Types, Frames, and
Applicative Structures

Def.: Types

Let $\mathcal{T}$ be the least set s.t:

$$\mathsf{o} \in \mathcal{T}$$

$$\iota \in \mathcal{T}$$

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- The set $\mathcal{T}$ is defined inductively.
- The set $\mathcal{T}$ is "freely generated".

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- $\forall \mathsf{n} : 0 \neq \mathsf{s}(\mathsf{n})$.
- $\forall \mathsf{m}, \mathsf{n} : \mathsf{s}(\mathsf{m}) = \mathsf{s}(\mathsf{n}) \Rightarrow \mathsf{m} = \mathsf{n}$.

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Contrast $\mathbb{N}$ to $\mathbb{Z} = \{..., -1, 0, 1, ...\}$.
Note that $\mathbb{Z}$ contains 0 and is closed under successor, but is not the least such set.

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But we can and will consider it shorthand by replacing missing parenthesis, associating to the left: $(\mathsf{o}\iota\iota) = ((\mathsf{o}\iota)\iota) \neq (\mathsf{o}(\iota\iota))$.

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- $A = \{0, 1\}, B = \{0, 1, 2\}$
- $|A^B| = 2 \cdot 2 \cdot 2 = 2^3 = 8$

Ex.: Sets of Functions

Let $F = \{f : B \rightarrow A | \forall x, y \in B : x \leq y \Rightarrow f(x) \leq f(y)\} \subseteq A^B$.

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$|F| = 4$

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$$|F_C| = 3 \cdot 4 = 12$$

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D: the standard frame with $\mathsf{D}_o = \{\bot, \top\}, \mathsf{D}_i = \{1\}$

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  Since $D_\alpha \neq \emptyset \Rightarrow \exists a \in D_\alpha$, hence $K_a \in D_{\alpha\beta}$.

  (Here K_a is the constant function which always returns a. We will often use this notation for constant functions.)

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Usually we write $\mathsf{f}@\mathsf{b}$ for $@^{\alpha\beta}(\mathsf{f}, \mathsf{b})$ when $\mathsf{f} \in \mathsf{D}_{\alpha\beta} \wedge \mathsf{b} \in \mathsf{D}_\beta$

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The application operator $@$ in an applicative structure is an abstract version of function application. It is no restriction to exclusively use a binary application operator, which corresponds to unary function application, since we can define higher-arity application operators from the binary one by setting $\mathsf{f}@(\mathsf{a}^1,\ldots,\mathsf{a}^n) := (\ldots(\mathsf{f}@\mathsf{a}^1)\ldots@\mathsf{a}^n)$ ("Currying").

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$$\forall f, g \in D_{\alpha\beta} \quad (\forall b \in D_\beta : f@b = g@b) \Rightarrow f = g.$$

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$$\forall \alpha, \beta \in \mathcal{T} : \forall \mathsf{f}, \mathsf{g} \in \mathsf{D}_{\alpha\beta}(\forall \mathsf{b} \in \mathsf{D}_{\beta} : \mathsf{f}@\mathsf{b} = \mathsf{g}@\mathsf{b}) \Rightarrow \mathsf{f} = \mathsf{g}$$

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Note that the definitions of functional, full, and standard impose restrictions on the domains for function types only.

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Furthermore, an applicative structure is standard iff it is a full frame.

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$1 \in D_{oo}$ but $1 \notin D_o^{D_o} \Rightarrow D_{oo} \not\subseteq D_o^{D_o}$

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- $\kappa_\alpha : \mathsf{D}^1_\alpha \longrightarrow \mathsf{D}^2_\alpha \quad \forall \alpha \in \mathcal{T}$
- $\forall \alpha, \beta \in \mathcal{T}, \quad \forall \mathsf{f} \in \mathsf{D}^1_{\alpha\beta}, \quad \forall \mathsf{b} \in \mathsf{D}^1_\beta$:

$$\kappa(\mathsf{f})@^2\kappa(\mathsf{b}) = \kappa(\mathsf{f}@^1\mathsf{b})$$

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- j is a homomorphism from $\langle D^2, @^2\rangle$ to $\langle D^1, @^1\rangle$
- i and j are inverses (i.e $i(j(a^2)) = a^2$ and $j(i(a^1)) = a^1$).

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- $\mathsf{x} \in \mathcal{V}$ then $\mathsf{x} \in \Lambda$

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- $(f\,\overline{A}^n) \rightsquigarrow (\ldots((f\,A^1)\,A^2)\ldots A^n)$

Def.: Positions in λ-Terms

Consider the following term:

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Consider the following term:

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The position $[212]$ points to the red y in

$$((\lambda x.x)((\lambda y.\color{red}{y})(\lambda z.z)))$$

. . . Graphics on Blackboard . . .

Def.: Position (Contd.)

The expression

$$A_p$$

refers to the subterm of A at position p.

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$$T_{[212]} = y$$

Def.: Replacement at Position

Replacement of A_p in A by a term B is denoted as

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Example:

$$T[(f\,x)]_{[212]} = ((\lambda x.x)((\lambda y.(fx))(\lambda z.z)))$$

Def.: Scope of λ-Term

$(\lambda x.A)$: We say that A is in the scope of λ-binder that binds x.

Def.: Free and Bound Variables

An occurrence of a variable x in a term A is called bound if it is in the scope of a λ-binder that binds x.

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We denote the set of all free variables in a λ-term as $FV(A)$.

Syntax: Simply Typed
λ-Calculus (Contd.)

Def.: Substitution

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4. $[N_\alpha/x_\alpha](\lambda x_\alpha.A_\gamma) = (\lambda x_\alpha A_\gamma)$

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4. $[N_\alpha/x_\alpha](\lambda x_\alpha.A_\gamma) = (\lambda x_\alpha A_\gamma)$
5. $[N_\alpha/x_\alpha](\lambda y_\beta.A_\gamma) = (\lambda y_\beta.[N_\alpha/x_\alpha]A_\gamma)$ if $x_\alpha \neq y_\beta \wedge (y_\beta \notin FV(N_\alpha) \vee x_\alpha \notin FV(A_\gamma))$

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5. $[N_\alpha/x_\alpha](\lambda y_\beta.A_\gamma) = (\lambda y_\beta.[N_\alpha/x_\alpha]A_\gamma)$ if $x_\alpha \neq y_\beta \land (y_\beta \notin FV(N_\alpha) \lor x_\alpha \notin FV(A_\gamma))$
6. $[N_\alpha/x_\alpha](\lambda y_\alpha.A_\gamma) = (\lambda z_\beta.[N_\alpha/x_\alpha][z_\beta/y_\beta]A_\gamma)$ if $x_\alpha \neq y_\beta \land$ $(y_\beta \in FV(N_\alpha) \land x_\alpha \in FV(A_\gamma))$ and z is a 'fresh' variable.

Ex.: Substitution

- $[y/x](\lambda y.x)$ — the occurrence of x is free
  $\neq (\lambda y.y)$ — if we replace x with y, the variable y becomes bound.

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  $= (\lambda z.y)$ — the occurrence of y is free
- Further Examples on Blackboard
- Claim: $[N/x]A = A$ if $x \notin FV(A)$
  Proof: Induction on A

Def.: α-Conversion

$$[\lambda x.\, M] \rightarrow_{\alpha} [\lambda y.\, [y/x]M]$$

where $y \notin FV(M)$

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if A can be converted to B by renaming the bound variables. We read $A =_\alpha B$ as A is α-equal to B.

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if A can be converted to B by renaming the bound variables. We read $A =_\alpha B$ as **A is α-equal to B**.

From now on $(\lambda y.\, y) = (\lambda z.\, z)$, that is, we will say that two terms are simply equal, if they are α-equal. Two terms are equal means that two terms are α-convertable.

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A β-redex is a term $((\lambda x.\, A)B)$. The β-reduct of this redex is $[B/x]A$.

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We say $M \twoheadrightarrow_\beta N$, ie. β-reduces in several steps, if $\exists M^1, \ldots, M^n$ for $n \geq 1$ such that $M = M^1$ and $N = M^n$ and $M^i \rightarrow_\beta M^{i+1}$.

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A term is called β-normal if it contains no β-redexes.

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Thm.: Church-Rosser Property for $\twoheadrightarrow_\beta$

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\begin{tikzcd}
 & \mathsf{T}_\alpha \arrow[dl, two heads, "\beta"'] \arrow[dr, two heads, "\beta"] & \\
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If T_α β-reduces in multiple steps with one strategy to L_α and with another strategy to R_α then there exists a term B_α such that L_α and R_α β-reduce in multiple steps to B_α.

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The Church-Rosser Property for $\twoheadrightarrow_\beta$ holds for Λ and Λ^α.

Ex.: Church-Rosser Property for $\twoheadrightarrow_\beta$

$$
\begin{tikzcd}
 & ((\lambda f.\, f)\, g)((\lambda x.\, x)\, a) \arrow[dl, two heads, "\beta"'] \arrow[dr, two heads, "\beta"] & \\
g((\lambda x.\, x)\, a) \arrow[dr, two heads, "\beta"'] & & ((\lambda f.\, f) g)\, a \arrow[dl, two heads, "\beta"] \\
 & g\, a &
\end{tikzcd}
$$

Termination

Do we always get a β-normal form as we apply β-reduction?

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Typed Case: Forall A_α there exists a unique (up to α-conversion) β-normal term B such that $A \twoheadrightarrow_\beta B$

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Untyped Case: Consider the term $\omega = (\lambda x.\, xx)$

$$(\lambda x.\, xx)(\lambda x.\, xx) \rightarrow^1_\beta \omega\omega$$

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If T_α η-reduces in multiple steps with one strategy to L_α and with another strategy to R_α then there exists a term B_α such that L_α and R_α η-reduce in multiple steps to B_α.

The Church-Rosser Property for $\twoheadrightarrow_\eta$ holds for Λ and Λ^α.

Def.: $\beta\eta$-Conversion

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We say $M \twoheadrightarrow_{\beta\eta} N$, ie. η-reduces in several steps, if $\exists M^1, \ldots, M^n$ for $n \geq 1$ such that $M = M^1$ and $N = M^n$ and $M^i \longrightarrow_{\beta\eta} M^{i+1}$.

Def.: $\beta\eta$-Normal Form

A term is $\beta\eta$-normal if it contains no β-redexes and no η-redexes.

Thm.: Church-Rosser Property for $\twoheadrightarrow_{\beta\eta}$

$$\begin{tikzcd}
 & T_\alpha \arrow[dl, dashed, two heads, "\beta\eta"'] \arrow[dr, dashed, two heads, "\beta\eta"] & \\
L_\alpha \arrow[dr, dashed, two heads, "\beta\eta"'] & & R_\alpha \arrow[dl, dashed, two heads, "\beta\eta"] \\
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The Church-Rosser Property for $\twoheadrightarrow_{\beta\eta}$ holds for Λ and Λ^α.

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In Λ^{α} (simply typed λ-calculus) the relations $\twoheadrightarrow_{\beta}$ and $\twoheadrightarrow_{\beta\eta}$ have the strong Church Rosser property:

Thm.: Strong Church-Rosser Property

In Λ^α (simply typed λ-calculus) the relations $\twoheadrightarrow_\beta$ and $\twoheadrightarrow_{\beta\eta}$ have the strong Church Rosser property: for very term A_τ there exists a unique (up to α-renaming) β-normal resp. $\beta\eta$-normal term B_τ such that $A_\tau \twoheadrightarrow_\beta B_\tau$ resp. $A_\tau \twoheadrightarrow_{\beta\eta} B_\tau$.

Def.: Long $\beta\eta$-Normal Form

Let $\mathsf{n} \geq 0$, $\alpha^1, \ldots, \alpha^\mathsf{n} \in \mathcal{T}$, and $\beta \in \{\mathsf{o}, \iota\}$. A term A of type $(\beta, \alpha^\mathsf{n}, \ldots, \alpha^1)$ is in long $\beta\eta$-normal form if it is of form

$$\lambda \mathsf{x}^1_{\alpha^1} \ldots \mathsf{x}^\mathsf{n}_{\alpha^\mathsf{n}}.(\mathsf{h}_{\beta\gamma^\mathsf{m}\ldots\gamma^1} \mathsf{A}^1_{\gamma^1} \ldots \mathsf{A}^\mathsf{m}_{\gamma^\mathsf{m}})$$

for a variable or constant $\mathsf{h}_{\beta\gamma^\mathsf{m}\ldots\gamma^1}$, $\mathsf{m} \geq 0$ and long $\beta\eta$-normal forms $\mathsf{A}^1_{\gamma^1}, \ldots, \mathsf{A}^\mathsf{m}_{\gamma^\mathsf{m}}$.

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for a variable or constant $\mathsf{h}_{\beta\gamma^\mathsf{m}\ldots\gamma^1}$, $\mathsf{m} \geq 0$ and long $\beta\eta$-normal forms $\mathsf{A}^1_{\gamma^1}, \ldots, \mathsf{A}^\mathsf{m}_{\gamma^\mathsf{m}}$. Note that this is an inductive definition; the base case is when $\mathsf{m} = 0$. Note that if $\lambda\overline{\mathsf{x}^\mathsf{n}}.(\mathsf{h}\overline{\mathsf{A}^\mathsf{m}})$ is in long $\beta\eta$-normal form then $(\mathsf{h}\overline{\mathsf{A}^\mathsf{m}})$ is of base type.

Ex.: Long $\beta\eta$-Normal Form

Consider the $\beta\eta$-normal term $\mathsf{f}_{\iota(\iota\iota)}$.

$$\mathsf{f}_{\iota(\iota\iota)}$$

$$\uparrow^{\eta}$$

$$\lambda \mathsf{w}_{\iota\iota}.\,(\mathsf{f}_{\iota(\iota\iota)}\mathsf{w}_{\iota\iota})$$

$$\uparrow^{\eta}$$

$$\lambda \mathsf{w}_{\iota\iota}.\,(\mathsf{f}(\lambda \mathsf{x}_{\iota}.\,\mathsf{w}_{\iota\iota}\mathsf{x}))$$

Thm.: Long $\beta\eta$-Normal Form

For every term A there is unique long $\beta\eta$-normal form B such that $A =^{\beta\eta} B$.

Rem.: $\beta\eta$-Head Normal Form

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Instead of terms in long $\beta\eta$-normal form we often use in practice terms in $\beta\eta$-head normal form. Definition is similar to long $\beta\eta$-normal, but we do not require the embedded terms $A^i_{\gamma^i}$ to be in normal form.

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- $A{\downarrow_\beta}$ is the β-normal form of A.

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Semantics: Σ-Evaluations

Ex.: An Interesting Applicative Structure

$\mathsf{D}_\alpha := \{\mathsf{A}_\alpha \in \Lambda_\alpha | \mathsf{A} \text{ is closed}\}$.

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- $\mathsf{D} = (\mathsf{D}_\alpha)_{\alpha \in \mathcal{T}}$ is not a frame!
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- $D = (D_\alpha)_{\alpha \in \mathcal{T}}$ is not a frame!
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- If Λ_α is non-empty for all $\alpha \in \mathcal{T}$, then $< D, @ >$ is an applicative structure.

Ex.: Interpretation of Terms

Syntax   Semantics $< \mathrm{D}, @ >$

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$(\lambda x_\iota . x) a_\iota$	$(\lambda x_\iota . x) @ a_\iota$	

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$a_\iota \in C$	a	$\in D_\iota$
$(\lambda x_\iota . x) a_\iota$	$(\lambda x_\iota . x) @ a_\iota$	$\in D_\iota$

Remark: The variable y_ι is a non-closed well-formed formula of type ι. We need an assignment $\varphi_\alpha : V_\alpha \longrightarrow D_\alpha$ to give it a meaning.

Ex.: Interesting Applicative Structures

- Let $\mathsf{D}_\alpha\downarrow_\beta := \{\mathsf{A}_\alpha \in \Lambda_\alpha | \, \mathsf{A} \text{ is closed and } \mathsf{A} \text{ is in } \beta\text{-normal form}\}$

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- Let $@^\beta_{\gamma\delta} : \mathsf{D}_{\gamma\delta} \times \mathsf{D}_\delta \longrightarrow \mathsf{D}_\gamma$ be defined by

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for all $\mathsf{F}_{\gamma\delta} \in \mathsf{D}_{\gamma\delta}$ and $\mathsf{G}_\delta \in \mathsf{D}_\delta$.

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- $@^\beta = (@^\beta_{\gamma\delta})_{\gamma\delta \in \mathcal{T}}$

Claim: If $\mathcal{C}_\iota \neq \emptyset$ and $\mathcal{C}_o \neq \emptyset$ (i.e., at least one constant for each base type is given), then $(\mathsf{D}, @^\beta)$ is an applicative structure.

Ex.: Interesting Applicative Structures

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- Is $\mathsf{F}_{\gamma\delta} @^{\beta}_{\gamma\delta} \mathsf{G}_\delta \in \mathsf{D}_\gamma \downarrow_\beta$?

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Proof:

- Is $D_\alpha \downarrow_\beta$ nonempty for all $\alpha \in \mathcal{T}$?
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- Is $F_{\gamma\delta} @^{\beta}_{\gamma\delta} G_\delta \in D_\gamma \downarrow_\beta$?
- Let's check: $F_{\gamma\delta} @^{\beta}_{\gamma\delta} G_\delta = (F\,G) \downarrow_\beta \in D_\gamma \downarrow_\beta$

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Claim: If $\mathcal{C}_\iota \neq \emptyset$ and $\mathcal{C}_o \neq \emptyset$ (i.e., at least one constant for each base type is given), then $(\mathsf{D}, @^{\beta\eta})$ is an applicative structure.

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- . . . analogous . . .

Def.: Variable Assignment

Let $\mathcal{A} := (\mathcal{D}, @)$ be an applicative structure.

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$$(\varphi, [\mathsf{a}/\mathsf{X}])(\mathsf{X}) = \mathsf{a}$$

and

$$(\varphi, [\mathsf{a}/\mathsf{X}])(\mathsf{Y}) = \varphi(\mathsf{Y})$$

for variables Y other than X.

Some Assumptions

From now on, we assume the signature $\Sigma_\alpha = (\mathcal{V}, \mathcal{C})$ to be infinite for each type α.

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From now on, we assume the signature $\Sigma_\alpha = (\mathcal{V}, \mathcal{C})$ to be infinite for each type α. Furthermore, we assume there is a particular cardinal $\aleph_s$ such that Σ_α has cardinality $\aleph_s$ for every type α. Since $\mathcal{V}$ is countable, this implies $\mathit{wff}_\alpha(\Sigma) := \Lambda^\alpha$ and $\mathit{cwff}_\alpha(\Sigma) := \{\mathsf{A} \in \Lambda^\alpha | \mathsf{A}\ \text{closed}\}$ have cardinality $\aleph_s$ for each type α.

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Σ-Evaluations

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In such models, a function is not uniquely determined by its behavior on all possible arguments.

Such models can be constructed, for example, by labeling for functions (e.g., a green and a red version of a function f) in order to differentiate between them, even though they are functionally equivalent.

Σ-Evaluations

Let $\mathcal{E}\colon \mathcal{F}_{\mathcal{T}}(\mathcal{V};\mathcal{D}) \longrightarrow \mathcal{F}_{\mathcal{T}}(\textit{wff}(\Sigma),\mathcal{D})$ be a total function, where $\mathcal{F}_{\mathcal{T}}(\mathcal{V};\mathcal{D})$ is the set of variable assignments and $\mathcal{F}_{\mathcal{T}}(\textit{wff}(\Sigma),\mathcal{D})$ is the set of typed functions mapping terms into objects in $\mathcal{D}$.

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What properties shall $\mathcal{E}$ fulfill?

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2. $\mathcal{E}_\varphi(\mathbf{FA}) = \mathcal{E}_\varphi(\mathbf{F})@\mathcal{E}_\varphi(\mathbf{A})$ for any $\mathbf{F} \in \mathit{wff}_{\alpha\to\beta}(\Sigma)$ and $\mathbf{A} \in \mathit{wff}_\alpha(\Sigma)$ and types α and β.

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We call $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$ a Σ-evaluation if $(\mathcal{D}, @)$ is an applicative structure and $\mathcal{E}$ is an evaluation function for $(\mathcal{D}, @)$. We call $\mathcal{E}_\varphi(\mathbf{A}_\alpha) \in \mathcal{D}_\alpha$ the denotation of $\mathbf{A}_\alpha$ in $\mathcal{J}$ for φ.

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Remark: since $\mathcal{E}$ is a function, the denotation in $\mathcal{J}$ is unique. However, for a given applicative structure $\mathcal{A}$, there may be many possible evaluation functions.

If $\mathbf{A}$ is a closed formula, then $\mathcal{E}_\varphi(\mathbf{A})$ is independent of φ, since $\mathbf{Free}(\mathbf{A}) = \emptyset$. In these cases we sometimes drop the reference to φ from $\mathcal{E}_\varphi(\mathbf{A})$ and simply write $\mathcal{E}(\mathbf{A})$.

Def.: Functional/Full/Standard Σ-Eval.

We call a Σ-evaluation $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$ functional [full, standard] if the applicative structure $(\mathcal{D}, @)$ is functional [full, standard].

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We say $\mathcal{J}$ is a Σ-evaluation over a frame if $(\mathcal{D}, @)$ is a frame.

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Σ-evaluations generalize Σ-evaluations over frames, which are the basis for Henkin models, to the non-functional case.

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The existence of an evaluation function that meets the conditions as presented seems to be the weakest situation where one would like to speak of a model.

We cannot in general assume the evaluation function is uniquely determined by its values on constants as this requires functionality. Example: two evaluation functions $\mathcal{E}$ and $\mathcal{E}'$ on the same applicative structure may agree on all constants, but give a different value to the term $(\lambda X_\iota.X)$.

Lemma: Σ-Evaluations respect β-Equality

Let $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$ be a Σ-evaluation and $\mathbf{A} =_\beta \mathbf{B}$. For all assignments φ into $(\mathcal{D}, @)$, we have

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$$\mathcal{E}_{\varphi,[\mathcal{E}_\varphi(\mathbf{B})/\mathsf{X}]}(\mathbf{A}) = \mathcal{E}_\varphi([\mathbf{B}/\mathsf{X}]\mathbf{A})$$

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- η-functionality simply means the evaluation respects η-conversion.
- ξ-functionality means we have functionality (only) with respect to λ-abstractions.

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Once More: Cantor’s Theorem

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Once More: Cantor's Theorem

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which is shorthand for:

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Note: for this term to be in the set $cwff_\alpha(\Sigma)$, the constants $\neg_{oo}$, $\Sigma^{o\iota\iota}_{o(o(o\iota\iota))}$, $\Pi^{o\iota}_{o(o(o\iota))}$, Σ^{ι} and $=^{o\iota}$ have to be in the set $\mathcal{C}$.

Once More: Cantor's Theorem

Proof:

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g is surjective,

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Let $\mathsf{f} = \{\mathsf{x} \mid \mathsf{x} \notin \mathsf{gx}\}$ that is $\mathsf{f} = (\lambda \mathsf{x}_\iota . \neg \mathsf{gxx})$.
g is surjective, hence

$$(\exists \mathsf{y}_\iota : \mathsf{gy} = [\lambda \mathsf{x} . \neg \mathsf{gxx}])$$

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Contradiction!

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Contradiction!

Note that the proof uses $\neg$.

Semantics: Σ-Models

Def.: Properties of Logical Constants

Let $(\mathrm{D}, @)$ be an applicative structure and let $\mathtt{v} : \mathcal{D}_{\mathsf{o}} \longrightarrow \{\mathtt{T}, \mathtt{F}\}$ be a function (for given $\mathtt{T} \neq \mathtt{F}$).

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c		

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c	β	

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c	β	$\mathfrak{L}_c(()a)$ holds when

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c	β	$\mathfrak{L}_c(()a)$ holds when
$\top$		

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$\top$	o	

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$\bot$	o	$v(a) = \mathtt{F}$
$\neg$		

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c	β	$\mathfrak{L}_c(()a)$ holds when
$\top$	o	$v(a) = \mathtt{T}$
$\perp$	o	$v(a) = \mathtt{F}$
$\neg$	oo	

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$\bot$	o	$v(a) = F$	
$\neg$	oo	$v(a@b) = T$	iff $v(b) = F$ $\forall b \in \mathcal{D}_o$
$\vee$	ooo	$v(a@b@c) = T$	iff $v(b) = T$ or $v(c) = T$ $\forall b, c \in \mathcal{D}_o$
$\wedge$	ooo	$v(a@b@c) = T$	iff $v(b) = T$ and $v(c) = T$ $\forall b, c \in \mathcal{D}_o$
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$=^\alpha$			

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$\vee$	ooo	$v(a@b@c) = \mathtt{T}$	iff $v(b) = \mathtt{T}$ or $v(c) = \mathtt{T}$ $\forall b, c \in \mathcal{D}_o$
$\wedge$	ooo	$v(a@b@c) = \mathtt{T}$	iff $v(b) = \mathtt{T}$ and $v(c) = \mathtt{T}$ $\forall b, c \in \mathcal{D}_o$
$\supset$	ooo	$v(a@b@c) = \mathtt{T}$	iff $v(b) = \mathtt{F}$ or $v(c) = \mathtt{T}$ $\forall b, c \in \mathcal{D}_o$
$\Leftrightarrow$	ooo	$v(a@b@c) = \mathtt{T}$	iff $v(b) = v(c)$ $\forall b, c \in \mathcal{D}_o$
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$\neg$	oo	$\mathsf{v}(\mathsf{a}@\mathsf{b}) = \mathtt{T}$	iff $\mathsf{v}(\mathsf{b}) = \mathtt{F}$ $\forall \mathsf{b} \in \mathcal{D}_o$
$\vee$	ooo	$\mathsf{v}(\mathsf{a}@\mathsf{b}@\mathsf{c}) = \mathtt{T}$	iff $\mathsf{v}(\mathsf{b}) = \mathtt{T}$ or $\mathsf{v}(\mathsf{c}) = \mathtt{T}$ $\forall \mathsf{b}, \mathsf{c} \in \mathcal{D}_o$
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We use $\stackrel{*}{=}$ in the following to refer to any of the above

Def.: Properties $\mathfrak{f}, \mathfrak{b}, \eta, \xi$

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  Note: In the [JSC04]-paper, b is defined as $\mathsf{D}_o = \{\mathtt{T}, \mathtt{F}\}$, but here we are using the injectivity criterion, because we are varying the signature. If the signature is too sparse, we could have a D_o with two elements which both valuate via v to $\mathtt{T}$. Another ill case would be D_o with just one element.

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  Note: This basically says that for each type α the identity relation over α is already present in the model. If we require $=_{o\alpha\alpha} \in \mathcal{C}$ with $\mathfrak{L}_{=^\alpha}(\mathcal{E}_\varphi(=_{o\alpha\alpha}))$, then this property is automatically ensured, but not for weaker signatures. See [Andrew71] for a detailed discussion of property q. Andrews constructs a Henkin model where Leibniz equality $\doteq$ does not evaluate to the intended identity relation. This is resolved by property q.

Lemma: Surjective v

Let $\mathcal{C}$ be a signature and $\mathcal{M} = (\mathsf{D}, @, \mathcal{E}, \mathsf{v})$ be a $\mathcal{C}$-model.

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Proof: Exercise.

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Thm.: Property $\mathfrak{b}$

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Then $\mathcal{M}$ satisfies property $\mathfrak{b}$ iff $|\mathsf{D}_{\mathsf{o}}| = 2$.

Thm.: Property $\mathfrak{b}$

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Proof: Exercise.

Semantics: HOL-CUBE

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Let $\mathcal{J} := (\mathrm{D}, @, \mathcal{E})$ be a Σ-evaluation and let $\mathsf{v} : \mathcal{D}_o \to \{\mathtt{T}, \mathtt{F}\}$ be a Σ-valuation w.r.t $\mathcal{J}$

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If $(\mathsf{D}, @, \mathcal{E})$ is functional (full, standard), we say $\mathcal{M}$ is functional (full, standard).

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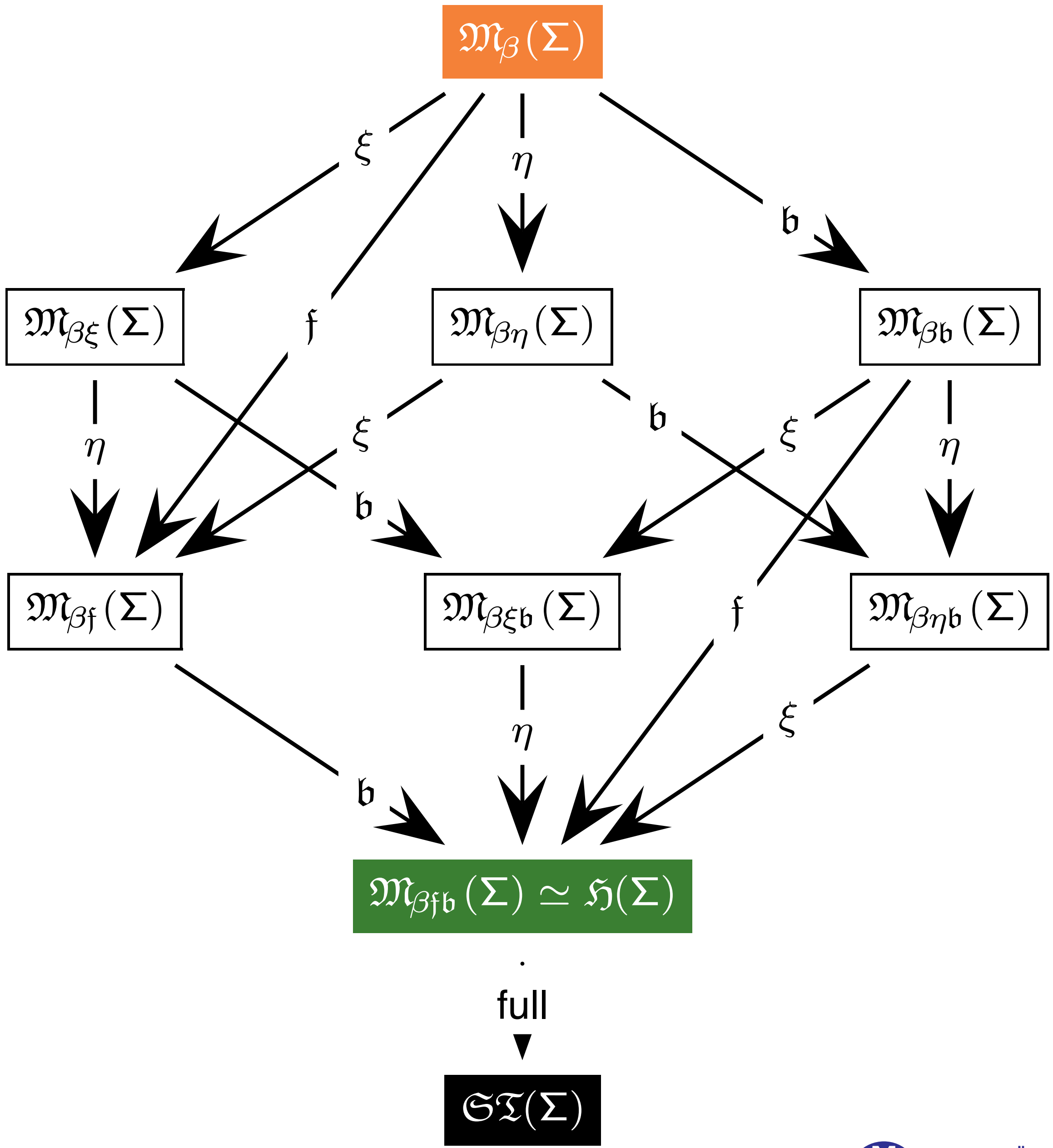
Finally, we say that $\mathcal{M}$ is a Σ-model for a set $\Phi \subseteq \mathit{cwff}_o(\Sigma)$ (we write $\mathcal{M} \models \Phi$) if $\mathcal{M} \models \mathbf{A}$ for all $\mathbf{A} \in \Phi$.

Semantics: HOL-CUBE



Landscape of HOL model classes

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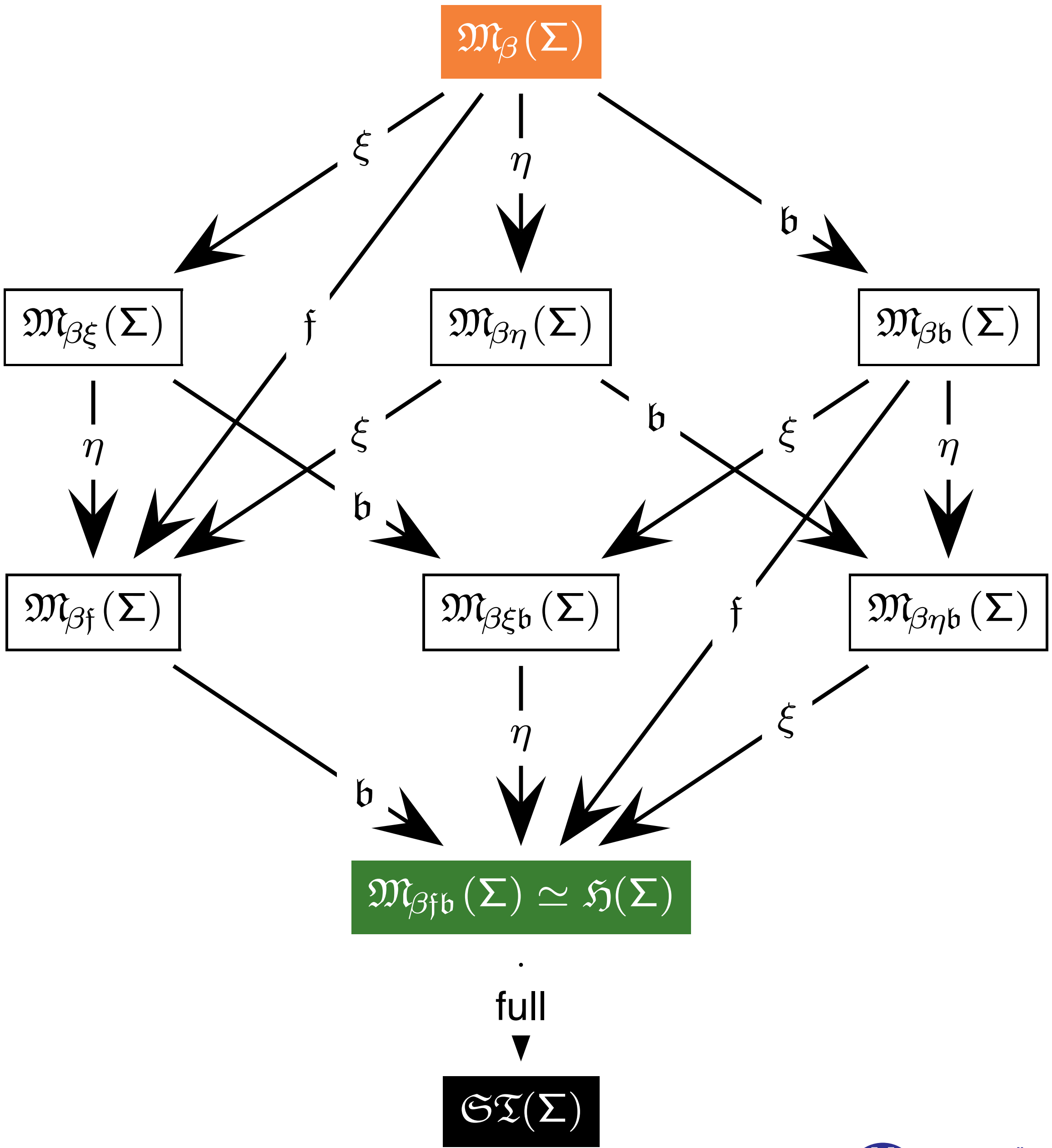

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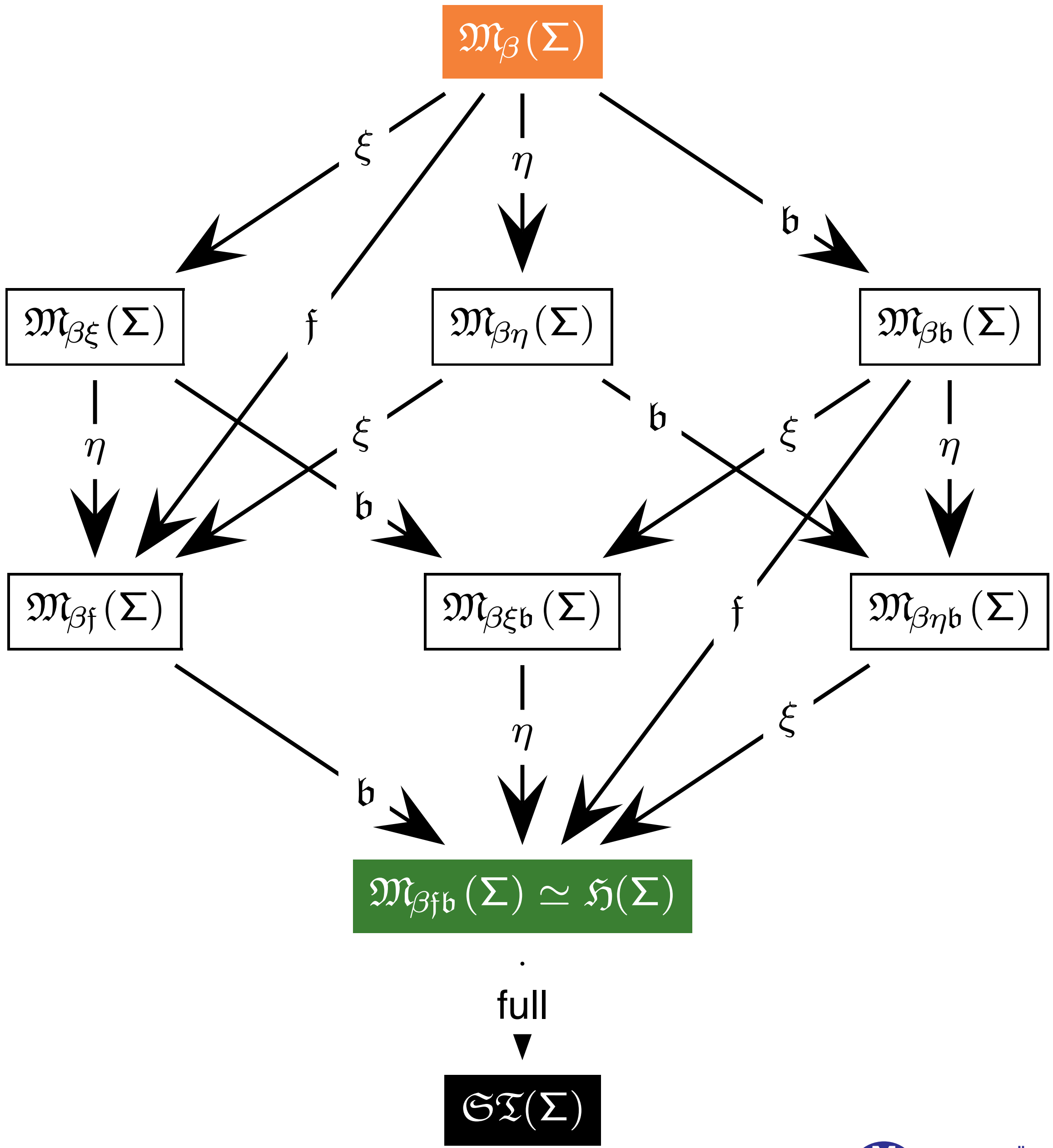
$\mathfrak{M}_{\beta}(\Sigma)$ model class for Σ-fragment of elementary type theory



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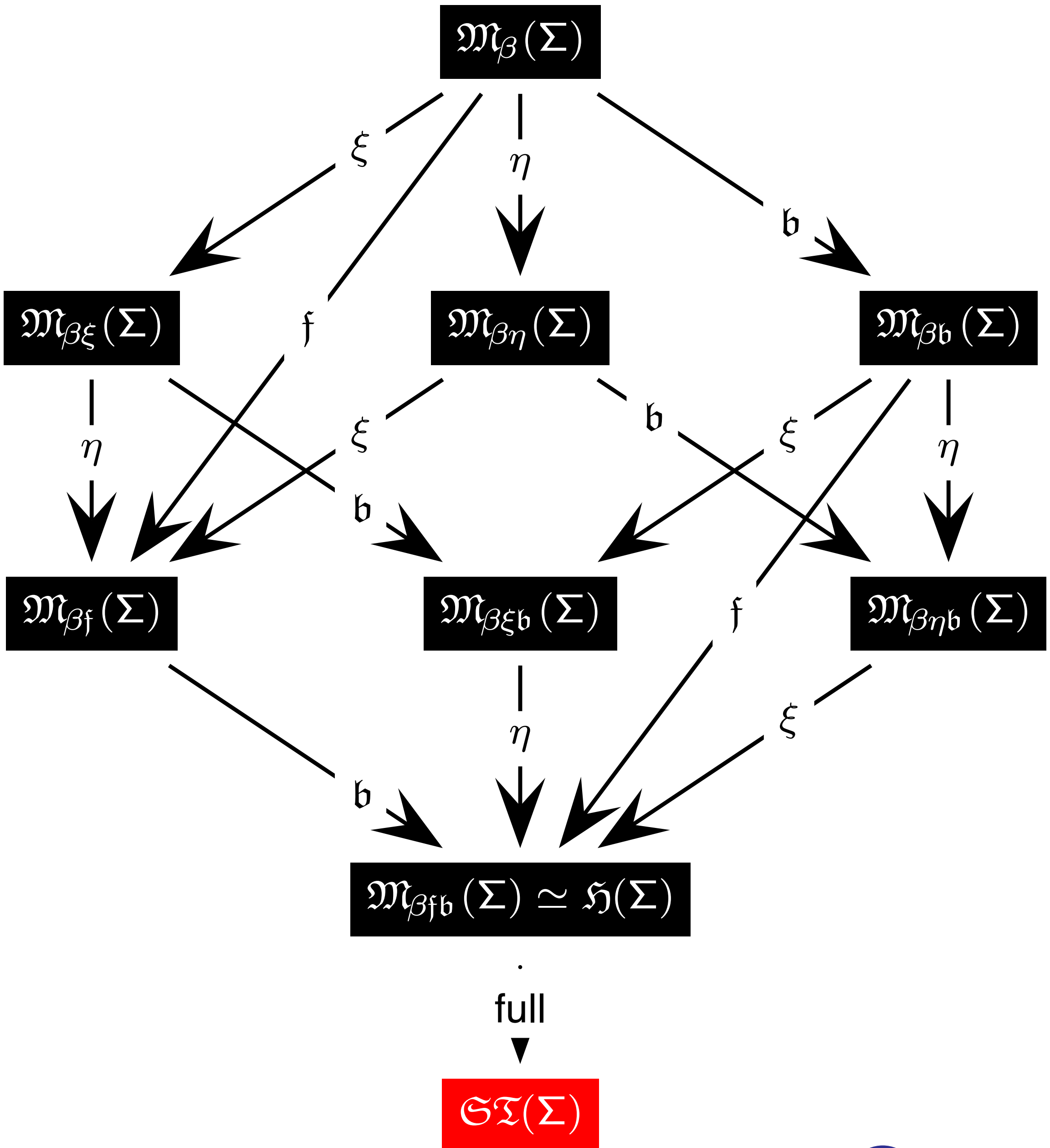
$\mathfrak{M}_{\beta}(\Sigma)$ model class for Σ-fragment of elementary type theory

$\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma)$ model class for Σ-fragment of extensional type theory (Henkin models)

Semantics: HOL-CUBE



β: models support β-equality

$\mathfrak{q}$: models provide identity relations

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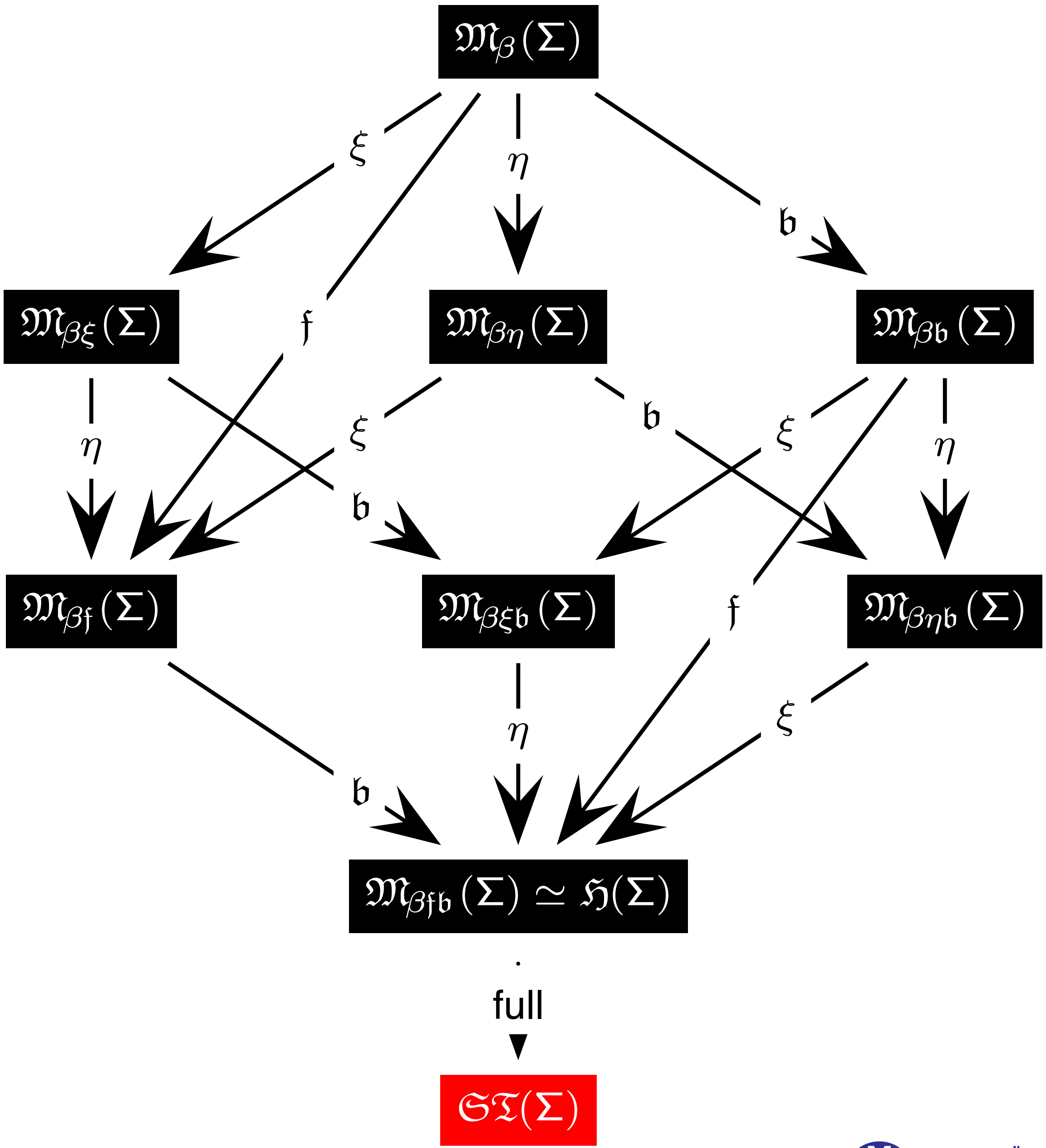

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- [Andrews72]: without property $\mathfrak{q}$ Leibniz equality $\doteq$ not necessarily evaluates to identity relation even in Henkin semantics ($\mathfrak{H}(\Sigma)$)

Standard Models and Henkin Models

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Instead of requiring $\mathcal{D}_{\alpha\beta}$ (and thus in particular, $\mathcal{D}_{o\iota}$) to be the full set of functions (predicates), it is sufficient to require that $\mathcal{D}_{\alpha\beta}$ has enough members that any well-formed formula can be evaluated (in other words, the domains of function types are rich enough to satisfy comprehension).

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Note that with this generalized notion of a model, there are fewer formulae that are valid in all models (intuitively, for any given formula there are more possibilities for counter-models).

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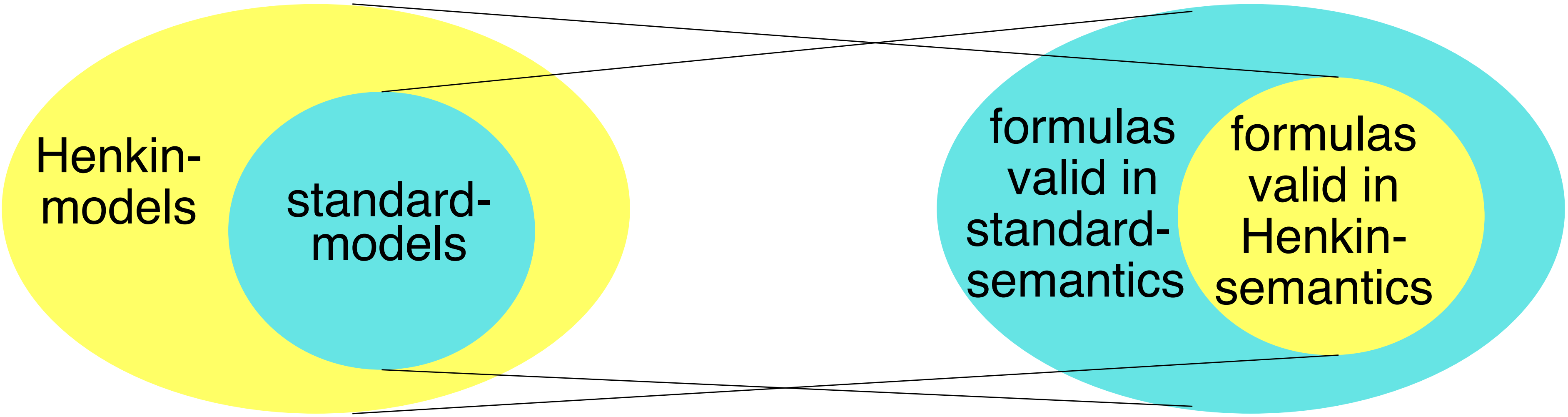

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Note that even though we can consider model classes with richer and richer function spaces, we can never reach standard models where function spaces are full while maintaining complete (recursively axiomatizable) calculi.

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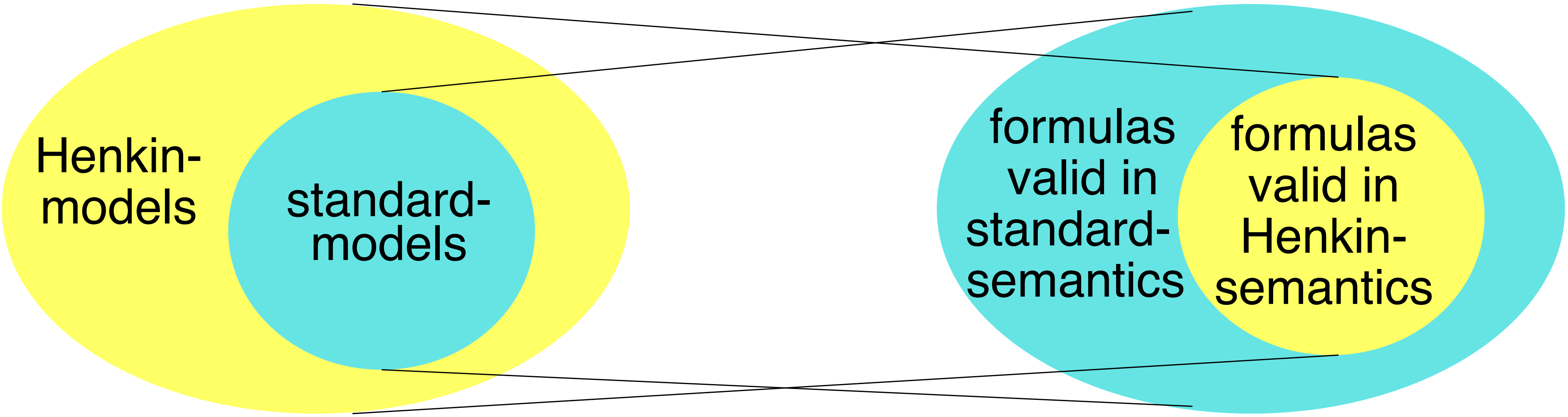

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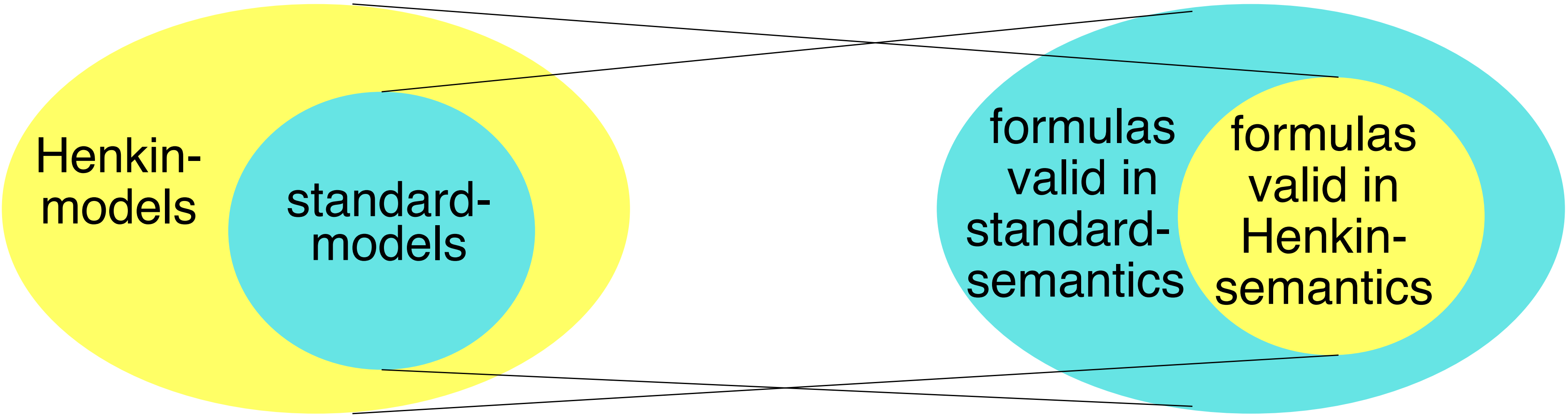

What has been our motivation for further generalization of Henkin semantics with respect to Boolean and functional extensionality?

Models without Functional Extensionality

Motivation: modeling programs as (higher-order) functions

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- $\mathbf{I} := \lambda X.X$ and $\mathbf{L} := \lambda X.\mathsf{rev}(\mathsf{rev}(X))$, where rev is the self-inverse function.
- The identity function has constant complexity, the function rev is linear in the length of its argument.

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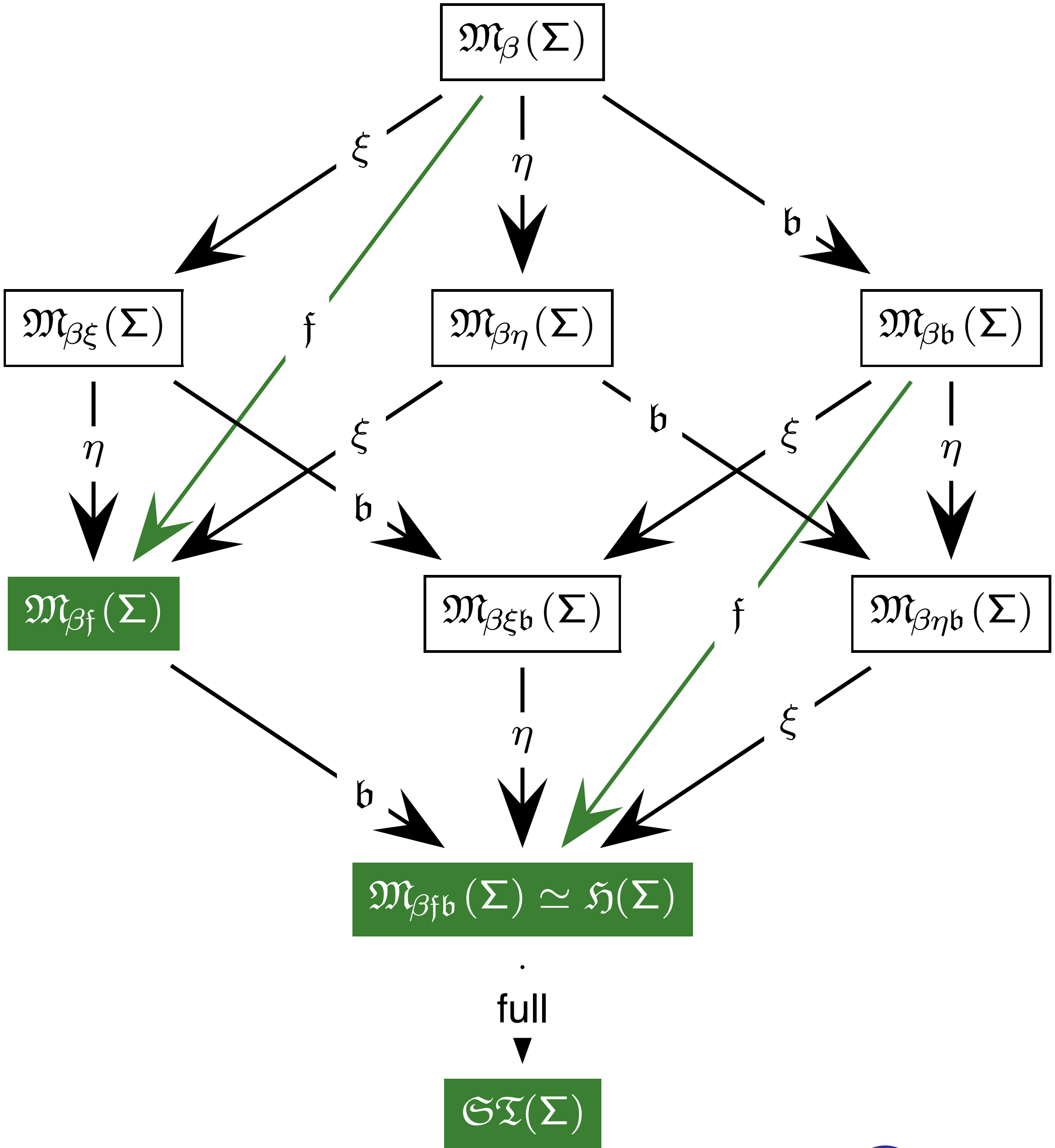
- We build on the notion of applicative structures to define Σ-evaluations, where the evaluation function is assumed to respect application and β-conversion.
- In such models, a function is not uniquely determined by its behavior on all possible arguments.

Semantics: HOL-CUBE



$\mathfrak{f}$: models are functional

$$\forall f, g \in \mathcal{D}_{\beta\alpha} :$$
$$f = g \text{ iff } f@a = g@a \; (\forall a \in \mathcal{D}_{\alpha})$$

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- Therefore, we integrated these two cases in our landscape.

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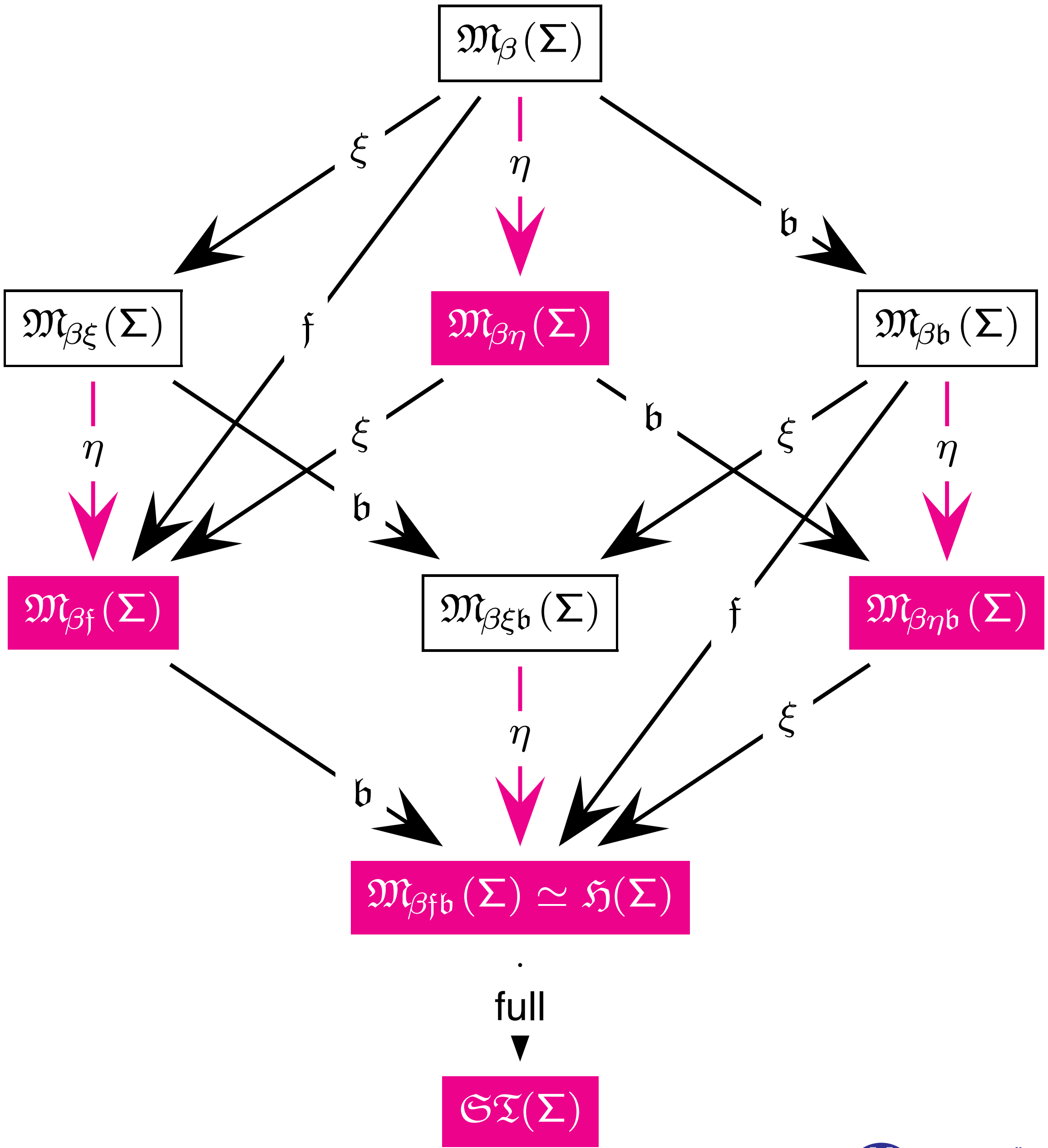

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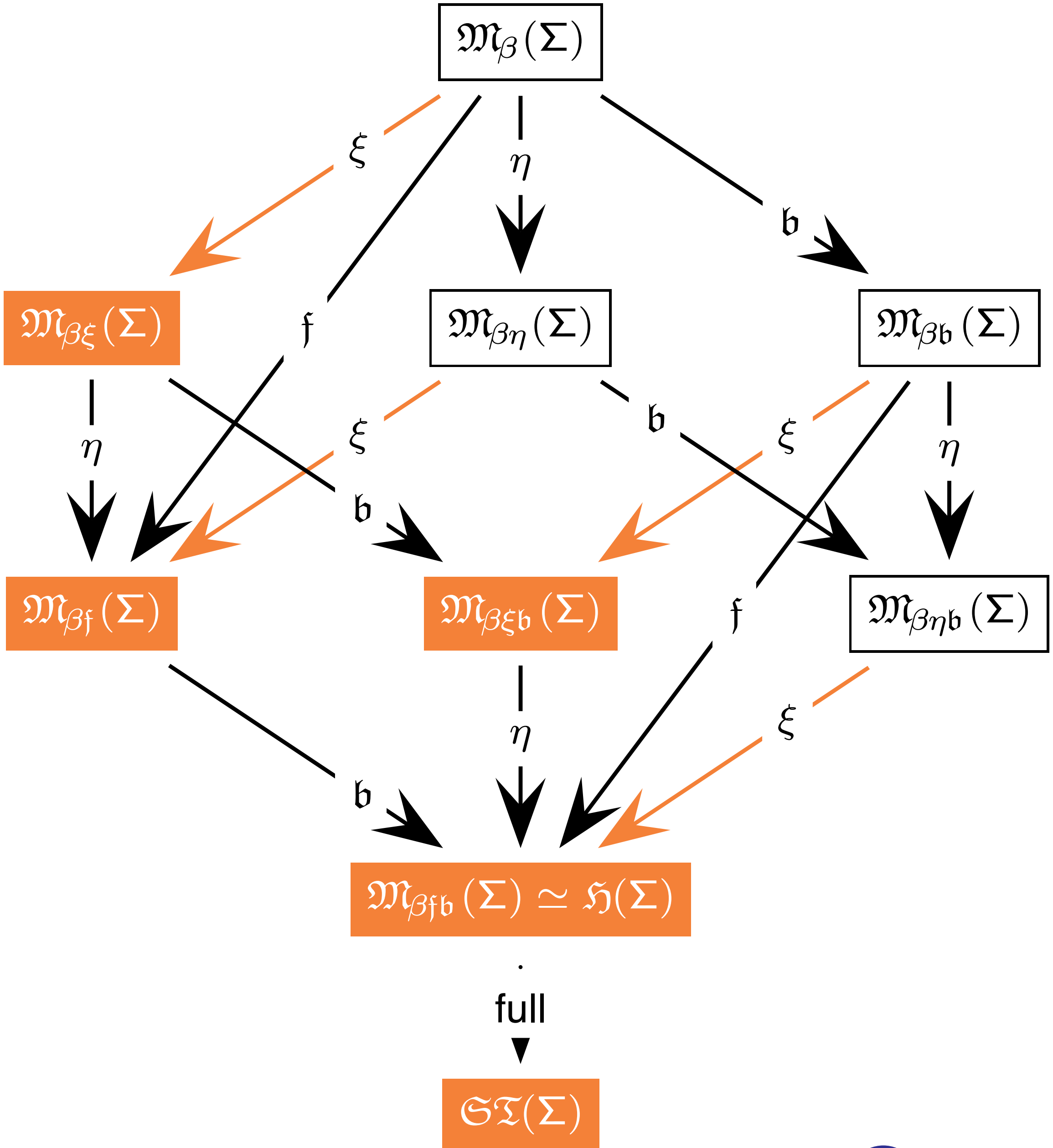

ξ: models are ξ-functional

$\mathcal{E}_\varphi(\lambda X_\alpha \mathbf{.} M_\beta) = \mathcal{E}_\varphi(\lambda X_\alpha \mathbf{.} N_\beta)$ iff
$\mathcal{E}_{\varphi,[a/X]}(M) = \mathcal{E}_{\varphi,[a/X]}(N)$ $(\forall a \in \mathcal{D}_\alpha)$

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- Another example: $\mathsf{obvious}(\mathbf{O})$ and $\mathsf{obvious}(\mathbf{F})$ where $\mathbf{O} := 2 + 2 = 4$ and $\mathbf{F} := \forall n > 2.x^n + y^n = z^n \Rightarrow x = y = z = 0$ should not be equivalent, even if their arguments are.

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- Such phenomena have been studied under the heading of "hyper-intensional semantics" in theoretical semantics.

Models without Boolean Extensionality

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- In our Σ-models without property $\mathfrak{b}$ we only insist that there is a division of the truth values into “good” and “bad” ones, which we express by insisting on the existence of a valuation υ of $\mathcal{D}_o$, i.e., a function $\upsilon\colon \mathcal{D}_o \longrightarrow \{\mathtt{T}, \mathtt{F}\}$ that is coordinated with the interpretations of the logical constants $\neg$, $\vee$, and Π^α (for each type α).

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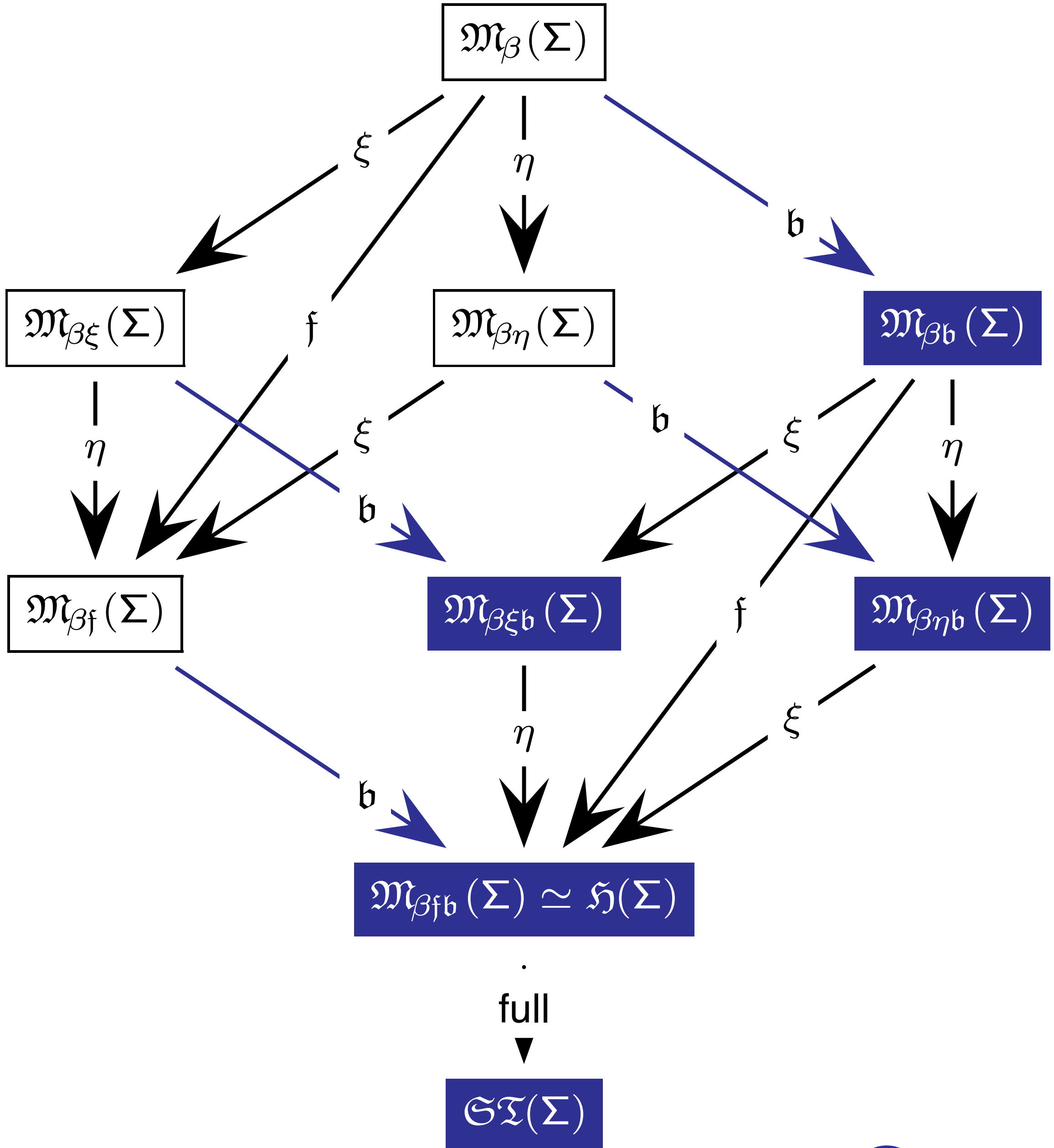
- Notion of validity: we call a sentence $\mathbf{A}$ valid in such a model if $\upsilon(\mathsf{a}) = \mathtt{T}$, where $\mathsf{a} \in \mathcal{D}_o$ is the denotation of the sentence $\mathbf{A}$.

Semantics: HOL-CUBE



$\mathfrak{b}$: models are Boolean extensional

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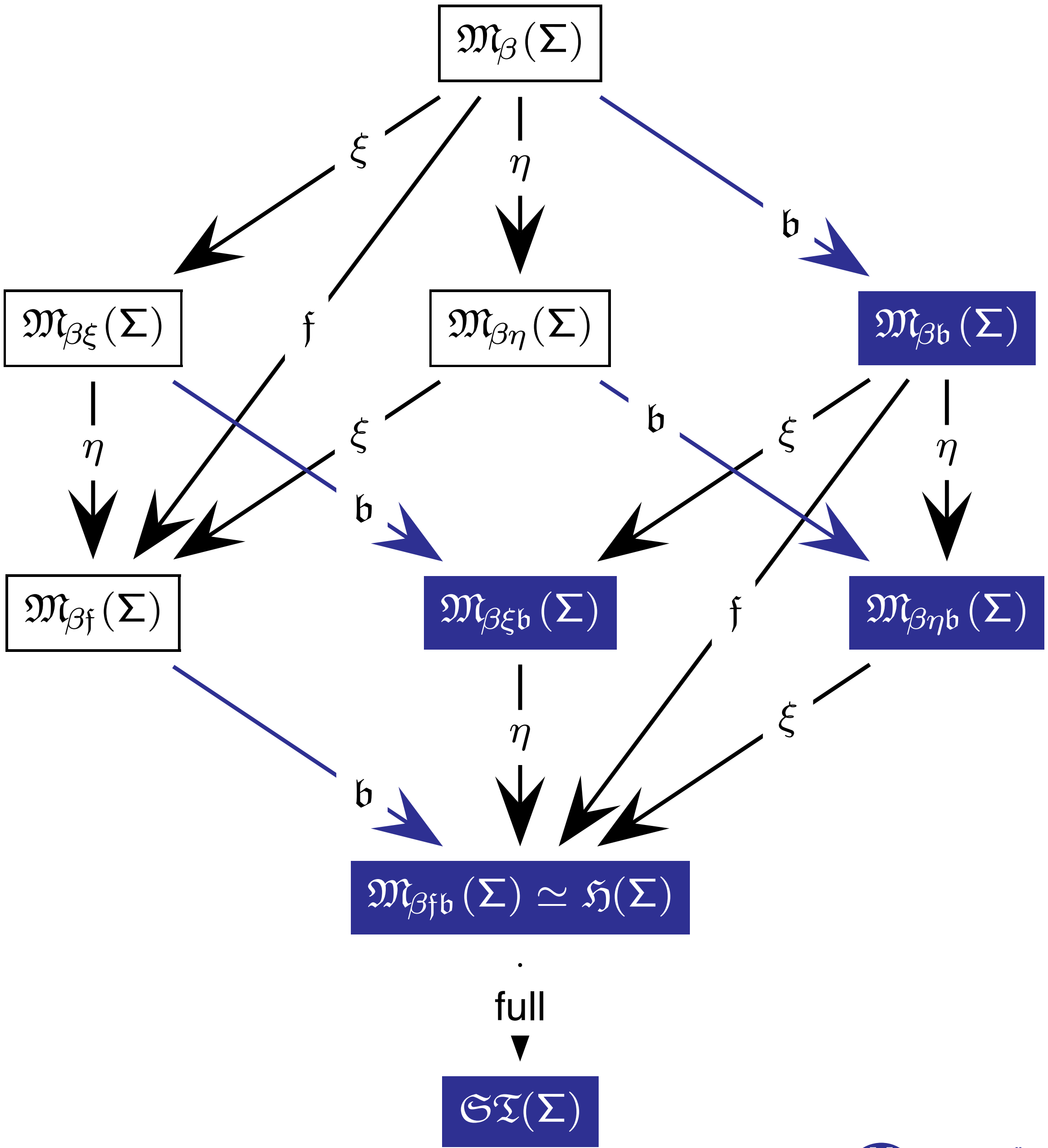

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If Σ contains sufficiently many logical constants:

$\mathcal{D}_o = \{\bot, \top\}$

Semantics and Theorem Proving: Test Problems for Theorem Provers

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  - many are collected from experience with LEO and TPS
- (Some more challenging examples are also added in [TPHOLS-05])

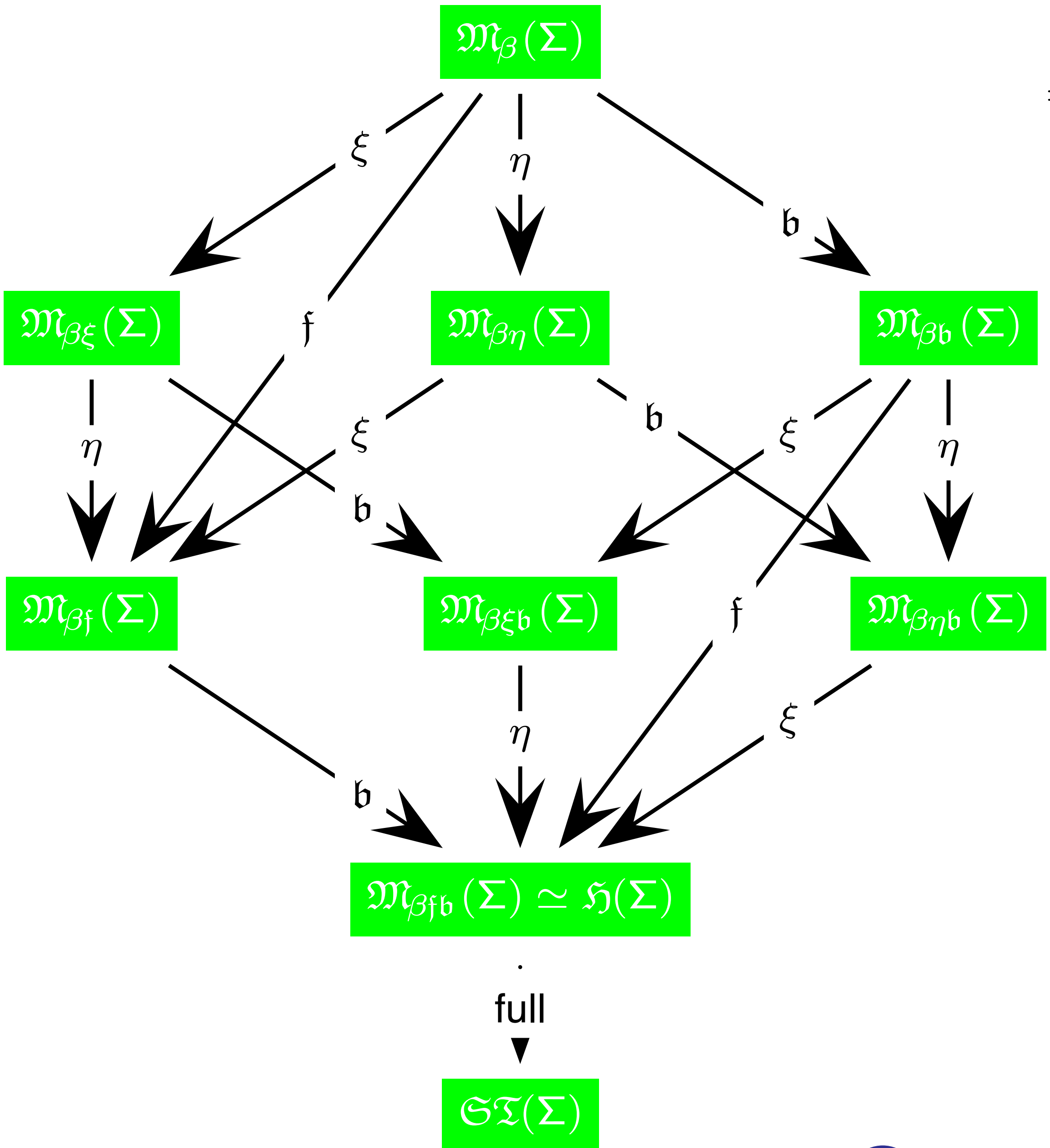
Remark: Signature

Unless stated otherwise we assume on the following slides that our signature Σ contains the following logical connectives:

$$\{\top, \bot, \neg, \wedge, \vee, \supset, \Leftrightarrow\} \cup \{\Pi^\alpha, \Sigma^\alpha, =^\alpha\}$$

(less logical connectives are possible)

HOL-Problems: β



$\stackrel{*}{\doteq}$ is equivalence relation

- $\forall X_\alpha. X \stackrel{*}{\doteq} X$
- $\forall X_\alpha, Y_\alpha. X \stackrel{*}{\doteq} Y \supset Y \stackrel{*}{\doteq} X$
- $\forall X_\alpha, Y_\alpha, Z_\alpha. (X \stackrel{*}{\doteq} Y \wedge Y \stackrel{*}{\doteq} Z) \supset X \stackrel{*}{\doteq} Z$

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$$
\begin{tikzcd}
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 & & \mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma) \arrow[d, dotted, "\text{full}"] & & \\
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$\stackrel{*}{\doteq}$ is congruence relation

- $\forall X_\alpha, Y_\alpha, F_{\alpha\alpha} . X \stackrel{*}{\doteq} Y \supset (FX) \stackrel{*}{\doteq} (FY)$
- $\forall X_\alpha, Y_\alpha, P_{o\alpha} . X \stackrel{*}{\doteq} Y \wedge (PX) \supset (PY)$

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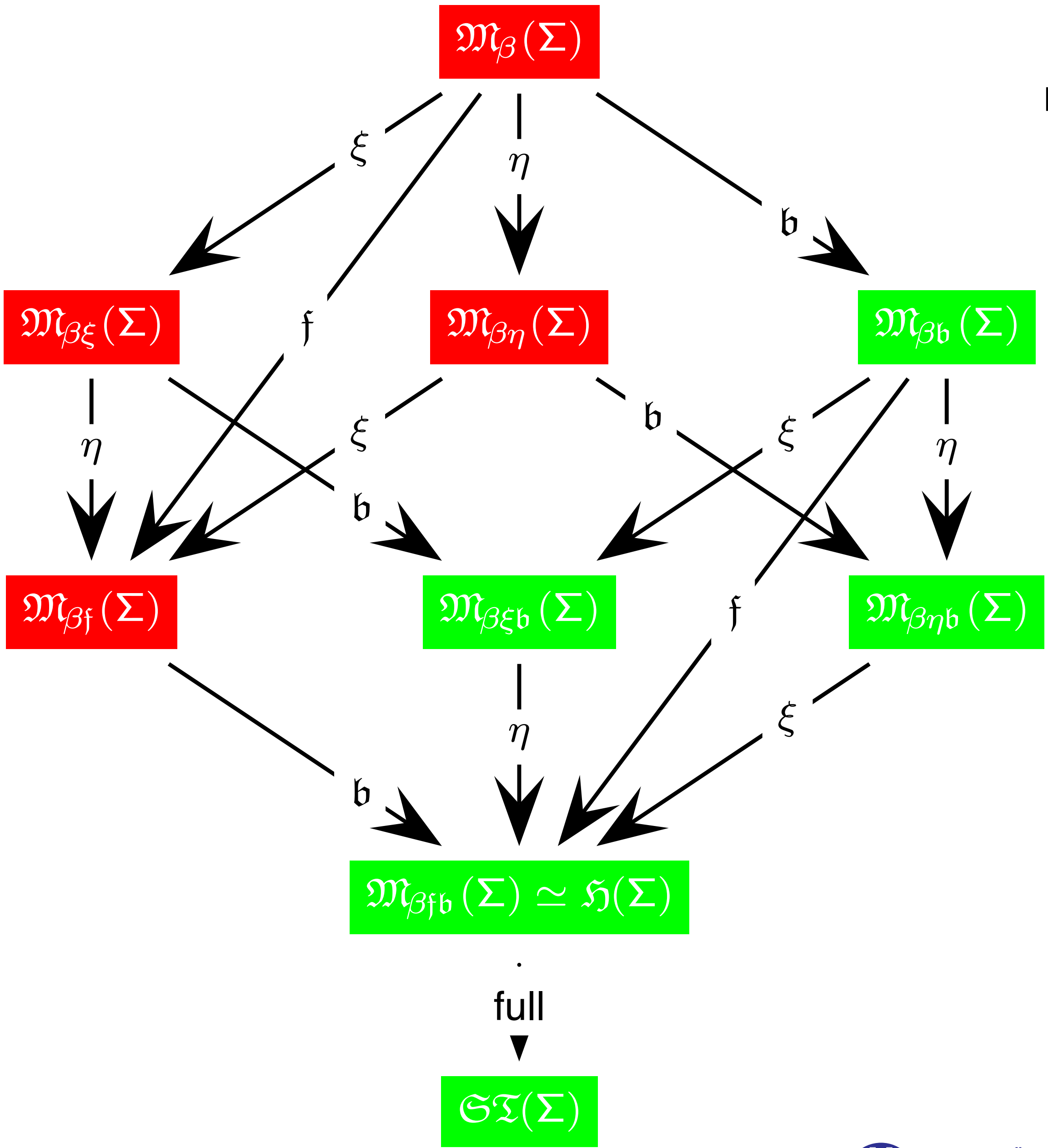
Trivial directions of Boolean and functional extensionality

- $\forall A_o, B_o . A \stackrel{*}{\doteq} B \supset (A \Leftrightarrow B)$
- $\forall F_{\beta\alpha}, G_{\beta\alpha} . F \stackrel{*}{\doteq} G \supset (\forall X_\alpha . FX \stackrel{*}{\doteq} GX)$

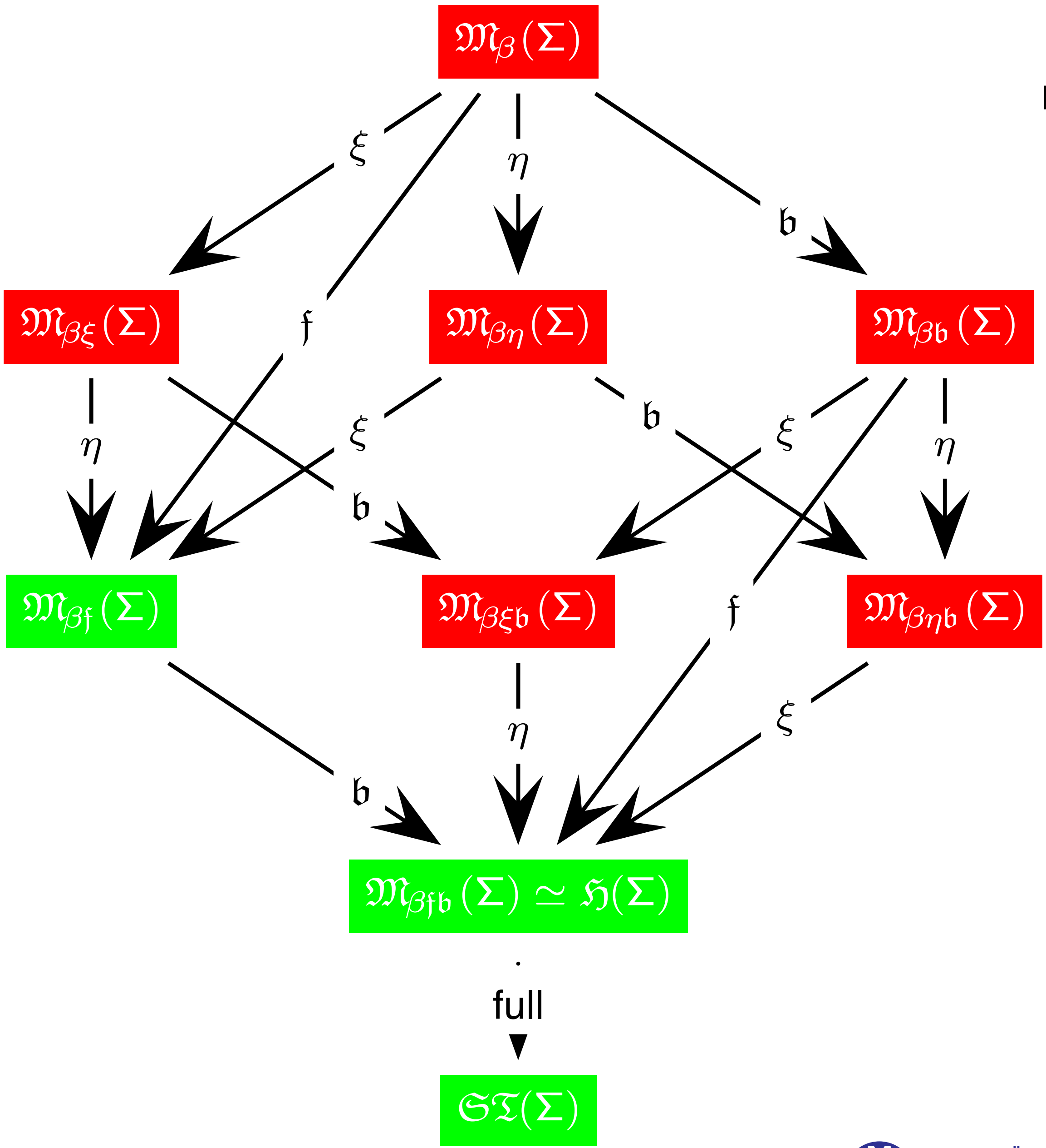
HOL-Problems: $\mathfrak{b}$



Non-trivial direction of Boolean extensionality

- $\forall A_o, B_o. (A \Leftrightarrow B) \supset A \stackrel{*}{\doteq} B$

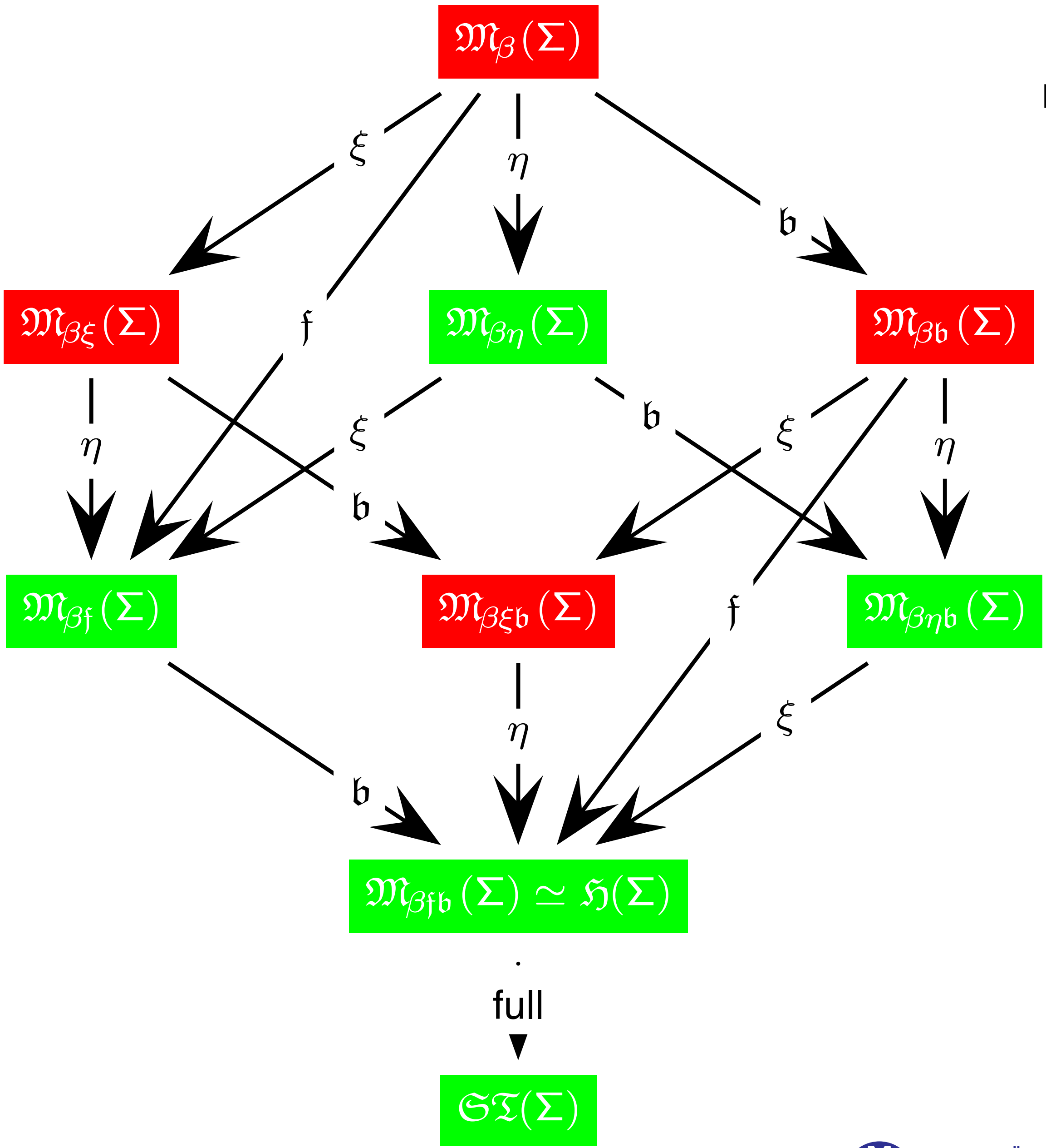
HOL-Problems: $\mathfrak{f}$



Non-trivial direct. of functional extensionality

- $\forall F_{\beta\alpha}, G_{\beta\alpha}.(\forall X_{\alpha}.FX \stackrel{*}{\doteq} GX) \supset F \stackrel{*}{\doteq} G$

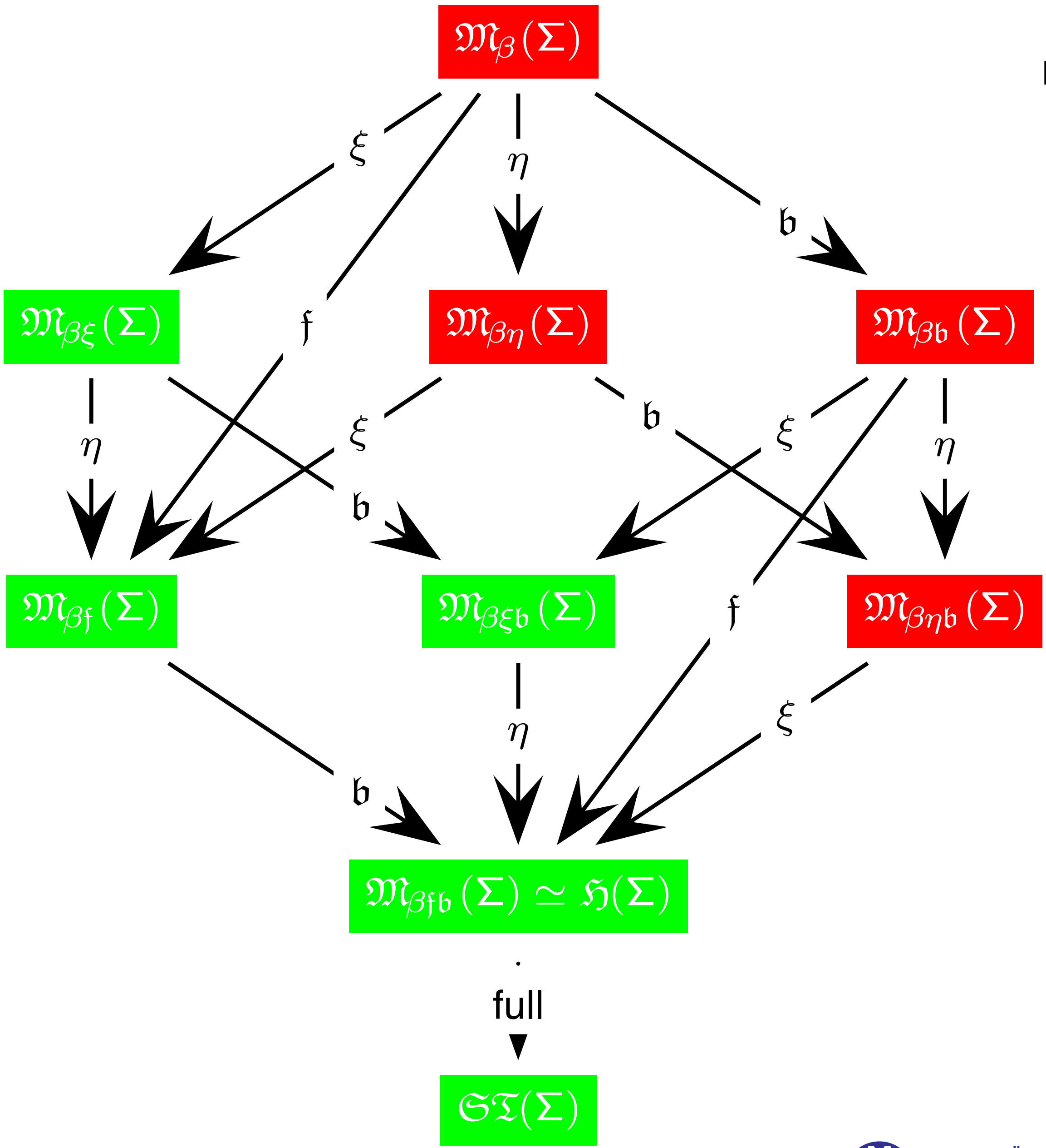
HOL-Problems: η



Example requiring property η

- $(p_{o(\iota\iota)}(\lambda X_\iota . f_{\iota\iota} X)) \supset (p\ f)$

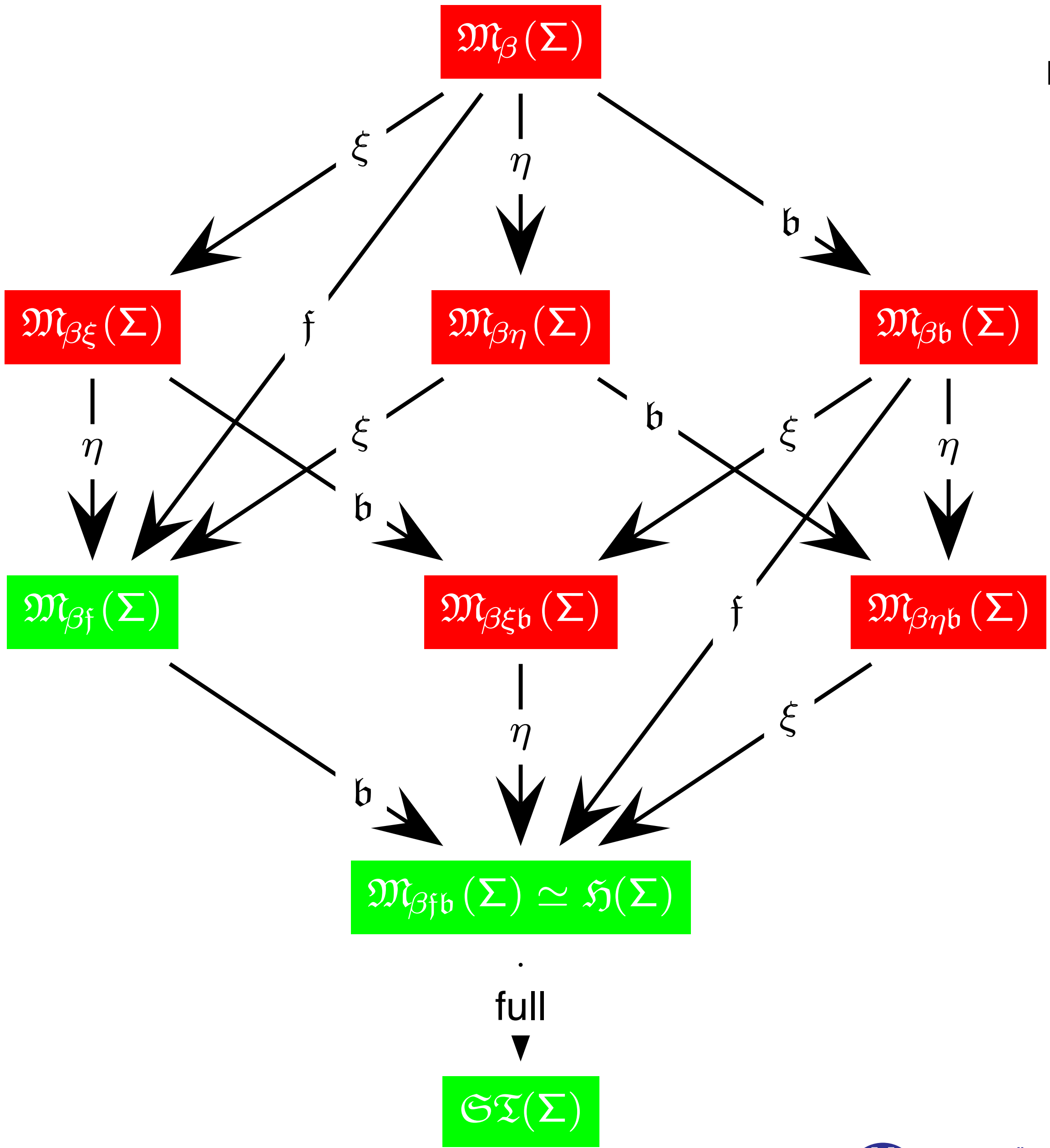
HOL-Problems: ξ



Example requiring property ξ (and q!)

- $(\forall X_\iota.(f_{\iota\iota}X) \stackrel{*}{\doteq} X) \wedge p_{o(\iota\iota)}(\lambda X_\iota.X)$
  $\supset p(\lambda X_\iota.fX)$

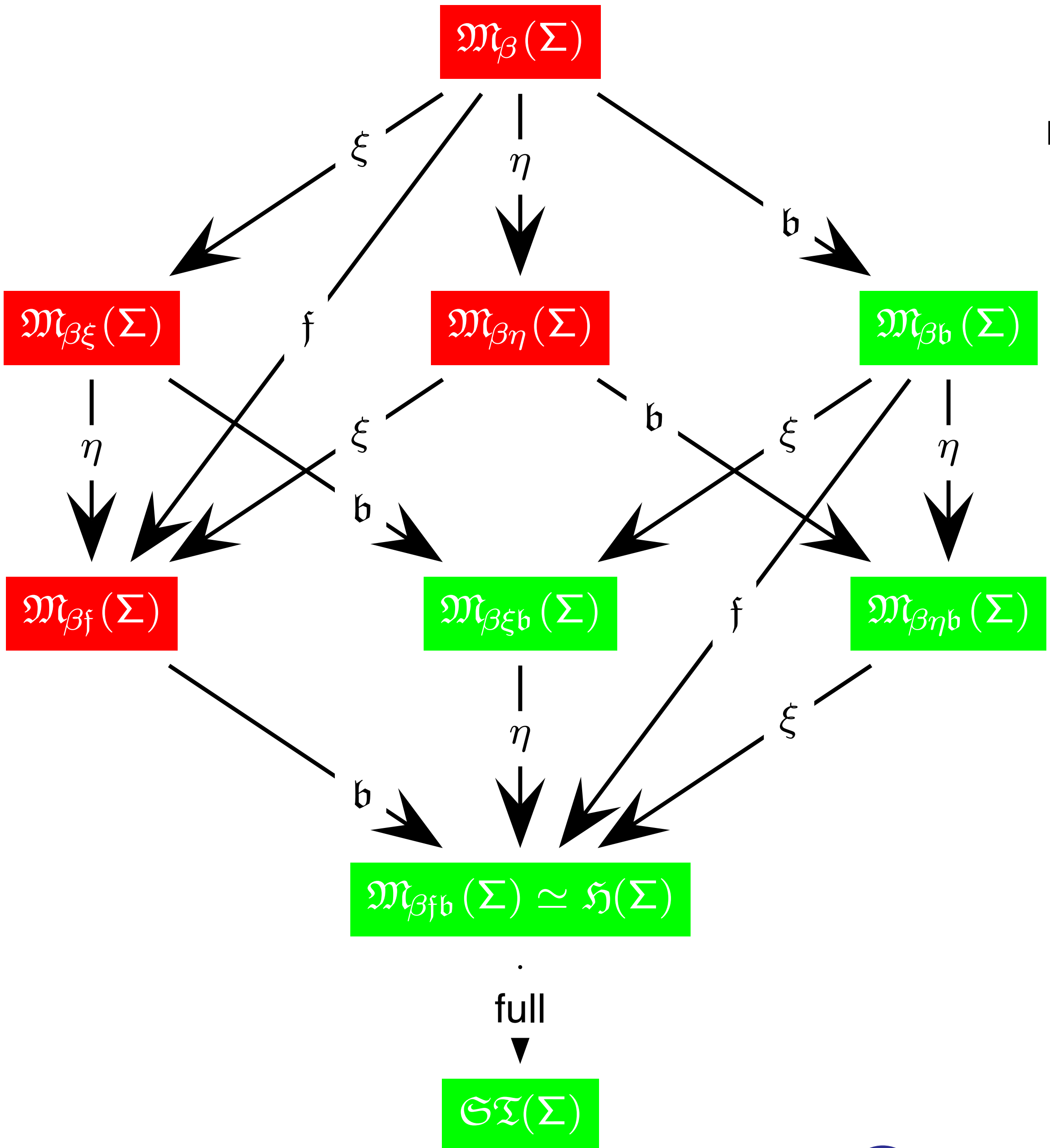
HOL-Problems: $\mathfrak{f}$



Example requiring property $\mathfrak{f}$ (and $\mathfrak{q}$!)

- $(\forall X_\iota.(f_{\iota\iota} X) \stackrel{*}{\doteq} X) \wedge p_{o(\iota\iota)}(\lambda X_\iota.X) \supset (p\ f)$

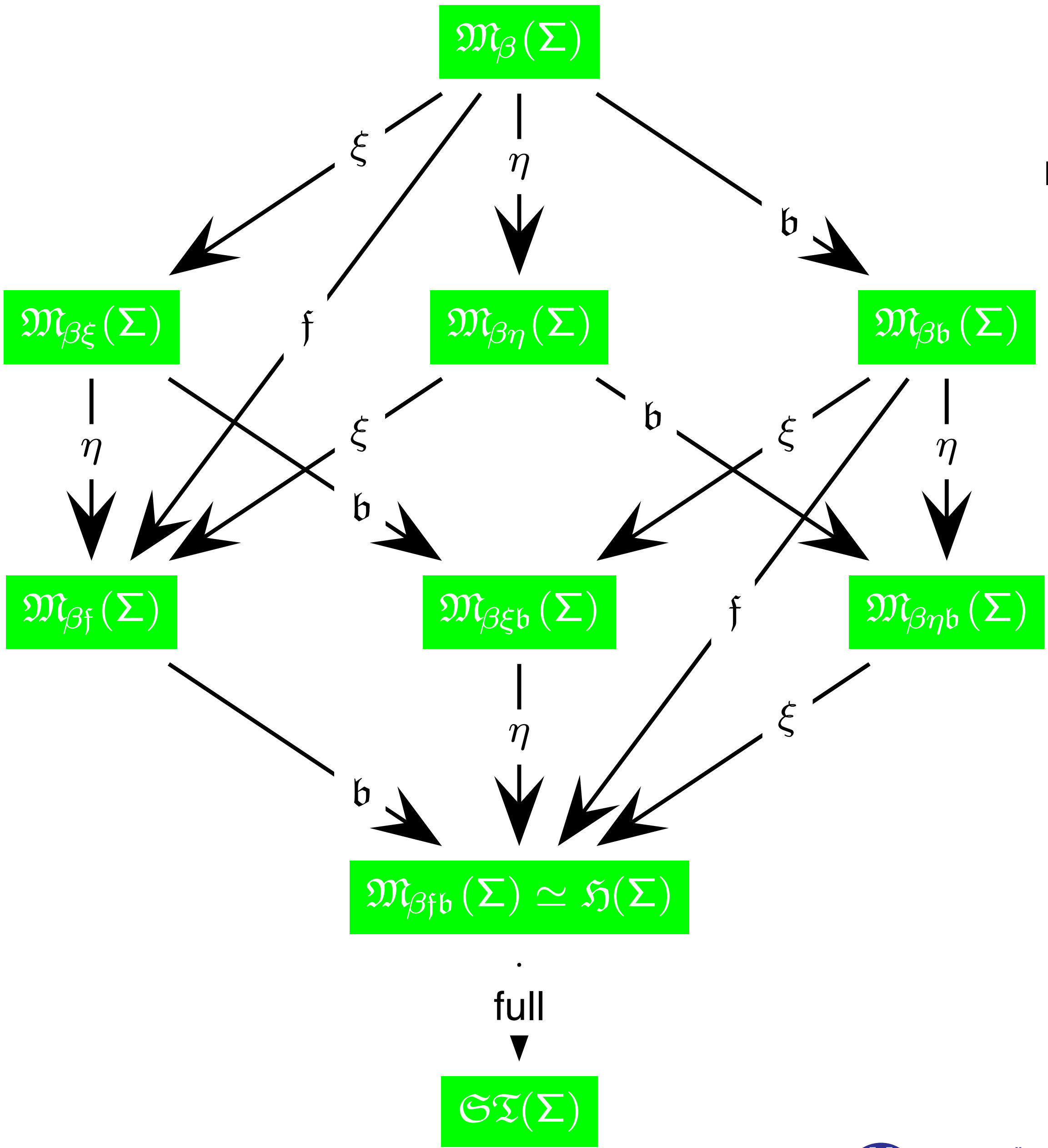
HOL-Problems: $\mathfrak{b}$



Examples requiring property $\mathfrak{b}$

- $(p_{oo}\ a_o) \wedge (p\ b_o) \Rightarrow (p\ (a \wedge b))$
- $\neg(a \stackrel{*}{\doteq} \neg a)$ (in particular $\neg(a = \neg a)$)
- $(h_{\iota o}((h \top) \stackrel{*}{\doteq} (h \bot))) \stackrel{*}{\doteq} (h \bot)$

HOL-Problems: Other Examples

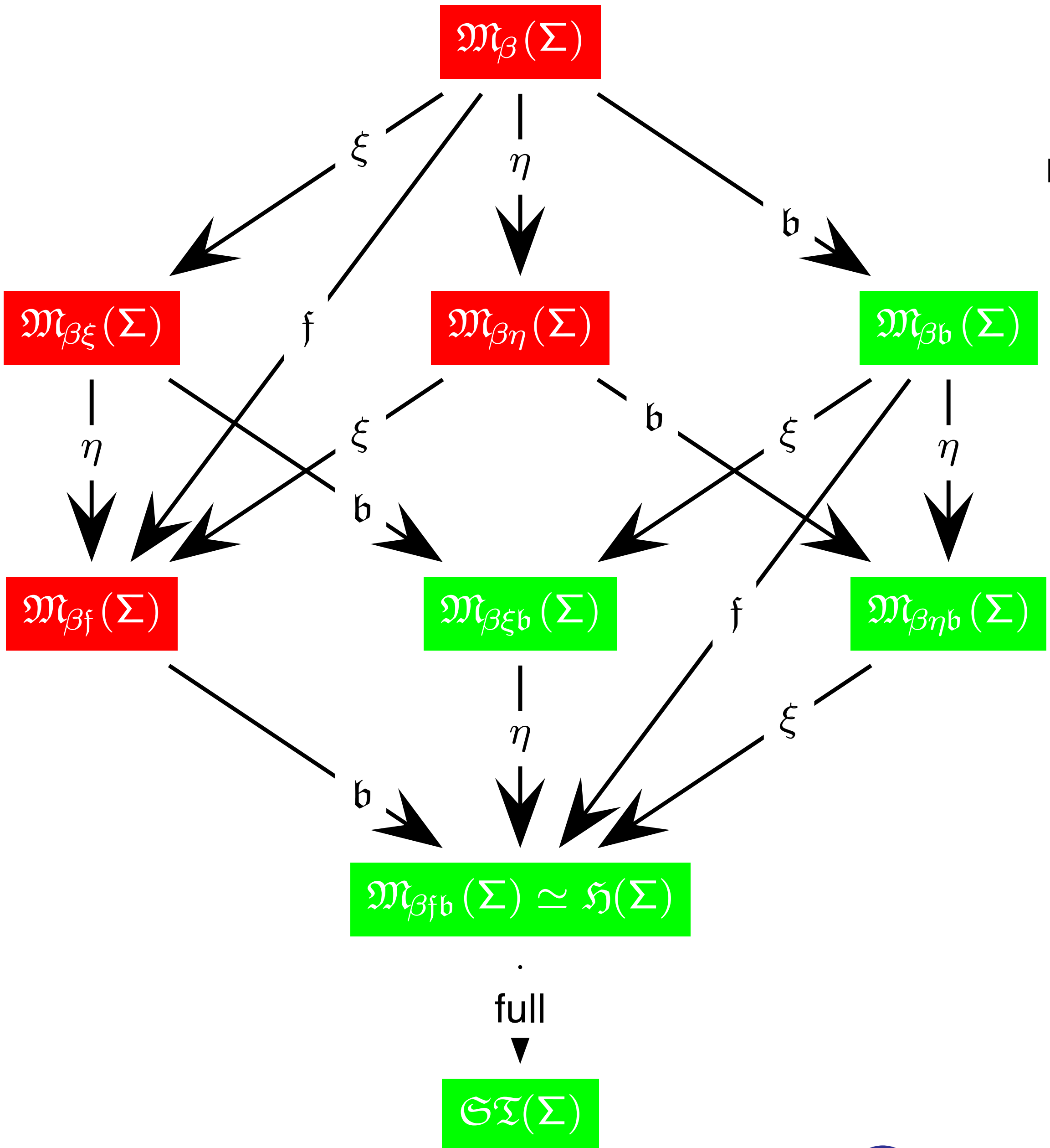


Playing with DeMorgan's Law:

- $\forall X, Y \boldsymbol{.} X \wedge Y \Leftrightarrow \neg(\neg X \vee \neg Y)$

'Ok' for all model classes

HOL-Problems: DeMorgan's Law



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- $\forall X, Y \boldsymbol{.} X \wedge Y \Leftrightarrow \neg(\neg X \vee \neg Y)$
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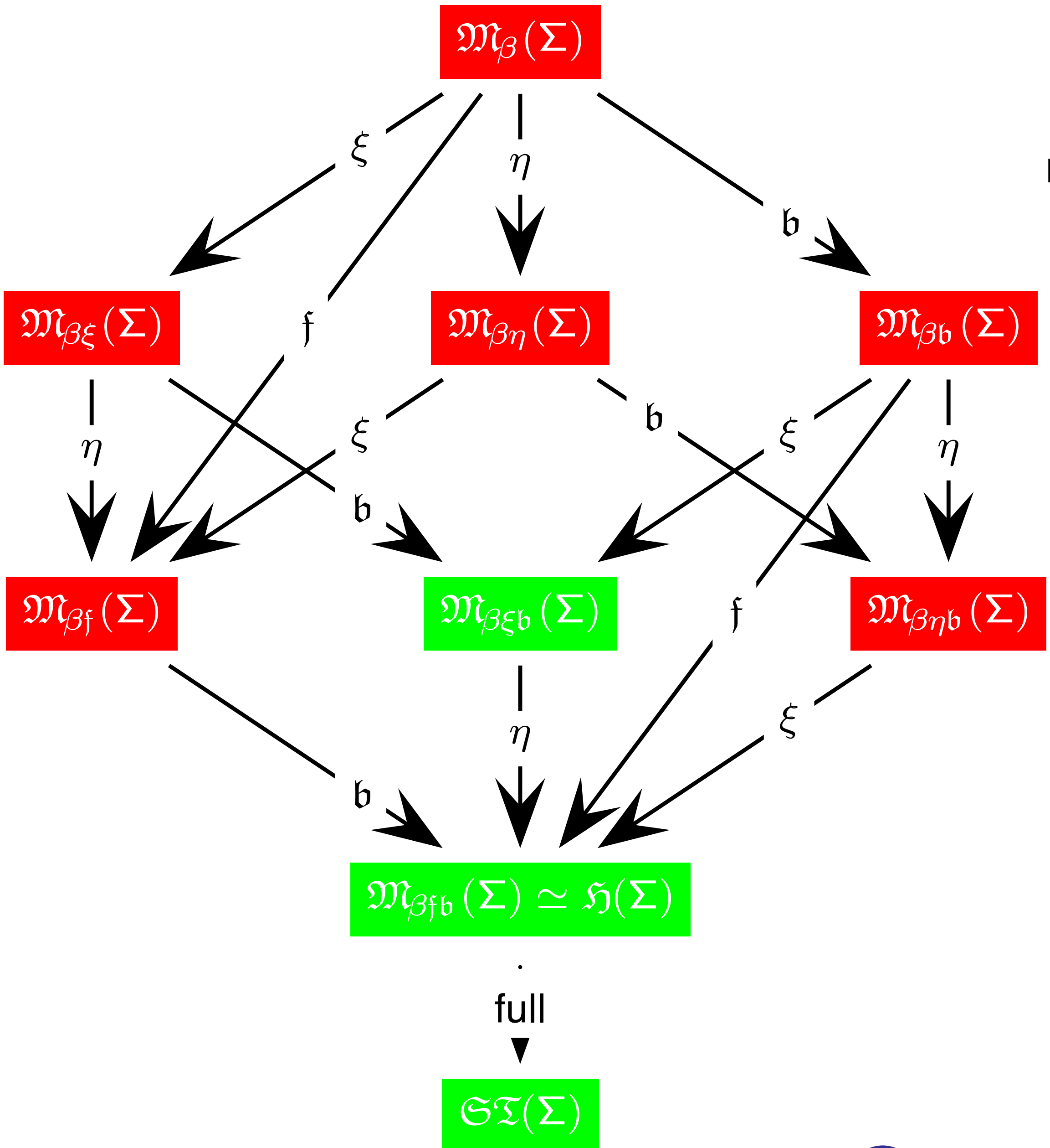
requires $\mathfrak{b}$

HOL-Problems: DeMorgan's Law

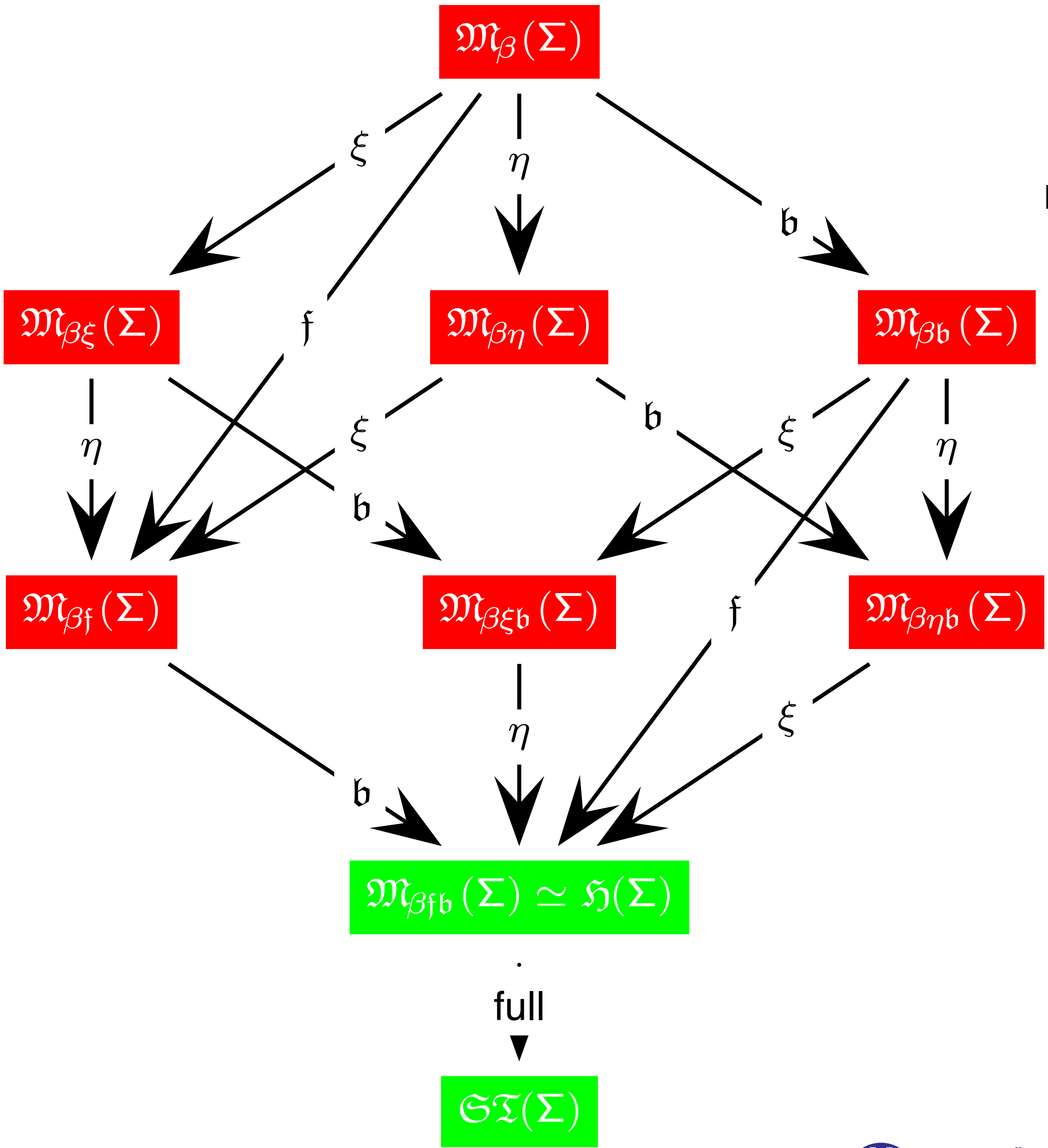


Playing with DeMorgan's Law:

- $\forall X, Y.X \land Y \Leftrightarrow \neg(\neg X \lor \neg Y)$
- $\forall X, Y.X \land Y \stackrel{*}{\doteq} \neg(\neg X \lor \neg Y)$
- $(\lambda U \lambda V.U \land V) \stackrel{*}{\doteq} (\lambda X \lambda Y.\neg(\neg X \lor \neg Y))$

requires $\mathfrak{b}$ and ξ

HOL-Problems: DeMorgan's Law



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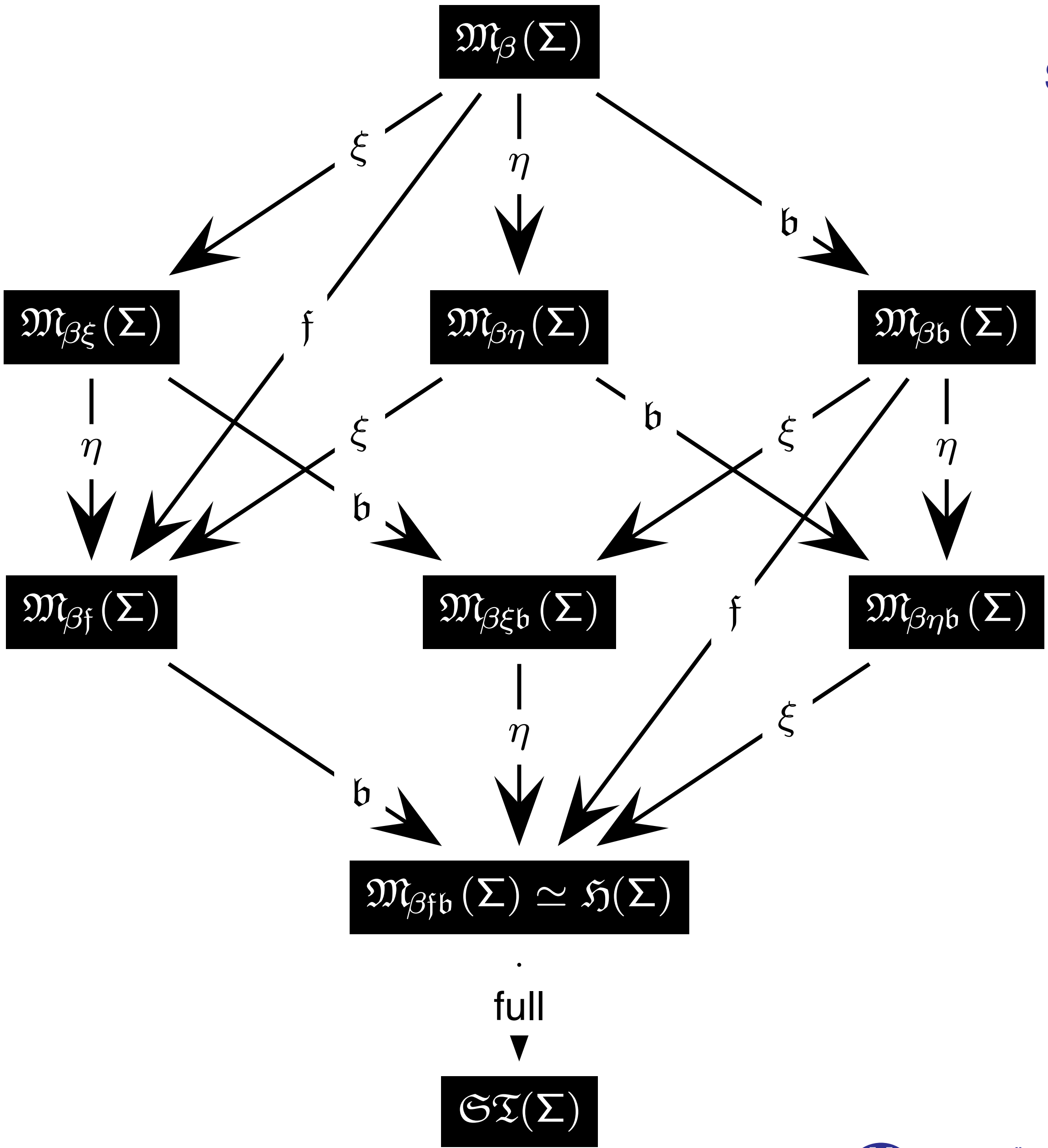
- $\wedge \stackrel{*}{\doteq} (\lambda X \lambda Y. \neg(\neg X \vee \neg Y))$

requires $\mathfrak{b}$ and $\mathfrak{f}$

HOL-Problems: Set Comprehension



Set comprehension

- big challenge for automation
- [Benzm.BrownKohlhase-Draft-05] set instantiations can be used to simulate cut-rule if one of the following axioms is given: comprehension, induction, extensionality, choice, description
- dependend on logical constants in Σ

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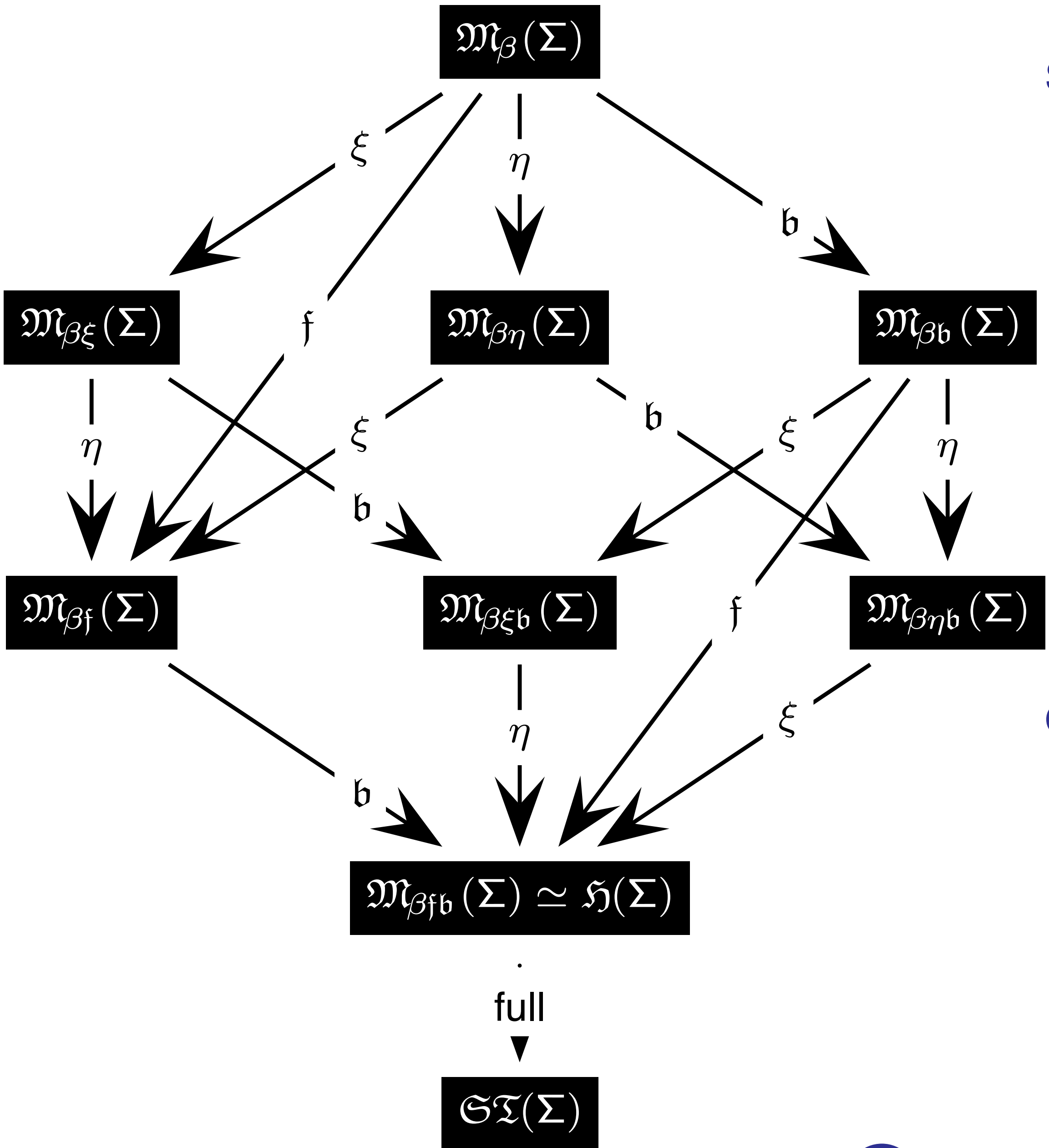

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- signature Σ varying
- no property $\mathfrak{q}$ assumed

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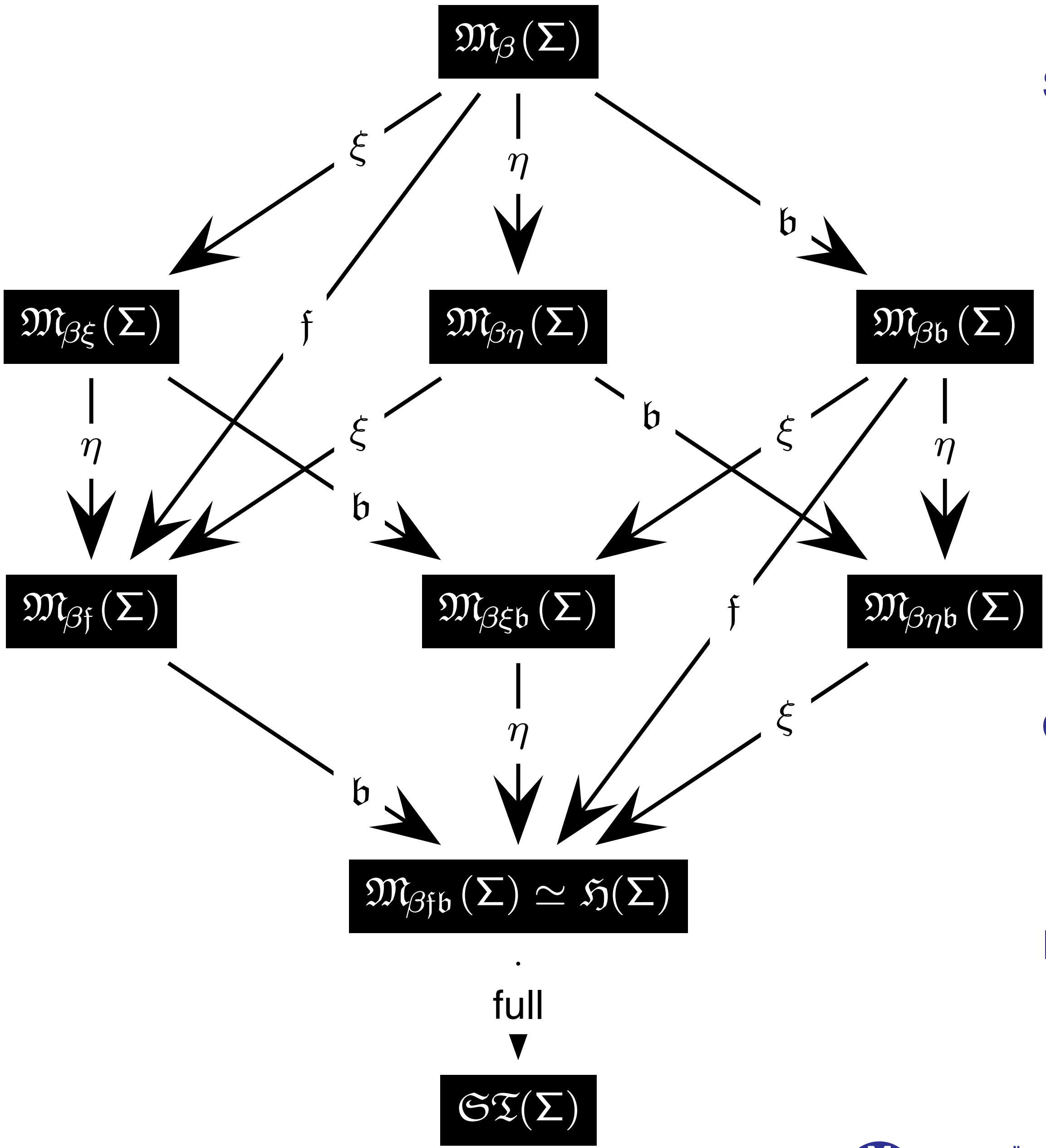

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External vs. internal logical constants

- if $\neg \notin \Sigma$:
  $\neg$ refers to 'external' symbol
  $\mathcal{M} \models \neg A$ means $\mathcal{M} \not\models A$

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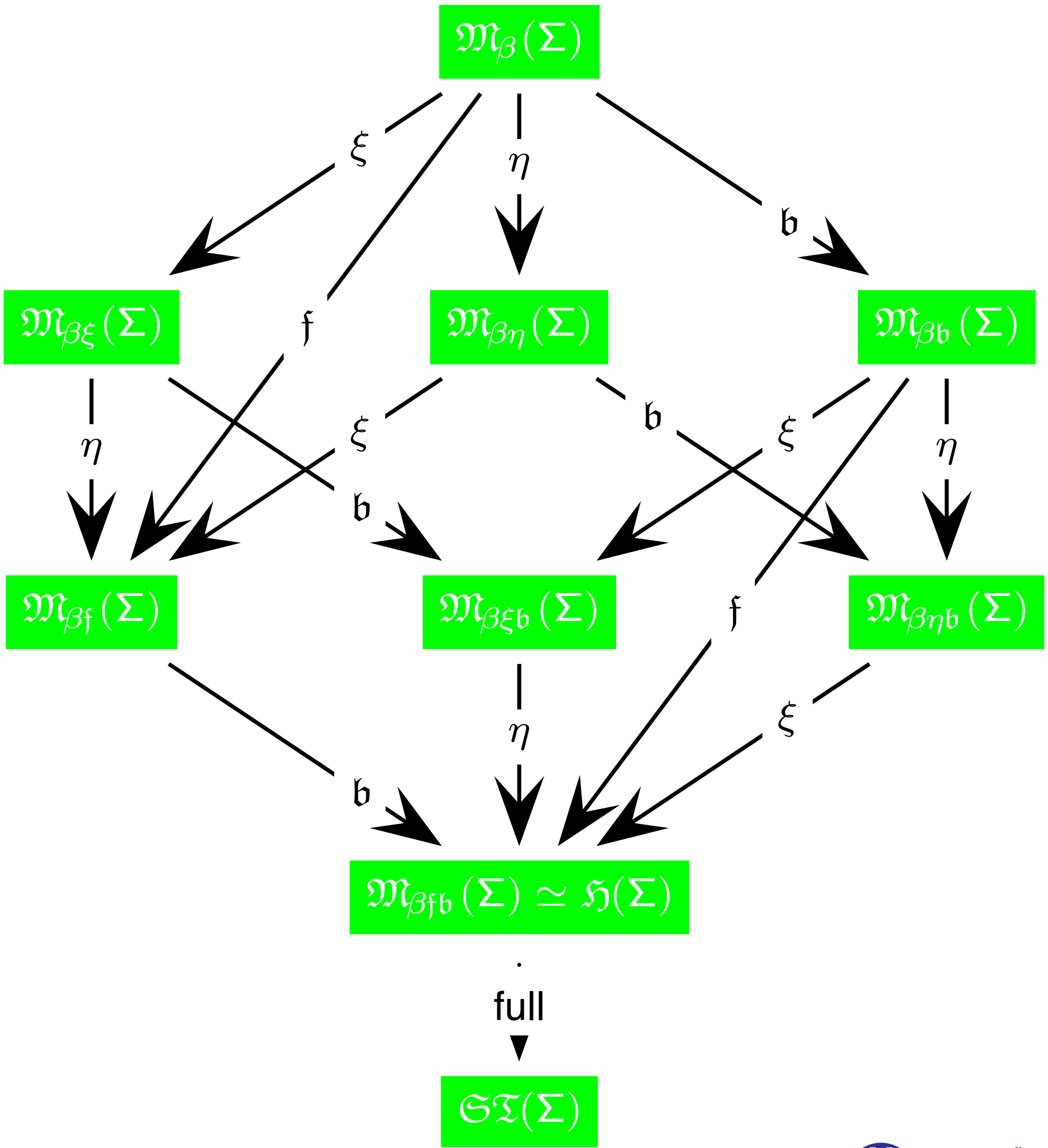

Set comprehension

- $\exists N_{oo} \forall P_o . NP \Leftrightarrow \neg P$
  - if $\neg \in \Sigma$ or $\{\bot, \supset\} \subseteq \Sigma$ or $\{\bot, \Leftrightarrow\} \subseteq \Sigma$
  - e.g.: $N_{oo} \longleftarrow \lambda X_o . \neg X$
    e.g.: $N_{oo} \longleftarrow \lambda X_o . X \supset \bot$

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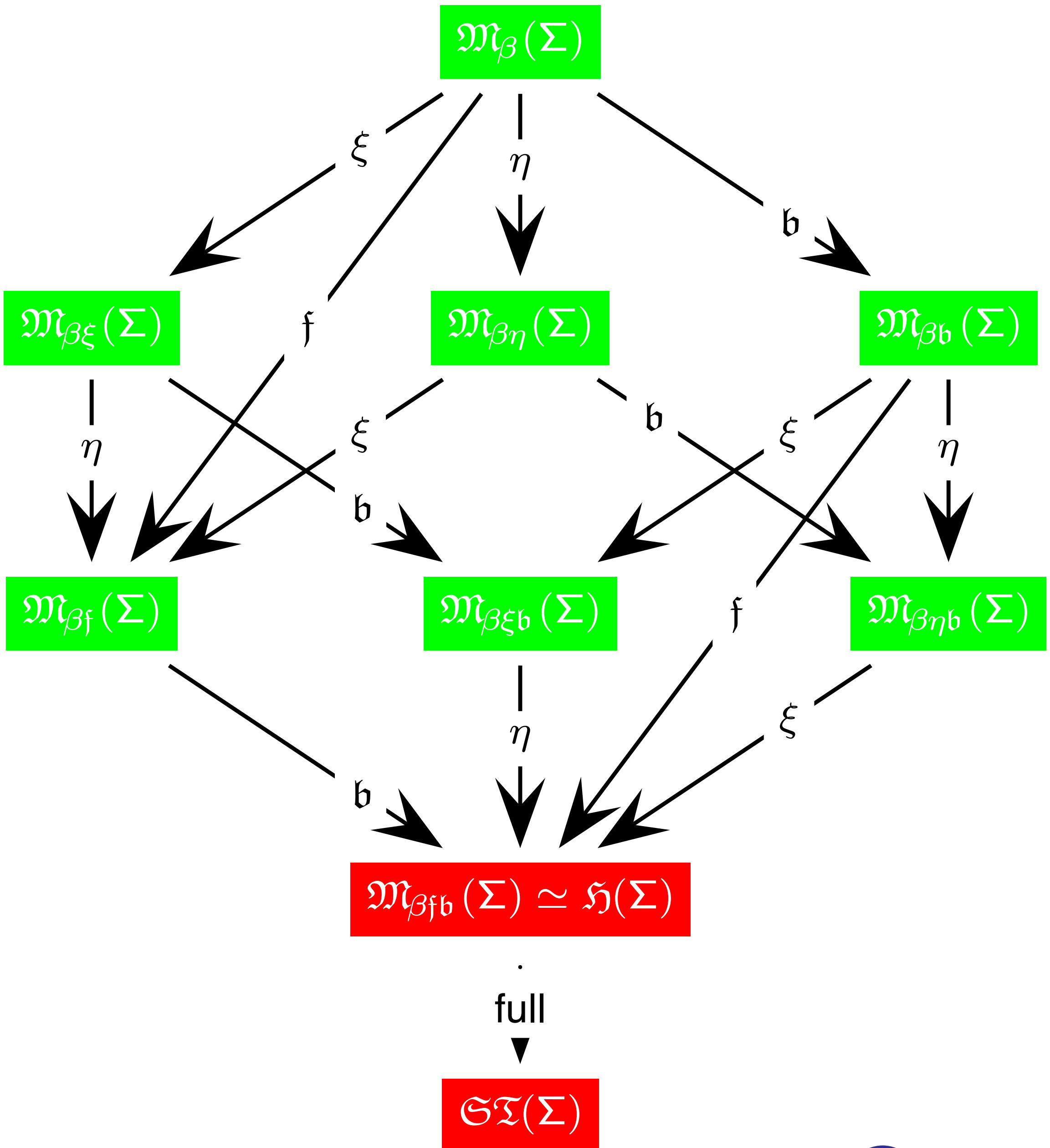

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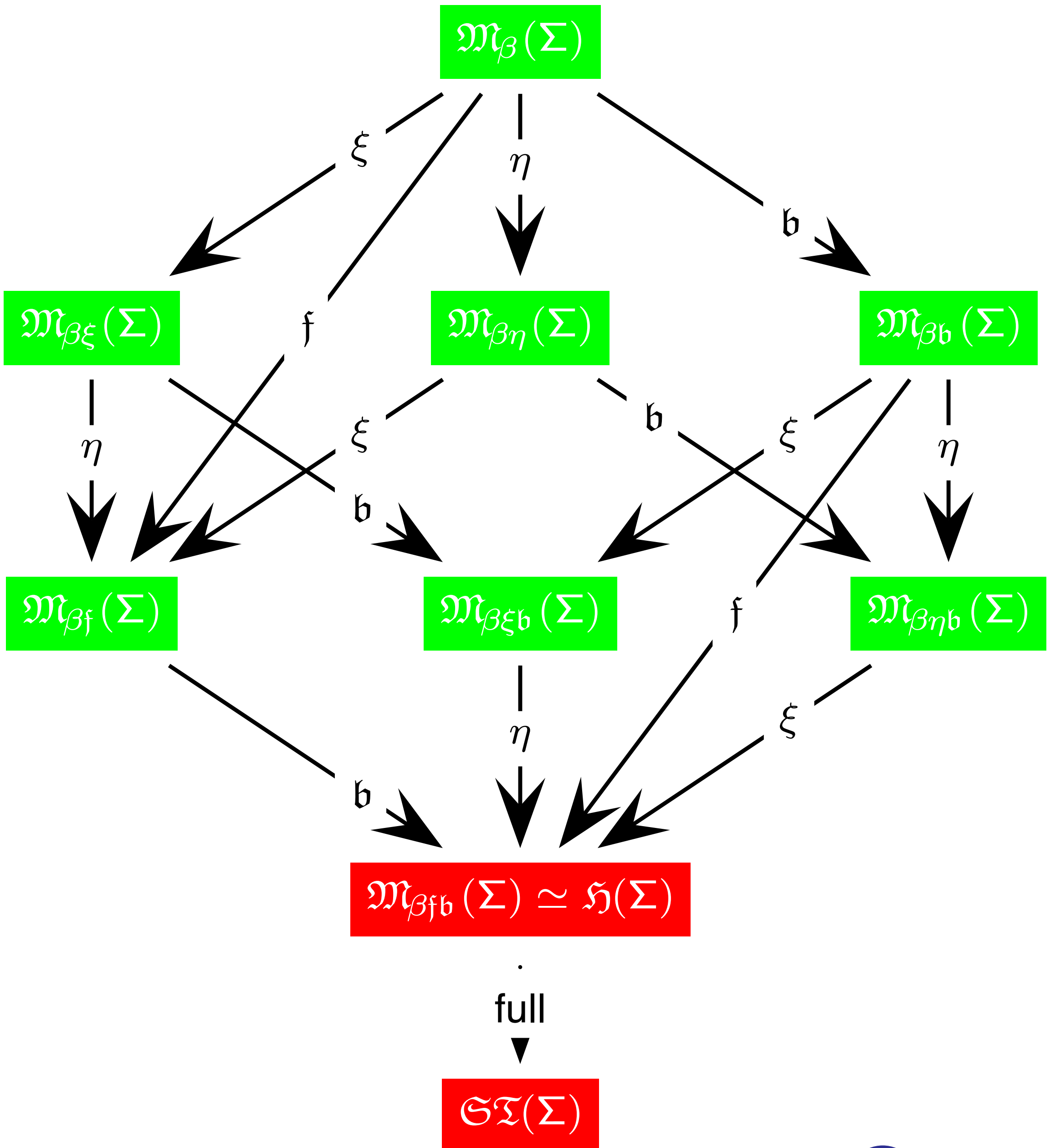

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Other examples from [Brown-PhD-04]

- Surjective Cantor Theorem
- Injective Cantor Theorem

Semantics: Examples of Σ-Models

Examples of Σ-Models

We now sketch the construction of models in the model classes $\mathfrak{M}_*(\Sigma)$ to demonstrate concretely how properties for Boolean, strong and weak functional extensionality can fail.

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We now sketch the construction of models in the model classes $\mathfrak{M}_*(\Sigma)$ to demonstrate concretely how properties for Boolean, strong and weak functional extensionality can fail.

We need this to show that the inclusions of the model classes in our landscape are proper, and we indeed need all of them.

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- Thus, $\mathcal{M}^{\beta\mathfrak{fb}} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ defines a Σ-model.

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- Thus, $\mathcal{M}^{\beta\mathfrak{f}\mathfrak{b}} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ defines a Σ-model.
- This model satisfies properties $\mathfrak{b}$, $\mathfrak{f}$ (hence η and ξ) and $\mathfrak{q}$ (since the frame is full).

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- This model satisfies properties $\mathfrak{b}$, $\mathfrak{f}$ (hence η and ξ) and $\mathfrak{q}$ (since the frame is full).
- So, $\mathcal{M}^{\beta\mathfrak{fb}} \in \mathfrak{ST}(\Sigma) \subseteq \mathfrak{H}(\Sigma) \subseteq \mathfrak{M}_{\beta\mathfrak{fb}}(\Sigma) \subseteq \ldots$.

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\end{tikzcd}
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- Assume Σ contains only the connectives $\neg, \vee, \Pi^{\alpha}$; other connectives defined as usual, e.g., $\forall X, Y. X \wedge Y \Leftrightarrow \neg(\neg X \vee \neg Y)$.

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$\mathcal{E}(\neg)$	a	b	c
	c	c	a

$\mathcal{E}(\vee)$	a	b	c
a	a	a	a
b	a	a	a
c	a	a	c

$$\mathcal{E}(\Pi^\alpha)@f = \begin{cases} a, & \text{if } f@g \in \{a, b\} \text{ for all } g \in \mathcal{D}_\alpha \\ c, & \text{if } f@g = c \text{ for some } g \in \mathcal{D}_\alpha \end{cases}$$

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b	a	a	a
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$$\mathcal{E}(\Pi^\alpha)@f = \begin{cases} a, & \text{if } f@g \in \{a, b\} \text{ for all } g \in \mathcal{D}_\alpha \\ c, & \text{if } f@g = c \text{ for some } g \in \mathcal{D}_\alpha \end{cases}$$

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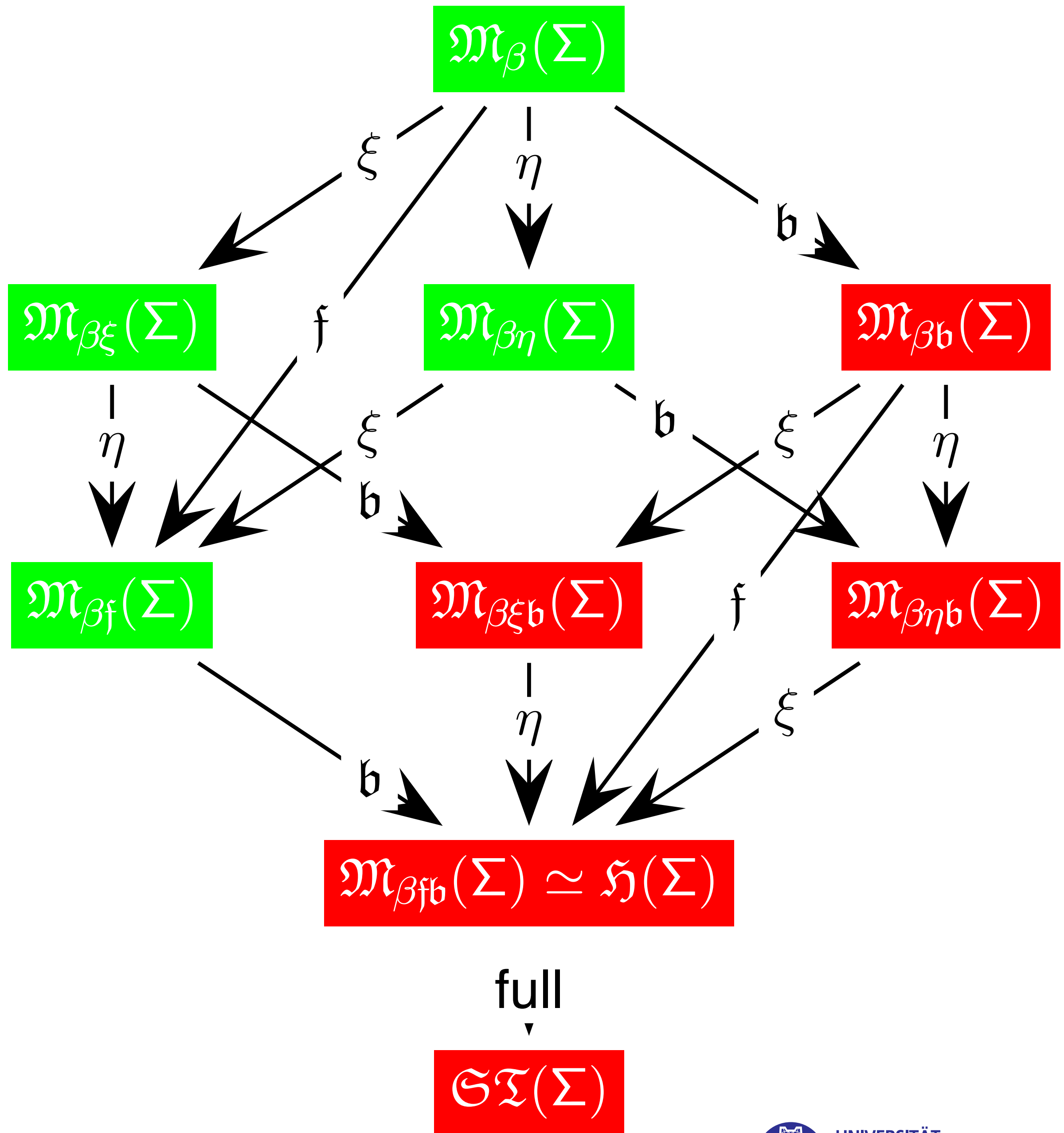
- Since this frame is full, we know property $\mathfrak{q}$ holds.
- Clearly property $\mathfrak{b}$ fails.
- So, $\mathcal{M}^{\beta\mathfrak{f}} \in \mathfrak{M}_{\beta\mathfrak{f}}(\Sigma) \setminus \mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma)$.

Ex.: Model without Boolean Extensionality



Ex.: Model without Boolean Extensionality

In the previous model one can easily verify, if $\mathsf{d} := \mathcal{E}_\varphi(\mathbf{D}_o)$ and $\mathsf{e} := \mathcal{E}_\varphi(\mathbf{E}_o)$, then the values $\mathcal{E}_\varphi(\mathbf{D} \wedge \mathbf{E})$, $\mathcal{E}_\varphi(\mathbf{D} \Rightarrow \mathbf{E})$, and $\mathcal{E}_\varphi(\mathbf{D} \Leftrightarrow \mathbf{E})$ are given by the following tables:

		e:		
$\mathcal{E}(\mathbf{D} \wedge \mathbf{E})$		a	b	c
d:	a	a	a	c
	b	a	a	c
	c	c	c	c

		e:		
$\mathcal{E}(\mathbf{D} \Rightarrow \mathbf{E})$		a	b	c
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Now we show that one can properly model the woodchuck/groundhog example.

Ex.: Groundhogs and Woodchucks

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- Let $\mathcal{E}(\mathsf{woodchuck})$ be the function $\mathsf{w} \in \mathcal{D}_{\iota\to o}$ with $\mathsf{w}(0) = \mathsf{b}$ and $\mathsf{w}(1) = \mathsf{c}$.

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- Let $\mathcal{E}(\mathsf{woodchuck})$ be the function $\mathsf{w} \in \mathcal{D}_{\iota\rightarrow o}$ with $\mathsf{w}(0) = \mathsf{b}$ and $\mathsf{w}(1) = \mathsf{c}$.
- Let $\mathcal{E}(\mathsf{groundhog})$ be the function $\mathsf{g} \in \mathcal{D}_{\iota\rightarrow o}$ with $\mathsf{g}(0) = \mathsf{a}$ and $\mathsf{g}(1) = \mathsf{c}$.
- One can show that the sentence $\forall X_{\iota}.(\mathsf{woodchuck}\ X) \Leftrightarrow (\mathsf{groundhog}\ X)$ is valid.
- Also, $\mathcal{E}(\mathsf{woodchuck}\ \mathsf{phil}) = \mathsf{b}$ and $\mathcal{E}(\mathsf{groundhog}\ \mathsf{phil}) = \mathsf{a}$, so the propositions $(\mathsf{woodchuck}\ \mathsf{phil})$ and $(\mathsf{groundhog}\ \mathsf{phil})$ are valid.

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- Suppose $\mathsf{believe}_{\iota\longrightarrow o\longrightarrow o} \in \Sigma$ and $\mathcal{E}(\mathsf{believe})$ is the (Curried) function $\mathsf{bel} \in \mathcal{D}_{\iota\longrightarrow o\longrightarrow o}$ such that $\mathsf{bel}(1)(\mathsf{b}) = \mathsf{b}$ and $\mathsf{bel}(1)(\mathsf{a}) = \mathsf{bel}(1)(\mathsf{c}) = \mathsf{bel}(0)(\mathsf{a}) = \mathsf{bel}(0)(\mathsf{b}) = \mathsf{bel}(0)(\mathsf{c}) = \mathsf{c}$.

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- So, $\mathsf{believes\ john(woodchuck\ phil)}$ is valid, while $\mathsf{believes\ john(groundhog\ phil)}$ is not.

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These semantic constructions are similar to those in multi-valued logics. In contrast to these logics where the logical connectives are adapted to talk about multiple truth values, in our setting we are mainly interested in multiple truth values as diverse υ-pre-images of $\mathtt{T}$ and $\mathtt{F}$.

Semantics: Examples of Σ-Models (Contd.)

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- For each function type $\beta\alpha$, let

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- We define application by

$$(\mathsf{i}, \mathsf{f})@\mathsf{a} := \mathsf{f}(\mathsf{a})$$

whenever $(\mathsf{i}, \mathsf{f}) \in \mathcal{D}_{\beta\alpha}$ and $\mathsf{a} \in \mathcal{D}_\alpha$

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  - $\mathcal{E}(\Pi^{\alpha}) := (0, \pi^{\alpha})$ where for each $(\mathsf{i}, \mathsf{f}) \in \mathcal{D}_{\mathsf{o}\alpha}$, $\pi^{\alpha}((\mathsf{i}, \mathsf{f})) := \mathtt{T}$ if $\mathsf{f}(\mathsf{a}) \in \mathcal{B}$ for all $\mathsf{a} \in \mathcal{D}_{\alpha}$ and $\pi^{\alpha}(\mathsf{i}, \mathsf{f}) := \mathtt{F}$ otherwise

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  - $\mathsf{q}^{\alpha} := (0, \mathsf{q}^{\alpha}) \in \mathcal{D}_{\mathsf{o}\alpha\alpha}$ where $\mathsf{q}^{\alpha}(\mathsf{a}) := (0, \mathsf{s}^{\mathsf{a}})$ and $\mathsf{s}^{\mathsf{a}}(\mathsf{b}) := \mathtt{T}$ if $\mathsf{a} = \mathsf{b}$ and $\mathsf{s}^{\mathsf{a}}(\mathsf{b}) := \mathtt{F}$ otherwise

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  - $\mathcal{E}(\mathsf{w}) \in \mathcal{D}_\alpha$ arbitrary for parameters $\mathsf{w} \in \Sigma_\alpha$.

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- Hence, $\mathcal{M}^{\mathcal{B}} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ is a Σ-model

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- Hence, $\mathcal{M}^\mathcal{B} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ is a Σ-model with property $\mathfrak{q}$

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  - For both $(0, \mathrm{u}), (1, \mathrm{u}) \in \mathcal{D}_{\iota\iota}$ we have

$$(\mathsf{i}, \mathrm{u})@* = *$$

  although $(0, \mathrm{u}) \neq (1, \mathrm{u})$

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for every $\mathsf{a} \in \mathcal{D}_\alpha$, then

$$\mathcal{E}_\varphi(\lambda \mathsf{X}_\alpha . \mathbf{M}) = (0, \mathsf{f}) = \mathcal{E}_\varphi(\lambda \mathsf{X} . \mathbf{N})$$

where $\mathsf{f}(\mathsf{a}) = \mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathbf{M}) = \mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathbf{N})$ for every $\mathsf{a} \in \mathcal{D}_\alpha$.

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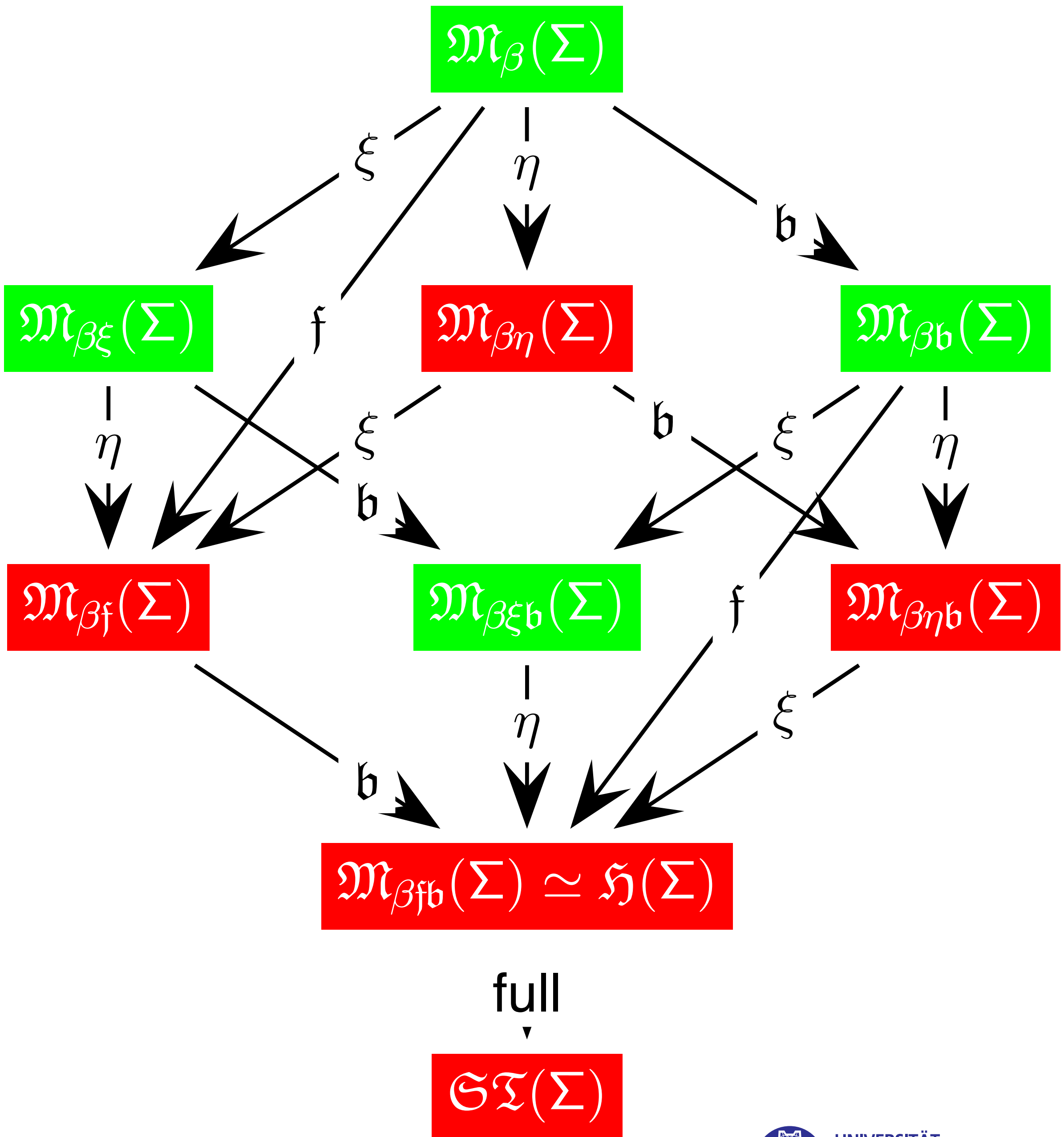

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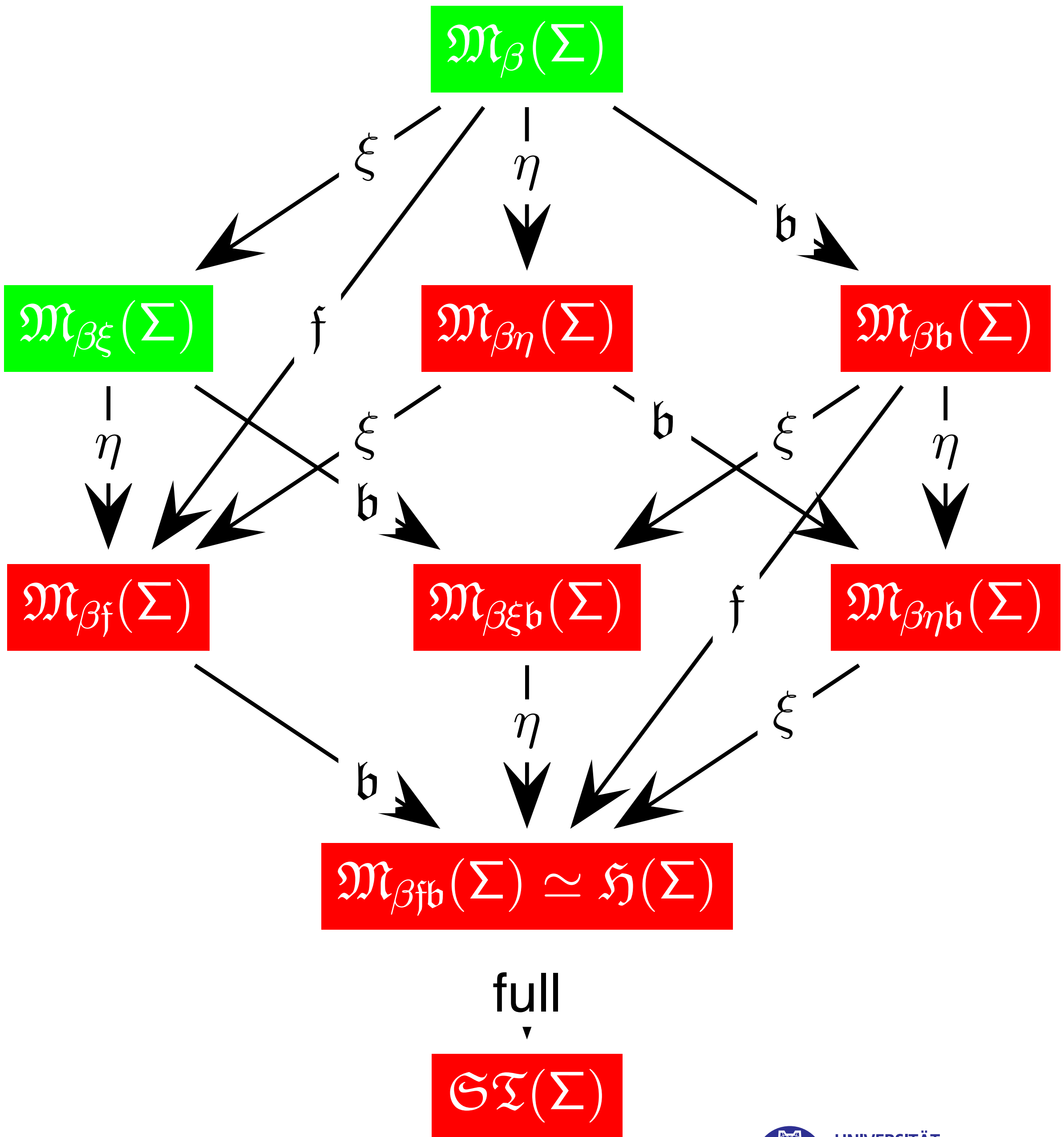

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- $\mathcal{E}$ and $\mathcal{E}'$ agree on all constants, they are different though:

$$\mathcal{E}(\lambda \mathsf{X}_\iota \mathbf{.} \mathsf{X}) = (0, \mathsf{id}) \neq (1, \mathsf{id}) = \mathcal{E}'(\lambda \mathsf{X}_\iota \mathbf{.} \mathsf{X})$$

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- Thus, in non-functional models evaluation functions are not uniquely determined by their values on constants

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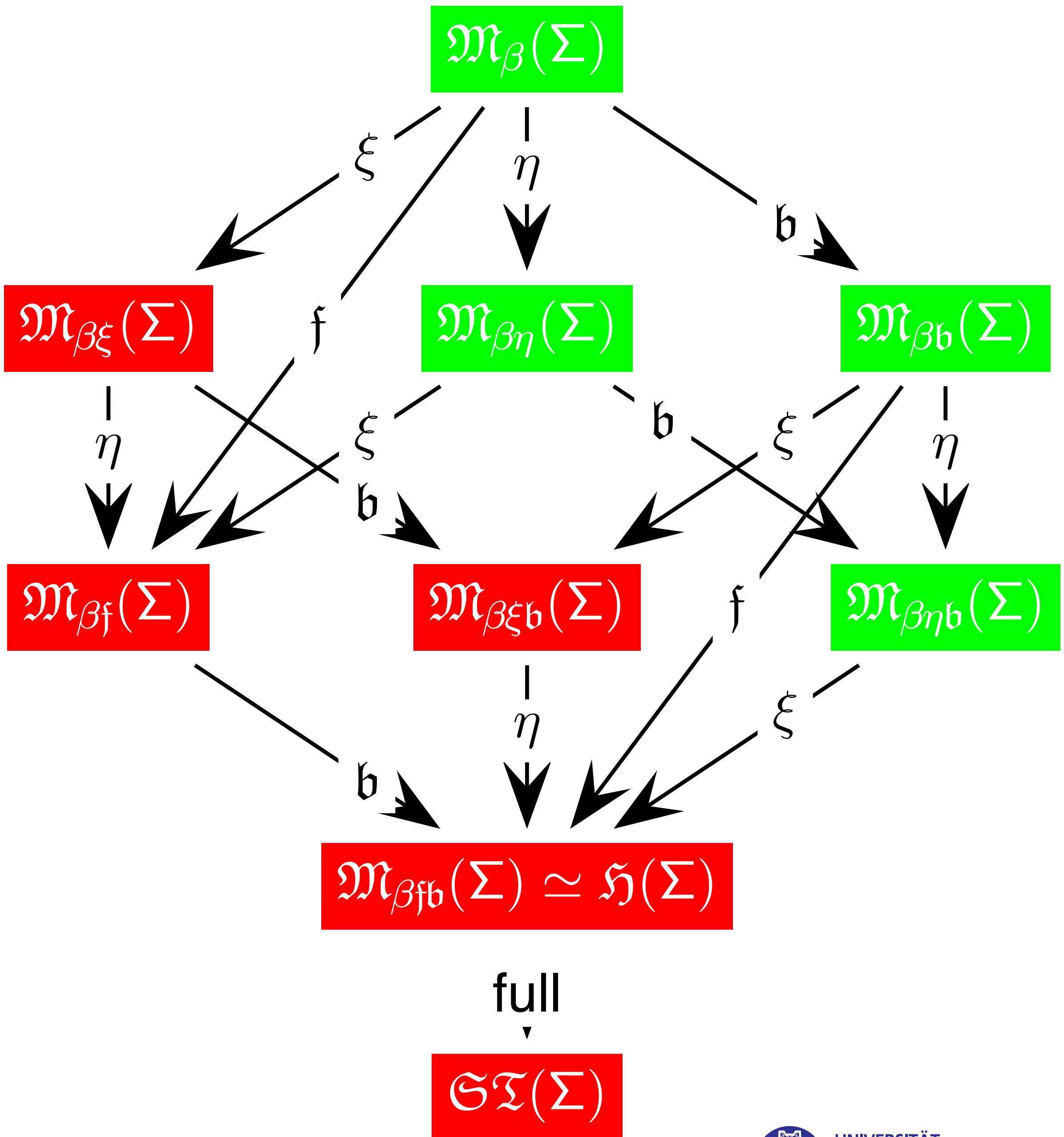

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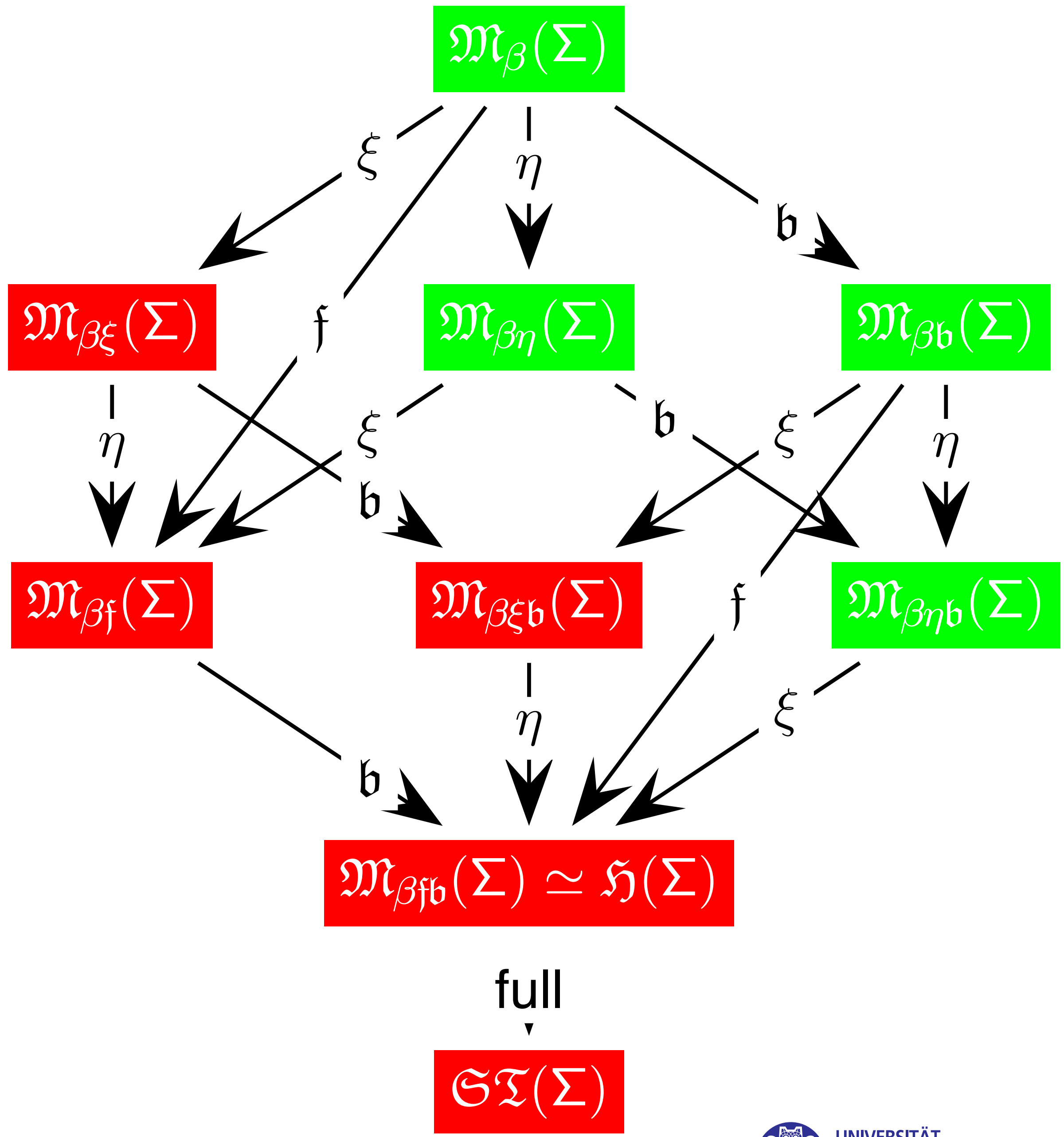

Not here!

See [JSL-04]

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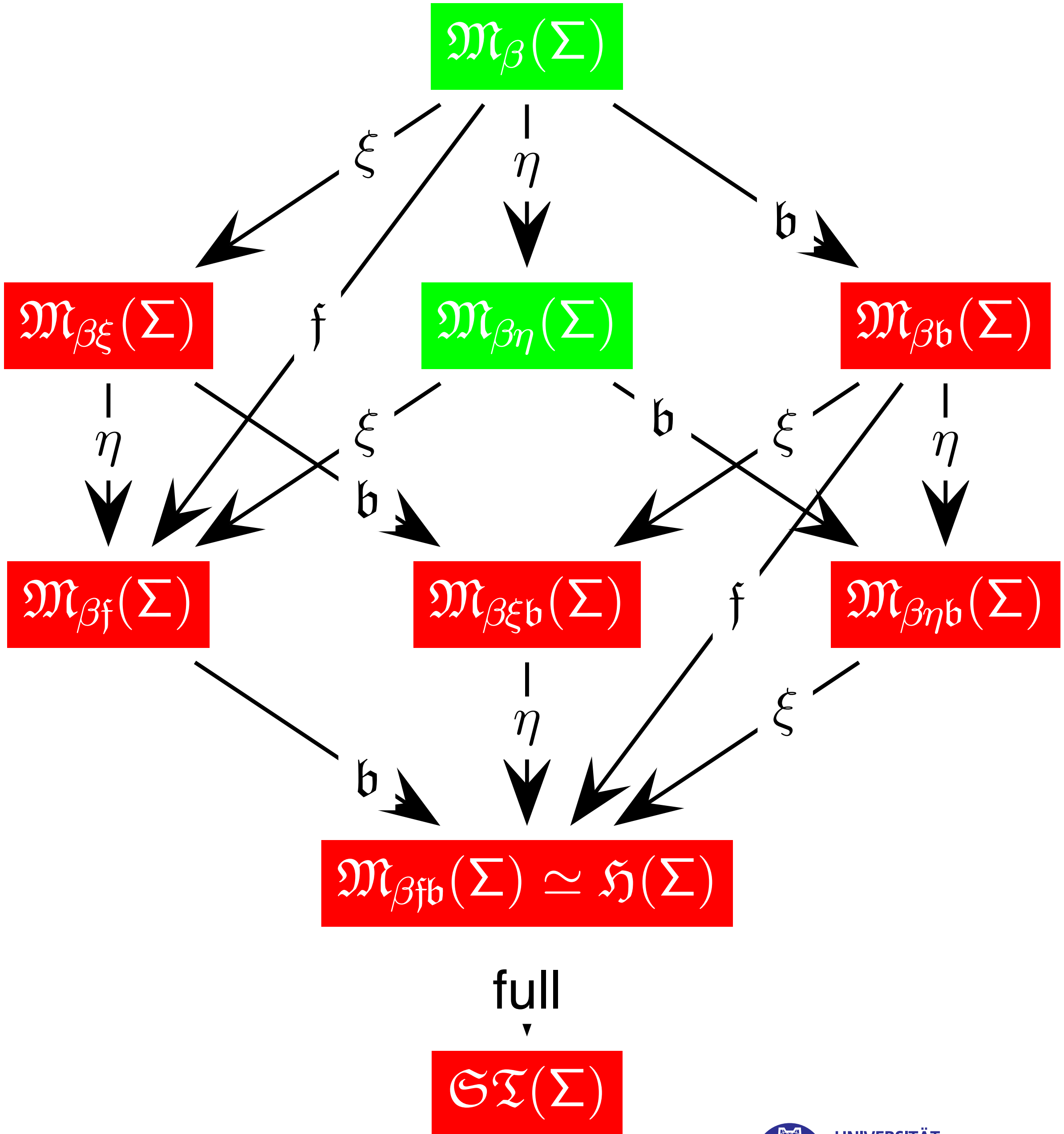

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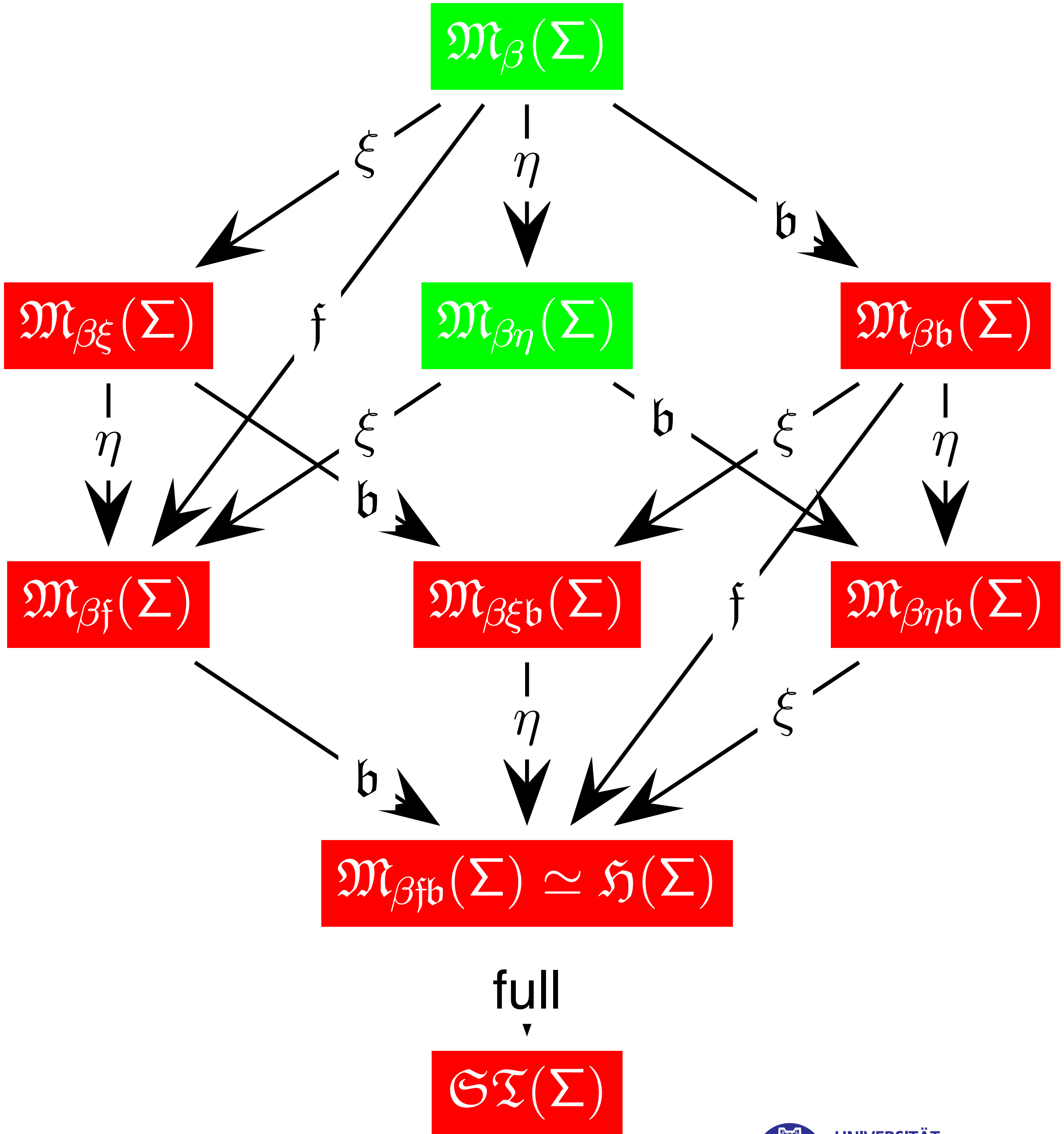

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