{c}
\textbf{A1b} \quad \forall \varphi. [-P(\varphi) \supset P(\neg \varphi)] \\
\textbf{A4} \quad \forall \varphi. [P(\varphi) \rightarrow \Box P(\varphi)] \\
\textbf{A5} \quad \Box P(E)
\end{array}
\]

\[
\begin{array}{c}
\textbf{T2:} \forall y. [G(y) \supset G \text{ ess. } y] \\
\textbf{L1:} \exists z. G(z) \supset \Box \exists x. G(x) \\
\Diamond \exists z. G(z) \supset \Diamond \Box \exists x. G(x)
\end{array}
\]

\[
\begin{array}{c}
\textbf{S5} \quad \forall \xi. [\Diamond \Box \xi \supset \Box \xi] \\
\textbf{C1:} \Diamond \exists z. G(z) \\
\textbf{L2:} \Diamond \exists z. G(z) \supset \Box \exists x. G(x)
\end{array}
\]

\[
\begin{array}{c}
\textbf{T3:} \Box \exists x. G(x)
\end{array}
\]
Proof Overview

D1: \( G(x) \equiv \forall \varphi.[P(\varphi) \to \varphi(x)] \)

D2: \( \varphi \text{ ess. } x \equiv \varphi(x) \land \forall \psi.(\psi(x) \supset \Box \forall x. (\varphi(x) \supset \psi(x))) \)

D3: \( E(x) \equiv \forall \varphi. [\varphi \text{ ess. } x \supset \Box \exists y. \varphi(y)] \)

A3 \( \overline{P(G)} \)

A2 \( \overline{\forall \varphi. \forall \psi. [(P(\varphi) \land \Box \forall x. (\varphi(x) \supset \psi(x)))] \supset P(\psi)] \)

A1a \( \overline{\forall \varphi. [P(\neg \varphi) \supset \neg P(\varphi)]} \)

C1: \( \Diamond \exists z. G(z) \)

A1b \( \overline{\forall \varphi. [\neg P(\varphi) \supset P(\neg \varphi)]} \)

A4 \( \overline{\forall \varphi. [P(\neg \varphi) \to \Box P(\varphi)]} \)

A5 \( \overline{P(E)} \)

T1: \( \forall \varphi. [P(\varphi) \supset \Diamond \exists x. \varphi(x)] \)

T2: \( \forall y. [G(y) \supset G \text{ ess. } y] \)

L1: \( \exists z. G(z) \supset \Box \exists x. G(x) \)

S5 \( \forall \xi. [\Diamond \Box \xi \supset \Box \xi] \)

L2: \( \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \)

C1: \( \Diamond \exists z. G(z) \)

T3: \( \Box \exists x. G(x) \)
Natural Deduction Calculus
Rules for Modalities

\[ \alpha : \begin{array}{c} \vdots \\ A \end{array} \quad \square \vdash \alpha \]

\[ \begin{array}{c} A \\ \vdots \end{array} \quad \square I \]

\[ t : \begin{array}{c} \vdots \\ A \end{array} \quad \Diamond \vdash t \]

\[ \begin{array}{c} \vdots \\ A \end{array} \quad \Diamond I \]

\[ \begin{array}{c} \Diamond A \end{array} \quad \Diamond E \]

\[ \Delta A \equiv \neg \neg A \]
Natural Deduction Proofs
T1 and C1

A2

$\forall \varphi. \forall \psi. \left[ \left( P(\varphi) \land \Box \forall x. [\varphi(x) \supset \psi(x)] \right) \supset P(\psi) \right]$ 

$\forall \psi. \left[ \left( P(\rho) \land \Box \forall x. [\rho(x) \supset \psi(x)] \right) \supset P(\psi) \right]$ 

$(P(\rho) \land \Box \forall x. [\rho(x) \supset \neg \rho(x)]) \supset P(\neg \rho)$ 

$(P(\rho) \land \Box \forall x. [\neg \rho(x)]) \supset \neg P(\rho)$ 

$P(\rho) \supset \Diamond \exists x. \rho(x)$ 

T1: $\forall \varphi. [P(\varphi) \supset \Diamond \exists x. \varphi(x)]$ 

A1a

$\forall \varphi. [P(\neg \varphi) \supset \neg P(\varphi)]$ 

$P(\neg \rho) \supset \neg P(\rho)$ 

A3

$\forall \varphi. [P(\varphi) \supset \Diamond \exists x. \varphi(x)]$ 

$P(G) \supset \Diamond \exists x. G(x)$ 

$\Diamond \exists x. G(x)$
Natural Deduction Proofs

T2 (Partial)

$\psi(x)^6 \quad \Pi_2 \quad \Pi_3$

$\psi(x) \rightarrow \Box P(\psi) \quad \rightarrow E$

$P(\psi) \quad \Box E \quad \Pi_3

\forall x. (G(x) \rightarrow \psi(x)) \quad \rightarrow E$

$\Box P(\psi) \rightarrow \Box \forall x. (G(x) \rightarrow \psi(x)) \quad \rightarrow^7 I$

$\forall x. (G(x) \rightarrow \psi(x)) \quad \rightarrow^7 E$

$\psi(x) \rightarrow \Box \forall x. (G(x) \rightarrow \psi(x)) \quad \rightarrow^6 I$
Implementations and Experiments

- Formal encodings (in HOL) of:
  - modal logic axioms
  - axioms, definitions, and theorems in Scott’s proof script
- Experiments using automated provers
  - LEO-II, Satallax, AgsyHOL
- Interactive proofs using proof assistants
  - Isabelle and Coq

Source files available at:

https://github.com/FormalTheology/GoedelGod/

Demos on request!
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Axioms and definitions are consistent.
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Logic K is sufficient for proving T1, C and T2.

Logic KB is sufficient for proving the final theorem T3.
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Logic K is sufficient for proving T1, C and T2.
Logic KB is sufficient for proving the final theorem T3.

Adresses criticisms: modal logic S5 is too strong

\[ \forall P. [\Diamond \Box P \supset \Box P] \]

If something is possibly necessary, then it is necessary.

S5 usually considered adequate
(But KB is sufficient! — shown by HOL ATPs)

\[ \forall P. [P \supset \Box \Diamond P] \]

If something is the case, then it is necessarily possible.
Axioms and definitions are consistent.
Logic K is sufficient for proving T1, C and T2.
Logic KB is sufficient for proving the final theorem T3.
HOL-ATPs prove T1, C, and T2 from axioms quickly; succeed in proving T3 from axioms, C and T2; but fail in proving T3 from axioms alone.
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$\exists x. G(x)$ can be proved without first proving $\Box \exists x. G(x)$. 
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• Gödel’s original axioms and definitions, omitting conjunct $\phi(x)$ in the definition of essence, seem inconsistent.
• $\exists x. G(x)$ can be proved without first proving $\Box \exists x. G(x)$.
• Equality is not necessary to prove T1.
Results

- Axioms and definitions are consistent.
- Logic K is sufficient for proving T1, C and T2.
- Logic KB is sufficient for proving the final theorem T3.
- HOL-ATPs prove T1, C, and T2 from axioms quickly; succeed in proving T3 from axioms, C and T2; but fail in proving T3 from axioms alone.
- Gödel’s original axioms and definitions, omitting conjunct $\phi(x)$ in the definition of essence, seem inconsistent.
- $\exists x. G(x)$ can be proved without first proving $\Box \exists x. G(x)$.
- Equality is not necessary to prove T1.
- A2 may be used only once to prove T1.
Gödel’s axioms imply the modal collapse: $\forall \phi. (\phi \supset \square \phi)$
Gödel’s axioms imply the modal collapse: $\forall \phi.(\phi \supset \Box \phi)$

Fundamental criticism against Gödel’s argument.

Everything that is the case is so necessarily.

Follows from T2, T3 and D2 (as shown by HOL ATPs).

There are no contingent “truths”.
Everything is determined.
There is no free will.

Many proposed solutions: Anderson, Fitting, Hájek, . . .
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God is **flawless**: $\forall x. G(x) \supset (\forall \varphi. \neg P(\varphi) \supset \neg \varphi(x))$.

**Monotheism**: $\forall x. \forall y. G(x) \land G(y) \supset x = y$.

All results hold for both
- constant domain semantics
- varying domain semantics
God is **flawless**: \( \forall x. G(x) \supset (\forall \varphi. \neg P(\varphi) \supset \neg \varphi(x)) \).

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Conclusions
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Achievements:

- Infra-structure for automated higher-order modal reasoning
- Verification of Gödel’s ontological argument with HOL provers
  - experiments with different parameters
- Novel results and insights
- Major step towards Computer-assisted Theoretical Philosophy
  - see also Ed Zalta’s *Computational Metaphysics* project at Stanford University
  - see also John Rushby’s recent verification of Anselm’s proof in PVS
  - remember Leibniz’ dictum — *Calculemus!*
- Interesting bridge between CS, Philosophy and Theology

Ongoing and future work

- Formalize and verify literature on ontological arguments
  - …in particular the criticisms and proposed improvements
- Own contributions — supported by theorem provers
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