Cut-Simulation in Impredicative Logics

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and

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UNIVERSITÄT DES SAARLANDES

Many Connected Stories in the Paper

- Study challenges for automating impredicate logics/HOL
  - Leibniz-equations
  - axioms of:
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- etc.

- unification not powerful enough $\Longrightarrow$ guessing is state of the art
- problem not limited to HOL

HOL: Simple Types

Simple Types $\mathcal{T}$:

o	(truth values)
ι	(individuals)
$(\alpha \longrightarrow \beta)$	(functions from α to β)

HOL: Simply Typed λ-Terms

Typed Terms:

X_α	Variables ($\mathcal{V}$)
a_α	Parameters ($\mathcal{P}$)
c_α	Constants (Σ)
$(\mathbf{F}_{\alpha\to\beta}\,\mathbf{B}_\alpha)_\beta$	Application
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Equality of terms:

α-conversion	Changing bound variables	
β-reduction	$((\lambda \mathsf{Y}_\beta\,\mathbf{A}_\alpha)\,\mathbf{B}) \xrightarrow{\beta} [\mathbf{B}/\mathsf{Y}]\mathbf{A}$	
η-reduction	$(\lambda \mathsf{Y}_\alpha\,(\mathbf{F}_{\alpha\rightarrow\beta}\,\mathsf{Y})) \xrightarrow{\eta} \mathbf{F}$	$(\mathsf{Y}_\beta \notin \mathbf{Free}(\mathbf{F}))$

HOL: Adding Logical Constants to Σ

- $\top_o$ – true
- $\bot_o$ – false
- $\neg_{o\to o}$ – negation
- $\vee_{o\to o\to o}$ – disjunction
- $\wedge_{o\to o\to o}$ – conjunction
- $\supset_{o\to o\to o}$ – implication
- $\Leftrightarrow_{o\to o\to o}$ – equivalence
- $\Pi^{\alpha}_{(\alpha\to o)\to o}$ – universal quantification over type α ($\forall$ types α)
- $\Sigma^{\alpha}_{(\alpha\to o)\to o}$ – existential quantification over type α ($\forall$ types α)
- $=^{\alpha}_{\alpha\to\alpha\to o}$ – equality at type α ($\forall$ types α)

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Our choice for signature Σ in this paper:

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Use abbreviations for other logical operators

$\mathbf{A} \vee \mathbf{B}$	means	$(\vee\, \mathbf{A}\, \mathbf{B})$
$\mathbf{A} \wedge \mathbf{B}$	means	$\neg(\neg\mathbf{A} \vee \neg\mathbf{B})$
$\mathbf{A} \supset \mathbf{B}$	means	$\neg\mathbf{A} \vee \mathbf{B}$
$\mathbf{A} \Leftrightarrow \mathbf{B}$	means	$(\mathbf{A} \supset \mathbf{B}) \wedge (\mathbf{B} \supset \mathbf{A})$
$\forall X_{\alpha}\, \mathbf{A}$	means	$\Pi^{\alpha}(\lambda X_{\alpha}\, \mathbf{A})$
$\exists X_{\alpha}\, \mathbf{A}$	means	$\neg(\forall X_{\alpha}\, \neg\mathbf{A})$

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Use Leibniz-equality to encode equality

$$\mathbf{A}_\alpha \doteq^\alpha \mathbf{B}_\alpha \quad \text{means} \quad \forall P_{\alpha \to o} (P\, \mathbf{A} \supset P\, \mathbf{B})$$

$$\text{resp.} \quad \Pi^\alpha (\lambda P_{\alpha \to o} \neg P \mathbf{A} \vee P \mathbf{B})$$

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The Sequent Calculus $\mathcal{G}_\beta$

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- Analysis of Soundness and Completeness:
  - we need appropriate notions of semantics for HOL
  - standard semantics not appropriate

Semantics: HOL-CUBE

$\mathfrak{M}_{\beta}(\Sigma)$ elementary type theory (Andrews)

$\mathfrak{b}$: **Boolean extensionality** $|\mathcal{D}_o| = 2$

$\mathfrak{f}(= \eta + \xi)$: **functional extensionality**

see [Journal of Symbolic Logic (2004) 69(4)]

$\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma)$ extensional type theory (Henkin semantics)

full function domains

$\mathfrak{S}\mathfrak{T}(\Sigma)$ **standard semantics**

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\begin{tikzcd}
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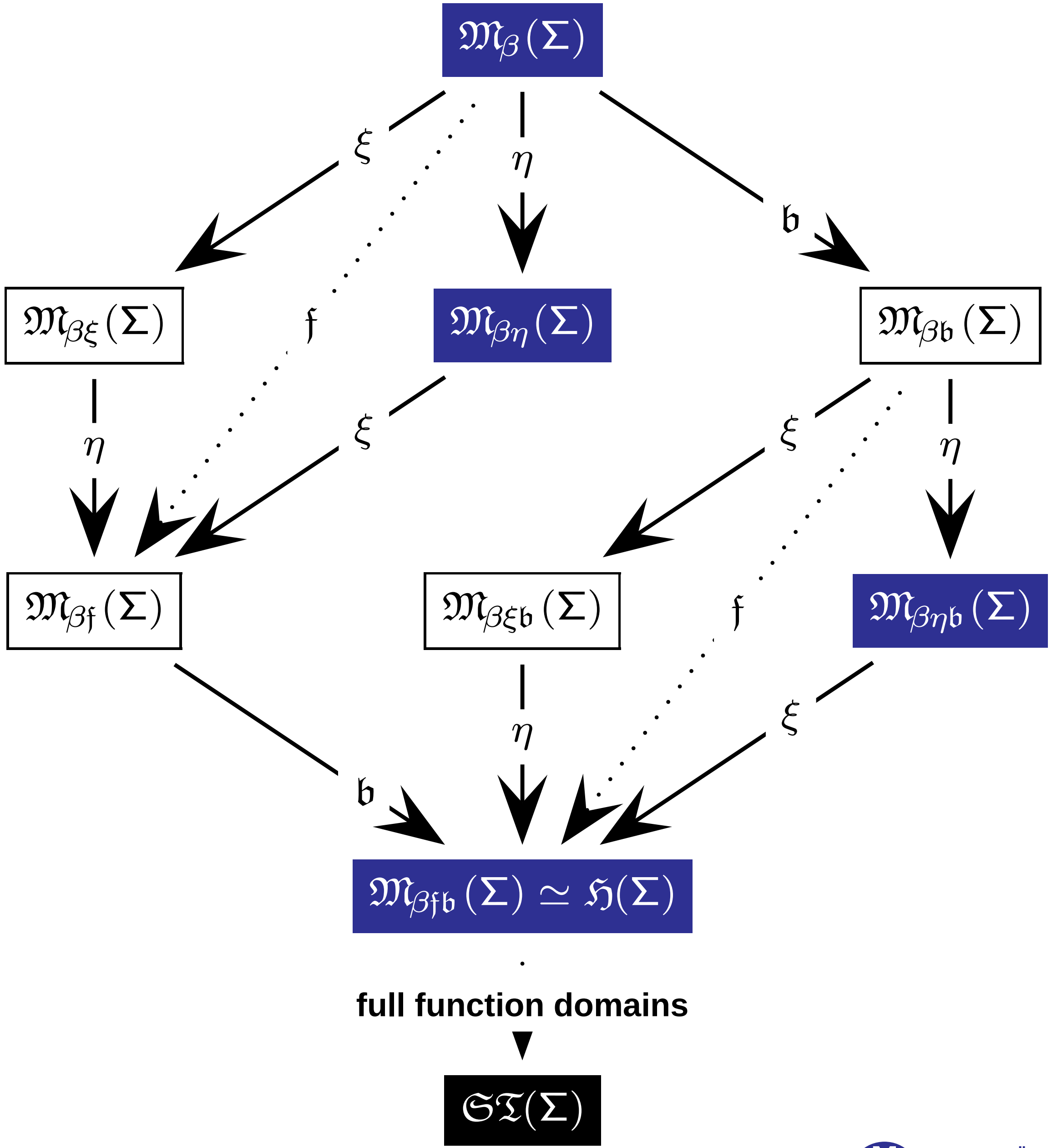

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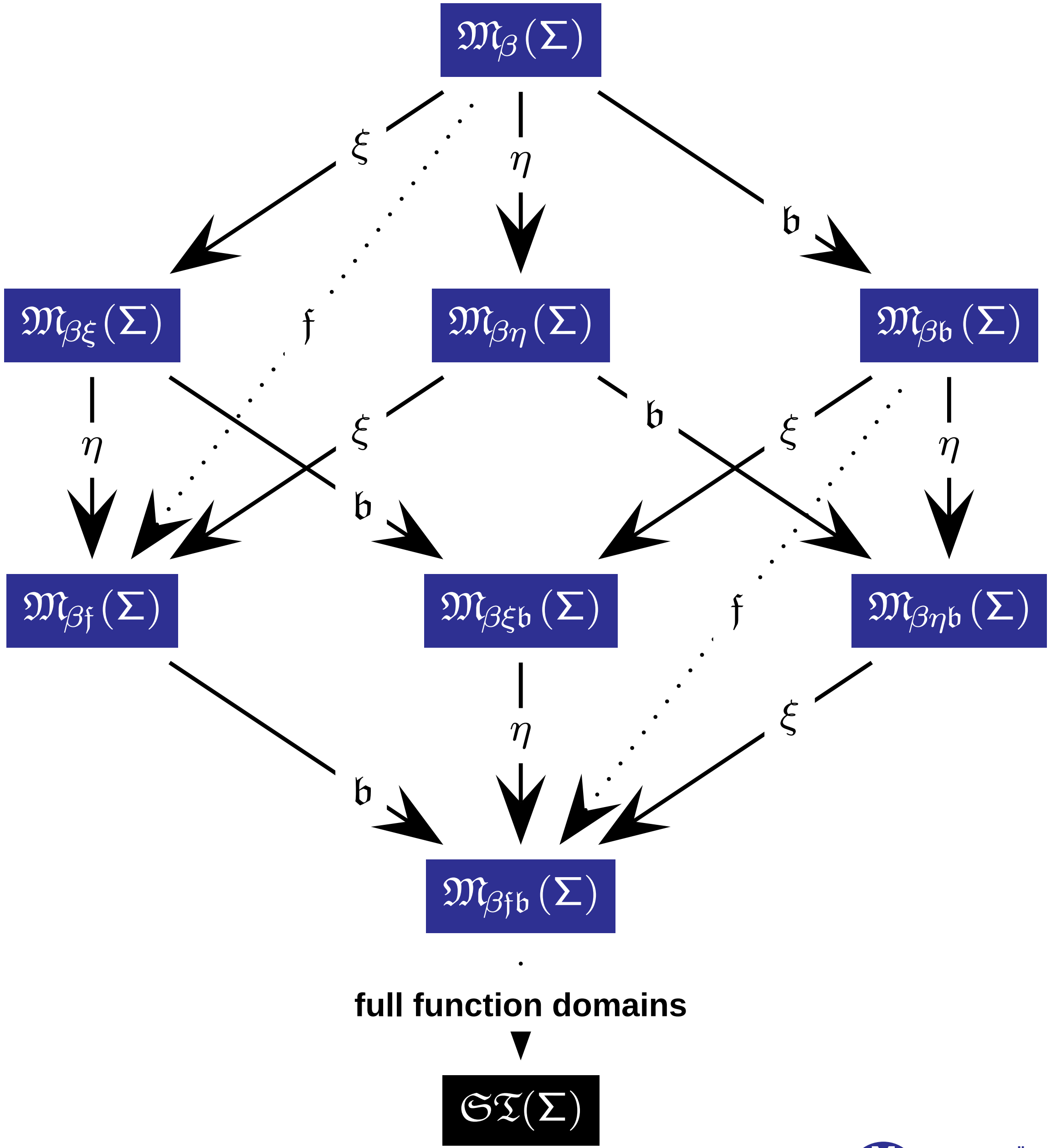

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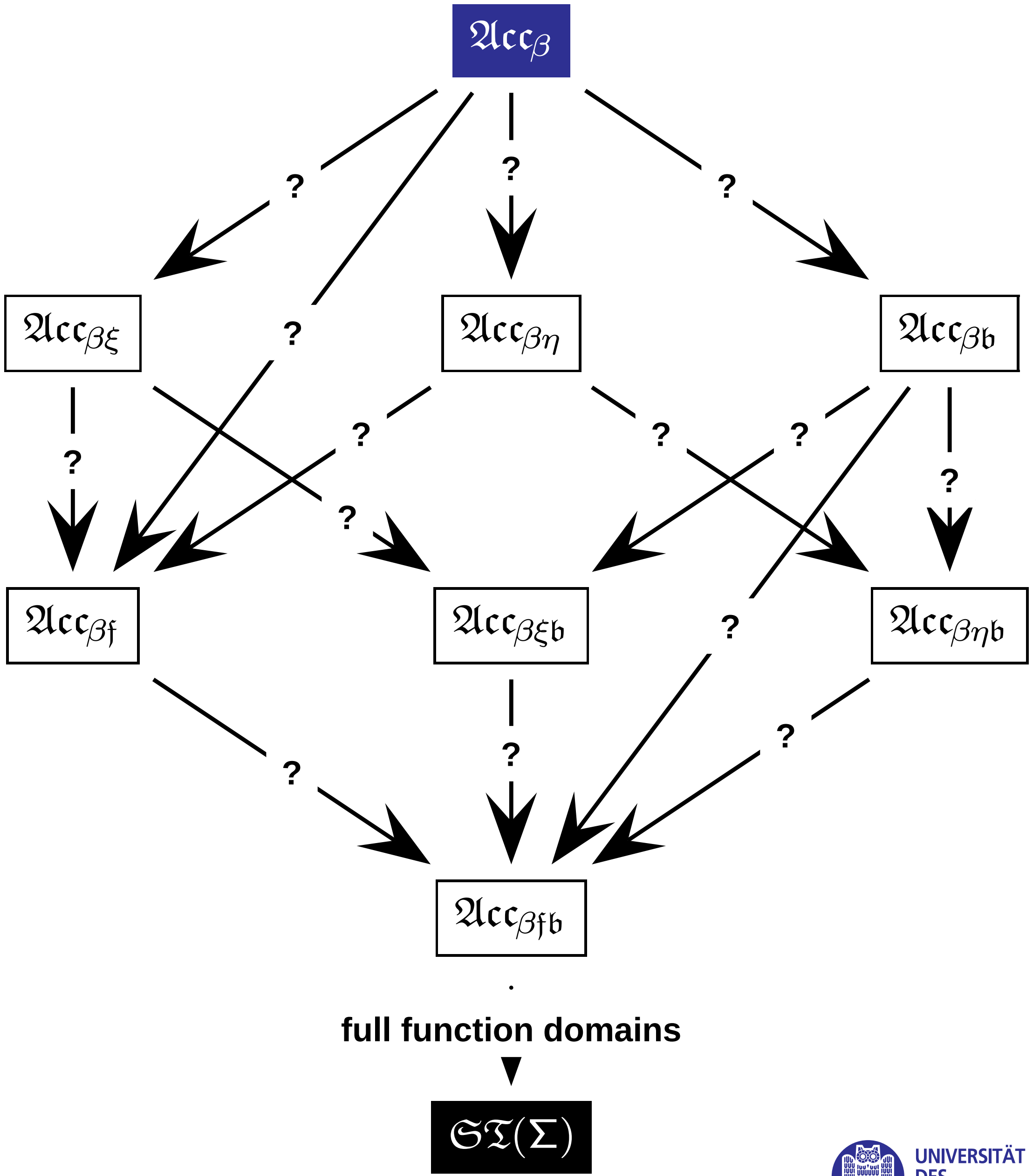
Abstract Consistency Proof Method



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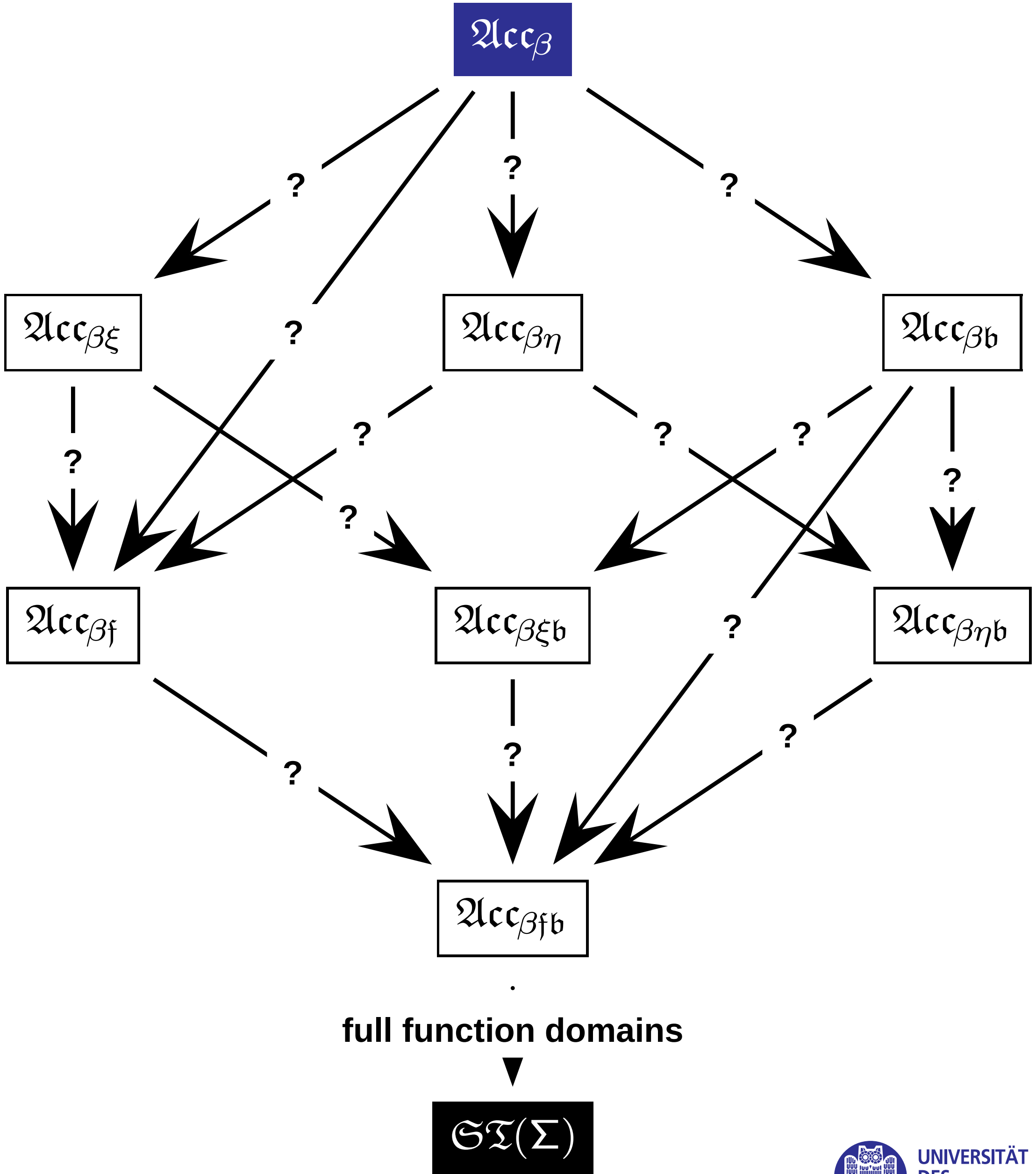


Properties for $\mathfrak{Acc}_\beta$: (Γ_Σ is class of sets of formulas; $\Phi \in \Gamma_\Sigma$)

∇_c If $\mathbf{A}$ is atomic, then $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$.

$\nabla_\neg$ If $\neg\neg\mathbf{A} \in \Phi$, then $\Phi,\mathbf{A} \in \Gamma_\Sigma$.

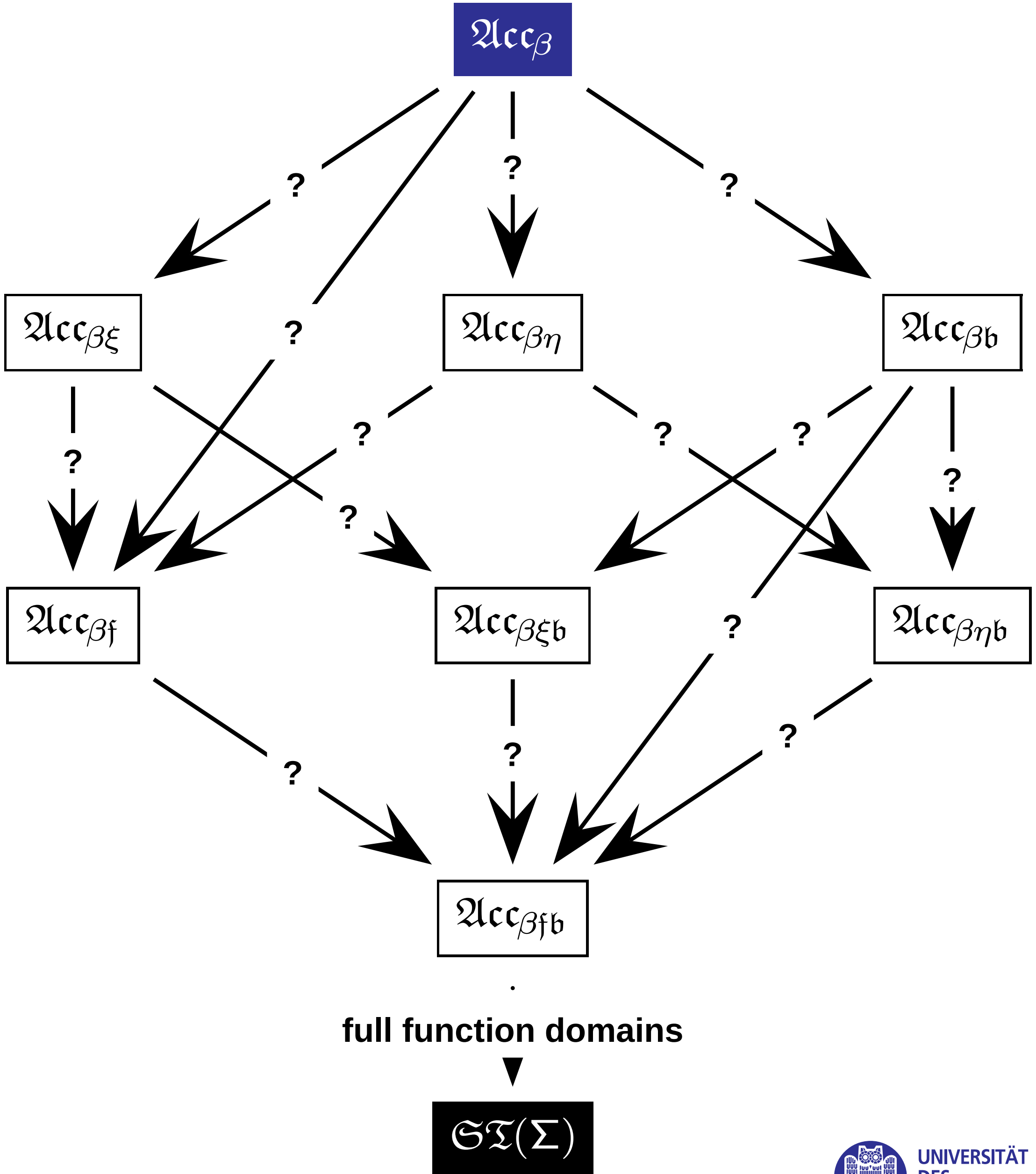
$\nabla_\vee$ If $\mathbf{A} \vee \mathbf{B} \in \Phi$, then $\Phi,\mathbf{A} \in \Gamma_\Sigma$ or $\Phi,\mathbf{B} \in \Gamma_\Sigma$.

$\nabla_\wedge$ If $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, then $\Phi,\neg\mathbf{A},\neg\mathbf{B} \in \Gamma_\Sigma$.

$\nabla_\forall$ If $\Pi^\alpha\mathbf{F} \in \Phi$, then $\Phi,\mathbf{F}\mathbf{W} \in \Gamma_\Sigma$ for each $\mathbf{W} \in cwff_\alpha(\Sigma)$.

$\nabla_\exists$ If $\neg\Pi^\alpha\mathbf{F} \in \Phi$, then $\Phi,\neg(\mathbf{F}w) \in \Gamma_\Sigma$ for any parameter $w_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

Abstract Consistency Proof Method



Properties for $\mathfrak{Acc}_\beta$: (Γ_Σ is class of sets of formulas; $\Phi \in \Gamma_\Sigma$)

- ∇_c If $\mathbf{A}$ is atomic, then $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$.
- $\nabla_\neg$ If $\neg\neg\mathbf{A} \in \Phi$, then $\Phi,\mathbf{A} \in \Gamma_\Sigma$.
- ∇_β If $\mathbf{A} =_\beta \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi,\mathbf{B} \in \Gamma_\Sigma$.
- $\nabla_\vee$ If $\mathbf{A} \vee \mathbf{B} \in \Phi$, then $\Phi,\mathbf{A} \in \Gamma_\Sigma$ or $\Phi,\mathbf{B} \in \Gamma_\Sigma$.
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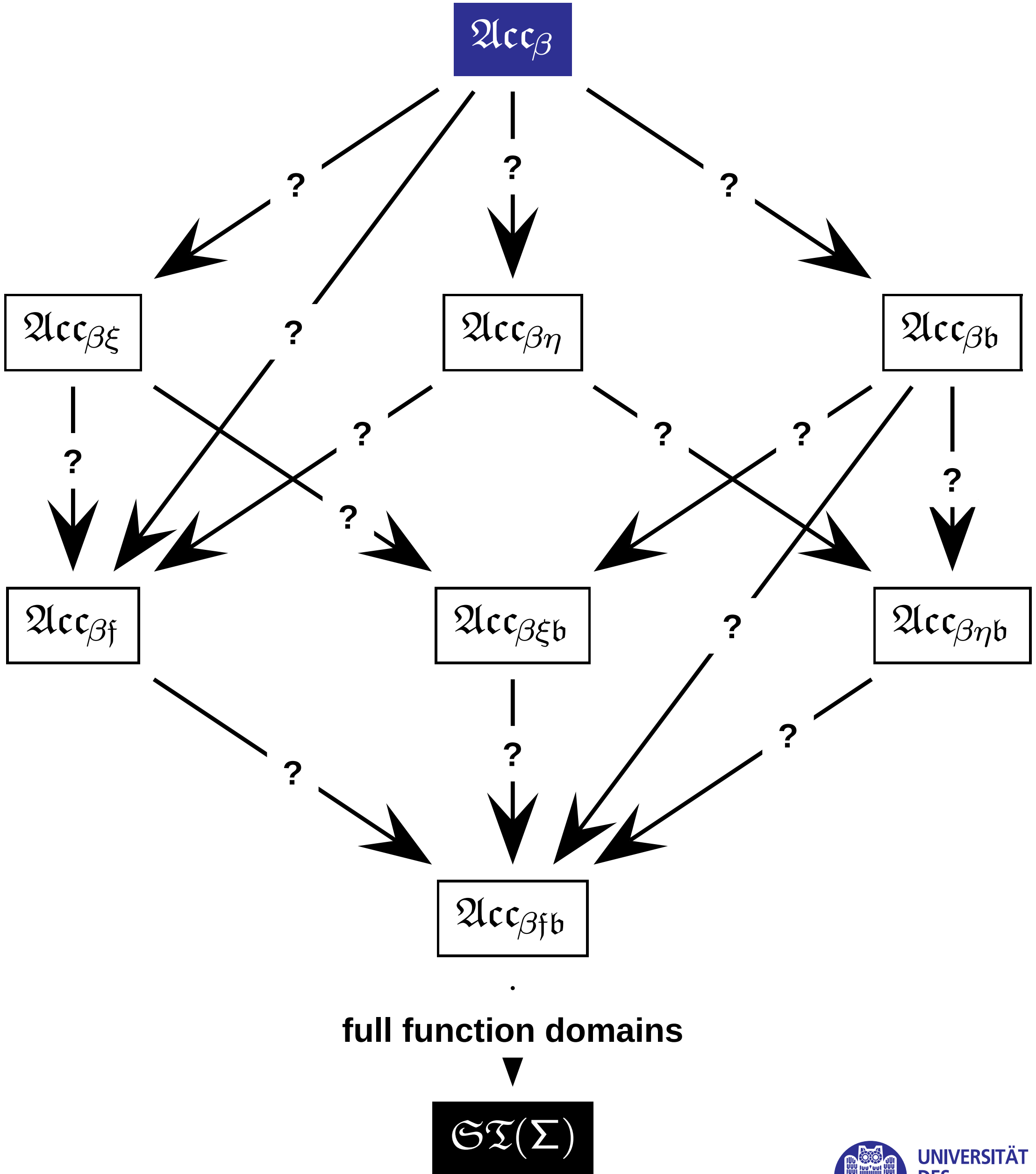
Abstract Consistency

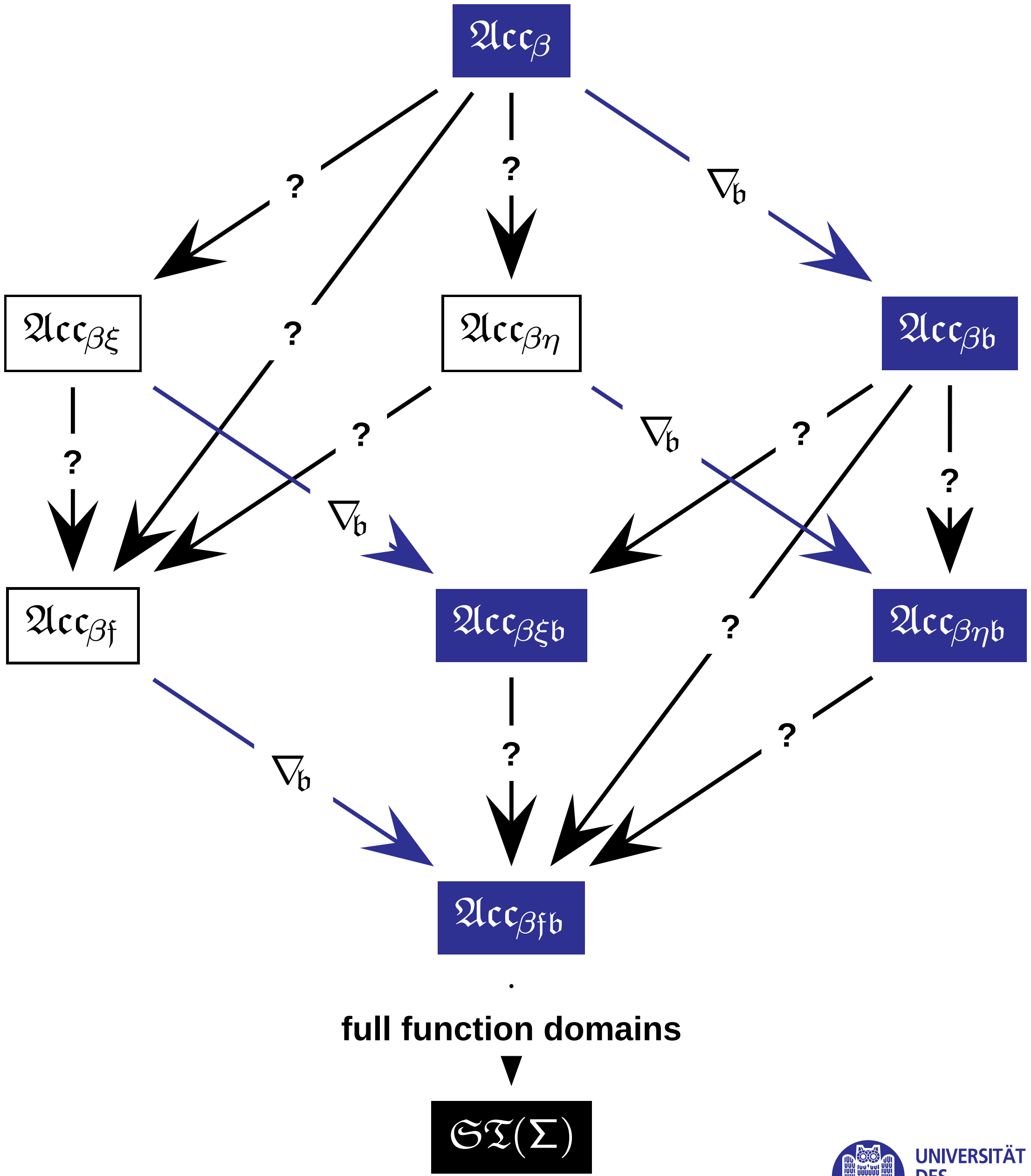


Properties for $\mathfrak{Acc}_\beta$

∇_c	...	$\nabla_\vee$	...
$\nabla_\neg$	...	$\nabla_\wedge$	...
∇_β	...	$\nabla_\forall$	...
		$\nabla_\exists$	...

Properties for Extensionality

Abstract Consistency



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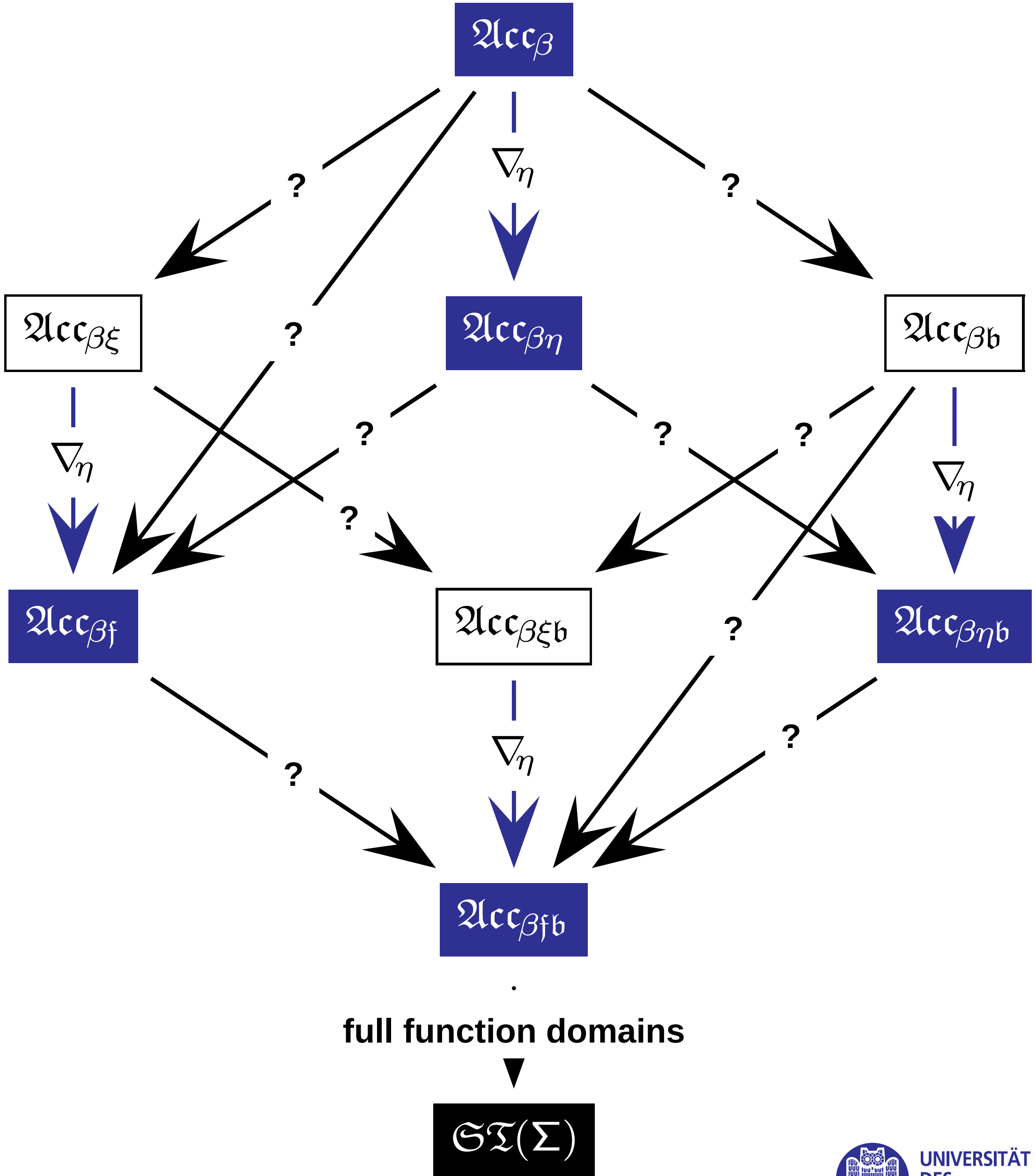

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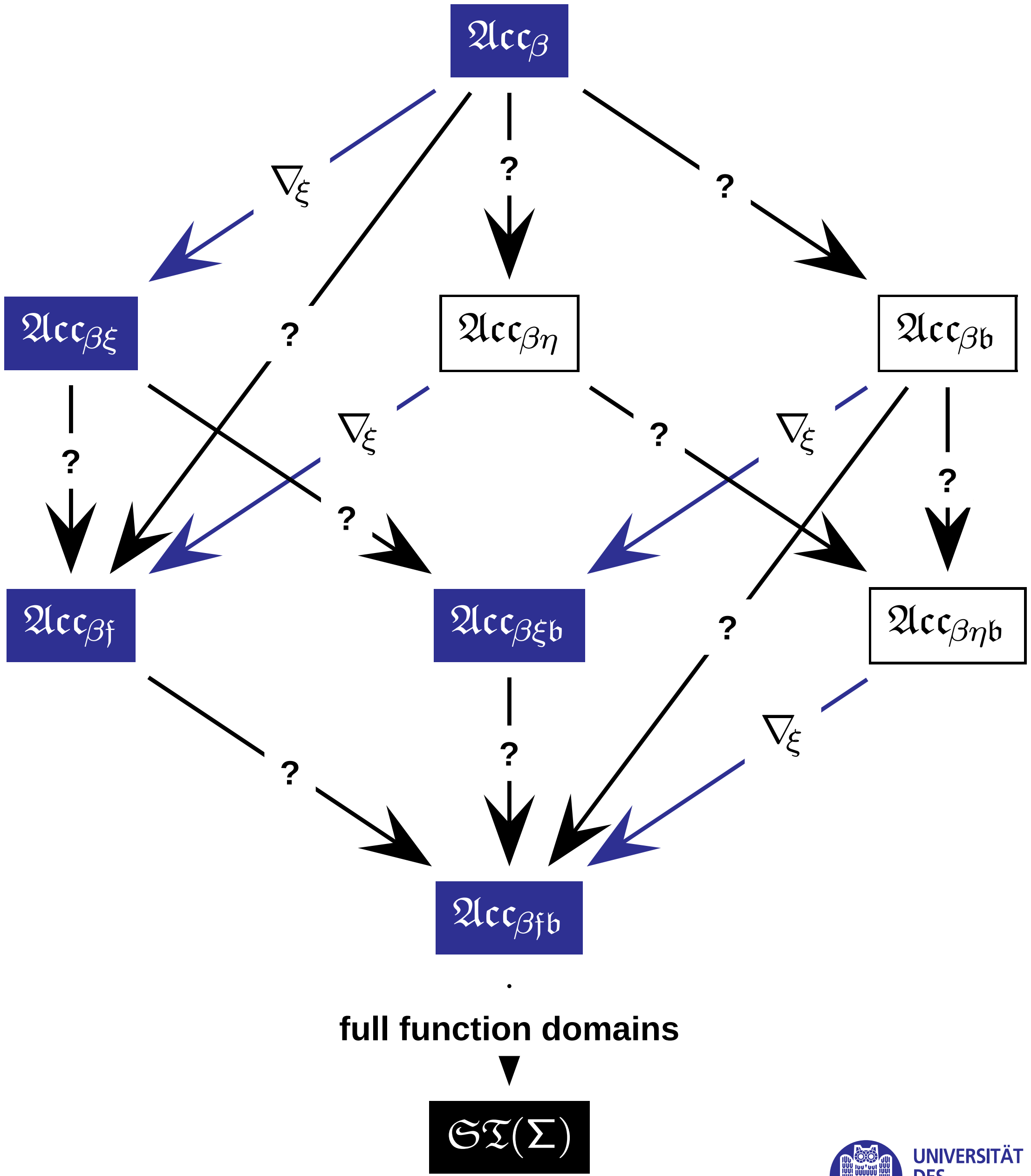

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∇_ξ If $\neg(\lambda X_\alpha.\mathbf{M} \doteq^{\alpha\to\beta} \lambda X_\alpha.\mathbf{N}) \in \Phi$, then $\Phi, \neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N}) \in \Gamma_\Sigma$ for any new $w_\alpha \in \Sigma_\alpha$.

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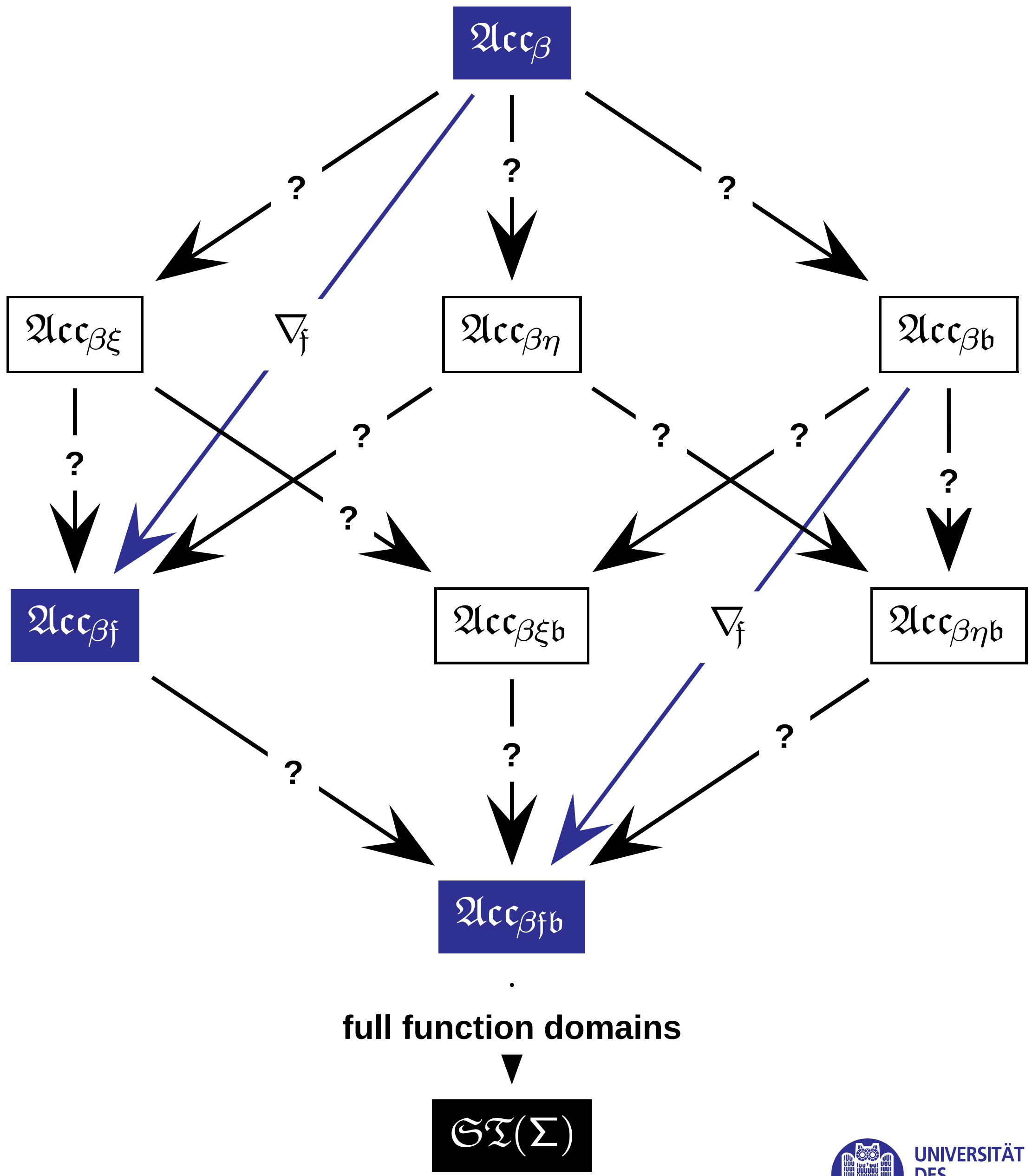

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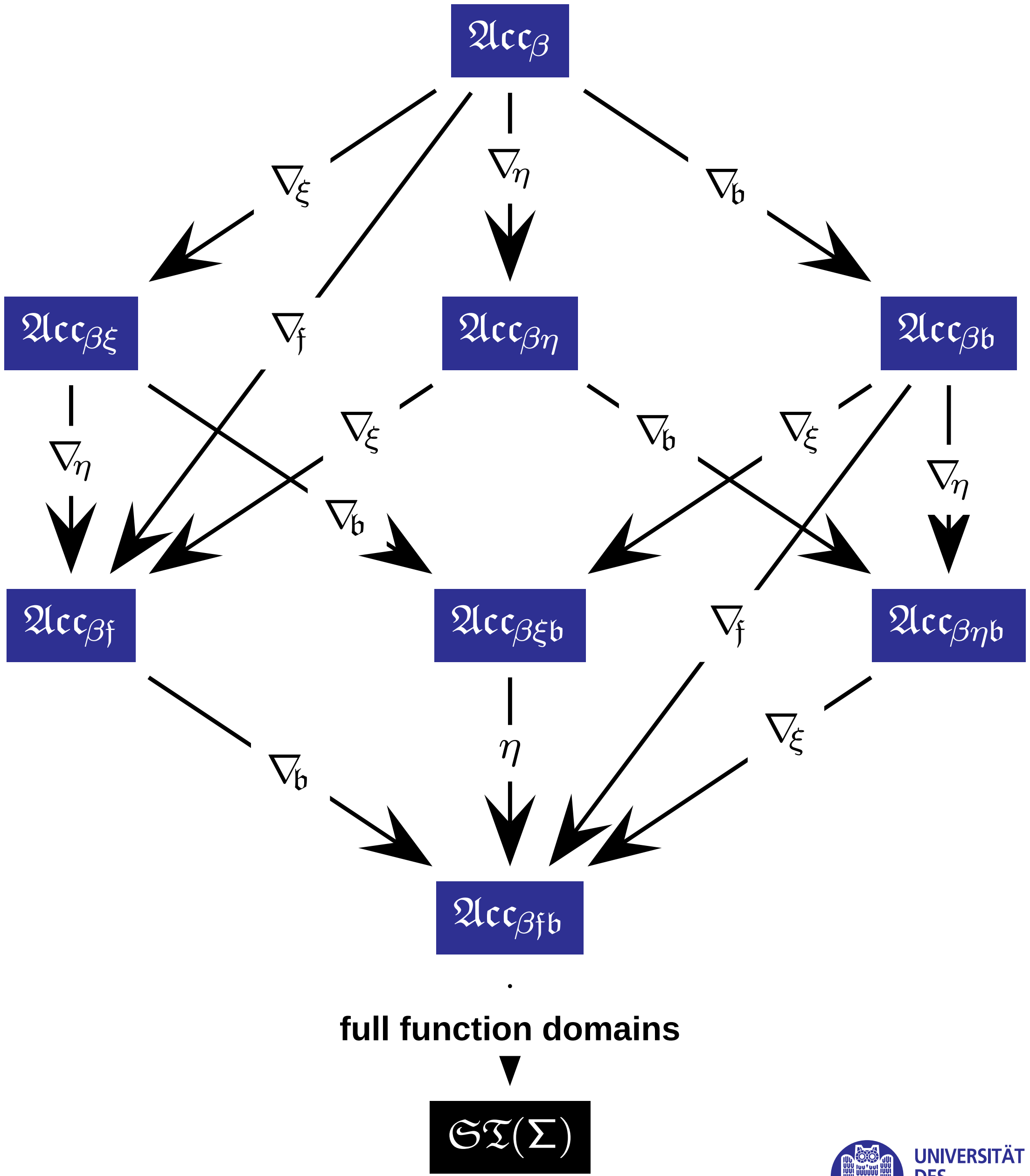

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Model Existence Theorem

Thm.: (Model Existence Theorem(s) for $\mathfrak{Acc}_*$)
If a class of sets of formulas Γ_Σ
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Cor.: (Cut-freeness of $\mathcal{G}_\beta$)
The cut rule $\mathcal{G}(cut)$ is admissible in $\mathcal{G}_\beta$.

k-Admissibility

Def.: We say a sequent calculus rule

$$\frac{\Delta_1 \quad \cdots \quad \Delta_n}{\Delta}\, r \text{ is admissible in } \mathcal{G}$$

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if elimination of any instance of r causes $\leq k$ additional steps.

Idea: k-admissible (or k-derivable) rules are effectively simulated by the calculus

Is $\mathcal{G}(cut)$ k-Admissible in $\mathcal{G}_\beta$?

Claim: $\mathcal{G}(cut)$ is not k-admissible (i.e. not effectively simulated) in $\mathcal{G}_\beta$ for any $\mathsf{k} \in \mathbb{N}$.

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Hence: Certain formulas allow for effective cut-simulation.

Thread to the suitability of a calculus for proof automation!

Cut-Simulation

Leibniz-equations support Cut-Simulation

Ex.: Available Leibniz-equations $\mathbf{M} \doteq^{\alpha} \mathbf{N}$ $(:= \forall \mathrm{P}_{o\alpha}. \neg \mathrm{P}\mathbf{M} \vee \mathrm{P}\mathbf{N}))$
support cut-simulation in $\mathcal{G}_{\beta}$ in only 3 steps.
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Proof:

$$\dfrac{\dfrac{\dfrac{\Delta, \mathbf{C}}{\Delta, \neg\neg\mathbf{C}}\ \mathcal{G}(\neg) \qquad \Delta, \neg\mathbf{C}}{\Delta, \neg(\neg\mathbf{C} \vee \mathbf{C})}\ \mathcal{G}(\vee_{-})}{\Delta' := \Delta, \neg\forall \mathrm{P}_{o\alpha} \centerdot \neg \mathrm{P}\mathbf{M} \vee \mathrm{P}\mathbf{N}}\ \mathcal{G}(\Pi_{-}^{\lambda X_{\alpha} \centerdot \mathbf{C}})$$

Extensionality Axioms

- The extensionality axioms are usually added to HOL systems (e.g. [Huet73]) to reach Henkin completeness.

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- Is adding these axioms a suitable option for proof automation?

Sequent Calculus $\mathcal{G}_\beta^{\mathsf{E}}$

Def.: (Sequent Calculus $\mathcal{G}_\beta^{\mathsf{E}}$)

We define $\mathcal{G}_\beta^{\mathsf{E}} := \mathcal{G}_\beta \cup \{\mathcal{G}(\mathcal{F}_{\alpha\beta}), \mathcal{G}(\mathcal{B})\}$

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$$\frac{\Delta, \neg\mathcal{F}_{\alpha\beta} \quad \alpha\beta \in \mathcal{T}}{\Delta}\ \mathcal{G}(\mathcal{F}_{\alpha\beta}) \qquad \frac{\Delta, \neg\mathcal{B}_o}{\Delta}\ \mathcal{G}(\mathcal{B})$$

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But: $\mathcal{G}_\beta^{\mathsf{E}}$ supports effective cut-simulation.

Ext. Axioms support Cut-Simulation

Ex.: The functional extensionality axioms support effective cut-simulation in $\mathcal{G}_\beta^{\mathsf{E}}$ in 12-steps.
($\mathcal{G}(cut)$ is 12-derivable, hence 12-admissible, in $\mathcal{G}_\beta^{\mathsf{E}}$)

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Proof:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\begin{array}{c}\text{3 steps; easy}\\ \vdots \\ \Delta, \mathsf{fa} \doteq^{\beta} \mathsf{fa}\end{array}}{\Delta, (\forall \mathsf{X}_\alpha \centerdot \mathsf{fX} \doteq^{\beta} \mathsf{fX})}\ \mathcal{G}(\Pi_+^{a_\alpha})
}{\Delta, \neg\neg\forall \mathsf{X}_\alpha \centerdot \mathsf{fX} \doteq^{\beta} \mathsf{fX}}\ \mathcal{G}(\neg)
\qquad
\begin{array}{c}\Delta, \mathbf{C} \quad \Delta, \neg\mathbf{C}\\ \vdots\ \text{3 steps; see before}\\ \Delta, \neg(\mathsf{f} \doteq^{\alpha\to\beta} \mathsf{f})\end{array}
}{\Delta, \neg(\neg(\forall \mathsf{X}_\alpha \centerdot \mathsf{fX} \doteq^{\beta} \mathsf{fX}) \vee \mathsf{f} \doteq^{\alpha\to\beta} \mathsf{f})}\ \mathcal{G}(\vee_-)
}{\Delta, \neg\mathcal{F}_{\alpha\beta}}\ 2 \times \mathcal{G}(\Pi_-^{f})
}{\Delta}\ \mathcal{G}(\mathcal{F}_{\alpha\beta})
$$

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Ex.: It also works with Boolean extensionality axiom – in 15 steps.

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Proof:

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \begin{array}{c}\text{7 steps; easy}\\ \vdots\\ \Delta, a \Leftrightarrow a\end{array}
      }{
        \Delta, \neg\neg(a \Leftrightarrow a)
      }\;\mathcal{G}(\neg)
      \qquad
      \begin{array}{c}\Delta, \mathbf{C} \quad \Delta, \neg\mathbf{C}\\ \vdots \;\text{3 steps; see before}\\ \Delta, \neg(a \doteq^{o} a)\end{array}
    }{
      \Delta, \neg(\neg(a \Leftrightarrow a) \lor a \doteq^{o} a)
    }\;\mathcal{G}(\lor_{-})
  }{
    \Delta, \neg\mathcal{B}_{o}
  }\;2 \times \mathcal{G}(\Pi^{a}_{-})
}{
  \Delta
}\;\mathcal{G}(\mathcal{B})
$$

Cut-Simulation with other Formulas

- Reflexivity definition of equality (Andrews) 4 steps

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Cut-Simulation with other Formulas

Reflexivity definition of equality (Andrews)	4 steps
Instances of Comprehension axioms	16 steps
Axiom of Induction	18 steps
Axiom of Choice	7 steps
Axiom of Description	25 steps

Cut-Simulation with other Formulas

- Reflexivity definition of equality (Andrews) 4 steps
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$$\frac{\Delta, (\forall \mathsf{X}_\alpha . \mathbf{F}\mathsf{X} \doteq^{\beta} \mathbf{G}\mathsf{X}){\downarrow_\beta}}{\Delta, (\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G})}\ \mathcal{G}(\mathfrak{f}) \qquad \frac{\Delta, \neg\mathbf{A}, \mathbf{B} \quad \Delta, \neg\mathbf{B}, \mathbf{A}}{\Delta, (\mathbf{A} \doteq^{o} \mathbf{B})}\ \mathcal{G}(\mathfrak{b})$$

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Claim: $\mathcal{G}_{\beta\mathfrak{f}\mathfrak{b}}$ does not support effective cut-simulation

Abstract Cut-Elimination Result

The rules $\mathcal{G}(Init^{\dot{=}}), \mathcal{G}(d)$ motivate corresponding

abstract consistency conditions for Henkin semantics

∇_m If $\mathbf{A}, \mathbf{B} \in cwff_o(\Sigma)$ are atomic and $\mathbf{A}, \neg\mathbf{B} \in \Phi$, then $\Phi * \neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Gamma_\Sigma$.

∇_d If $\neg(h\overline{\mathbf{A}^n} \doteq^\beta h\overline{\mathbf{B}^n}) \in \Phi$ for some types α_i where $\beta \in \{o, \iota\}$ and $h_{\overline{\alpha^n} \to \beta} \in \Sigma$ is a parameter, then there is an i ($1 \leq i \leq n$) such that $\Phi * \neg(\mathbf{A}^i \doteq^{\alpha^i} \mathbf{B}^i) \in \Gamma_\Sigma$.

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- Further work:
  - . . . research is only at its very beginning . . .