Completeness (of $\mathfrak{N}\mathfrak{K}_*$)

Completeness (of $\mathfrak{NK}_*$)

- How to prove completeness?

Completeness (of $\mathfrak{NK}_*$)

- How to prove completeness?
- Completeness can be proven rather easily for propositional logic calculi.

Completeness (of $\mathfrak{NK}_*$)

- How to prove completeness?
- Completeness can be proven rather easily for propositional logic calculi.
- For first-order and especially higher-order logic completeness proofs become increasingly difficult and technical.

Completeness (of $\mathfrak{NK}_*$)

- How to prove completeness?
- Completeness can be proven rather easily for propositional logic calculi.
- For first-order and especially higher-order logic completeness proofs become increasingly difficult and technical.
- Here we will introduce a strong proof tool that uniformly supports completeness proofs (and many other things): abstract consistency.

Completeness (of $\mathfrak{NK}_*$)

- How to prove completeness?
- Completeness can be proven rather easily for propositional logic calculi.
- For first-order and especially higher-order logic completeness proofs become increasingly difficult and technical.
- Here we will introduce a strong proof tool that uniformly supports completeness proofs (and many other things): abstract consistency.
- This proof tool is based on a strong theorem which connects syntax and semantics: model existence theorem.

Abstract Consistency

Abstract Consistency: History

- Technique was developed for first-order logic by Jaakko Hintikka and Raymond Smullyan [Hintikka55,Smullyan63,Smullyan68]. It is well explained in Fitting's textbook [Fitting96].

Abstract Consistency: History

- Technique was developed for first-order logic by Jaakko Hintikka and Raymond Smullyan [Hintikka55,Smullyan63,Smullyan68]. It is well explained in Fitting's textbook [Fitting96].
- The technidue has been (partly) extended to higher-order logic by Peter Andrews' in [Andrews71]; Peter Andrews only achieves a generalization for his rather weak semantical v-complexes (corresponding to our $\mathfrak{M}_\beta(\Sigma)$) and not, for instance, for Henkin Semantics. This extension is well explained in Peter Andrews's textbook [Andrews02].

Abstract Consistency: History

- Technique was developed for first-order logic by Jaakko Hintikka and Raymond Smullyan [Hintikka55,Smullyan63,Smullyan68]. It is well explained in Fitting's textbook [Fitting96].
- The technidue has been (partly) extended to higher-order logic by Peter Andrews' in [Andrews71]; Peter Andrews only achieves a generalization for his rather weak semantical v-complexes (corresponding to our $\mathfrak{M}_\beta(\Sigma)$) and not, for instance, for Henkin Semantics. This extension is well explained in Peter Andrews's textbook [Andrews02].
- The technique has been extended to our landscape of HOL model classes in [Benzmueller-PhD-99,JSL04].

Abstract Consistency: Idea

- A model existence theorem for a logical system (i.e., a logical language L together with a consequence relation $\models$) is a theorem of the form:

Abstract Consistency: Idea

- A model existence theorem for a logical system (i.e., a logical language L together with a consequence relation $\models$) is a theorem of the form:

  If a set of sentences Φ of L is a member of an (saturated) abstract consistency class Γ, then there exists a model for Φ.

Abstract Consistency: Idea

- Employing the model existence theorem we can prove completeness of a calculus $\mathcal{C}$ (i.e., the derivability rel. $\vdash_{\mathcal{C}}$) by

Abstract Consistency: Idea

- Employing the model existence theorem we can prove completeness of a calculus $\mathcal{C}$ (i.e., the derivability rel. $\vdash_{\mathcal{C}}$) by

  proving that the class Γ of sets of sentences Φ that are $\mathcal{C}$-consistent (i.e., cannot be refuted in $\mathcal{C}$) is an (saturated) abstract consistency class.

Abstract Consistency: Idea

- Employing the model existence theorem we can prove completeness of a calculus $\mathcal{C}$ (i.e., the derivability rel. $\vdash_{\mathcal{C}}$) by

  proving that the class Γ of sets of sentences Φ that are $\mathcal{C}$-consistent (i.e., cannot be refuted in $\mathcal{C}$) is an (saturated) abstract consistency class.

- Why does this work?

Abstract Consistency: Idea

- Employing the model existence theorem we can prove completeness of a calculus $\mathcal{C}$ (i.e., the derivability rel. $\vdash_{\mathcal{C}}$) by

  proving that the class Γ of sets of sentences Φ that are $\mathcal{C}$-consistent (i.e., cannot be refuted in $\mathcal{C}$) is an (saturated) abstract consistency class.

- Why does this work?
  - The model existence theorem tells us that $\mathcal{C}$-consistent sets of sentences are satisfiable.

Abstract Consistency: Idea

- Employing the model existence theorem we can prove completeness of a calculus $\mathcal{C}$ (i.e., the derivability rel. $\vdash_{\mathcal{C}}$) by

  proving that the class Γ of sets of sentences Φ that are $\mathcal{C}$-consistent (i.e., cannot be refuted in $\mathcal{C}$) is an (saturated) abstract consistency class.

- Why does this work?
  - The model existence theorem tells us that $\mathcal{C}$-consistent sets of sentences are satisfiable.
  - Now we assume that a sentence $\mathbf{A}$ is valid, so $\neg\mathbf{A}$ does not have a model and is therefore $\mathcal{C}$-inconsistent.

Abstract Consistency: Idea

- Employing the model existence theorem we can prove completeness of a calculus C (i.e., the derivability rel. $\vdash_C$) by

  proving that the class Γ of sets of sentences Φ that are C-consistent (i.e., cannot be refuted in C) is an (saturated) abstract consistency class.

- Why does this work?
  - The model existence theorem tells us that C-consistent sets of sentences are satisfiable.
  - Now we assume that a sentence A is valid, so $\neg A$ does not have a model and is therefore C-inconsistent.
  - Hence, $\neg A$ is refutable in C.

Abstract Consistency: Idea

- Employing the model existence theorem we can prove completeness of a calculus $\mathcal{C}$ (i.e., the derivability rel. $\vdash_{\mathcal{C}}$) by

  proving that the class Γ of sets of sentences Φ that are $\mathcal{C}$-consistent (i.e., cannot be refuted in $\mathcal{C}$) is an (saturated) abstract consistency class.

- Why does this work?
  - The model existence theorem tells us that $\mathcal{C}$-consistent sets of sentences are satisfiable.
  - Now we assume that a sentence $\mathbf{A}$ is valid, so $\neg\mathbf{A}$ does not have a model and is therefore $\mathcal{C}$-inconsistent.
  - Hence, $\neg\mathbf{A}$ is refutable in $\mathcal{C}$.
  - This shows refutation completeness of $\mathcal{C}$.

Abstract Consistency: Idea

- Employing the model existence theorem we can prove completeness of a calculus $\mathcal{C}$ (i.e., the derivability rel. $\vdash_{\mathcal{C}}$) by

  proving that the class Γ of sets of sentences Φ that are $\mathcal{C}$-consistent (i.e., cannot be refuted in $\mathcal{C}$) is an (saturated) abstract consistency class.

- Why does this work?
  - The model existence theorem tells us that $\mathcal{C}$-consistent sets of sentences are satisfiable.
  - Now we assume that a sentence $\mathbf{A}$ is valid, so $\neg\mathbf{A}$ does not have a model and is therefore $\mathcal{C}$-inconsistent.
  - Hence, $\neg\mathbf{A}$ is refutable in $\mathcal{C}$.
  - This shows refutation completeness of $\mathcal{C}$.
  - For many calculi $\mathcal{C}$, this also shows $\mathbf{A}$ is provable, thus establishing completeness of $\mathcal{C}$.

Def.: Closed under Subsets / Compact

Let $\mathcal{C}$ be a class of sets then $\mathcal{C}$ is called closed under subset if for all sets $\mathcal{S}$ and $\mathcal{T}$ it holds that

Def.: Closed under Subsets / Compact

Let C be a class of sets then C is called closed under subset if for all sets S and T it holds that

from $S \subseteq T$ and $T \in C$ it follows that $S \in C$.

Def.: Closed under Subsets / Compact

Let C be a class of sets then C is called closed under subset if for all sets S and T it holds that

from $S \subseteq T$ and $T \in C$ it follows that $S \in C$.

Let C be a class of sets. C is called compact or of finite character if and only if for every set S holds:

Def.: Closed under Subsets / Compact

Let C be a class of sets then C is called closed under subset if for all sets S and T it holds that

> from $S \subseteq T$ and $T \in C$ it follows that $S \in C$.

Let C be a class of sets. C is called compact or of finite character if and only if for every set S holds:

> $S \in C$ if and only if every finite subset of S is a member of C.

Ex.: Closed under Subsets / Compact

- not closed under subsets: $\{\{\neg(A \vee B), \neg A, C\}, \{\neg A\}\}$

Ex.: Closed under Subsets / Compact

- not closed under subsets: $\{\{\neg(A \lor B), \neg A, C\}, \{\neg A\}\}$
- closed under subsets: $\{\{\neg(A \lor B), \neg A, C\}, \{\neg(A \lor B), \neg A\}, \{\neg(A \lor B), C\}, \{\neg A, C\}, \{\neg(A \lor B)\}, \{\neg A\}, \{C\}, \{\}\}$

Ex.: Closed under Subsets / Compact

- not closed under subsets: $\{\{\neg(A \vee B), \neg A, C\}, \{\neg A\}\}$
- closed under subsets: $\{\{\neg(A \vee B), \neg A, C\}, \{\neg(A \vee B), \neg A\}, \{\neg(A \vee B), C\}, \{\neg A, C\}, \{\neg(A \vee B)\}, \{\neg A\}, \{C\}, \{\}\}$
- We define two classes of sets

Ex.: Closed under Subsets / Compact

- not closed under subsets: $\{\{\neg(A \vee B), \neg A, C\}, \{\neg A\}\}$
- closed under subsets: $\{\{\neg(A \vee B), \neg A, C\}, \{\neg(A \vee B), \neg A\}, \{\neg(A \vee B), C\}, \{\neg A, C\}, \{\neg(A \vee B)\}, \{\neg A\}, \{C\}, \{\}\}$
- We define two classes of sets
  - $C := \{\varphi \mid \varphi$ is finite subset of $\mathbb{N}\,\}$

Ex.: Closed under Subsets / Compact

- not closed under subsets: $\{\{\neg(A \lor B), \neg A, C\}, \{\neg A\}\}$
- closed under subsets: $\{\{\neg(A \lor B), \neg A, C\}, \{\neg(A \lor B), \neg A\}, \{\neg(A \lor B), C\}, \{\neg A, C\}, \{\neg(A \lor B)\}, \{\neg A\}, \{C\}, \{\}\}$
- We define two classes of sets
  - $C := \{\varphi \mid \varphi$ is finite subset of $\mathbb{N}\,\}$
  - $D := 2^{\mathbb{N}}$

Ex.: Closed under Subsets / Compact

- not closed under subsets: $\{\{\neg(A \vee B), \neg A, C\}, \{\neg A\}\}$
- closed under subsets: $\{\{\neg(A \vee B), \neg A, C\}, \{\neg(A \vee B), \neg A\}, \{\neg(A \vee B), C\}, \{\neg A, C\}, \{\neg(A \vee B)\}, \{\neg A\}, \{C\}, \{\}\}$
- We define two classes of sets
  - $C := \{\varphi \mid \varphi$ is finite subset of $\mathbb{N}\}$
  - $D := 2^{\mathbb{N}}$
  - C is closed under subsets but not compact.

Ex.: Closed under Subsets / Compact

- not closed under subsets: $\{\{\neg(A \vee B), \neg A, C\}, \{\neg A\}\}$
- closed under subsets: $\{\{\neg(A \vee B), \neg A, C\}, \{\neg(A \vee B), \neg A\}, \{\neg(A \vee B), C\}, \{\neg A, C\}, \{\neg(A \vee B)\}, \{\neg A\}, \{C\}, \{\}\}$
- We define two classes of sets
  - $C := \{\varphi \mid \varphi$ is finite subset of $\mathbb{N}\}$
  - $D := 2^{\mathbb{N}}$
  - C is closed under subsets but not compact.
  - D is closed under subsets and compact.

Lemma: Closed under Subsets / Compact

Lemma:

If C is compact then C is closed under subsets.

Lemma: Closed under Subsets / Compact

Lemma:

If $\mathcal{C}$ is compact then $\mathcal{C}$ is closed under subsets.

Proof:

Lemma: Closed under Subsets / Compact

Lemma:

If C is compact then C is closed under subsets.

Proof:

Let $T \in C$ and $S \subseteq T$.

Lemma: Closed under Subsets / Compact

Lemma:

If C is compact then C is closed under subsets.

Proof:

Let $T \in C$ and $S \subseteq T$.
We have to show that $S \in C$.

Lemma: Closed under Subsets / Compact

Lemma:

If C is compact then C is closed under subsets.

Proof:

Let $T \in C$ and $S \subseteq T$.
We have to show that $S \in C$.
Every finite subset A of S is also a finite subset of T.

Lemma: Closed under Subsets / Compact

Lemma:

If C is compact then C is closed under subsets.

Proof:

Let $T \in C$ and $S \subseteq T$.
We have to show that $S \in C$.
Every finite subset A of S is also a finite subset of T.
Since C is compact and $T \in C$ we get that all $A \in C$.

Lemma: Closed under Subsets / Compact

Lemma:

If C is compact then C is closed under subsets.

Proof:

Let $T \in C$ and $S \subseteq T$.
We have to show that $S \in C$.
Every finite subset A of S is also a finite subset of T.
Since C is compact and $T \in C$ we get that all $A \in C$.
Thus, $S \in C$ by compactness.

Def.: Sufficiently Σ-Pure

We introduce a technical side-condition that ensures that we always have enough witness constants.

Def.: Sufficiently Σ-Pure

We introduce a technical side-condition that ensures that we always have enough witness constants.

> Let Σ be a signature and Φ be a set of Σ-sentences. Φ is called sufficiently Σ-pure if for each type α there is a set $\mathcal{P}_\alpha \subseteq \Sigma_\alpha$ of parameters with equal cardinality to $\mathit{wff}_\alpha(\Sigma)$, such that the elements of $\mathcal{P}_\alpha$ do not occur in the sentences of Φ.

Def.: Sufficiently Σ-Pure

We introduce a technical side-condition that ensures that we always have enough witness constants.

> Let Σ be a signature and Φ be a set of Σ-sentences. Φ is called sufficiently Σ-pure if for each type α there is a set $\mathcal{P}_\alpha \subseteq \Sigma_\alpha$ of parameters with equal cardinality to $\mathit{wff}_\alpha(\Sigma)$, such that the elements of $\mathcal{P}_\alpha$ do not occur in the sentences of Φ.

This can be obtained in practice by enriching the signature with spurious parameters.

Abstract Consistency: Conventions

Remember the conventions for this part of the lecture:

- signature Σ contains only the logical constants $\neg, \lor, \Pi^{\alpha}$ unless stated otherwise

Abstract Consistency: Conventions

Remember the conventions for this part of the lecture:

- signature Σ contains only the logical constants $\neg, \vee, \Pi^\alpha$ unless stated otherwise
- as a matter of convenience we will write $\varphi * \mathrm{A}$ for $\varphi \cup \{\mathrm{A}\}$.

Def.: Abstract Consistency Properties

Let Γ_Σ be a class of sets of Σ-sentences. We define (where $\Phi \in \Gamma_\Sigma$, $\alpha, \beta \in \mathcal{T}$, $\mathbf{A}, \mathbf{B} \in \mathit{cwff}_o(\Sigma)$, $\mathbf{F} \in \mathit{cwff}_{\alpha\rightarrow o}(\Sigma)$ are arbitrary):

Def.: Abstract Consistency Properties

Let Γ_Σ be a class of sets of Σ-sentences. We define (where $\Phi \in \Gamma_\Sigma$, $\alpha, \beta \in \mathcal{T}$, $\mathbf{A}, \mathbf{B} \in cwff_o(\Sigma)$, $\mathbf{F} \in cwff_{\alpha\to o}(\Sigma)$ are arbitrary):

∇_c If $\mathbf{A}$ is atomic, then $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$.

Def.: Abstract Consistency Properties

Let Γ_Σ be a class of sets of Σ-sentences. We define (where $\Phi \in \Gamma_\Sigma$, $\alpha, \beta \in \mathcal{T}$, $\mathbf{A}, \mathbf{B} \in \mathit{cwff}_o(\Sigma)$, $\mathbf{F} \in \mathit{cwff}_{\alpha \to o}(\Sigma)$ are arbitrary):

∇_c If $\mathbf{A}$ is atomic, then $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$.

$\nabla_\neg$ If $\neg\neg\mathbf{A} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$.

Def.: Abstract Consistency Properties

Let Γ_Σ be a class of sets of Σ-sentences. We define (where $\Phi \in \Gamma_\Sigma$, $\alpha, \beta \in \mathcal{T}$, $\mathbf{A}, \mathbf{B} \in \mathit{cwff}_o(\Sigma)$, $\mathbf{F} \in \mathit{cwff}_{\alpha \to o}(\Sigma)$ are arbitrary):

∇_c If $\mathbf{A}$ is atomic, then $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$.

$\nabla_\neg$ If $\neg\neg\mathbf{A} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$.

∇_β If $\mathbf{A} =_\beta \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{B} \in \Gamma_\Sigma$.

Def.: Abstract Consistency Properties

Let Γ_Σ be a class of sets of Σ-sentences. We define (where $\Phi \in \Gamma_\Sigma$, $\alpha, \beta \in \mathcal{T}$, $\mathbf{A}, \mathbf{B} \in \mathit{cwff}_o(\Sigma)$, $\mathbf{F} \in \mathit{cwff}_{\alpha\to o}(\Sigma)$ are arbitrary):

∇_c If $\mathbf{A}$ is atomic, then $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$.

$\nabla_\neg$ If $\neg\neg\mathbf{A} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$.

∇_β If $\mathbf{A} =_\beta \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{B} \in \Gamma_\Sigma$.

$\nabla_\vee$ If $\mathbf{A} \vee \mathbf{B} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$ or $\Phi * \mathbf{B} \in \Gamma_\Sigma$.

Def.: Abstract Consistency Properties

Let Γ_Σ be a class of sets of Σ-sentences. We define (where $\Phi \in \Gamma_\Sigma$, $\alpha, \beta \in \mathcal{T}$, $\mathbf{A}, \mathbf{B} \in cwff_o(\Sigma)$, $\mathbf{F} \in cwff_{\alpha\rightarrow o}(\Sigma)$ are arbitrary):

∇_c If $\mathbf{A}$ is atomic, then $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$.

$\nabla_\neg$ If $\neg\neg\mathbf{A} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$.

∇_β If $\mathbf{A} =_\beta \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{B} \in \Gamma_\Sigma$.

$\nabla_\vee$ If $\mathbf{A} \vee \mathbf{B} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$ or $\Phi * \mathbf{B} \in \Gamma_\Sigma$.

$\nabla_\wedge$ If $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, then $\Phi * \neg\mathbf{A} * \neg\mathbf{B} \in \Gamma_\Sigma$.

Def.: Abstract Consistency Properties

Let Γ_Σ be a class of sets of Σ-sentences. We define (where $\Phi \in \Gamma_\Sigma$, $\alpha, \beta \in \mathcal{T}$, $\mathbf{A}, \mathbf{B} \in \mathit{cwff}_o(\Sigma)$, $\mathbf{F} \in \mathit{cwff}_{\alpha\to o}(\Sigma)$ are arbitrary):

- ∇_c If $\mathbf{A}$ is atomic, then $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$.
- $\nabla_\neg$ If $\neg\neg\mathbf{A} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$.
- ∇_β If $\mathbf{A} =_\beta \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{B} \in \Gamma_\Sigma$.
- $\nabla_\vee$ If $\mathbf{A} \vee \mathbf{B} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$ or $\Phi * \mathbf{B} \in \Gamma_\Sigma$.
- $\nabla_\wedge$ If $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, then $\Phi * \neg\mathbf{A} * \neg\mathbf{B} \in \Gamma_\Sigma$.
- $\nabla_\forall$ If $\Pi^\alpha \mathbf{F} \in \Phi$, then $\Phi * \mathbf{F}\mathbf{W} \in \Gamma_\Sigma$ for each $\mathbf{W} \in \mathit{cwff}_\alpha(\Sigma)$.

Def.: Abstract Consistency Properties

Let Γ_Σ be a class of sets of Σ-sentences. We define (where $\Phi \in \Gamma_\Sigma$, $\alpha, \beta \in \mathcal{T}$, $\mathbf{A}, \mathbf{B} \in \mathit{cwff}_o(\Sigma)$, $\mathbf{F} \in \mathit{cwff}_{\alpha\to o}(\Sigma)$ are arbitrary):

∇_c If $\mathbf{A}$ is atomic, then $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$.

$\nabla_\neg$ If $\neg\neg\mathbf{A} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$.

∇_β If $\mathbf{A} =_\beta \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{B} \in \Gamma_\Sigma$.

$\nabla_\vee$ If $\mathbf{A} \vee \mathbf{B} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$ or $\Phi * \mathbf{B} \in \Gamma_\Sigma$.

$\nabla_\wedge$ If $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, then $\Phi * \neg\mathbf{A} * \neg\mathbf{B} \in \Gamma_\Sigma$.

$\nabla_\forall$ If $\Pi^\alpha\mathbf{F} \in \Phi$, then $\Phi * \mathbf{F}\mathbf{W} \in \Gamma_\Sigma$ for each $\mathbf{W} \in \mathit{cwff}_\alpha(\Sigma)$.

$\nabla_\exists$ If $\neg\Pi^\alpha\mathbf{F} \in \Phi$, then $\Phi * \neg(\mathbf{F}w) \in \Gamma_\Sigma$ for any parameter $w_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

Def.: Abstract Consistency Properties

Let Γ_Σ be a class of sets of Σ-sentences. We define (where $\Phi \in \Gamma_\Sigma$, $\alpha, \beta \in \mathcal{T}$, $\mathbf{A}, \mathbf{B} \in \mathit{cwff}_o(\Sigma)$, $\mathbf{F} \in \mathit{cwff}_{\alpha\to o}(\Sigma)$ are arbitrary):

∇_c If $\mathbf{A}$ is atomic, then $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$.

$\nabla_\neg$ If $\neg\neg\mathbf{A} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$.

∇_β If $\mathbf{A} =_\beta \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{B} \in \Gamma_\Sigma$.

$\nabla_\vee$ If $\mathbf{A} \vee \mathbf{B} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$ or $\Phi * \mathbf{B} \in \Gamma_\Sigma$.

$\nabla_\wedge$ If $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, then $\Phi * \neg\mathbf{A} * \neg\mathbf{B} \in \Gamma_\Sigma$.

$\nabla_\forall$ If $\Pi^\alpha \mathbf{F} \in \Phi$, then $\Phi * \mathbf{F}\mathbf{W} \in \Gamma_\Sigma$ for each $\mathbf{W} \in \mathit{cwff}_\alpha(\Sigma)$.

$\nabla_\exists$ If $\neg\Pi^\alpha \mathbf{F} \in \Phi$, then $\Phi * \neg(\mathbf{F}\mathsf{w}) \in \Gamma_\Sigma$ for any parameter $\mathsf{w}_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

(These properties are going back to Hintikka, Smullyan, and Andrews)

Def.: Abstract Consistency Properties

Let Γ_Σ be a class of sets of Σ-sentences. We define (where $\Phi \in \Gamma_\Sigma$, $\alpha, \beta \in \mathcal{T}$, $\mathbf{A}, \mathbf{B} \in cwff_o(\Sigma)$, $\mathbf{G}, \mathbf{H}, (\lambda X_\alpha \mathbf{.} \mathbf{M}), (\lambda X_\alpha \mathbf{.} \mathbf{N}) \in cwff_{\alpha\rightarrow\beta}(\Sigma)$ are arbitrary):

Def.: Abstract Consistency Properties

Let Γ_Σ be a class of sets of Σ-sentences. We define (where $\Phi \in \Gamma_\Sigma$, $\alpha, \beta \in \mathcal{T}$, $\mathbf{A}, \mathbf{B} \in cwff_o(\Sigma)$, $\mathbf{G}, \mathbf{H}, (\lambda X_\alpha \mathbf{.} \mathbf{M}), (\lambda X_\alpha \mathbf{.} \mathbf{N}) \in cwff_{\alpha \to \beta}(\Sigma)$ are arbitrary):

$\nabla_\mathfrak{b}$ If $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$, then $\Phi * \mathbf{A} * \neg \mathbf{B} \in \Gamma_\Sigma$ or $\Phi * \neg \mathbf{A} * \mathbf{B} \in \Gamma_\Sigma$.

Def.: Abstract Consistency Properties

Let Γ_Σ be a class of sets of Σ-sentences. We define (where $\Phi \in \Gamma_\Sigma$, $\alpha, \beta \in \mathcal{T}$, $\mathbf{A}, \mathbf{B} \in \mathit{cwff}_o(\Sigma)$, $\mathbf{G}, \mathbf{H}, (\lambda X_\alpha \mathbf{.} \mathbf{M}), (\lambda X_\alpha \mathbf{.} \mathbf{N}) \in \mathit{cwff}_{\alpha \to \beta}(\Sigma)$ are arbitrary):

$\nabla_{\mathfrak{b}}$ If $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$, then $\Phi * \mathbf{A} * \neg \mathbf{B} \in \Gamma_\Sigma$ or $\Phi * \neg \mathbf{A} * \mathbf{B} \in \Gamma_\Sigma$.

∇_η If $\mathbf{A} \overset{\beta\eta}{=} \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{B} \in \Gamma_\Sigma$.

Def.: Abstract Consistency Properties

Let Γ_Σ be a class of sets of Σ-sentences. We define (where $\Phi \in \Gamma_\Sigma$, $\alpha, \beta \in \mathcal{T}$, $\mathbf{A}, \mathbf{B} \in \mathit{cwff}_o(\Sigma)$, $\mathbf{G}, \mathbf{H}, (\lambda X_\alpha \mathbf{.} \mathbf{M}), (\lambda X_\alpha \mathbf{.} \mathbf{N}) \in \mathit{cwff}_{\alpha \to \beta}(\Sigma)$ are arbitrary):

$\nabla_\mathfrak{b}$ If $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$, then $\Phi * \mathbf{A} * \neg\mathbf{B} \in \Gamma_\Sigma$ or $\Phi * \neg\mathbf{A} * \mathbf{B} \in \Gamma_\Sigma$.

∇_η If $\mathbf{A} \overset{\beta\eta}{=} \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{B} \in \Gamma_\Sigma$.

∇_ξ If $\neg(\lambda X_\alpha \mathbf{.} \mathbf{M} \doteq^{\alpha \to \beta} \lambda X_\alpha \mathbf{.} \mathbf{N}) \in \Phi$, then $\Phi * \neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N}) \in \Gamma_\Sigma$ for any parameter $w_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

Def.: Abstract Consistency Properties

Let Γ_Σ be a class of sets of Σ-sentences. We define (where $\Phi \in \Gamma_\Sigma$, $\alpha, \beta \in \mathcal{T}$, $\mathbf{A}, \mathbf{B} \in cwff_o(\Sigma)$, $\mathbf{G}, \mathbf{H}, (\lambda X_\alpha . \mathbf{M}), (\lambda X_\alpha . \mathbf{N}) \in cwff_{\alpha \to \beta}(\Sigma)$ are arbitrary):

$\nabla_\mathfrak{b}$ If $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$, then $\Phi * \mathbf{A} * \neg\mathbf{B} \in \Gamma_\Sigma$ or $\Phi * \neg\mathbf{A} * \mathbf{B} \in \Gamma_\Sigma$.

∇_η If $\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{B} \in \Gamma_\Sigma$.

∇_ξ If $\neg(\lambda X_\alpha . \mathbf{M} \doteq^{\alpha \to \beta} \lambda X_\alpha . \mathbf{N}) \in \Phi$, then $\Phi * \neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N}) \in \Gamma_\Sigma$ for any parameter $w_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

$\nabla_\mathfrak{f}$ If $\neg(\mathbf{G} \doteq^{\alpha \to \beta} \mathbf{H}) \in \Phi$, then $\Phi * \neg(\mathbf{G}w \doteq^\beta \mathbf{H}w) \in \Gamma_\Sigma$ for any parameter $w_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

Def.: Abstract Consistency Properties

Let Γ_Σ be a class of sets of Σ-sentences. We define (where $\Phi \in \Gamma_\Sigma$, $\alpha, \beta \in \mathcal{T}$, $\mathbf{A}, \mathbf{B} \in cwff_o(\Sigma)$, $\mathbf{G}, \mathbf{H}, (\lambda X_\alpha \mathbf{.} \mathbf{M}), (\lambda X_\alpha \mathbf{.} \mathbf{N}) \in cwff_{\alpha \to \beta}(\Sigma)$ are arbitrary):

$\nabla_\mathfrak{b}$ If $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$, then $\Phi * \mathbf{A} * \neg\mathbf{B} \in \Gamma_\Sigma$ or $\Phi * \neg\mathbf{A} * \mathbf{B} \in \Gamma_\Sigma$.

∇_η If $\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{B} \in \Gamma_\Sigma$.

∇_ξ If $\neg(\lambda X_\alpha \mathbf{.} \mathbf{M} \doteq^{\alpha \to \beta} \lambda X_\alpha \mathbf{.} \mathbf{N}) \in \Phi$, then $\Phi * \neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N}) \in \Gamma_\Sigma$ for any parameter $w_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

$\nabla_\mathfrak{f}$ If $\neg(\mathbf{G} \doteq^{\alpha \to \beta} \mathbf{H}) \in \Phi$, then $\Phi * \neg(\mathbf{G}w \doteq^\beta \mathbf{H}w) \in \Gamma_\Sigma$ for any parameter $w_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

(These properties are new in [Benzmueller-PhD-99,JSL04])

Def.: Abstract Consistency Classes

Let Σ be a signature and Γ_Σ be a class of sets of Σ-sentences that is closed under subsets.

Def.: Abstract Consistency Classes

Let Σ be a signature and Γ_Σ be a class of sets of Σ-sentences that is closed under subsets.

If $\nabla_c, \nabla_\neg, \nabla_\beta, \nabla_\vee, \nabla_\wedge, \nabla_\forall$ and $\nabla_\exists$ are valid for Γ_Σ, then Γ_Σ is called an abstract consistency class for Σ-models.

Def.: Abstract Consistency Classes

Let Σ be a signature and Γ_Σ be a class of sets of Σ-sentences that is closed under subsets.

If $\nabla_c, \nabla_\neg, \nabla_\beta, \nabla_\vee, \nabla_\wedge, \nabla_\forall$ and $\nabla_\exists$ are valid for Γ_Σ, then Γ_Σ is called an abstract consistency class for Σ-models.

We will denote the collection of abstract consistency classes by $\mathfrak{Acc}_\beta$.

Def.: Abstract Consistency Classes

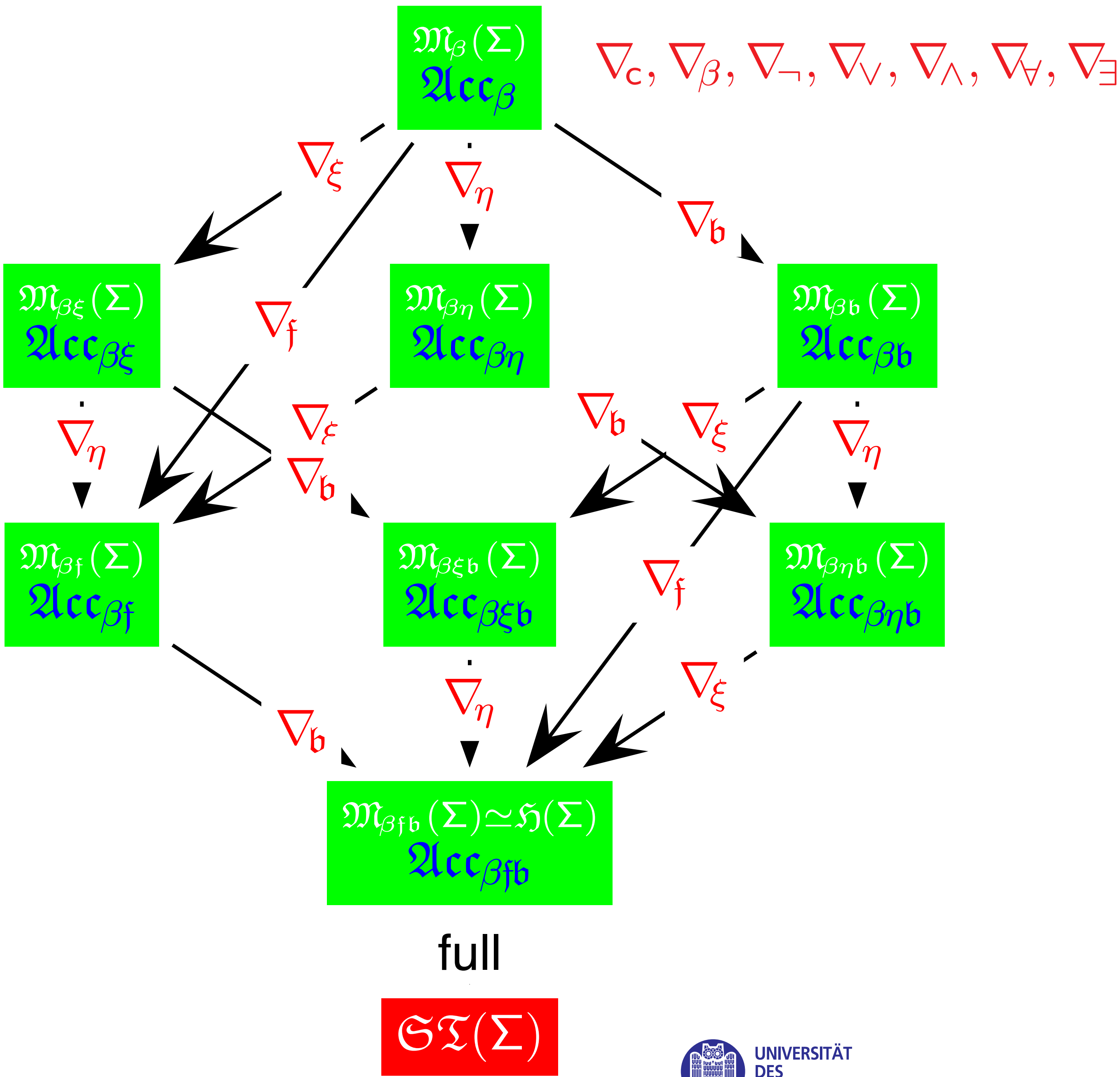
Let Σ be a signature and Γ_Σ be a class of sets of Σ-sentences that is closed under subsets.

If $\nabla_c, \nabla_\neg, \nabla_\beta, \nabla_\vee, \nabla_\wedge, \nabla_\forall$ and $\nabla_\exists$ are valid for Γ_Σ, then Γ_Σ is called an abstract consistency class for Σ-models.

We will denote the collection of abstract consistency classes by $\mathfrak{Acc}_\beta$.

Similarly, we introduce the following collections of specialized abstract consistency classes (with primitive equality): $\mathfrak{Acc}_{\beta\eta}, \mathfrak{Acc}_{\beta\xi}, \mathfrak{Acc}_{\beta\mathfrak{f}}, \mathfrak{Acc}_{\beta\mathfrak{b}}, \mathfrak{Acc}_{\beta\eta\mathfrak{b}}, \mathfrak{Acc}_{\beta\xi\mathfrak{b}}, \mathfrak{Acc}_{\beta\mathfrak{f}\mathfrak{b}}$, where we indicate by indices which additional properties from $\{\nabla_\eta, \nabla_\xi, \nabla_\mathfrak{f}, \nabla_\mathfrak{b}\}$ are required.

Abstract Consistency Classes



Ex.: Abstract Consistency Class

- not an abstract consistency class:
  $\{\{\neg(A \lor B), \neg A\}, \{\neg(A \lor B)\}, \{\neg A\}, \{\}\}$

Ex.: Abstract Consistency Class

- not an abstract consistency class:
  $\{\{\neg(A \vee B), \neg A\}, \{\neg(A \vee B)\}, \{\neg A\}, \{\}\}$
- still not:
  $\{\{\neg(A \vee B), \neg A, \neg B\}, \{\neg(A \vee B), \neg A\}, \{\neg(A \vee B)\}, \{\neg A\}, \{\}\}$

Ex.: Abstract Consistency Class

- not an abstract consistency class:
  $\{\{\neg(A \lor B), \neg A\}, \{\neg(A \lor B)\}, \{\neg A\}, \{\}\}$
- still not:
  $\{\{\neg(A \lor B), \neg A, \neg B\}, \{\neg(A \lor B), \neg A\}, \{\neg(A \lor B)\}, \{\neg A\}, \{\}\}$
- how about this one:
  $\Gamma := \{\{\neg(A \lor B), \neg A, \neg B\}, \{\neg(A \lor B), \neg A\}, \{\neg(A \lor B), \neg B\}, \{\neg A, \neg B\}, \{\neg(A \lor B)\}, \{\neg A\}, \{\neg B\}, \{\}\}$

Ex.: Abstract Consistency Class

- not an abstract consistency class:
  $\{\{\neg(A \lor B), \neg A\}, \{\neg(A \lor B)\}, \{\neg A\}, \{\}\}$
- still not:
  $\{\{\neg(A \lor B), \neg A, \neg B\}, \{\neg(A \lor B), \neg A\}, \{\neg(A \lor B)\}, \{\neg A\}, \{\}\}$
- how about this one:
  $\Gamma := \{\{\neg(A \lor B), \neg A, \neg B\}, \{\neg(A \lor B), \neg A\}, \{\neg(A \lor B), \neg B\}, \{\neg A, \neg B\}, \{\neg(A \lor B)\}, \{\neg A\}, \{\neg B\}, \{\}\}$
- and how about this:
  $\Gamma_0 := \Gamma$
  $\Phi \in \Gamma_i \land A \in \Phi \land B =_{\beta\eta} A \land B \neq A \land (\Phi * B) \notin \Gamma_i \longrightarrow$
  $\Gamma_{i+1} :=$ close-under-subsets$(\Gamma_i * (\Phi * B))$
  $\Gamma^* := \Gamma_\infty$

Rem.: Possible Generalization

The work presented here is based on the choice of the primitive logical connectives $\neg$, $\vee$ and Π^α.

Rem.: Possible Generalization

The work presented here is based on the choice of the primitive logical connectives $\neg, \vee$ and Π^{α}. A means to generalize the framework over the concrete choice of logical primitives is provided by the uniform notation approach as, for instance, given in [Fitting96].

Rem.: Possible Generalization

The work presented here is based on the choice of the primitive logical connectives $\neg, \vee$ and Π^{α}. A means to generalize the framework over the concrete choice of logical primitives is provided by the uniform notation approach as, for instance, given in [Fitting96]. This can be done in straightforward manner: $\nabla_{\wedge}$ becomes an α-property, $\nabla_{\vee}$ becomes a β-property, $\nabla_{\forall}$ becomes a γ-property, and $\nabla_{\exists}$ becomes a δ-property.

Rem.: Possible Generalization

The work presented here is based on the choice of the primitive logical connectives $\neg, \vee$ and Π^{α}. A means to generalize the framework over the concrete choice of logical primitives is provided by the uniform notation approach as, for instance, given in [Fitting96]. This can be done in straightforward manner: $\nabla_{\wedge}$ becomes an α-property, $\nabla_{\vee}$ becomes a β-property, $\nabla_{\forall}$ becomes a γ-property, and $\nabla_{\exists}$ becomes a δ-property. Thus they will have the following form:

Rem.: Possible Generalization

The work presented here is based on the choice of the primitive logical connectives $\neg, \vee$ and Π^{α}. A means to generalize the framework over the concrete choice of logical primitives is provided by the uniform notation approach as, for instance, given in [Fitting96]. This can be done in straightforward manner: $\nabla_{\wedge}$ becomes an α-property, $\nabla_{\vee}$ becomes a β-property, $\nabla_{\forall}$ becomes a γ-property, and $\nabla_{\exists}$ becomes a δ-property. Thus they will have the following form:

α**-case** If $\alpha \in \Phi$, then $\Phi * \alpha_1 * \alpha_2 \in \Gamma_{\Sigma}$.

Rem.: Possible Generalization

The work presented here is based on the choice of the primitive logical connectives $\neg$, $\vee$ and Π^{α}. A means to generalize the framework over the concrete choice of logical primitives is provided by the uniform notation approach as, for instance, given in [Fitting96]. This can be done in straightforward manner: $\nabla_{\wedge}$ becomes an α-property, $\nabla_{\vee}$ becomes a β-property, $\nabla_{\forall}$ becomes a γ-property, and $\nabla_{\exists}$ becomes a δ-property. Thus they will have the following form:

α-case If $\alpha \in \Phi$, then $\Phi * \alpha_1 * \alpha_2 \in \Gamma_{\Sigma}$.

β-case If $\beta \in \Phi$, then $\Phi * \beta_1 \in \Gamma_{\Sigma}$ or $\Phi * \beta_2 \in \Gamma_{\Sigma}$.

Rem.: Possible Generalization

The work presented here is based on the choice of the primitive logical connectives $\neg, \vee$ and Π^α. A means to generalize the framework over the concrete choice of logical primitives is provided by the uniform notation approach as, for instance, given in [Fitting96]. This can be done in straightforward manner: $\nabla_\wedge$ becomes an α-property, $\nabla_\vee$ becomes a β-property, $\nabla_\forall$ becomes a γ-property, and $\nabla_\exists$ becomes a δ-property. Thus they will have the following form:

α-case If $\alpha \in \Phi$, then $\Phi * \alpha_1 * \alpha_2 \in \Gamma_\Sigma$.

β-case If $\beta \in \Phi$, then $\Phi * \beta_1 \in \Gamma_\Sigma$ or $\Phi * \beta_2 \in \Gamma_\Sigma$.

γ-case If $\gamma \in \Phi$, then $\Phi * \gamma \mathbf{W} \in \Gamma_\Sigma$ for each $\mathbf{W} \in \mathit{cwff}_\alpha(\Sigma)$.

Rem.: Possible Generalization

The work presented here is based on the choice of the primitive logical connectives $\neg$, $\vee$ and Π^α. A means to generalize the framework over the concrete choice of logical primitives is provided by the uniform notation approach as, for instance, given in [Fitting96]. This can be done in straightforward manner: $\nabla_\wedge$ becomes an α-property, $\nabla_\vee$ becomes a β-property, $\nabla_\forall$ becomes a γ-property, and $\nabla_\exists$ becomes a δ-property. Thus they will have the following form:

α-case If $\alpha \in \Phi$, then $\Phi * \alpha_1 * \alpha_2 \in \Gamma_\Sigma$.

β-case If $\beta \in \Phi$, then $\Phi * \beta_1 \in \Gamma_\Sigma$ or $\Phi * \beta_2 \in \Gamma_\Sigma$.

γ-case If $\gamma \in \Phi$, then $\Phi * \gamma\mathbf{W} \in \Gamma_\Sigma$ for each $\mathbf{W} \in \mathit{cwff}_\alpha(\Sigma)$.

δ-case If $\delta \in \Phi$, then $\Phi * \delta \mathrm{w} \in \Gamma_\Sigma$ for any parameter $\mathrm{w}_\alpha \in \Sigma$ which does not occur in any sentence of Φ.

Def.: Saturated

Consider the following property (where $\Phi \in \Gamma_\Sigma$, $\mathbf{A} \in \mathit{cwff}_o(\Sigma)$):

Def.: Saturated

Consider the following property (where $\Phi \in \Gamma_\Sigma$, $\mathbf{A} \in \mathit{cwff}_o(\Sigma)$):

∇_{sat} Either $\Phi * \mathbf{A} \in \Gamma_\Sigma$ or $\Phi * \neg\mathbf{A} \in \Gamma_\Sigma$.

Def.: Saturated

Consider the following property (where $\Phi \in \Gamma_\Sigma$, $\mathbf{A} \in cwff_o(\Sigma)$):

∇_{sat} Either $\Phi * \mathbf{A} \in \Gamma_\Sigma$ or $\Phi * \neg\mathbf{A} \in \Gamma_\Sigma$.

We call an abstract consistency class Γ_Σ atomically saturated if ∇_{sat} holds for all atomic A.

Def.: Saturated

Consider the following property (where $\Phi \in \Gamma_\Sigma$, $\mathbf{A} \in cwff_o(\Sigma)$):

∇_{sat} Either $\Phi * \mathbf{A} \in \Gamma_\Sigma$ or $\Phi * \neg\mathbf{A} \in \Gamma_\Sigma$.

We call an abstract consistency class Γ_Σ atomically saturated if ∇_{sat} holds for all atomic A.

We call an abstract consistency class Γ_Σ saturated if ∇_{sat} holds for all A.

Ex.: Saturated

- consider Γ (and Γ^*) from before:
  $\{\{\neg(A \lor B), \neg A, \neg B\}, \{\neg(A \lor B), \neg A\}, \{\neg(A \lor B), \neg B\}, \{\neg A, \neg B\}, \{\neg(A \lor B)\}, \{\neg A\}, \{\neg B\}, \{\}\}$

Ex.: Saturated

- consider Γ (and Γ^*) from before: $\{\{\neg(A \vee B), \neg A, \neg B\}, \{\neg(A \vee B), \neg A\}, \{\neg(A \vee B), \neg B\}, \{\neg A, \neg B\}, \{\neg(A \vee B)\}, \{\neg A\}, \{\neg B\}, \{\}\}$
- Γ (and Γ^*) is atomically saturated in case our signature contains no further constants besides A_o and B_o and the logical connectives.

Ex.: Saturated

- consider Γ (and Γ^*) from before:
  $\{\{\neg(A \vee B), \neg A, \neg B\}, \{\neg(A \vee B), \neg A\}, \{\neg(A \vee B), \neg B\}, \{\neg A, \neg B\}, \{\neg(A \vee B)\}, \{\neg A\}, \{\neg B\}, \{\}\}$
- Γ (and Γ^*) is atomically saturated in case our signature contains no further constants besides A_o and B_o and the logical connectives.
- if there is another symbol C_o in the signature, then Γ (and Γ^*) is not atomically saturated anymore

Ex.: Saturated

- consider Γ (and Γ^*) from before: $\{\{\neg(A \lor B), \neg A, \neg B\}, \{\neg(A \lor B), \neg A\}, \{\neg(A \lor B), \neg B\}, \{\neg A, \neg B\}, \{\neg(A \lor B)\}, \{\neg A\}, \{\neg B\}, \{\}\}$
- Γ (and Γ^*) is atomically saturated in case our signature contains no further constants besides A_o and B_o and the logical connectives.
- if there is another symbol C_o in the signature, then Γ (and Γ^*) is not atomically saturated anymore
- Γ (and Γ^*) is not saturated: for instance, it does not provide information on the formulas $(\neg A \lor B) \lor A$ and $\Pi^o(\lambda X_o.X)$

Thm.: Model Existence Theorem

Let Γ_Σ be a saturated abstract consistency class and let $\Phi \in \Gamma_\Sigma$ be a sufficiently Σ-pure set of sentences.
For all $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ we have:

Thm.: Model Existence Theorem

Let Γ_Σ be a saturated abstract consistency class and let $\Phi \in \Gamma_\Sigma$ be a sufficiently Σ-pure set of sentences.
For all $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ we have:

> If Γ_Σ is an $\mathfrak{Acc}_*$, then there exists a model $\mathcal{M} \in \mathfrak{M}_*$ that satisfies Φ.

Thm.: Model Existence Theorem

Let Γ_Σ be a saturated abstract consistency class and let $\Phi \in \Gamma_\Sigma$ be a sufficiently Σ-pure set of sentences.
For all $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ we have:

> If Γ_Σ is an $\mathfrak{Acc}_*$, then there exists a model $\mathcal{M} \in \mathfrak{M}_*$ that satisfies Φ.

Furthermore, each domain of $\mathcal{M}$ has cardinality at most $\aleph_s$.

Thm.: Model Existence Theorem

Let Γ_Σ be a saturated abstract consistency class and let $\Phi \in \Gamma_\Sigma$ be a sufficiently Σ-pure set of sentences.
For all $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ we have:

> If Γ_Σ is an $\mathfrak{Acc}_*$, then there exists a model $\mathcal{M} \in \mathfrak{M}_*$ that satisfies Φ.

Furthermore, each domain of $\mathcal{M}$ has cardinality at most $\aleph_s$.

Proof: . . . we are not yet ready for this . . .

Thm.: Model Existence for Henkin Models

Let Γ_Σ be a saturated abstract consistency class in $\mathfrak{Acc}_{\beta\mathfrak{fb}}$ and let $\Phi \in \Gamma_\Sigma$ be a sufficiently Σ-pure set of sentences.

Thm.: Model Existence for Henkin Models

Let Γ_Σ be a saturated abstract consistency class in $\mathfrak{Acc}_{\beta\mathfrak{fb}}$ and let $\Phi \in \Gamma_\Sigma$ be a sufficiently Σ-pure set of sentences.

Then there is a Henkin Model that satisfies Φ.

Thm.: Model Existence for Henkin Models

Let Γ_Σ be a saturated abstract consistency class in $\mathfrak{Acc}_{\beta\mathfrak{fb}}$ and let $\Phi \in \Gamma_\Sigma$ be a sufficiently Σ-pure set of sentences.

Then there is a Henkin Model that satisfies Φ.

Furthermore, each domain of the model has cardinality at most $\aleph_\mathsf{s}$.

Thm.: Model Existence for Henkin Models

Let Γ_Σ be a saturated abstract consistency class in $\mathfrak{Acc}_{\beta\mathfrak{fb}}$ and let $\Phi \in \Gamma_\Sigma$ be a sufficiently Σ-pure set of sentences.

Then there is a Henkin Model that satisfies Φ.

Furthermore, each domain of the model has cardinality at most $\aleph_s$.

Proof: ... we are not yet ready for this ...

Completeness of $\mathfrak{NK}_*$ via Abstract Consistency

Def.: $\mathfrak{NK}_*$-Consistent/Inconsistent

A set of sentences Φ is $\mathfrak{NK}_*$-inconsistent if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{F}_o$, and $\mathfrak{NK}_*$-consistent otherwise.

Lemma: Saturated $\mathfrak{Acc}_*$

The class $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \text{ is } \mathfrak{NK}_*\text{-consistent}\}$ is a saturated $\mathfrak{Acc}_*$.

Lemma: Saturated $\mathfrak{Acc}_*$

The class $\Gamma_\Sigma^* := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \text{ is } \mathfrak{NK}_*\text{-consistent}\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Obviously Γ_Σ^* is closed under subsets, since any subset of an $\mathfrak{NK}_*$-consistent set is $\mathfrak{NK}_*$-consistent.

Lemma: Saturated $\mathfrak{Acc}_*$

The class $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \text{ is } \mathfrak{NK}_*\text{-consistent}\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Obviously Γ^*_Σ is closed under subsets, since any subset of an $\mathfrak{NK}_*$-consistent set is $\mathfrak{NK}_*$-consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

Lemma: Saturated $\mathfrak{Acc}_*$

The class $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \text{ is } \mathfrak{NK}_*\text{-consistent}\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Obviously Γ^*_Σ is closed under subsets, since any subset of an $\mathfrak{NK}_*$-consistent set is $\mathfrak{NK}_*$-consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

Lemma: Saturated $\mathfrak{Acc}_*$

The class $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi$ is $\mathfrak{NK}_*$-consistent$\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Obviously Γ^*_Σ is closed under subsets, since any subset of an $\mathfrak{NK}_*$-consistent set is $\mathfrak{NK}_*$-consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

∇_c Suppose $\mathbf{A}, \neg\mathbf{A} \in \Phi$.

Lemma: Saturated $\mathfrak{Acc}_*$

The class $\Gamma^*_\Sigma := \{\Phi \subseteq \textit{cwff}_o(\Sigma) \mid \Phi \text{ is } \mathfrak{NK}_*\text{-consistent}\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Obviously Γ^*_Σ is closed under subsets, since any subset of an $\mathfrak{NK}_*$-consistent set is $\mathfrak{NK}_*$-consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

∇_c Suppose $\mathbf{A}, \neg\mathbf{A} \in \Phi$. We have $\Phi \Vdash \mathbf{F}_o$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\neg E)$.

Lemma: Saturated $\mathfrak{Acc}_*$

The class $\Gamma^*_\Sigma := \{\Phi \subseteq \textit{cwff}_o(\Sigma) \mid \Phi \text{ is } \mathfrak{NK}_*\text{-consistent}\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Obviously Γ^*_Σ is closed under subsets, since any subset of an $\mathfrak{NK}_*$-consistent set is $\mathfrak{NK}_*$-consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

∇_c Suppose $\mathbf{A}, \neg\mathbf{A} \in \Phi$. We have $\Phi \Vdash \mathbf{F}_o$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\neg E)$. Hence Φ is $\mathfrak{NK}_*$-inconsistent.

Lemma: Saturated $\mathfrak{Acc}_*$

The class $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \text{ is } \mathfrak{NK}_*\text{-consistent}\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Obviously Γ^*_Σ is closed under subsets, since any subset of an $\mathfrak{NK}_*$-consistent set is $\mathfrak{NK}_*$-consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

∇_c Suppose $\mathbf{A}, \neg\mathbf{A} \in \Phi$. We have $\Phi \Vdash \mathbf{F}_o$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\neg E)$. Hence Φ is $\mathfrak{NK}_*$-inconsistent.

Lemma: Saturated $\mathfrak{Acc}_*$

The class $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \text{ is } \mathfrak{NK}_*\text{-consistent}\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Obviously Γ^*_Σ is closed under subsets, since any subset of an $\mathfrak{NK}_*$-consistent set is $\mathfrak{NK}_*$-consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

∇_c Suppose $\mathbf{A}, \neg\mathbf{A} \in \Phi$. We have $\Phi \Vdash \mathbf{F}_o$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\neg E)$. Hence Φ is $\mathfrak{NK}_*$-inconsistent.

∇_β Let $\mathbf{A} \in \Phi$, $\mathbf{A} =_\beta \mathbf{B}$ and $\Phi * \mathbf{B}$ be $\mathfrak{NK}_*$-inconsistent.

Lemma: Saturated $\mathfrak{Acc}_*$

The class $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \text{ is } \mathfrak{NK}_*\text{-consistent}\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Obviously Γ^*_Σ is closed under subsets, since any subset of an $\mathfrak{NK}_*$-consistent set is $\mathfrak{NK}_*$-consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

∇_c Suppose $\mathbf{A}, \neg\mathbf{A} \in \Phi$. We have $\Phi \Vdash \mathbf{F}_o$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\neg E)$. Hence Φ is $\mathfrak{NK}_*$-inconsistent.

∇_β Let $\mathbf{A} \in \Phi$, $\mathbf{A} =_\beta \mathbf{B}$ and $\Phi * \mathbf{B}$ be $\mathfrak{NK}_*$-inconsistent. That is, $\Phi * \mathbf{B} \Vdash \mathbf{F}_o$.

Lemma: Saturated $\mathfrak{Acc}_*$

The class $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \text{ is } \mathfrak{NK}_*\text{-consistent}\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Obviously Γ^*_Σ is closed under subsets, since any subset of an $\mathfrak{NK}_*$-consistent set is $\mathfrak{NK}_*$-consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

∇_c Suppose $\mathbf{A}, \neg\mathbf{A} \in \Phi$. We have $\Phi \Vdash \mathbf{F}_o$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\neg E)$. Hence Φ is $\mathfrak{NK}_*$-inconsistent.

∇_β Let $\mathbf{A} \in \Phi$, $\mathbf{A} =_\beta \mathbf{B}$ and $\Phi * \mathbf{B}$ be $\mathfrak{NK}_*$-inconsistent. That is, $\Phi * \mathbf{B} \Vdash \mathbf{F}_o$. By $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{B}$.

Lemma: Saturated $\mathfrak{Acc}_*$

The class $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \text{ is } \mathfrak{NK}_*\text{-consistent}\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Obviously Γ^*_Σ is closed under subsets, since any subset of an $\mathfrak{NK}_*$-consistent set is $\mathfrak{NK}_*$-consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

∇_c Suppose $\mathbf{A}, \neg\mathbf{A} \in \Phi$. We have $\Phi \Vdash \mathbf{F}_o$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\neg E)$. Hence Φ is $\mathfrak{NK}_*$-inconsistent.

∇_β Let $\mathbf{A} \in \Phi$, $\mathbf{A} =_\beta \mathbf{B}$ and $\Phi * \mathbf{B}$ be $\mathfrak{NK}_*$-inconsistent. That is, $\Phi * \mathbf{B} \Vdash \mathbf{F}_o$. By $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{B}$. Since $\mathbf{A} \in \Phi$, we know $\Phi \Vdash \mathbf{B}$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\beta)$.

Lemma: Saturated $\mathfrak{Acc}_*$

The class $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \text{ is } \mathfrak{NK}_*\text{-consistent}\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Obviously Γ^*_Σ is closed under subsets, since any subset of an $\mathfrak{NK}_*$-consistent set is $\mathfrak{NK}_*$-consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

∇_c Suppose $\mathbf{A}, \neg\mathbf{A} \in \Phi$. We have $\Phi \Vdash \mathbf{F}_o$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\neg E)$. Hence Φ is $\mathfrak{NK}_*$-inconsistent.

∇_β Let $\mathbf{A} \in \Phi$, $\mathbf{A} =_\beta \mathbf{B}$ and $\Phi * \mathbf{B}$ be $\mathfrak{NK}_*$-inconsistent. That is, $\Phi * \mathbf{B} \Vdash \mathbf{F}_o$. By $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{B}$. Since $\mathbf{A} \in \Phi$, we know $\Phi \Vdash \mathbf{B}$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\beta)$. Using $\mathfrak{NK}(\neg E)$, we know $\Phi \Vdash \mathbf{F}_o$.

Lemma: Saturated $\mathfrak{Acc}_*$

The class $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \text{ is } \mathfrak{NK}_*\text{-consistent}\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Obviously Γ^*_Σ is closed under subsets, since any subset of an $\mathfrak{NK}_*$-consistent set is $\mathfrak{NK}_*$-consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

∇_c Suppose $\mathbf{A}, \neg\mathbf{A} \in \Phi$. We have $\Phi \Vdash \mathbf{F}_o$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\neg E)$. Hence Φ is $\mathfrak{NK}_*$-inconsistent.

∇_β Let $\mathbf{A} \in \Phi$, $\mathbf{A} =_\beta \mathbf{B}$ and $\Phi * \mathbf{B}$ be $\mathfrak{NK}_*$-inconsistent. That is, $\Phi * \mathbf{B} \Vdash \mathbf{F}_o$. By $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{B}$. Since $\mathbf{A} \in \Phi$, we know $\Phi \Vdash \mathbf{B}$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\beta)$. Using $\mathfrak{NK}(\neg E)$, we know $\Phi \Vdash \mathbf{F}_o$.

Lemma: Saturated $\mathfrak{Acc}_*$

The class $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \text{ is } \mathfrak{NK}_*\text{-consistent}\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Obviously Γ^*_Σ is closed under subsets, since any subset of an $\mathfrak{NK}_*$-consistent set is $\mathfrak{NK}_*$-consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

∇_c Suppose $\mathbf{A}, \neg\mathbf{A} \in \Phi$. We have $\Phi \Vdash \mathbf{F}_o$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\neg E)$. Hence Φ is $\mathfrak{NK}_*$-inconsistent.

∇_β Let $\mathbf{A} \in \Phi$, $\mathbf{A} =_\beta \mathbf{B}$ and $\Phi * \mathbf{B}$ be $\mathfrak{NK}_*$-inconsistent. That is, $\Phi * \mathbf{B} \Vdash \mathbf{F}_o$. By $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{B}$. Since $\mathbf{A} \in \Phi$, we know $\Phi \Vdash \mathbf{B}$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\beta)$. Using $\mathfrak{NK}(\neg E)$, we know $\Phi \Vdash \mathbf{F}_o$.

$\nabla_\neg$ Suppose $\neg\neg\mathbf{A} \in \Phi$ and $\Phi * \mathbf{A}$ is $\mathfrak{NK}_*$-inconsistent.

Lemma: Saturated $\mathfrak{Acc}_*$

The class $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \text{ is } \mathfrak{NK}_*\text{-consistent}\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Obviously Γ^*_Σ is closed under subsets, since any subset of an $\mathfrak{NK}_*$-consistent set is $\mathfrak{NK}_*$-consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

∇_c Suppose $\mathbf{A}, \neg\mathbf{A} \in \Phi$. We have $\Phi \Vdash \mathbf{F}_o$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\neg E)$. Hence Φ is $\mathfrak{NK}_*$-inconsistent.

∇_β Let $\mathbf{A} \in \Phi$, $\mathbf{A} =_\beta \mathbf{B}$ and $\Phi * \mathbf{B}$ be $\mathfrak{NK}_*$-inconsistent. That is, $\Phi * \mathbf{B} \Vdash \mathbf{F}_o$. By $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{B}$. Since $\mathbf{A} \in \Phi$, we know $\Phi \Vdash \mathbf{B}$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\beta)$. Using $\mathfrak{NK}(\neg E)$, we know $\Phi \Vdash \mathbf{F}_o$.

$\nabla_\neg$ Suppose $\neg\neg\mathbf{A} \in \Phi$ and $\Phi * \mathbf{A}$ is $\mathfrak{NK}_*$-inconsistent. From $\Phi * \mathbf{A} \Vdash \mathbf{F}_o$ and $\mathfrak{NK}(\neg I)$, we have $\Phi \Vdash \neg\mathbf{A}$.

Lemma: Saturated $\mathfrak{Acc}_*$

The class $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \text{ is } \mathfrak{NK}_*\text{-consistent}\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Obviously Γ^*_Σ is closed under subsets, since any subset of an $\mathfrak{NK}_*$-consistent set is $\mathfrak{NK}_*$-consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

∇_c Suppose $\mathbf{A}, \neg\mathbf{A} \in \Phi$. We have $\Phi \Vdash \mathbf{F}_o$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\neg E)$. Hence Φ is $\mathfrak{NK}_*$-inconsistent.

∇_β Let $\mathbf{A} \in \Phi$, $\mathbf{A} =_\beta \mathbf{B}$ and $\Phi * \mathbf{B}$ be $\mathfrak{NK}_*$-inconsistent. That is, $\Phi * \mathbf{B} \Vdash \mathbf{F}_o$. By $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{B}$. Since $\mathbf{A} \in \Phi$, we know $\Phi \Vdash \mathbf{B}$ by $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\beta)$. Using $\mathfrak{NK}(\neg E)$, we know $\Phi \Vdash \mathbf{F}_o$.

$\nabla_\neg$ Suppose $\neg\neg\mathbf{A} \in \Phi$ and $\Phi * \mathbf{A}$ is $\mathfrak{NK}_*$-inconsistent. From $\Phi * \mathbf{A} \Vdash \mathbf{F}_o$ and $\mathfrak{NK}(\neg I)$, we have $\Phi \Vdash \neg\mathbf{A}$. Since $\neg\neg\mathbf{A} \in \Phi$, we can apply $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\neg E)$ to obtain $\Phi \Vdash \mathbf{F}_o$.

Lemma: Saturated $\mathfrak{Acc}_*$

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_\vee$ Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent.

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_\vee$ Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\vee E)$, we have $\Phi \Vdash \mathbf{F}_\circ$.

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_\vee$ Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\vee E)$, we have $\Phi \Vdash \mathbf{F}_o$.

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_\vee$ Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\vee E)$, we have $\Phi \Vdash \mathbf{F}_o$.

$\nabla_\wedge$ Suppose $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and $\Phi * \neg\mathbf{A} * \neg\mathbf{B}$ is $\mathfrak{NK}_*$-inconsistent.

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_\vee$ Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\vee E)$, we have $\Phi \Vdash \mathbf{F}_\circ$.

$\nabla_\wedge$ Suppose $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and $\Phi * \neg\mathbf{A} * \neg\mathbf{B}$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_R)$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{A} \vee \mathbf{B}$.

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_\vee$ Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\vee E)$, we have $\Phi \Vdash \mathbf{F}_o$.

$\nabla_\wedge$ Suppose $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and $\Phi * \neg\mathbf{A} * \neg\mathbf{B}$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_R)$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{F}_o$.

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_\vee$ Suppose $(\mathbf{A} \lor \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\lor E)$, we have $\Phi \Vdash \mathbf{F}_\circ$.

$\nabla_\wedge$ Suppose $\neg(\mathbf{A} \lor \mathbf{B}) \in \Phi$ and $\Phi * \neg\mathbf{A} * \neg\mathbf{B}$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\lor I_R)$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{A} \lor \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \lor \mathbf{B}) \in \Phi$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{F}_\circ$. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\lor I_L)$, we have $\Phi \Vdash \mathbf{A} \lor \mathbf{B}$.

Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla_\vee$ Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\vee E)$, we have $\Phi \Vdash \mathbf{F}_o$.

- $\nabla_\wedge$ Suppose $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and $\Phi * \neg\mathbf{A} * \neg\mathbf{B}$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_R)$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{F}_o$. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_L)$, we have $\Phi \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, Φ is $\mathfrak{NK}_*$-inconsistent.

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_\vee$ Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\vee E)$, we have $\Phi \Vdash \mathbf{F}_o$.

$\nabla_\wedge$ Suppose $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and $\Phi * \neg\mathbf{A} * \neg\mathbf{B}$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_R)$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{F}_o$. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_L)$, we have $\Phi \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, Φ is $\mathfrak{NK}_*$-inconsistent.

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_\vee$ Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\vee E)$, we have $\Phi \Vdash \mathbf{F}_\circ$.

$\nabla_\wedge$ Suppose $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and $\Phi * \neg\mathbf{A} * \neg\mathbf{B}$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_R)$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{F}_\circ$. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_L)$, we have $\Phi \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, Φ is $\mathfrak{NK}_*$-inconsistent.

$\nabla_\forall$ Suppose $(\Pi^\alpha \mathbf{G}) \in \Phi$ and $\Phi * (\mathbf{G}\mathbf{A})$ is $\mathfrak{NK}_*$-inconsistent.

Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla_\vee$ Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\vee E)$, we have $\Phi \Vdash \mathbf{F}_o$.

- $\nabla_\wedge$ Suppose $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and $\Phi * \neg\mathbf{A} * \neg\mathbf{B}$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_R)$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{F}_o$. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_L)$, we have $\Phi \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, Φ is $\mathfrak{NK}_*$-inconsistent.

- $\nabla_\forall$ Suppose $(\Pi^\alpha \mathbf{G}) \in \Phi$ and $\Phi * (\mathbf{G}\mathbf{A})$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(\neg I)$, $\Phi \Vdash \neg(\mathbf{G}\mathbf{A})$.

Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla_\vee$ Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\vee E)$, we have $\Phi \Vdash \mathbf{F}_o$.
- $\nabla_\wedge$ Suppose $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and $\Phi * \neg\mathbf{A} * \neg\mathbf{B}$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_R)$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{F}_o$. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_L)$, we have $\Phi \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, Φ is $\mathfrak{NK}_*$-inconsistent.
- $\nabla_\forall$ Suppose $(\Pi^\alpha \mathbf{G}) \in \Phi$ and $\Phi * (\mathbf{G}\mathbf{A})$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(\neg I)$, $\Phi \Vdash \neg(\mathbf{G}\mathbf{A})$. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\Pi E)$, $\Phi \Vdash \mathbf{G}\mathbf{A}$.

Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla_\vee$ Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\vee E)$, we have $\Phi \Vdash \mathbf{F}_o$.

- $\nabla_\wedge$ Suppose $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and $\Phi * \neg\mathbf{A} * \neg\mathbf{B}$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_R)$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{F}_o$. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_L)$, we have $\Phi \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, Φ is $\mathfrak{NK}_*$-inconsistent.

- $\nabla_\forall$ Suppose $(\Pi^\alpha \mathbf{G}) \in \Phi$ and $\Phi * (\mathbf{G}\mathbf{A})$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(\neg I)$, $\Phi \Vdash \neg(\mathbf{G}\mathbf{A})$. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\Pi E)$, $\Phi \Vdash \mathbf{G}\mathbf{A}$. Finally, $\mathfrak{NK}(\neg E)$ implies $\Phi \Vdash \mathbf{F}_o$.

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_\vee$ Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\vee E)$, we have $\Phi \Vdash \mathbf{F}_o$.

$\nabla_\wedge$ Suppose $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and $\Phi * \neg\mathbf{A} * \neg\mathbf{B}$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_R)$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{F}_o$. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_L)$, we have $\Phi \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, Φ is $\mathfrak{NK}_*$-inconsistent.

$\nabla_\forall$ Suppose $(\Pi^\alpha \mathbf{G}) \in \Phi$ and $\Phi * (\mathbf{G}\mathbf{A})$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(\neg I)$, $\Phi \Vdash \neg(\mathbf{G}\mathbf{A})$. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\Pi E)$, $\Phi \Vdash \mathbf{G}\mathbf{A}$. Finally, $\mathfrak{NK}(\neg E)$ implies $\Phi \Vdash \mathbf{F}_o$.

$\nabla_\exists$ Suppose $\neg(\Pi^\alpha \mathbf{G}) \in \Phi$, w_α is a parameter which does not occur in Φ, and $\Phi * \neg(\mathbf{G}\mathsf{w})$ is $\mathfrak{NK}_*$-inconsistent.

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_\vee$ Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\vee E)$, we have $\Phi \Vdash \mathbf{F}_o$.

$\nabla_\wedge$ Suppose $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and $\Phi * \neg\mathbf{A} * \neg\mathbf{B}$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_R)$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{F}_o$. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_L)$, we have $\Phi \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, Φ is $\mathfrak{NK}_*$-inconsistent.

$\nabla_\forall$ Suppose $(\Pi^\alpha \mathbf{G}) \in \Phi$ and $\Phi * (\mathbf{G}\mathbf{A})$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(\neg I)$, $\Phi \Vdash \neg(\mathbf{G}\mathbf{A})$. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\Pi E)$, $\Phi \Vdash \mathbf{G}\mathbf{A}$. Finally, $\mathfrak{NK}(\neg E)$ implies $\Phi \Vdash \mathbf{F}_o$.

$\nabla_\exists$ Suppose $\neg(\Pi^\alpha \mathbf{G}) \in \Phi$, w_α is a parameter which does not occur in Φ, and $\Phi * \neg(\mathbf{G}w)$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Contr)$, $\Phi \Vdash \mathbf{G}w$.

Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla_\vee$ Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\vee E)$, we have $\Phi \Vdash \mathbf{F}_o$.

- $\nabla_\wedge$ Suppose $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and $\Phi * \neg\mathbf{A} * \neg\mathbf{B}$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_R)$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{F}_o$. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_L)$, we have $\Phi \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, Φ is $\mathfrak{NK}_*$-inconsistent.

- $\nabla_\forall$ Suppose $(\Pi^\alpha \mathbf{G}) \in \Phi$ and $\Phi * (\mathbf{G}\,\mathbf{A})$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(\neg I)$, $\Phi \Vdash \neg(\mathbf{G}\,\mathbf{A})$. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\Pi E)$, $\Phi \Vdash \mathbf{G}\,\mathbf{A}$. Finally, $\mathfrak{NK}(\neg E)$ implies $\Phi \Vdash \mathbf{F}_o$.

- $\nabla_\exists$ Suppose $\neg(\Pi^\alpha \mathbf{G}) \in \Phi$, w_α is a parameter which does not occur in Φ, and $\Phi * \neg(\mathbf{G}\mathsf{w})$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Contr)$, $\Phi \Vdash \mathbf{G}\mathsf{w}$. By $\mathfrak{NK}(\Pi I)^{\mathsf{w}}$, $\Phi \Vdash (\Pi^\alpha \mathbf{G})$.

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_\vee$ Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\vee E)$, we have $\Phi \Vdash \mathbf{F}_o$.

$\nabla_\wedge$ Suppose $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and $\Phi * \neg\mathbf{A} * \neg\mathbf{B}$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_R)$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, we have $\Phi, \neg\mathbf{A} \Vdash \mathbf{F}_o$. By $\mathfrak{NK}(Contr)$ and $\mathfrak{NK}(\vee I_L)$, we have $\Phi \Vdash \mathbf{A} \vee \mathbf{B}$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, Φ is $\mathfrak{NK}_*$-inconsistent.

$\nabla_\forall$ Suppose $(\Pi^\alpha \mathbf{G}) \in \Phi$ and $\Phi * (\mathbf{G}\,\mathbf{A})$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(\neg I)$, $\Phi \Vdash \neg(\mathbf{G}\,\mathbf{A})$. By $\mathfrak{NK}(Hyp)$ and $\mathfrak{NK}(\Pi E)$, $\Phi \Vdash \mathbf{G}\,\mathbf{A}$. Finally, $\mathfrak{NK}(\neg E)$ implies $\Phi \Vdash \mathbf{F}_o$.

$\nabla_\exists$ Suppose $\neg(\Pi^\alpha \mathbf{G}) \in \Phi$, w_α is a parameter which does not occur in Φ, and $\Phi * \neg(\mathbf{G}\mathsf{w})$ is $\mathfrak{NK}_*$-inconsistent. By $\mathfrak{NK}(Contr)$, $\Phi \Vdash \mathbf{G}\mathsf{w}$. By $\mathfrak{NK}(\Pi I)^{\mathsf{w}}$, $\Phi \Vdash (\Pi^\alpha \mathbf{G})$. Using $\mathfrak{NK}(\neg E)$ with $\neg(\Pi^\alpha \mathbf{G}) \in \Phi$, Φ is $\mathfrak{NK}_*$-inconsistent.

Lemma: Saturated $\mathfrak{Acc}_*$

Lemma: Saturated $\mathfrak{Acc}_*$

∇_{sat} Let $\Phi * \mathbf{A}$ and $\Phi * \neg\mathbf{A}$ be $\mathfrak{NK}_*$-inconsistent.

Lemma: Saturated $\mathfrak{Acc}_*$

∇_{sat} Let $\Phi * \mathbf{A}$ and $\Phi * \neg\mathbf{A}$ be $\mathfrak{NK}_*$-inconsistent. We show that Φ is $\mathfrak{NK}_*$-inconsistent.

Lemma: Saturated $\mathfrak{Acc}_*$

∇_{sat} Let $\Phi * \mathbf{A}$ and $\Phi * \neg\mathbf{A}$ be $\mathfrak{NK}_*$-inconsistent. We show that Φ is $\mathfrak{NK}_*$-inconsistent. Using $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{A}$ and $\Phi \Vdash \neg\neg\mathbf{A}$.

Lemma: Saturated $\mathfrak{Acc}_*$

∇_{sat} Let $\Phi * \mathbf{A}$ and $\Phi * \neg\mathbf{A}$ be $\mathfrak{NK}_*$-inconsistent. We show that Φ is $\mathfrak{NK}_*$-inconsistent. Using $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{A}$ and $\Phi \Vdash \neg\neg\mathbf{A}$. By $\mathfrak{NK}(\neg E)$, we have $\Phi \Vdash \mathbf{F}_o$.

Lemma: Saturated $\mathfrak{Acc}_*$

∇_{sat} Let $\Phi * \mathbf{A}$ and $\Phi * \neg\mathbf{A}$ be $\mathfrak{NK}_*$-inconsistent. We show that Φ is $\mathfrak{NK}_*$-inconsistent. Using $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{A}$ and $\Phi \Vdash \neg\neg\mathbf{A}$. By $\mathfrak{NK}(\neg E)$, we have $\Phi \Vdash \mathbf{F}_o$.

Thus we have shown that Γ_Σ^β is saturated and in $\mathfrak{Acc}_\beta$.

Lemma: Saturated $\mathfrak{Acc}_*$

∇_{sat} Let $\Phi * \mathbf{A}$ and $\Phi * \neg\mathbf{A}$ be $\mathfrak{NK}_*$-inconsistent. We show that Φ is $\mathfrak{NK}_*$-inconsistent. Using $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{A}$ and $\Phi \Vdash \neg\neg\mathbf{A}$. By $\mathfrak{NK}(\neg E)$, we have $\Phi \Vdash \mathbf{F}_o$.

Thus we have shown that Γ_Σ^β is saturated and in $\mathfrak{Acc}_\beta$.

Now let us check the conditions for the additional properties η, ξ, $\mathfrak{f}$, and $\mathfrak{b}$.

Lemma: Saturated $\mathfrak{Acc}_*$

∇_{sat} Let $\Phi * \mathbf{A}$ and $\Phi * \neg\mathbf{A}$ be $\mathfrak{NK}_*$-inconsistent. We show that Φ is $\mathfrak{NK}_*$-inconsistent. Using $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{A}$ and $\Phi \Vdash \neg\neg\mathbf{A}$. By $\mathfrak{NK}(\neg E)$, we have $\Phi \Vdash \mathbf{F}_o$.

Thus we have shown that Γ_Σ^β is saturated and in $\mathfrak{Acc}_\beta$.

Now let us check the conditions for the additional properties η, ξ, $\mathfrak{f}$, and $\mathfrak{b}$.

Lemma: Saturated $\mathfrak{Acc}_*$

∇_{sat} Let $\Phi * \mathbf{A}$ and $\Phi * \neg\mathbf{A}$ be $\mathfrak{NK}_*$-inconsistent. We show that Φ is $\mathfrak{NK}_*$-inconsistent. Using $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{A}$ and $\Phi \Vdash \neg\neg\mathbf{A}$. By $\mathfrak{NK}(\neg E)$, we have $\Phi \Vdash \mathbf{F}_o$.

Thus we have shown that Γ_Σ^β is saturated and in $\mathfrak{Acc}_\beta$.
Now let us check the conditions for the additional properties η, ξ, $\mathfrak{f}$, and $\mathfrak{b}$.

∇_η If $*$ includes η, then the proof proceeds as in ∇_β above, but with the rule $\mathfrak{NK}(\eta)$.

Lemma: Saturated $\mathfrak{Acc}_*$

∇_{sat} Let $\Phi * \mathbf{A}$ and $\Phi * \neg\mathbf{A}$ be $\mathfrak{NK}_*$-inconsistent. We show that Φ is $\mathfrak{NK}_*$-inconsistent. Using $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{A}$ and $\Phi \Vdash \neg\neg\mathbf{A}$. By $\mathfrak{NK}(\neg E)$, we have $\Phi \Vdash \mathbf{F}_o$.

Thus we have shown that Γ_Σ^β is saturated and in $\mathfrak{Acc}_\beta$.
Now let us check the conditions for the additional properties η, ξ, $\mathfrak{f}$, and $\mathfrak{b}$.

∇_η If $*$ includes η, then the proof proceeds as in ∇_β above, but with the rule $\mathfrak{NK}(\eta)$.

Lemma: Saturated $\mathfrak{Acc}_*$

∇_{sat} Let $\Phi * \mathbf{A}$ and $\Phi * \neg\mathbf{A}$ be $\mathfrak{NK}_*$-inconsistent. We show that Φ is $\mathfrak{NK}_*$-inconsistent. Using $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{A}$ and $\Phi \Vdash \neg\neg\mathbf{A}$. By $\mathfrak{NK}(\neg E)$, we have $\Phi \Vdash \mathbf{F}_o$.

Thus we have shown that Γ_Σ^β is saturated and in $\mathfrak{Acc}_\beta$.

Now let us check the conditions for the additional properties η, ξ, $\mathfrak{f}$, and $\mathfrak{b}$.

∇_η If $*$ includes η, then the proof proceeds as in ∇_β above, but with the rule $\mathfrak{NK}(\eta)$.

∇_ξ Suppose $*$ includes ξ, $\neg(\lambda X_\bullet \mathbf{M} \doteq^{\alpha\to\beta} \lambda X_\bullet \mathbf{N}) \in \Phi$, and $\Phi * \neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N})$ is $\mathfrak{NK}_*$-inconsistent for some parameter w_α which does not occur in any sentence of Φ.

Lemma: Saturated $\mathfrak{Acc}_*$

∇_{sat} Let $\Phi * \mathbf{A}$ and $\Phi * \neg\mathbf{A}$ be $\mathfrak{NK}_*$-inconsistent. We show that Φ is $\mathfrak{NK}_*$-inconsistent. Using $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{A}$ and $\Phi \Vdash \neg\neg\mathbf{A}$. By $\mathfrak{NK}(\neg E)$, we have $\Phi \Vdash \mathbf{F}_o$.

Thus we have shown that Γ_Σ^β is saturated and in $\mathfrak{Acc}_\beta$.
Now let us check the conditions for the additional properties η, ξ, $\mathfrak{f}$, and $\mathfrak{b}$.

∇_η If $*$ includes η, then the proof proceeds as in ∇_β above, but with the rule $\mathfrak{NK}(\eta)$.

∇_ξ Suppose $*$ includes ξ, $\neg(\lambda \mathsf{X}.\mathbf{M} \doteq^{\alpha\to\beta} \lambda \mathsf{X}.\mathbf{N}) \in \Phi$, and $\Phi * \neg([\mathsf{w}/\mathsf{X}]\mathbf{M} \doteq^\beta [\mathsf{w}/\mathsf{X}]\mathbf{N})$ is $\mathfrak{NK}_*$-inconsistent for some parameter w_α which does not occur in any sentence of Φ. By $\mathfrak{NK}(Contr)$, we have $\Phi \Vdash ([\mathsf{w}/\mathsf{X}]\mathbf{M} \doteq^\beta [\mathsf{w}/\mathsf{X}]\mathbf{N})$.

Lemma: Saturated $\mathfrak{Acc}_*$

∇_{sat} Let $\Phi * \mathbf{A}$ and $\Phi * \neg\mathbf{A}$ be $\mathfrak{NK}_*$-inconsistent. We show that Φ is $\mathfrak{NK}_*$-inconsistent. Using $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{A}$ and $\Phi \Vdash \neg\neg\mathbf{A}$. By $\mathfrak{NK}(\neg E)$, we have $\Phi \Vdash \mathbf{F}_o$.

Thus we have shown that Γ_Σ^β is saturated and in $\mathfrak{Acc}_\beta$.
Now let us check the conditions for the additional properties η, ξ, $\mathfrak{f}$, and $\mathfrak{b}$.

∇_η If $*$ includes η, then the proof proceeds as in ∇_β above, but with the rule $\mathfrak{NK}(\eta)$.

∇_ξ Suppose $*$ includes ξ, $\neg(\lambda X.\mathbf{M} \doteq^{\alpha\to\beta} \lambda X.\mathbf{N}) \in \Phi$, and $\Phi * \neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N})$ is $\mathfrak{NK}_*$-inconsistent for some parameter w_α which does not occur in any sentence of Φ. By $\mathfrak{NK}(Contr)$, we have $\Phi \Vdash ([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N})$. By $\mathfrak{NK}(\beta)$, we have $\Phi \Vdash ((\lambda X.\mathbf{M} \doteq^\beta \mathbf{N})w)$.

Lemma: Saturated $\mathfrak{Acc}_*$

∇_{sat} Let $\Phi * \mathbf{A}$ and $\Phi * \neg\mathbf{A}$ be $\mathfrak{NK}_*$-inconsistent. We show that Φ is $\mathfrak{NK}_*$-inconsistent. Using $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{A}$ and $\Phi \Vdash \neg\neg\mathbf{A}$. By $\mathfrak{NK}(\neg E)$, we have $\Phi \Vdash \mathbf{F}_o$.

Thus we have shown that Γ_Σ^β is saturated and in $\mathfrak{Acc}_\beta$.

Now let us check the conditions for the additional properties η, ξ, $\mathfrak{f}$, and $\mathfrak{b}$.

∇_η If $*$ includes η, then the proof proceeds as in ∇_β above, but with the rule $\mathfrak{NK}(\eta)$.

∇_ξ Suppose $*$ includes ξ, $\neg(\lambda X.\mathbf{M} \doteq^{\alpha\to\beta} \lambda X.\mathbf{N}) \in \Phi$, and $\Phi * \neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N})$ is $\mathfrak{NK}_*$-inconsistent for some parameter w_α which does not occur in any sentence of Φ. By $\mathfrak{NK}(Contr)$, we have $\Phi \Vdash ([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N})$. By $\mathfrak{NK}(\beta)$, we have $\Phi \Vdash ((\lambda X.\mathbf{M} \doteq^\beta \mathbf{N})w)$. By $\mathfrak{NK}(\Pi I)$, $\Phi \Vdash (\forall X.\mathbf{M} \doteq^\beta \mathbf{N})$.

Lemma: Saturated $\mathfrak{Acc}_*$

∇_{sat} Let $\Phi * \mathbf{A}$ and $\Phi * \neg\mathbf{A}$ be $\mathfrak{NK}_*$-inconsistent. We show that Φ is $\mathfrak{NK}_*$-inconsistent. Using $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{A}$ and $\Phi \Vdash \neg\neg\mathbf{A}$. By $\mathfrak{NK}(\neg E)$, we have $\Phi \Vdash \mathbf{F}_o$.

Thus we have shown that Γ_Σ^β is saturated and in $\mathfrak{Acc}_\beta$.
Now let us check the conditions for the additional properties η, ξ, $\mathfrak{f}$, and $\mathfrak{b}$.

∇_η If $*$ includes η, then the proof proceeds as in ∇_β above, but with the rule $\mathfrak{NK}(\eta)$.

∇_ξ Suppose $*$ includes ξ, $\neg(\lambda X.\mathbf{M} \doteq^{\alpha\to\beta} \lambda X.\mathbf{N}) \in \Phi$, and $\Phi * \neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N})$ is $\mathfrak{NK}_*$-inconsistent for some parameter w_α which does not occur in any sentence of Φ. By $\mathfrak{NK}(Contr)$, we have $\Phi \Vdash ([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N})$. By $\mathfrak{NK}(\beta)$, we have $\Phi \Vdash ((\lambda X.\mathbf{M} \doteq^\beta \mathbf{N})w)$. By $\mathfrak{NK}(\Pi I)$, $\Phi \Vdash (\forall X.\mathbf{M} \doteq^\beta \mathbf{N})$. By $\mathfrak{NK}(\xi)$, $\Phi \Vdash (\lambda X.\mathbf{M} \doteq^{\alpha\to\beta} \lambda X.\mathbf{N})$.

Lemma: Saturated $\mathfrak{Acc}_*$

∇_{sat} Let $\Phi * \mathbf{A}$ and $\Phi * \neg\mathbf{A}$ be $\mathfrak{NK}_*$-inconsistent. We show that Φ is $\mathfrak{NK}_*$-inconsistent. Using $\mathfrak{NK}(\neg I)$, we know $\Phi \Vdash \neg\mathbf{A}$ and $\Phi \Vdash \neg\neg\mathbf{A}$. By $\mathfrak{NK}(\neg E)$, we have $\Phi \Vdash \mathbf{F}_o$.

Thus we have shown that Γ_Σ^β is saturated and in $\mathfrak{Acc}_\beta$.
Now let us check the conditions for the additional properties η, ξ, $\mathfrak{f}$, and $\mathfrak{b}$.

∇_η If $*$ includes η, then the proof proceeds as in ∇_β above, but with the rule $\mathfrak{NK}(\eta)$.

∇_ξ Suppose $*$ includes ξ, $\neg(\lambda \mathsf{X}.\mathbf{M} \doteq^{\alpha\to\beta} \lambda \mathsf{X}.\mathbf{N}) \in \Phi$, and $\Phi * \neg([\mathsf{w}/\mathsf{X}]\mathbf{M} \doteq^\beta [\mathsf{w}/\mathsf{X}]\mathbf{N})$ is $\mathfrak{NK}_*$-inconsistent for some parameter w_α which does not occur in any sentence of Φ. By $\mathfrak{NK}(Contr)$, we have $\Phi \Vdash ([\mathsf{w}/\mathsf{X}]\mathbf{M} \doteq^\beta [\mathsf{w}/\mathsf{X}]\mathbf{N})$. By $\mathfrak{NK}(\beta)$, we have $\Phi \Vdash ((\lambda \mathsf{X}.\mathbf{M} \doteq^\beta \mathbf{N})\mathsf{w})$. By $\mathfrak{NK}(\Pi I)$, $\Phi \Vdash (\forall \mathsf{X}.\mathbf{M} \doteq^\beta \mathbf{N})$. By $\mathfrak{NK}(\xi)$, $\Phi \Vdash (\lambda \mathsf{X}.\mathbf{M} \doteq^{\alpha\to\beta} \lambda \mathsf{X}.\mathbf{N})$. By $\mathfrak{NK}(\neg E)$, Φ is $\mathfrak{NK}_*$-inconsistent.

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\mathfrak{f}}$ This case is analogous to the previous one, generalizing $\lambda X_{\bullet}\mathbf{M} \doteq \lambda X_{\bullet}\mathbf{N}$ to arbitrary $\mathbf{G} \doteq \mathbf{H}$ and using the extensionality rule $\mathfrak{NK}(\mathfrak{f})$ instead of $\mathfrak{NK}(\xi)$.

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\mathfrak{f}}$ This case is analogous to the previous one, generalizing $\lambda X_\bullet \mathbf{M} \doteq \lambda X_\bullet \mathbf{N}$ to arbitrary $\mathbf{G} \doteq \mathbf{H}$ and using the extensionality rule $\mathfrak{NK}(\mathfrak{f})$ instead of $\mathfrak{NK}(\xi)$.

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_\mathfrak{f}$ This case is analogous to the previous one, generalizing $\lambda X_\bullet \mathbf{M} \doteq \lambda X_\bullet \mathbf{N}$ to arbitrary $\mathbf{G} \doteq \mathbf{H}$ and using the extensionality rule $\mathfrak{NK}(\mathfrak{f})$ instead of $\mathfrak{NK}(\xi)$.

$\nabla_\mathfrak{b}$ Suppose $*$ includes $\mathfrak{b}$.

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_\mathfrak{f}$ This case is analogous to the previous one, generalizing $\lambda X_\bullet \mathbf{M} \doteq \lambda X_\bullet \mathbf{N}$ to arbitrary $\mathbf{G} \doteq \mathbf{H}$ and using the extensionality rule $\mathfrak{NR}(\mathfrak{f})$ instead of $\mathfrak{NR}(\xi)$.

$\nabla_\mathfrak{b}$ Suppose $*$ includes $\mathfrak{b}$. Assume that $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$ but both $\Phi * \neg\mathbf{A} * \mathbf{B} \notin \Gamma^*_\Sigma$ and $\Phi * \mathbf{A} * \neg\mathbf{B} \notin \Gamma^*_\Sigma$.

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\mathfrak{f}}$ This case is analogous to the previous one, generalizing $\lambda X_\bullet \mathbf{M} \doteq \lambda X_\bullet \mathbf{N}$ to arbitrary $\mathbf{G} \doteq \mathbf{H}$ and using the extensionality rule $\mathfrak{NK}(\mathfrak{f})$ instead of $\mathfrak{NK}(\xi)$.

$\nabla_{\mathfrak{b}}$ Suppose $*$ includes $\mathfrak{b}$. Assume that $\neg(\mathbf{A} \doteq^{o} \mathbf{B}) \in \Phi$ but both $\Phi * \neg\mathbf{A} * \mathbf{B} \notin \Gamma_{\Sigma}^{*}$ and $\Phi * \mathbf{A} * \neg\mathbf{B} \notin \Gamma_{\Sigma}^{*}$. So both are $\mathfrak{NK}_*$-inconsistent and we have $\Phi * \mathbf{A} \Vdash \mathbf{B}$ and $\Phi * \mathbf{B} \Vdash \mathbf{A}$ by $\mathfrak{NK}(Contr)$.

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\mathfrak{f}}$ This case is analogous to the previous one, generalizing $\lambda X.\mathbf{M} \doteq \lambda X.\mathbf{N}$ to arbitrary $\mathbf{G} \doteq \mathbf{H}$ and using the extensionality rule $\mathfrak{NK}(\mathfrak{f})$ instead of $\mathfrak{NK}(\xi)$.

$\nabla_{\mathfrak{b}}$ Suppose $*$ includes $\mathfrak{b}$. Assume that $\neg(\mathbf{A} \doteq^{o} \mathbf{B}) \in \Phi$ but both $\Phi * \neg\mathbf{A} * \mathbf{B} \notin \Gamma_{\Sigma}^{*}$ and $\Phi * \mathbf{A} * \neg\mathbf{B} \notin \Gamma_{\Sigma}^{*}$. So both are $\mathfrak{NK}_*$-inconsistent and we have $\Phi * \mathbf{A} \Vdash \mathbf{B}$ and $\Phi * \mathbf{B} \Vdash \mathbf{A}$ by $\mathfrak{NK}(Contr)$. By $\mathfrak{NK}(\mathfrak{b})$, we have $\Phi \Vdash (\mathbf{A} \doteq^{o} \mathbf{B})$.

Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\mathfrak{f}}$ This case is analogous to the previous one, generalizing $\lambda X.\mathbf{M} \doteq \lambda X.\mathbf{N}$ to arbitrary $\mathbf{G} \doteq \mathbf{H}$ and using the extensionality rule $\mathfrak{NK}(\mathfrak{f})$ instead of $\mathfrak{NK}(\xi)$.

$\nabla_{\mathfrak{b}}$ Suppose $*$ includes $\mathfrak{b}$. Assume that $\neg(\mathbf{A} \doteq^{o} \mathbf{B}) \in \Phi$ but both $\Phi * \neg\mathbf{A} * \mathbf{B} \notin \Gamma_{\Sigma}^{*}$ and $\Phi * \mathbf{A} * \neg\mathbf{B} \notin \Gamma_{\Sigma}^{*}$. So both are $\mathfrak{NK}_*$-inconsistent and we have $\Phi * \mathbf{A} \Vdash \mathbf{B}$ and $\Phi * \mathbf{B} \Vdash \mathbf{A}$ by $\mathfrak{NK}(Contr)$. By $\mathfrak{NK}(\mathfrak{b})$, we have $\Phi \Vdash (\mathbf{A} \doteq^{o} \mathbf{B})$. Since $\neg(\mathbf{A} \doteq^{o} \mathbf{B}) \in \Phi$, Φ is $\mathfrak{NK}_*$-inconsistent.

Thm.: Henkin's Theorem for $\mathfrak{NK}_*$

Let $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Every sufficiently Σ-pure $\mathfrak{NK}_*$-consistent set of sentences has an $\mathfrak{M}_*(\Sigma)$-model.

Proof:

Thm.: Henkin's Theorem for $\mathfrak{NK}_*$

Let $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Every sufficiently Σ-pure $\mathfrak{NK}_*$-consistent set of sentences has an $\mathfrak{M}_*(\Sigma)$-model.

Proof: Let Φ be a sufficiently Σ-pure $\mathfrak{NK}_*$-consistent set of sentences.

Thm.: Henkin's Theorem for $\mathfrak{NK}_*$

Let $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Every sufficiently Σ-pure $\mathfrak{NK}_*$-consistent set of sentences has an $\mathfrak{M}_*(\Sigma)$-model.

Proof: Let Φ be a sufficiently Σ-pure $\mathfrak{NK}_*$-consistent set of sentences. By the previous lemma we know that the class of sets of $\mathfrak{NK}_*$-consistent sentences constitute a saturated $\mathfrak{Acc}_*$,

Thm.: Henkin's Theorem for $\mathfrak{NK}_*$

Let $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Every sufficiently Σ-pure $\mathfrak{NK}_*$-consistent set of sentences has an $\mathfrak{M}_*(\Sigma)$-model.

Proof: Let Φ be a sufficiently Σ-pure $\mathfrak{NK}_*$-consistent set of sentences. By the previous lemma we know that the class of sets of $\mathfrak{NK}_*$-consistent sentences constitute a saturated $\mathfrak{Acc}_*$, thus the Model Existence Theorem guarantees an $\mathfrak{M}_*(\Sigma)$ model for Φ.

Thm.: Completeness Theorem for $\mathfrak{NK}_*$

Let Φ be a sufficiently Σ-pure set of sentences, $\mathbf{A}$ be a sentence, and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. If $\mathbf{A}$ is valid in all models $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ that satisfy Φ, then $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{A}$.

Proof:

Thm.: Completeness Theorem for $\mathfrak{NK}_*$

Let Φ be a sufficiently Σ-pure set of sentences, $\mathbf{A}$ be a sentence, and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. If $\mathbf{A}$ is valid in all models $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ that satisfy Φ, then $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{A}$.

Proof: Let $\mathbf{A}$ be given such that $\mathbf{A}$ is valid in all $\mathfrak{M}_*(\Sigma)$ models that satisfy Φ.

Thm.: Completeness Theorem for $\mathfrak{NK}_*$

Let Φ be a sufficiently Σ-pure set of sentences, $\mathbf{A}$ be a sentence, and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. If $\mathbf{A}$ is valid in all models $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ that satisfy Φ, then $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{A}$.

Proof: Let $\mathbf{A}$ be given such that $\mathbf{A}$ is valid in all $\mathfrak{M}_*(\Sigma)$ models that satisfy Φ. So, $\Phi * \neg\mathbf{A}$ is unsatisfiable in $\mathfrak{M}_*(\Sigma)$. Since only finitely many constants occur in $\neg\mathbf{A}$, $\Phi * \neg\mathbf{A}$ is sufficiently Σ-pure.

Thm.: Completeness Theorem for $\mathfrak{NK}_*$

Let Φ be a sufficiently Σ-pure set of sentences, $\mathbf{A}$ be a sentence, and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. If $\mathbf{A}$ is valid in all models $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ that satisfy Φ, then $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{A}$.

Proof: Let $\mathbf{A}$ be given such that $\mathbf{A}$ is valid in all $\mathfrak{M}_*(\Sigma)$ models that satisfy Φ. So, $\Phi * \neg\mathbf{A}$ is unsatisfiable in $\mathfrak{M}_*(\Sigma)$. Since only finitely many constants occur in $\neg\mathbf{A}$, $\Phi * \neg\mathbf{A}$ is sufficiently Σ-pure. So, $\Phi * \neg\mathbf{A}$ must be $\mathfrak{NK}_*$-inconsistent by Henkin's theorem above.

Thm.: Completeness Theorem for $\mathfrak{NK}_*$

Let Φ be a sufficiently Σ-pure set of sentences, $\mathbf{A}$ be a sentence, and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. If $\mathbf{A}$ is valid in all models $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ that satisfy Φ, then $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{A}$.

Proof: Let $\mathbf{A}$ be given such that $\mathbf{A}$ is valid in all $\mathfrak{M}_*(\Sigma)$ models that satisfy Φ. So, $\Phi * \neg\mathbf{A}$ is unsatisfiable in $\mathfrak{M}_*(\Sigma)$. Since only finitely many constants occur in $\neg\mathbf{A}$, $\Phi * \neg\mathbf{A}$ is sufficiently Σ-pure. So, $\Phi * \neg\mathbf{A}$ must be $\mathfrak{NK}_*$-inconsistent by Henkin's theorem above. Thus, $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{A}$ by $\mathfrak{NK}(Contr)$.

Compactness

We can use the completeness theorems obtained so far to prove a compactness theorem for our semantics:

> Let Φ be a sufficiently Σ-pure set of sentences and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Φ has an $\mathfrak{M}_*(\Sigma)$-model iff every finite subset of Φ has an $\mathfrak{M}_*(\Sigma)$-model.

Proof:

Compactness

We can use the completeness theorems obtained so far to prove a compactness theorem for our semantics:

> Let Φ be a sufficiently Σ-pure set of sentences and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Φ has an $\mathfrak{M}_*(\Sigma)$-model iff every finite subset of Φ has an $\mathfrak{M}_*(\Sigma)$-model.

Proof: If Φ has no $\mathfrak{M}_*(\Sigma)$-model, then by the previous Henkin Theorem Φ is $\mathfrak{NK}_*$-inconsistent.

Compactness

We can use the completeness theorems obtained so far to prove a compactness theorem for our semantics:

> Let Φ be a sufficiently Σ-pure set of sentences and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Φ has an $\mathfrak{M}_*(\Sigma)$-model iff every finite subset of Φ has an $\mathfrak{M}_*(\Sigma)$-model.

Proof: If Φ has no $\mathfrak{M}_*(\Sigma)$-model, then by the previous Henkin Theorem Φ is $\mathfrak{NK}_*$-inconsistent. Since every $\mathfrak{NK}_*$-proof is finite, this means some finite subset Ψ of Φ is $\mathfrak{NK}_*$-inconsistent.

Compactness

We can use the completeness theorems obtained so far to prove a compactness theorem for our semantics:

> Let Φ be a sufficiently Σ-pure set of sentences and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Φ has an $\mathfrak{M}_*(\Sigma)$-model iff every finite subset of Φ has an $\mathfrak{M}_*(\Sigma)$-model.

Proof: If Φ has no $\mathfrak{M}_*(\Sigma)$-model, then by the previous Henkin Theorem Φ is $\mathfrak{NK}_*$-inconsistent. Since every $\mathfrak{NK}_*$-proof is finite, this means some finite subset Ψ of Φ is $\mathfrak{NK}_*$-inconsistent. Hence, Ψ has no $\mathfrak{M}_*(\Sigma)$-model.