Quantified Conditional Logics are Fragments of HOL

This work extends

[BenzmüllerGenoveseGabbayRispoli, submitted (arXiv:1106.3685v3)]

[BenzmüllerPaulson, Logica Universalis, to appear (arXiv:0905.2435v1)]

Quantified Conditional Logics – Motivation

Theory for (Reasoning with) Counterfactual Conditionals

If I had continued with competitive long-distance running in 1992,
I would have won the Olympic Games in 2000.

Problem: non-truth-functionality of counterfactual conditional statements

Solution (Stalnaker and Thomason)

- selection function semantics (a possible world semantics, extension of modal logics) [Stalnaker68]

$$\underbrace{\text{'If } A \text{ then } B\text{'}}_{(A \Rightarrow B)} \textit{ is true in world } w \quad \textit{iff} \quad B \textit{ is true for all } v \in f(w, A)$$

- idea: f selects worlds that are very *similar/close* to the actual world w
- many closely related theories: [Lewis73, Pollock76, Chellas75]

Quantified Conditional Logic – Syntax

$$\varphi, \psi ::= P \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi$$

Quantified Conditional Logic – Syntax

Propositional Variables (PV) Individual Variables (IV) Constants (Sym)

$$\varphi, \psi ::= P \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi \mid \forall P.\varphi \mid \forall X.\varphi \mid k(X^1, \ldots, X^n)$$

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Logical Connectives and Quantifiers (others may be defined as usual)

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Conditional (modal) operator

Quantified Conditional Logic – Semantics

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Interpretation

- is a structure $M = \langle S, f, D, Q, I \rangle$ with
  - S set of possible worlds
  - $f : S \times 2^S \mapsto 2^S$ is the selection function
  - D is a non-empty set of individuals (the first-order domain)
  - Q is a non-empty collection of subsets of S (the propositional domain)
  - I is a classical interpretation function where for each n-ary predicate symbol k, $I(k, w) \subseteq D^n$

Variable Assignment

- $g = \langle g^{iv}, g^{pv} \rangle$
  - $g^{iv} : IV \mapsto D$ maps individual variables to objects in D
  - $g^{pv} : PV \mapsto Q$ maps propositional variables to sets of worlds in Q

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Satisfiability $M, g, s \models \varphi$ defined as:

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Validity

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Quantified Conditional Logic – Normality

Above semantics of $\Rightarrow$ enforces normality property:

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The axiomatic counterpart of the normality condition given by rule (RCEA)

$$\frac{\varphi \leftrightarrow \varphi'}{(\varphi \Rightarrow \psi) \leftrightarrow (\varphi' \Rightarrow \psi)}\,(RCEA)$$

Above semantics forces also the following rules to hold:

$$\frac{(\varphi_1 \wedge \ldots \wedge \varphi_n) \leftrightarrow \psi}{(\varphi_0 \Rightarrow \varphi_1 \wedge \ldots \wedge \varphi_0 \Rightarrow \varphi_n) \rightarrow (\varphi_0 \Rightarrow \psi)}\,(RCK) \qquad \frac{\varphi \leftrightarrow \varphi'}{(\psi \Rightarrow \varphi) \leftrightarrow (\psi \Rightarrow \varphi')}\,(RCEC)$$

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Logic CK: minimal logic closed under rules RCEA, RCEC and RCK.
In what follows only logic CK and its extensions are considered.

Quantified Conditional Logics as Fragments of HOL

Kripke style semantics (higher-order) selection function!

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\begin{array}{lll}
M, g, s \models P & \text{iff} & s \in g(P) \\
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M, g, s \models \forall X_\centerdot \varphi & \text{iff} & M, [d/X]g, s \models \varphi \text{ for all } d \in D \\
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Semantic embedding: ML $\longrightarrow$ HOL terms of type $\iota \to o$

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\vee &= \lambda \varphi_{\iota\to o} . \lambda \psi_{\iota\to o} . \lambda W_\iota . (\varphi\, W) \vee (\psi\, W) \\
\Rightarrow &= \lambda \varphi_{\iota\to o} . \lambda \psi_{\iota\to o} . \lambda W_\iota . \forall V_\iota . \neg(f\, W\, \varphi\, V) \vee (\psi\, V) \\
\forall^\mu(\Pi^\mu) &= \lambda Q_{\mu\to(\iota\to o)} . \lambda W_\iota . \forall X_\mu . (Q\, X\, W) \\
\forall^p(\Pi^p) &= \lambda Q_{(\iota\to o)\to(\iota\to o)} . \lambda W_\iota . \forall P_{\iota\to o} . (Q\, P\, W)
\end{aligned}
$$

Kripke style semantics (higher-order) selection function!

$$
\begin{array}{lll}
M,g,s \models P & \text{iff} & s \in g(P) \\
M,g,s \models k(X^1,\ldots,X^n) & \text{iff} & s \in \langle g(X^1),\ldots,g(X^n)\rangle \in I(k,w) \\
M,g,s \models \neg\varphi & \text{iff} & \text{not } M,g,s \models \varphi \\
M,g,s \models \varphi \vee \psi & \text{iff} & M,g,s \models \varphi \text{ or } M,g,s \models \psi \\
M,g,s \models \varphi \Rightarrow \psi & \text{iff} & M,g,v \models \psi \text{ for all } v \in f(s,\overbrace{\{u \mid M,g,u \models \varphi\}}^{[\varphi]}) \\
M,g,s \models \forall X.\varphi & \text{iff} & M,[d/X]g,s \models \varphi \text{ for all } d \in D \\
M,g,s \models \forall P.\varphi & \text{iff} & M,[p/P]g,s \models \varphi \text{ for all } p \in Q
\end{array}
$$

Semantic embedding: ML $\longrightarrow$ HOL terms of type $\iota \to o$

$$
\begin{array}{rcl}
P & = & \lambda W_\iota.(P_{\iota\to o}\, W) = P_{\iota\to o} \\
k(X^1,\ldots,X^n) & = & \lambda W_\iota.(k_{\mu^n\to(\iota\to o)}\, X^1_\mu \ldots X^n_\mu)\, W \\
\neg & = & \lambda\varphi_{\iota\to o}.\lambda W_\iota.\neg(\varphi\, W) \\
\vee & = & \lambda\varphi_{\iota\to o}.\lambda\psi_{\iota\to o}.\lambda W_\iota.(\varphi\, W) \vee (\psi\, W) \\
\Rightarrow & = & \lambda\varphi_{\iota\to o}.\lambda\psi_{\iota\to o}.\lambda W_\iota.\forall V_\iota.\neg(f\, W\, \varphi\, V) \vee (\psi\, V) \\
\forall^\mu(\Pi^\mu) & = & \lambda Q_{\mu\to(\iota\to o)}.\lambda W_\iota.\forall X_\mu.(Q\, X\, W) \\
\forall^p(\Pi^p) & = & \lambda Q_{(\iota\to o)\to(\iota\to o)}.\lambda W_\iota.\forall P_{\iota\to o}.(Q\, P\, W)
\end{array}
$$

Soundness and Completeness

Validity defined as before

$$\text{valid} \quad = \quad \lambda\varphi_{\iota\to o}\boldsymbol{.}\forall W_{\iota}\boldsymbol{.}\,\varphi\, W$$

Soundness and Completeness Theorem

$$\models^{QCL} \varphi \quad \text{iff} \quad \models^{HOL} \text{valid}\, \varphi_{\iota\to o}$$

Proof Idea:

Explicate and analyze the relation between selection functions semantics and corresponding Henkin models; see paper for details.

For Propositional Conditional Logics see

[BenzmüllerGenoveseGabbayRispoli, submitted (arXiv:1106.3685v3)]

For Quantified Multimodal Logics see

[BenzmüllerPaulson, Logica Universalis, to appear (arXiv:0905.2435v1)]

Instances of (Converse) Barcan Formula:

$$\text{valid } \forall^{*}x(\varphi \Rightarrow \psi(x)) \rightarrow (\varphi \Rightarrow \forall^{*}x\psi(x)) \qquad \text{(BF)}$$

$$\text{valid } (\varphi \Rightarrow \forall^{*}x\psi(x)) \rightarrow \forall^{*}x(\varphi \Rightarrow \psi(x)) \qquad \text{(CBF)}$$

BF:

if $*$ = varying domain then HOL-P: CounterSatisfiable
if $*$ = constant domain then HOL-P: Theorem

CBF:

if $*$ = varying domain then HOL-P: CounterSatisfiable
if $*$ = constant domain then HOL-P: Theorem

Automating Quantified Conditional Logics

The following examples are taken from [Delgrande, Artif.Intell., 1998]

$\phi \Rightarrow_x \psi$ stands for $(\exists^{va} x \phi) \Rightarrow \forall^{va} x(\phi \rightarrow \psi)$

"Birds (b) normally fly (f), but Opus (o) is a bird that normally does not fly."

$$b(x) \Rightarrow_x f(x), \quad b(o), \quad b(o) \Rightarrow \neg f(o)$$

HOL-P: Satisfiable (constant domain HOL-P: Unsatisfiable)

"Birds normally fly and necessarily Opus the bird does not fly."

$$b(x) \Rightarrow_x f(x), \quad \Box(b(o) \wedge \neg f(o))$$

HOL-P: Satisfiable (constant domain HOL-P: Unsatisfiable)

"Birds normally fly, penguins normally do not fly and all penguins are necessarily birds."

$$b(x) \Rightarrow_x f(x), \quad p(x) \Rightarrow_x \neg f(x), \quad \forall^{va} \Box(p(x) \rightarrow b(x))$$

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for more see [Benzmüller, IJCAI, 2013]

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Freie Universität Berlin

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