Calculi: First-Order Natural Deduction and Sequent Calculus

From Natural Deduction to Sequent Calculus and Back

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Remark: We first illustrate the correspondence between natural deduction and sequent calculus in first-order logic. Later we will present natural deduction calculi for HOL. More precisely we will present one sound and complete calculus for each class in our landscape of semantics as presented before.

Reading

- F. Pfenning: Automated Theorem Proving, Course at Carnegie Mellon University. Draft. 1999.
- A.S. Troelstra and H. Schwichtenberg: Basic Proof Theory. Cambridge. 2nd Edition 2000.
- John Byrnes: Proof Search and Normal Forms in Natural Deduction. PhD Thesis. Carnegie Mellon University. 1999.
- . . . many more books on Proof Theory . . .

Natural Deduction: Motivation

- Frege, Russel, Hilbert: Predicate calculus and type theory as formal basis for mathematics

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The formalization of logical deduction, especially as it has been developed by Frege, Russel, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. . . . In contrast I intended first to set up a formal system which comes as close as possible to actual reasoning. The result was a calculus of natural deduction (NJ for intuitionist, NK for classical predicate logic).

[Gentzen: Investigations into logical deduction]

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  - prove of the Hauptsatz (all sequent proofs can be found with a simple strategy)
  - corollary: consistency of formal system(s)

The Hauptsatz says that every purely logical proof can be reduced to a definite, though not unique, normal form. Perhaps we may express the essential properties of such a normal proof by saying: it is not round-about. . . .

In order to be able to prove the Hauptsatz in a convenient form, I had to provide a logical calculus especially for the purpose. For this the natural calculus proved unsuitable.

[Gentzen: Investigations into logical deduction]

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  - Backward: tableaux, connection methods, matrix methods, some forms of resolution
  - Forward: classical resolution, inverse method
- Don't be afraid of the many variants of sequent calculi.
- Choose the one that is most suited for you.

Natural Deduction

Natural deduction rules operate on proof trees.

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- Conjunction: $$\dfrac{\begin{matrix} \mathcal{D}_1 & & \mathcal{D}_2 \\ \mathbf{A} & & \mathbf{B} \end{matrix}}{\mathbf{A} \wedge \mathbf{B}}\ \wedge\mathrm{I} \qquad \dfrac{\begin{matrix} \mathcal{D}_1 \\ \mathbf{A} \wedge \mathbf{B} \end{matrix}}{\mathbf{A}}\ \wedge\mathrm{E_l} \qquad \dfrac{\begin{matrix} \mathcal{D}_1 \\ \mathbf{A} \wedge \mathbf{B} \end{matrix}}{\mathbf{B}}\ \wedge\mathrm{E_r}$$

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The presentation on the next slides treats the proof tree aspects implicit.
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- Implication: $\dfrac{\begin{array}{c}[\mathbf{A}]_1 \\ \vdots \\ \mathbf{B}\end{array}}{\mathbf{A} \Rightarrow \mathbf{B}}\ \Rightarrow\mathsf{I}^1 \qquad \dfrac{\mathbf{A} \Rightarrow \mathbf{B} \quad \mathbf{A}}{\mathbf{B}}\ \Rightarrow\mathsf{E}$

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- Truth and Falsehood: $$\frac{}{\top}\ \top\mathsf{I} \qquad \frac{\bot}{\mathbf{C}}\ \bot\mathsf{E}$$

Natural Deduction Rules IIa

- Negation:

$$\dfrac{\begin{array}{c}[\mathbf{A}]_1 \\ \vdots \\ \bot\end{array}}{\neg\mathbf{A}}\ \neg\mathrm{I}^1 \qquad \dfrac{\neg\mathbf{A} \quad \mathbf{A}}{\bot}\ \neg\mathrm{E}$$

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- Universal Quantif.:

$$\dfrac{\mathbf{A}[x/P^*]}{\forall x.\mathbf{A}}\ \forall\mathrm{I} \qquad \dfrac{\forall x.\mathbf{A}}{\mathbf{A}[x/\mathbf{T}]}\ \forall\mathrm{E}$$

(*: parameter P must be new in context)

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$$\dfrac{\mathbf{A}[\mathsf{x}/\mathsf{P}^*]}{\forall \mathsf{x}.\mathbf{A}}\ \forall\mathrm{I} \qquad \dfrac{\forall \mathsf{x}.\mathbf{A}}{\mathbf{A}[\mathsf{x}/\mathbf{T}]}\ \forall\mathrm{E}$$

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- Existential Quantif.:

$$\dfrac{\mathbf{A}[\mathsf{x}/\mathbf{T}]}{\exists \mathsf{x}.\mathbf{A}}\ \exists\mathrm{I} \qquad \dfrac{\exists \mathsf{x}.\mathbf{A} \quad \begin{matrix}[\mathbf{A}[\mathsf{x}/\mathsf{P}^*]]\\ \vdots \\ \mathbf{C}\end{matrix}}{\mathbf{C}}\ \exists\mathrm{E}$$

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  - Excluded Middle $$\frac{}{\mathbf{A} \lor \neg \mathbf{A}}\ XM$$
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  - Proof by Contradiction $$\frac{\begin{array}{c}[\neg \mathbf{A}] \\ \vdots \\ \bot\end{array}}{\mathbf{A}}\ \bot_c$$

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Natural Deduction Proofs

$$\dfrac{\dfrac{\dfrac{[\mathbf{A}]_1 \quad [\mathbf{A}]_2}{\mathbf{A} \wedge \mathbf{A}}\,\wedge I}{\mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A})}\,\Rightarrow I^2}{\mathbf{A} \Rightarrow (\mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A}))}\,\Rightarrow I^1 \qquad \text{or} \qquad \dfrac{\dfrac{\dfrac{[\mathbf{A}]_1 \quad [\mathbf{A}]_1}{\mathbf{A} \wedge \mathbf{A}}\,\wedge I}{\mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A})}\,\Rightarrow I^2}{\mathbf{A} \Rightarrow (\mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A}))}\,\Rightarrow I^1$$

$$\dfrac{\dfrac{\dfrac{[\mathbf{A} \wedge \mathbf{B}]_1}{\mathbf{B}}\,\wedge E_r \qquad \dfrac{\dfrac{[\mathbf{A} \wedge \mathbf{B}]_1}{\mathbf{A}}\,\wedge E_l}{\mathbf{C} \vee \mathbf{A}}\,\vee I_r}{\mathbf{B} \wedge (\mathbf{C} \vee \mathbf{A})}\,\wedge I}{(\mathbf{A} \wedge \mathbf{B}) \Rightarrow (\mathbf{B} \wedge (\mathbf{C} \vee \mathbf{A}))}\,\Rightarrow I^1$$

Natural Deduction with Contexts

- FO-Soundness of ND: Let F be a first-order formula such that there is a ND proof of F. Then F is valid. $(\vdash \mathsf{F} \;\Rightarrow\; \models \mathsf{F})$
  (Proof: Standard textbooks)

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- FO-Completeness of ND: Let F be a valid first-order formula then there is a ND proof of F $(\models F \;\Rightarrow\; \vdash F)$.
  (Proof: Standard textbooks)

Natural Deduction with Contexts

Idea: Localizing hypotheses; explicit representation of the available assumptions for each formula occurrence in a ND proof:

$$\Gamma \vdash \mathbf{A}$$

Γ is a multiset of the (uncanceled) assumptions on which formula $\mathbf{A}$ depends. Γ is called context.

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Example proof in context notation:

$$\cfrac{\cfrac{\cfrac{\overline{\mathbf{A}_1 \vdash \mathbf{A}} \qquad \overline{\mathbf{A}_2 \vdash \mathbf{A}}}{\mathbf{A}_1, \mathbf{A}_2 \vdash \mathbf{A} \wedge \mathbf{A}} \wedge I}{\mathbf{A}_1 \vdash \mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A})} \Rightarrow I_2}{\vdash \mathbf{A} \Rightarrow (\mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A}))} \Rightarrow I_1$$

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- Implication: $$\frac{\Gamma, \mathbf{A} \vdash \mathbf{B}}{\Gamma \vdash \mathbf{A} \Rightarrow \mathbf{B}} \Rightarrow\mathrm{I} \qquad \frac{\Gamma \vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Gamma \vdash \mathbf{A}}{\Gamma \vdash \mathbf{B}} \Rightarrow\mathrm{E}$$

Natural Deduction Rules IIb

- Truth and Falsehood:

$$\frac{}{\Gamma \vdash \top}\ \top\mathrm{I} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \mathbf{C}}\ \bot\mathrm{E}$$

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- Truth and Falsehood: $$\frac{}{\Gamma \vdash \top}\ \top\mathrm{I} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \mathbf{C}}\ \bot\mathrm{E}$$

- Negation: $$\frac{\Gamma, \mathbf{A} \vdash \bot}{\Gamma \vdash \neg\mathbf{A}}\ \neg\mathrm{I} \qquad \frac{\Gamma \vdash \neg\mathbf{A} \quad \Gamma \vdash \mathbf{A}}{\Gamma \vdash \bot}\ \neg\mathrm{E}$$

Natural Deduction Rules IIb

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- Universal Quantif.: $$\frac{\Gamma \vdash \mathbf{A}[\mathsf{x}/\mathsf{P}^*]}{\Gamma \vdash \forall \mathsf{x}_{\blacksquare}\mathbf{A}}\ \forall I \qquad \frac{\Gamma \vdash \forall \mathsf{x}_{\blacksquare}\mathbf{A}}{\Gamma \vdash \mathbf{A}[\mathsf{x}/\mathbf{T}]}\ \forall E$$

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- Existential Quantif.:

$$\frac{\Gamma \vdash \mathbf{A}[\mathsf{x}/\mathbf{T}]}{\Gamma \vdash \exists \mathsf{x}.\mathbf{A}}\ \exists I \qquad \frac{\Gamma \vdash \exists \mathsf{x}.\mathbf{A} \quad \Gamma, \mathbf{A}[\mathsf{x}/\mathsf{P}^*] \vdash \mathbf{C}}{\Gamma \vdash \mathbf{C}}\ \exists E$$

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Natural Deduction Rules IIIb

For classical logic add:

- Proof by Contradiction:

$$\frac{\Gamma, \neg \mathbf{A} \vdash \bot}{\Gamma \vdash \mathbf{A}} \bot_c$$

Intercalation

- Idea (Prawitz, Sieg & Scheines, Byrnes & Sieg): Detour free proofs: strictly use introduction rules bottom up (from proposed theorem to hypothesis) and elimination rules top down (from assumptions to proposed theorem). When they meet in the middle we have found a proof in normal form.

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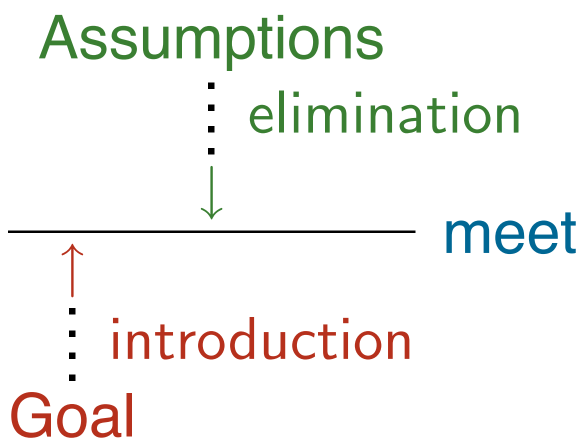

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \mathbf{A}\end{matrix} \qquad \begin{matrix}\vdots\\ \mathbf{B}\end{matrix}}{\mathbf{A}\wedge\mathbf{B}}\,\wedge\mathrm{I}}{\begin{matrix}\mathbf{A}\\ \vdots\end{matrix}}\,\wedge\mathrm{E_l}$$

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- Example:

$$\frac{\Gamma, \mathbf{A} \vdash_{\mathrm{IC}} \mathbf{B} \uparrow}{\Gamma \vdash_{\mathrm{IC}} \mathbf{A} \Rightarrow \mathbf{B} \uparrow} \Rightarrow\mathrm{I} \qquad \frac{\Gamma \vdash_{\mathrm{IC}} \mathbf{A} \Rightarrow \mathbf{B} \downarrow \qquad \Gamma \vdash_{\mathrm{IC}} \mathbf{A} \uparrow}{\Gamma \vdash_{\mathrm{IC}} \mathbf{B} \uparrow} \Rightarrow\mathrm{E}$$

ND Intercalation Rules I

- Hypotheses: $$\overline{\Gamma, \mathbf{A}, \Delta \vdash_{\mathsf{ic}} \mathbf{A} \downarrow}$$

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$$\overline{\Gamma, \mathbf{A}, \Delta \vdash_{\mathsf{ic}} \mathbf{A} \downarrow}$$

- Conjunction:

$$\frac{\Gamma \vdash_{\mathsf{ic}} \mathbf{A} \uparrow \quad \Gamma \vdash_{\mathsf{ic}} \mathbf{B} \uparrow}{\Gamma \vdash_{\mathsf{ic}} \mathbf{A} \wedge \mathbf{B} \uparrow} \wedge\mathsf{I} \qquad \frac{\Gamma \vdash_{\mathsf{ic}} \mathbf{A} \wedge \mathbf{B} \downarrow}{\Gamma \vdash_{\mathsf{ic}} \mathbf{A} \downarrow} \wedge\mathsf{E}_{\mathsf{l}} \qquad \frac{\Gamma \vdash_{\mathsf{ic}} \mathbf{A} \wedge \mathbf{B} \downarrow}{\Gamma \vdash_{\mathsf{ic}} \mathbf{B} \downarrow} \wedge\mathsf{E}_{\mathsf{r}}$$

- Disjunction:

$$\frac{\Gamma \vdash_{\mathsf{ic}} \mathbf{A} \uparrow}{\Gamma \vdash_{\mathsf{ic}} \mathbf{A} \vee \mathbf{B} \uparrow} \vee\mathsf{I}_{\mathsf{l}} \qquad \frac{\Gamma \vdash_{\mathsf{ic}} \mathbf{B} \uparrow}{\Gamma \vdash_{\mathsf{ic}} \mathbf{A} \vee \mathbf{B} \uparrow} \vee\mathsf{I}_{\mathsf{r}}$$

$$\frac{\Gamma \vdash_{\mathsf{ic}} \mathbf{A} \vee \mathbf{B} \downarrow \quad \Gamma, \mathbf{A} \vdash_{\mathsf{ic}} \mathbf{C} \uparrow \quad \Gamma, \mathbf{B} \vdash_{\mathsf{ic}} \mathbf{C} \uparrow}{\Gamma \vdash_{\mathsf{ic}} \mathbf{C} \uparrow} \vee\mathsf{E}$$

ND Intercalation Rules I

- Hypotheses: $$\overline{\Gamma, \mathbf{A}, \Delta \vdash_{\mathsf{IC}} \mathbf{A} \downarrow}$$
- Conjunction:

$$\frac{\Gamma \vdash_{\mathsf{IC}} \mathbf{A} \uparrow \quad \Gamma \vdash_{\mathsf{IC}} \mathbf{B} \uparrow}{\Gamma \vdash_{\mathsf{IC}} \mathbf{A} \wedge \mathbf{B} \uparrow} \wedge I \qquad \frac{\Gamma \vdash_{\mathsf{IC}} \mathbf{A} \wedge \mathbf{B} \downarrow}{\Gamma \vdash_{\mathsf{IC}} \mathbf{A} \downarrow} \wedge E_l \qquad \frac{\Gamma \vdash_{\mathsf{IC}} \mathbf{A} \wedge \mathbf{B} \downarrow}{\Gamma \vdash_{\mathsf{IC}} \mathbf{B} \downarrow} \wedge E_r$$

- Disjunction:

$$\frac{\Gamma \vdash_{\mathsf{IC}} \mathbf{A} \uparrow}{\Gamma \vdash_{\mathsf{IC}} \mathbf{A} \vee \mathbf{B} \uparrow} \vee I_l \qquad \frac{\Gamma \vdash_{\mathsf{IC}} \mathbf{B} \uparrow}{\Gamma \vdash_{\mathsf{IC}} \mathbf{A} \vee \mathbf{B} \uparrow} \vee I_r$$

$$\frac{\Gamma \vdash_{\mathsf{IC}} \mathbf{A} \vee \mathbf{B} \downarrow \quad \Gamma, \mathbf{A} \vdash_{\mathsf{IC}} \mathbf{C} \uparrow \quad \Gamma, \mathbf{B} \vdash_{\mathsf{IC}} \mathbf{C} \uparrow}{\Gamma \vdash_{\mathsf{IC}} \mathbf{C} \uparrow} \vee E$$

- Implication:

$$\frac{\Gamma, \mathbf{A} \vdash_{\mathsf{IC}} \mathbf{B} \uparrow}{\Gamma \vdash_{\mathsf{IC}} \mathbf{A} \Rightarrow \mathbf{B} \uparrow} \Rightarrow I \qquad \frac{\Gamma \vdash_{\mathsf{IC}} \mathbf{A} \Rightarrow \mathbf{B} \downarrow \quad \Gamma \vdash_{\mathsf{IC}} \mathbf{A} \uparrow}{\Gamma \vdash_{\mathsf{IC}} \mathbf{B} \uparrow} \Rightarrow E$$

ND Intercalation Rules II

- Truth and Falsehood: $$\frac{}{\Gamma \vdash_{ic} \top \uparrow}\ \top I \qquad \frac{\Gamma \vdash_{ic} \bot \downarrow}{\Gamma \vdash_{ic} \mathbf{C} \uparrow}\ \bot E$$

ND Intercalation Rules II

- Truth and Falsehood: $$\frac{}{\Gamma \vdash_{\text{ic}} \top \uparrow}\ \top\text{I} \qquad \frac{\Gamma \vdash_{\text{ic}} \bot \downarrow}{\Gamma \vdash_{\text{ic}} \mathbf{C} \uparrow}\ \bot\text{E}$$
- Negation: $$\frac{\Gamma, \mathbf{A} \vdash_{\text{ic}} \bot \uparrow}{\Gamma \vdash_{\text{ic}} \neg\mathbf{A} \uparrow}\ \neg\text{I} \qquad \frac{\Gamma \vdash_{\text{ic}} \neg\mathbf{A} \downarrow \quad \Gamma \vdash_{\text{ic}} \mathbf{A} \uparrow}{\Gamma \vdash_{\text{ic}} \bot \uparrow}\ \neg\text{E}$$

ND Intercalation Rules II

- Truth and Falsehood:

$$\frac{}{\Gamma \vdash_{ic} \top \uparrow} \top I \qquad \frac{\Gamma \vdash_{ic} \bot \downarrow}{\Gamma \vdash_{ic} \mathbf{C} \uparrow} \bot E$$

- Negation:

$$\frac{\Gamma, \mathbf{A} \vdash_{ic} \bot \uparrow}{\Gamma \vdash_{ic} \neg\mathbf{A} \uparrow} \neg I \qquad \frac{\Gamma \vdash_{ic} \neg\mathbf{A} \downarrow \quad \Gamma \vdash_{ic} \mathbf{A} \uparrow}{\Gamma \vdash_{ic} \bot \uparrow} \neg E$$

- Universal Quantif.:

$$\frac{\Gamma \vdash_{ic} \mathbf{A}[x/P^*] \uparrow}{\Gamma \vdash_{ic} \forall x_\bullet \mathbf{A} \uparrow} \forall I \qquad \frac{\Gamma \vdash_{ic} \forall x_\bullet \mathbf{A} \downarrow}{\Gamma \vdash_{ic} \mathbf{A}[x/\mathbf{T}] \downarrow} \forall E$$

(*: parameter P must be new in context)

ND Intercalation Rules II

- Truth and Falsehood:

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$$\frac{\Gamma, \mathbf{A} \vdash_{ic} \bot \uparrow}{\Gamma \vdash_{ic} \neg\mathbf{A} \uparrow}\ \neg I \qquad \frac{\Gamma \vdash_{ic} \neg\mathbf{A} \downarrow \quad \Gamma \vdash_{ic} \mathbf{A} \uparrow}{\Gamma \vdash_{ic} \bot \uparrow}\ \neg E$$

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$$\frac{\Gamma \vdash_{ic} \mathbf{A}[x/P^*] \uparrow}{\Gamma \vdash_{ic} \forall x_{\bullet}\mathbf{A} \uparrow}\ \forall I \qquad \frac{\Gamma \vdash_{ic} \forall x_{\bullet}\mathbf{A} \downarrow}{\Gamma \vdash_{ic} \mathbf{A}[x/\mathbf{T}] \downarrow}\ \forall E$$

(*: parameter P must be new in context)

- Existential Quantif.:

$$\frac{\Gamma \vdash_{ic} \mathbf{A}[x/\mathbf{T}] \uparrow}{\Gamma \vdash_{ic} \exists x_{\bullet}\mathbf{A} \uparrow}\ \exists I \qquad \frac{\Gamma \vdash_{ic} \exists x_{\bullet}\mathbf{A} \downarrow \quad \Gamma, \mathbf{A}[x/P^*] \vdash_{ic} \mathbf{C} \uparrow}{\Gamma \vdash \mathbf{C} \uparrow}\ \exists E$$

(*: parameter P must be new in context)

ND Intercalation Rules III

For classical logic add:

- Proof by Contradiction:

$$\frac{\Gamma, \neg \mathbf{A} \vdash_{\mathsf{ic}} \bot \uparrow}{\Gamma \vdash_{\mathsf{ic}} \mathbf{A} \uparrow} \; \bot_{\mathsf{c}}$$

Intercalation and ND

- Normal form proofs

Intercalation and ND

- Normal form proofs

Assumptions

$\vdots$ elimination

$\downarrow$

$\overline{}$ meet

$\uparrow$

$\vdots$ introduction

Goal

guaranteed by $$\frac{\Gamma \vdash_{ic} \mathbf{A} \downarrow}{\Gamma \vdash_{ic} \mathbf{A} \uparrow} \text{ meet}$$

Intercalation and ND

- Normal form proofs

Assumptions

⋮ elimination

$\downarrow$

———————— meet

$\uparrow$

⋮ introduction

Goal

guaranteed by $\dfrac{\Gamma \vdash_{ic} \mathbf{A} \downarrow}{\Gamma \vdash_{ic} \mathbf{A} \uparrow}$ meet

. . . proofs without detour . . .

Intercalation and ND

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. . . proofs without detour . . .

- To model all ND proofs add $\dfrac{\Gamma \vdash_{ic} \mathbf{A} \uparrow}{\Gamma \vdash_{ic} \mathbf{A} \downarrow}$ roundabout

Example Proofs

- In normal form

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{M} \wedge \mathbf{Q} \vdash_{ic} \mathbf{M} \wedge \mathbf{Q} \downarrow}}{\mathbf{M} \wedge \mathbf{Q} \vdash_{ic} \mathbf{Q} \downarrow} \wedge E_r}{\mathbf{M} \wedge \mathbf{Q} \vdash_{ic} \mathbf{Q} \uparrow} \text{meet}}{\dfrac{\mathbf{M} \wedge \mathbf{Q} \vdash_{ic} \mathbf{Q} \vee \mathbf{S} \uparrow}{\vdash_{ic} (\mathbf{M} \wedge \mathbf{Q}) \Rightarrow (\mathbf{Q} \vee \mathbf{S}) \uparrow} \Rightarrow I} \vee I_l$$

Example Proofs

- In normal form

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\mathbf{M}\wedge\mathbf{Q}\vdash_{ic}\mathbf{M}\wedge\mathbf{Q}\downarrow}}{\mathbf{M}\wedge\mathbf{Q}\vdash_{ic}\mathbf{Q}\downarrow}\;\wedge E_r}{\mathbf{M}\wedge\mathbf{Q}\vdash_{ic}\mathbf{Q}\uparrow}\;\text{meet}}{\mathbf{M}\wedge\mathbf{Q}\vdash_{ic}\mathbf{Q}\vee\mathbf{S}\uparrow}\;\vee I_l}{\vdash_{ic}(\mathbf{M}\wedge\mathbf{Q})\Rightarrow(\mathbf{Q}\vee\mathbf{S})\uparrow}\;\Rightarrow I$$

- With detour

$$\begin{array}{c}\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \mathbf{M}\wedge\mathbf{Q}\vdash_{ic}\mathbf{Q}\uparrow\end{array}\qquad\begin{array}{c}\vdots\\ \mathbf{M}\wedge\mathbf{Q}\vdash_{ic}\mathbf{M}\uparrow\end{array}}{\mathbf{M}\wedge\mathbf{Q}\vdash_{ic}\mathbf{Q}\wedge\mathbf{M}\uparrow}\;\wedge I}{\mathbf{M}\wedge\mathbf{Q}\vdash_{ic}\mathbf{Q}\wedge\mathbf{M}\downarrow}\;\text{roundabout}}{\mathbf{M}\wedge\mathbf{Q}\vdash_{ic}\mathbf{Q}\downarrow}\;\wedge E_l}{\mathbf{M}\wedge\mathbf{Q}\vdash_{ic}\mathbf{Q}\uparrow}\;\text{meet}\\ \vdots\end{array}$$

Soundness and Completeness

Let $\vdash^{\pm}_{\mathrm{ic}}$ denote the intercalation calculus with rule roundabout and $\vdash_{\mathrm{ic}}$ the calculus without this rule.

Soundness and Completeness

Let $\vdash^{\pm}_{ic}$ denote the intercalation calculus with rule roundabout and $\vdash_{ic}$ the calculus without this rule.

- Theorem 1 (Soundness of $\Gamma \vdash^{\pm}_{ic}$ relative to $\vdash$): If $\Gamma \vdash^{\pm}_{ic} \mathbf{A} \uparrow$ then $\Gamma \vdash \mathbf{A}$.

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- Theorem 2 (Completeness of $\Gamma \Vdash^{\pm}_{\text{ic}}$ relative to $\vdash$): If $\Gamma \vdash \mathbf{A}$ then $\Gamma \Vdash^{\pm}_{\text{ic}} \mathbf{A} \uparrow$.

Soundness and Completeness

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- Theorem 2 (Completeness of $\Gamma \models^{\pm}_{ic}$ relative to $\vdash$): If $\Gamma \vdash \mathbf{A}$ then $\Gamma \models^{\pm}_{ic} \mathbf{A} \uparrow$.
- Is normal form proof search also complete?:

  If $\Gamma \models^{\pm}_{ic} \mathbf{A} \uparrow$ then $\Gamma \vdash_{ic} \mathbf{A} \uparrow$?

  We will investigate this question within the sequent calculus.

Soundness and Completeness

Let $\stackrel{\pm}{\vdash}_{ic}$ denote the intercalation calculus with rule roundabout and $\vdash_{ic}$ the calculus without this rule.

- Theorem 1 (Soundness of $\Gamma \stackrel{\pm}{\vdash}_{ic}$ relative to $\vdash$): If $\Gamma \stackrel{\pm}{\vdash}_{ic} \mathbf{A} \uparrow$ then $\Gamma \vdash \mathbf{A}$.
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- Is normal form proof search also complete?:

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  We will investigate this question within the sequent calculus.

From ND to Sequent Calculus

Normal form ND proofs

Sequent proofs

From ND to Sequent Calculus

Normal form ND proofs

Sequent proofs

Assumptions

⋮ elim

↓

meet

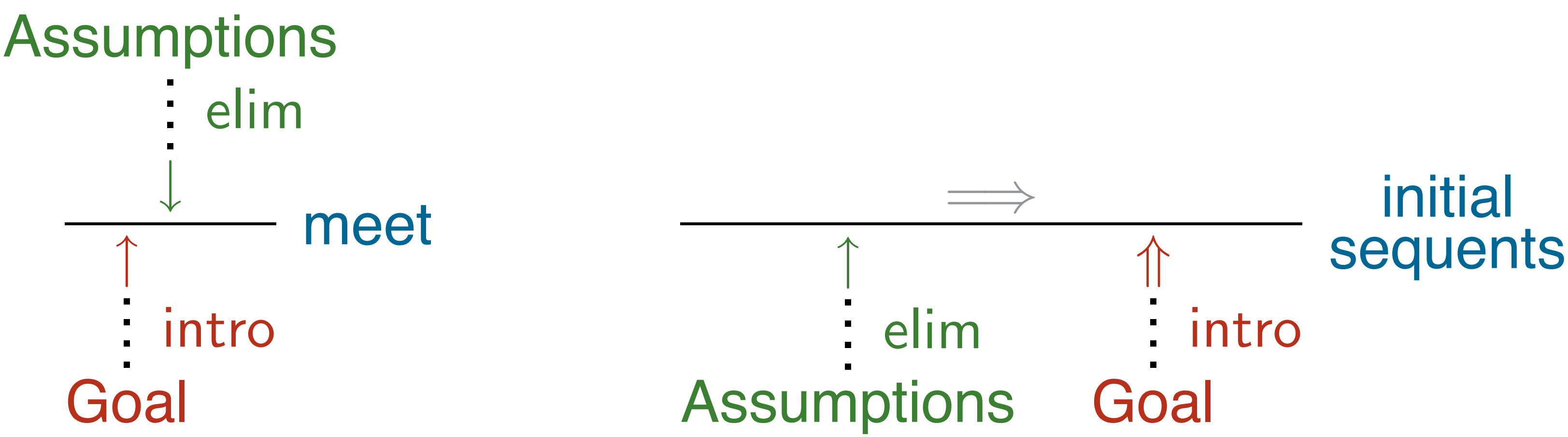
↑

⋮ intro

Goal

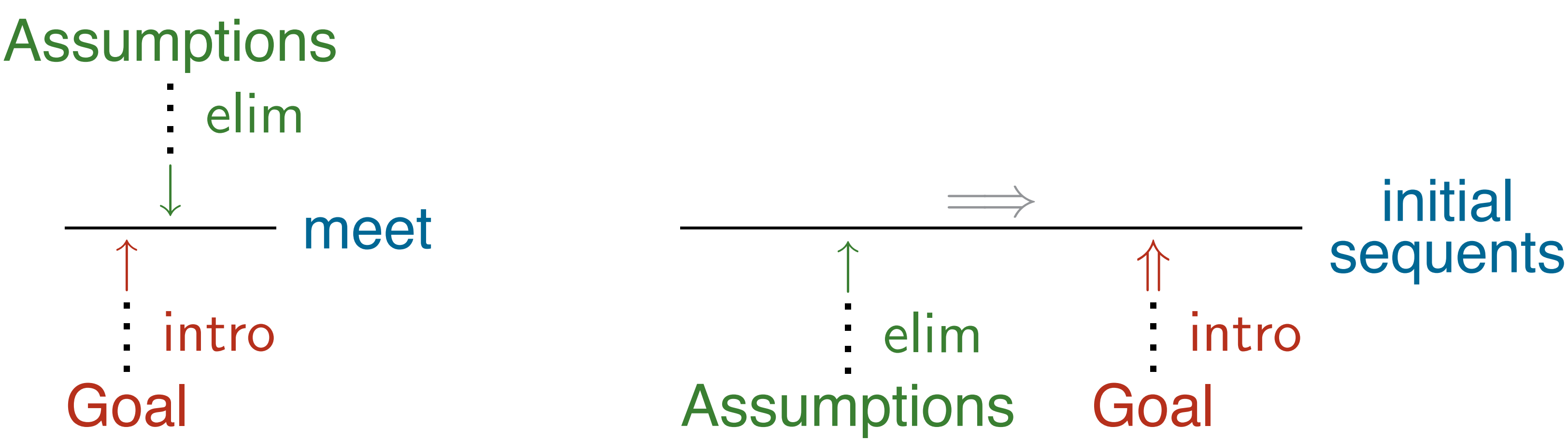
From ND to Sequent Calculus

Normal form ND proofs | Sequent proofs



From ND to Sequent Calculus

Normal form ND proofs | Sequent proofs



Sequents pair $<\Gamma, \Delta>$ of finite lists, multisets, or sets of formulas

Notation: $\Gamma \Longrightarrow \Delta$ Γ conjunctiv and Δ disjunctive

Intuitive: a kind of implication, Δ “follows from” Γ

Sequent Calculus Rules I

- Initial Sequents: $$\frac{}{\Gamma, \mathbf{A} \Longrightarrow \Delta, \mathbf{A}}\ \text{init} \qquad (\mathbf{A} \text{ atomic})$$

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$$\frac{\Gamma, \mathbf{A}, \mathbf{B} \Longrightarrow \Delta}{\Gamma, \mathbf{A} \wedge \mathbf{B} \Longrightarrow \Delta} \wedge\text{L} \qquad \frac{\Gamma \Longrightarrow \Delta, \mathbf{A} \quad \Gamma \Longrightarrow \Delta, \mathbf{B}}{\Gamma \Longrightarrow \Delta, \mathbf{A} \wedge \mathbf{B}} \wedge\text{R}$$

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- Implication
$$\frac{\Gamma \Longrightarrow \Delta, \mathbf{A} \quad \Gamma, \mathbf{B} \Longrightarrow \Delta}{\Gamma, \mathbf{A} \Rightarrow \mathbf{B} \Longrightarrow \Delta}\Rightarrow\mathrm{L} \qquad \frac{\Gamma, \mathbf{A} \Longrightarrow \Delta, \mathbf{B}}{\Gamma \Longrightarrow \Delta, \mathbf{A} \Rightarrow \mathbf{B}}\Rightarrow\mathrm{R}$$

Sequent Calculus Rules I

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$$\frac{\Gamma, \mathbf{A}, \mathbf{B} \Longrightarrow \Delta}{\Gamma, \mathbf{A} \wedge \mathbf{B} \Longrightarrow \Delta}\ \wedge L \qquad \frac{\Gamma \Longrightarrow \Delta, \mathbf{A} \quad \Gamma \Longrightarrow \Delta, \mathbf{B}}{\Gamma \Longrightarrow \Delta, \mathbf{A} \wedge \mathbf{B}}\ \wedge R$$

- Implication

$$\frac{\Gamma \Longrightarrow \Delta, \mathbf{A} \quad \Gamma, \mathbf{B} \Longrightarrow \Delta}{\Gamma, \mathbf{A} \Rightarrow \mathbf{B} \Longrightarrow \Delta} \Rightarrow L \qquad \frac{\Gamma, \mathbf{A} \Longrightarrow \Delta, \mathbf{B}}{\Gamma \Longrightarrow \Delta, \mathbf{A} \Rightarrow \mathbf{B}} \Rightarrow R$$

- Truth and Falsehood $$\frac{}{\Gamma, \bot \Longrightarrow \Delta}\ \bot L \qquad \frac{}{\Gamma \Longrightarrow \Delta, \top}\ \top R$$

Sequent Calculus Rules II

- Negation: $$\frac{\Gamma \Longrightarrow \Delta, \mathbf{A}}{\Gamma, \neg\mathbf{A} \Longrightarrow \Delta}\ \neg\mathrm{L} \qquad \frac{\Gamma, \mathbf{A} \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, \neg\mathbf{A}}\ \neg\mathrm{R}$$

Sequent Calculus Rules II

- Negation: $$\frac{\Gamma \Longrightarrow \Delta, \mathbf{A}}{\Gamma, \neg\mathbf{A} \Longrightarrow \Delta}\ \neg\mathrm{L} \qquad \frac{\Gamma, \mathbf{A} \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, \neg\mathbf{A}}\ \neg\mathrm{R}$$
- Disjunction:

$$\frac{\Gamma \Longrightarrow \Delta, \mathbf{A}, \mathbf{B}}{\Gamma \Longrightarrow \Delta, \mathbf{A} \lor \mathbf{B}}\ \lor\mathrm{R} \qquad \frac{\Gamma, \mathbf{A} \Longrightarrow \Delta \quad \Gamma, \mathbf{B} \Longrightarrow \Delta}{\Gamma, \mathbf{A} \lor \mathbf{B} \Longrightarrow \Delta}\ \lor\mathrm{L}$$

Sequent Calculus Rules II

- Negation:
$$\frac{\Gamma \Longrightarrow \Delta, \mathbf{A}}{\Gamma, \neg\mathbf{A} \Longrightarrow \Delta}\ \neg\text{L} \qquad \frac{\Gamma, \mathbf{A} \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, \neg\mathbf{A}}\ \neg\text{R}$$

- Disjunction:
$$\frac{\Gamma \Longrightarrow \Delta, \mathbf{A}, \mathbf{B}}{\Gamma \Longrightarrow \Delta, \mathbf{A} \vee \mathbf{B}}\ \vee\text{R} \qquad \frac{\Gamma, \mathbf{A} \Longrightarrow \Delta \quad \Gamma, \mathbf{B} \Longrightarrow \Delta}{\Gamma, \mathbf{A} \vee \mathbf{B} \Longrightarrow \Delta}\ \vee\text{L}$$

- Universal Quantification:
$$\frac{\Gamma, \forall x_\bullet \mathbf{A}, \mathbf{A}[x/\mathbf{T}] \Longrightarrow \Delta}{\Gamma, \forall x_\bullet \mathbf{A} \Longrightarrow \Delta}\ \forall\text{L} \qquad \frac{\Gamma \Longrightarrow \Delta, \mathbf{A}[x/P^*]}{\Gamma \Longrightarrow \Delta, \forall x_\bullet \mathbf{A}}\ \forall\text{R}$$

(*: parameter P must be new in context)

Sequent Calculus Rules II

- Negation: $\dfrac{\Gamma \Longrightarrow \Delta, \mathbf{A}}{\Gamma, \neg\mathbf{A} \Longrightarrow \Delta}\ \neg L \qquad \dfrac{\Gamma, \mathbf{A} \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, \neg\mathbf{A}}\ \neg R$
- Disjunction:

$$\dfrac{\Gamma \Longrightarrow \Delta, \mathbf{A}, \mathbf{B}}{\Gamma \Longrightarrow \Delta, \mathbf{A} \vee \mathbf{B}}\ \vee R \qquad \dfrac{\Gamma, \mathbf{A} \Longrightarrow \Delta \quad \Gamma, \mathbf{B} \Longrightarrow \Delta}{\Gamma, \mathbf{A} \vee \mathbf{B} \Longrightarrow \Delta}\ \vee L$$

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(*: parameter P must be new in context)

- Existential Quantification:

$$\dfrac{\Gamma, \mathbf{A}[x/P^*] \Longrightarrow \Delta}{\Gamma, \exists x_{\bullet}\mathbf{A} \Longrightarrow \Delta}\ \exists L \qquad \dfrac{\Gamma \Longrightarrow \Delta, \exists x_{\bullet}\mathbf{A}, \mathbf{A}[x/\mathbf{T}]}{\Gamma \Longrightarrow \Delta, \exists x_{\bullet}\mathbf{A}}\ \exists R$$

(*: parameter P must be new in context)

Example Proof

$$
\dfrac{\dfrac{\overline{\mathbf{A},\mathbf{B} \Longrightarrow \mathbf{B}}\;\text{init}}{\mathbf{A}\wedge\mathbf{B} \Longrightarrow \mathbf{B}}\;\wedge L
\qquad
\dfrac{\dfrac{\dfrac{}{\mathbf{A},\mathbf{B} \Longrightarrow \mathbf{C},\mathbf{A}}\;\text{init}}{\mathbf{A}\wedge\mathbf{B} \Longrightarrow \mathbf{C},\mathbf{A}}\;\wedge L}{\mathbf{A}\wedge\mathbf{B} \Longrightarrow \mathbf{C}\vee\mathbf{A}}\;\vee R}
{\dfrac{\mathbf{A}\wedge\mathbf{B} \Longrightarrow \mathbf{B}\wedge(\mathbf{C}\vee\mathbf{A})}{\Longrightarrow (\mathbf{A}\wedge\mathbf{B}) \Rightarrow \mathbf{B}\wedge(\mathbf{C}\vee\mathbf{A})}\;\Rightarrow R}\;\wedge R
$$

Sequent Calculus: Cut-rule

- To map natural deductions (in $\vdash$ and $\models^{\pm}_{ic}$) to sequent calculus derivations we add the so called cut-rule:

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Sequent Calculus: Cut-rule

- To map natural deductions (in $\vdash$ and $\vdash^{\pm}_{ic}$) to sequent calculus derivations we add the so called cut-rule:

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- The question whether normal form proof search ($\vdash_{ic}$) is complete corresponds to the question whether the cut-rule can be eliminated (is *admissible*) in sequent calculus.

Sequent Calculus

Let $\Longrightarrow^{+}$ denote the sequent calculus with cut-rule and $\Longrightarrow$ the sequent calculus without the cut-rule.

Sequent Calculus

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Theorem 3 (Soundness of $\Longrightarrow$ relative to $\Gamma \vdash_{\mathrm{ic}}$ and $\Gamma \models^{\pm}_{\mathrm{ic}}$)

(a) If $\Gamma \Longrightarrow \mathbf{C}$ then $\Gamma \vdash_{\mathrm{ic}} \mathbf{C} \uparrow$.

(b) If $\Gamma \Longrightarrow^{+} \mathbf{C}$ then $\Gamma \models^{\pm}_{\mathrm{ic}} \mathbf{C} \uparrow$.

Sequent Calculus

Let $\Longrightarrow^{+}$ denote the sequent calculus with cut-rule and $\Longrightarrow$ the sequent calculus without the cut-rule.

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(b) If $\Gamma \Longrightarrow^{+} \mathbf{C}$ then $\Gamma \Vdash^{\pm}_{\mathrm{ic}} \mathbf{C} \uparrow$.

Theorem 4 (Completeness of $\Longrightarrow$ relative to $\Gamma \vdash_{\mathrm{ic}}$ and $\Gamma \Vdash^{\pm}_{\mathrm{ic}}$)

(a) If $\Gamma \vdash_{\mathrm{ic}} \mathbf{C} \uparrow$ then $\Gamma \Longrightarrow \mathbf{C}$.

(b) If $\Gamma \Vdash^{\pm}_{\mathrm{ic}} \mathbf{C} \uparrow$ then $\Gamma \Longrightarrow^{+} \mathbf{C}$.

Gentzen's Hauptsatz

Theorem 5 (Cut-Elimination): Cut-elimination holds for the sequent calculus. In other words: The cut rule is *admissible* in the sequent calculus.

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$$\text{If } \Gamma \Longrightarrow^{+} \Delta \text{ then } \Gamma \Longrightarrow \Delta$$

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Proof non-trivial; main means: nested inductions and case distinctions over rule applications

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Proof non-trivial; main means: nested inductions and case distinctions over rule applications

This result qualifies the sequent calculus as suitable for automating proof search.

Applications of Cut-Elimination

Theorem (Normalization for ND):

If $\Gamma \vdash \mathbf{C}$ then $\Gamma \vdash_{\mathrm{ic}} \mathbf{C} \uparrow$.

Applications of Cut-Elimination

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$$\text{If } \Gamma \vdash \mathbf{C} \text{ then } \Gamma \vdash_{\text{ic}} \mathbf{C} \uparrow .$$

Proof sketch:

Applications of Cut-Elimination

Theorem (Normalization for ND):

If $\Gamma \vdash \mathbf{C}$ then $\Gamma \vdash_{ic} \mathbf{C} \uparrow$.

Proof sketch:

- Assume $\Gamma \vdash \mathbf{C}$.

Applications of Cut-Elimination

Theorem (Normalization for ND):

If $\Gamma \vdash \mathbf{C}$ then $\Gamma \vdash_{ic} \mathbf{C} \uparrow$.

Proof sketch:

- Assume $\Gamma \vdash \mathbf{C}$.
- Then $\Gamma \models^{\pm}_{ic} \mathbf{C} \uparrow$ by completeness of $\models^{\pm}_{ic}$.

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Proof sketch:

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- Then $\Gamma \Longrightarrow^{+} \mathbf{C}$ by completeness of $\Longrightarrow^{+}$.
- Then $\Gamma \Longrightarrow \mathbf{C}$ by cut-elimination.

Applications of Cut-Elimination

Theorem (Normalization for ND):

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- Assume $\Gamma \vdash \mathbf{C}$.
- Then $\Gamma \vdash^{\pm}_{\mathsf{ic}} \mathbf{C} \uparrow$ by completeness of $\vdash^{\pm}_{\mathsf{ic}}$.
- Then $\Gamma \Longrightarrow^{+} \mathbf{C}$ by completeness of $\Longrightarrow^{+}$.
- Then $\Gamma \Longrightarrow \mathbf{C}$ by cut-elimination.
- Then $\Gamma \vdash_{\mathsf{ic}} \mathbf{C} \uparrow$ by soundness of $\Longrightarrow$.

What have we done?

Natural Deduction		
$\vdash$ (with detours) $\longrightarrow$		

What have we done?

Natural Deduction	Intercalation	
$\vdash$ (with detours) $\longrightarrow$	$\longrightarrow$ $\models^{\pm}_{ic}$ (with roundabout) $\longrightarrow$	

What have we done?

Natural Deduction	Intercalation	Sequent Calculus
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Applications of Cut-Elimination

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Proof sketch:

- Assume there is a proof of $\vdash \bot$.
- Then $\Longrightarrow^{+} \bot$ by completeness of $\Longrightarrow^{+}$ and $\models^{\pm}_{ic}$.
- Then $\Longrightarrow \bot$ by cut-elimination.
- But $\Longrightarrow \bot$ cannot be the conclusion of any sequent rule.

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- We have illustrated the connection of
  - natural deduction and sequent calculus
  - normal form natural deductions and cut-free sequent calculus.
- Fact: Sequent calculus often employed as meta-theory for specialized proof search calculi and strategies.
- Question: Can these calculi and strategies be transformed to natural deduction proof search?

Calculi: Higher-Order Natural Deduction

ND Calculi for HOL

Some conventions for this part:

- signature Σ contains only the logical constants $\neg, \vee, \Pi^{\alpha}$ unless stated otherwise.

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- signature Σ contains only the logical constants $\neg, \vee, \Pi^{\alpha}$ unless stated otherwise.
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- context representation of ND calculi

ND Calculi for HOL

Inference rules for $\mathfrak{NK}_\beta$

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}}\;\mathfrak{NK}(Hyp)$$

ND Calculi for HOL

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ND Calculi for HOL

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$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg \mathbf{A}}\ \mathfrak{NK}(\neg I)$$

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$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg\mathbf{A}}\,\mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\neg E)$$

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$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\ \mathfrak{NK}(\vee I_L)$$

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$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\ \mathfrak{NK}(\vee I_L) \qquad \frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\ \mathfrak{NK}(\vee I_R)$$

ND Calculi for HOL

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$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}}\ \mathfrak{NK}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\ \mathfrak{NK}(\beta)$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg\mathbf{A}}\ \mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\ \mathfrak{NK}(\neg E)$$

$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\ \mathfrak{NK}(\vee I_L) \qquad \frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\ \mathfrak{NK}(\vee I_R)$$

$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}}\ \mathfrak{NK}(\vee E)$$

ND Calculi for HOL

Inference rules for $\mathfrak{NK}_\beta$

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}}\ \mathfrak{NK}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\ \mathfrak{NK}(\beta)$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg\mathbf{A}}\ \mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\ \mathfrak{NK}(\neg E)$$

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$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}}\ \mathfrak{NK}(\vee E)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathsf{w}_\alpha \quad \mathsf{w} \text{ new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}}\ \mathfrak{NK}(\Pi I)^{\mathsf{w}}$$

ND Calculi for HOL

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$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg\mathbf{A}}\ \mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\ \mathfrak{NK}(\neg E)$$

$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\ \mathfrak{NK}(\vee I_L) \qquad \frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\ \mathfrak{NK}(\vee I_R)$$

$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}}\ \mathfrak{NK}(\vee E)$$

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ND Calculi for HOL

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$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg\mathbf{A}}\,\mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\neg E)$$

$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\,\mathfrak{NK}(\vee I_L) \qquad \frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\,\mathfrak{NK}(\vee I_R)$$

$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\vee E)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathsf{w}_\alpha \quad \text{w new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}}\,\mathfrak{NK}(\Pi I)^{\mathsf{w}}$$

$$\frac{\Phi \Vdash \Pi^\alpha \mathbf{G}}{\Phi \Vdash \mathbf{G}\mathbf{A}}\,\mathfrak{NK}(\Pi E) \qquad \frac{\Phi * \neg\mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(Contr)$$

ND Calculi for HOL

Inference rules for $\mathfrak{NK}_\beta$ (for richer signatures)

$$\frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{A}}\ \mathfrak{NK}(\wedge E_L)$$

ND Calculi for HOL

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$$\frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{A}}\ \mathfrak{NK}(\wedge E_L) \qquad \frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{B}}\ \mathfrak{NK}(\wedge E_R)$$

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$$\frac{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\Rightarrow E)$$

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$$\frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(\wedge E_L) \qquad \frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\wedge E_R) \qquad \frac{\Phi \Vdash \mathbf{A} \quad \Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}\,\mathfrak{NK}(\wedge I)$$

$$\frac{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\Rightarrow E) \qquad \frac{\Phi, \mathbf{A} \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B}}\,\mathfrak{NK}(\Rightarrow I)$$

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$$\frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{A}}\ \mathfrak{NK}(\wedge E_L) \qquad \frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{B}}\ \mathfrak{NK}(\wedge E_R) \qquad \frac{\Phi \Vdash \mathbf{A} \quad \Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}\ \mathfrak{NK}(\wedge I)$$

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$$\frac{\Phi \Vdash \mathbf{G}\mathbf{T}_\alpha}{\Phi \Vdash \Sigma^\alpha \mathbf{G}}\ \mathfrak{NK}(\Sigma I)$$

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$$\frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(\wedge E_L) \qquad \frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\wedge E_R) \qquad \frac{\Phi \Vdash \mathbf{A} \quad \Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}\,\mathfrak{NK}(\wedge I)$$

$$\frac{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\Rightarrow E) \qquad \frac{\Phi, \mathbf{A} \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B}}\,\mathfrak{NK}(\Rightarrow I)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathbf{T}_\alpha}{\Phi \Vdash \Sigma^\alpha \mathbf{G}}\,\mathfrak{NK}(\Sigma I) \qquad \frac{\Phi \Vdash \Sigma^\alpha \mathbf{G} \quad \Phi * \mathbf{G}\mathrm{w}_\alpha \Vdash \mathbf{C} \quad \text{w new parameter}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\Sigma E)$$

ND Calculi for HOL

Inference rules for $\mathfrak{NK}_\beta$ (for richer signatures)

$$\frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(\wedge E_L) \qquad \frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\wedge E_R) \qquad \frac{\Phi \Vdash \mathbf{A} \quad \Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}\,\mathfrak{NK}(\wedge I)$$

$$\frac{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\Rightarrow E) \qquad \frac{\Phi, \mathbf{A} \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B}}\,\mathfrak{NK}(\Rightarrow I)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathbf{T}_\alpha}{\Phi \Vdash \Sigma^\alpha \mathbf{G}}\,\mathfrak{NK}(\Sigma I) \qquad \frac{\Phi \Vdash \Sigma^\alpha \mathbf{G} \quad \Phi * \mathbf{G}\mathrm{w}_\alpha \Vdash \mathbf{C} \quad \text{w new parameter}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\Sigma E)$$

$$\frac{\Phi \Vdash \mathbf{T} =^\alpha \mathbf{W} \quad \Phi \Vdash \mathbf{A}[\mathbf{T}]}{\Phi \Vdash \mathbf{A}[\mathbf{W}]}\,\mathfrak{NK}(= Subst)$$

ND Calculi for HOL

Inference rules for $\mathfrak{NK}_\beta$ (for richer signatures)

$$\frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(\wedge E_L) \qquad \frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\wedge E_R) \qquad \frac{\Phi \Vdash \mathbf{A} \quad \Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}\,\mathfrak{NK}(\wedge I)$$

$$\frac{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\Rightarrow E) \qquad \frac{\Phi, \mathbf{A} \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B}}\,\mathfrak{NK}(\Rightarrow I)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathbf{T}_\alpha}{\Phi \Vdash \Sigma^\alpha \mathbf{G}}\,\mathfrak{NK}(\Sigma I) \qquad \frac{\Phi \Vdash \Sigma^\alpha \mathbf{G} \quad \Phi * \mathbf{G}w_\alpha \Vdash \mathbf{C} \quad w \text{ new parameter}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\Sigma E)$$

$$\frac{\Phi \Vdash \mathbf{T} =^\alpha \mathbf{W} \quad \Phi \Vdash \mathbf{A}[\mathbf{T}]}{\Phi \Vdash \mathbf{A}[\mathbf{W}]}\,\mathfrak{NK}(= Subst) \qquad \frac{}{\Phi \Vdash \mathbf{A} = \mathbf{A}}\,\mathfrak{NK}(= Refl)$$

ND Calculi for HOL

Inference rules for $\mathfrak{NK}_\beta$ (for richer signatures)

$$\frac{\Phi \Vdash \mathbf{A} \land \mathbf{B}}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(\land E_L) \qquad \frac{\Phi \Vdash \mathbf{A} \land \mathbf{B}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\land E_R) \qquad \frac{\Phi \Vdash \mathbf{A} \quad \Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \land \mathbf{B}}\,\mathfrak{NK}(\land I)$$

$$\frac{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\Rightarrow E) \qquad \frac{\Phi, \mathbf{A} \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B}}\,\mathfrak{NK}(\Rightarrow I)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathbf{T}_\alpha}{\Phi \Vdash \Sigma^\alpha \mathbf{G}}\,\mathfrak{NK}(\Sigma I) \qquad \frac{\Phi \Vdash \Sigma^\alpha \mathbf{G} \quad \Phi * \mathbf{G}\mathsf{w}_\alpha \Vdash \mathbf{C} \quad \mathsf{w}\text{ new parameter}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\Sigma E)$$

$$\frac{\Phi \Vdash \mathbf{T} =^\alpha \mathbf{W} \quad \Phi \Vdash \mathbf{A}[\mathbf{T}]}{\Phi \Vdash \mathbf{A}[\mathbf{W}]}\,\mathfrak{NK}(= Subst) \qquad \frac{}{\Phi \Vdash \mathbf{A} = \mathbf{A}}\,\mathfrak{NK}(= Refl)$$

Alternative: Define logical constants $\land, \Rightarrow, \Sigma$, etc. in terms of $\neg, \lor, \Pi$ as usual and strictly use Leibniz equality instead of primitive equality; then the above rules are not needed.

ND Calculi for HOL

Inference rules for extensionality (rules for $\xi, \eta, \mathfrak{f}, \mathfrak{b}$)

$$\frac{\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\eta)$$

ND Calculi for HOL

Inference rules for extensionality (rules for $\xi, \eta, \mathfrak{f}, \mathfrak{b}$)

$$\frac{\mathbf{A} \overset{\beta\eta}{=} \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}} \mathfrak{NK}(\eta) \qquad \frac{\Phi \Vdash \forall \mathsf{x}_\alpha \centerdot \mathbf{M} \doteq^{\beta} \mathbf{N}}{\Phi \Vdash (\lambda \mathsf{x}_\alpha \centerdot \mathbf{M}) \doteq^{\beta\alpha} (\lambda \mathsf{x}_\alpha \centerdot \mathbf{N})} \mathfrak{NK}(\xi)$$

ND Calculi for HOL

Inference rules for extensionality (rules for $\xi, \eta, \mathfrak{f}, \mathfrak{b}$)

$$\frac{\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\, \mathfrak{NK}(\eta) \qquad \frac{\Phi \Vdash \forall \mathsf{x}_\alpha \mathbf{.} \mathbf{M} \doteq^{\beta} \mathbf{N}}{\Phi \Vdash (\lambda \mathsf{x}_\alpha \mathbf{.} \mathbf{M}) \doteq^{\beta\alpha} (\lambda \mathsf{x}_\alpha \mathbf{.} \mathbf{N})}\, \mathfrak{NK}(\xi)$$

$$\frac{\Phi \Vdash \forall \mathsf{x}_\alpha \mathbf{.} \mathbf{G}\mathsf{x} \doteq^{\beta} \mathbf{H}\mathsf{x}}{\Phi \Vdash \mathbf{G} \doteq^{\beta\alpha} \mathbf{H}}\, \mathfrak{NK}(\mathfrak{f})$$

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$$\frac{\Phi \Vdash \forall \mathsf{x}_\alpha\text{.}\mathbf{G}\mathsf{x} \doteq^{\beta} \mathbf{H}\mathsf{x}}{\Phi \Vdash \mathbf{G} \doteq^{\beta\alpha} \mathbf{H}}\ \mathfrak{NK}(\mathfrak{f})$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{B} \quad \Phi * \mathbf{B} \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \doteq^{o} \mathbf{B}}\ \mathfrak{NK}(\mathfrak{b})$$

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$$\frac{\Phi \Vdash \forall \mathsf{x}_\alpha.\mathbf{G}\mathsf{x} \doteq^{\beta} \mathbf{H}\mathsf{x}}{\Phi \Vdash \mathbf{G} \doteq^{\beta\alpha} \mathbf{H}}\,\mathfrak{NK}(\mathfrak{f})$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{B} \quad \Phi * \mathbf{B} \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \doteq^{o} \mathbf{B}}\,\mathfrak{NK}(\mathfrak{b})$$

In case of a primitive notion of equality we define respective extensionality rules also for $=$.

ND Calculi for HOL

- The Calculi $\mathfrak{NK}_*$

ND Calculuses for HOL

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  - The calculus $\mathfrak{NK}_\beta$ consists of the inference rules for $\mathfrak{NK}_\beta$ for the provability judgment $\Vdash$ between sets of sentences Φ and sentences $\mathbf{A}$. (We write $\Vdash \mathbf{A}$ for $\emptyset \Vdash \mathbf{A}$.) The rule $\mathfrak{NK}(\beta)$ incorporates β-equality into $\Vdash$. The others characterize ‘the semantics of the connectives and quantifiers.

ND Calculi for HOL

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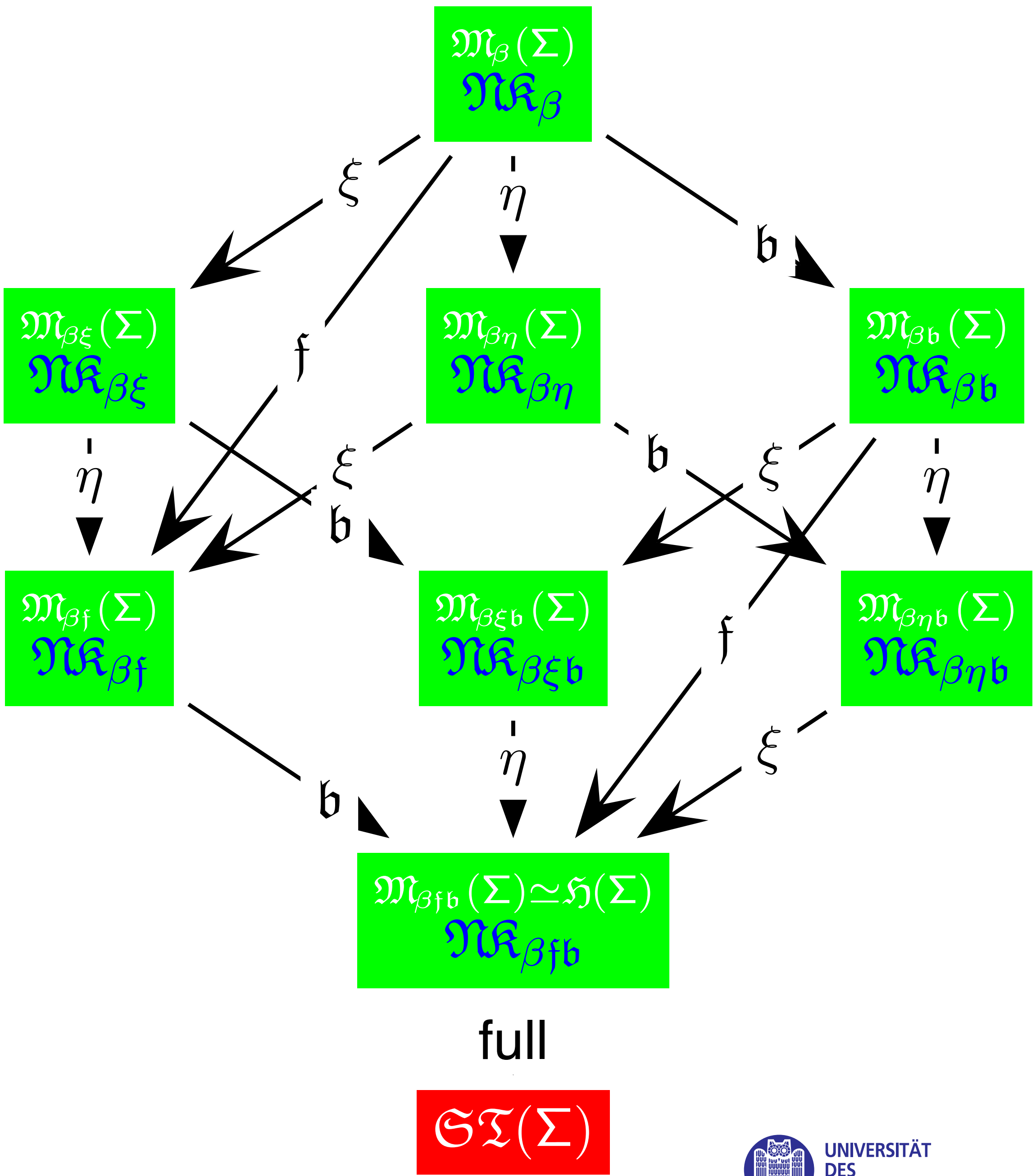
  - The calculus $\mathfrak{NK}_\beta$ consists of the inference rules for $\mathfrak{NK}_\beta$ for the provability judgment $\Vdash$ between sets of sentences Φ and sentences $\mathbf{A}$. (We write $\Vdash \mathbf{A}$ for $\emptyset \Vdash \mathbf{A}$.) The rule $\mathfrak{NK}(\beta)$ incorporates β-equality into $\Vdash$. The others characterize ‘the semantics of the connectives and quantifiers.
  - For $* \in \{\beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ we obtain the calculus $\mathfrak{NK}_*$ by adding the respective extensionality rules when specified in $*$.

ND Calculi for HOL



ND Calculi for HOL

- Note that $\mathfrak{NK}_\beta$ and $\mathfrak{NK}_{\beta\mathfrak{fb}}$ correspond to the extremes of the model classes in our landscape of model classes. For example, $\mathfrak{NK}_{\beta\mathfrak{fb}}$ will be proven sound and complete for Henkin models, and $\mathfrak{NK}_\beta$ will be proven sound and complete for $\mathfrak{M}_\beta(\Sigma)$.

ND Calculi for HOL

- Note that $\mathfrak{NK}_\beta$ and $\mathfrak{NK}_{\beta\mathfrak{fb}}$ correspond to the extremes of the model classes in our landscape of model classes. For example, $\mathfrak{NK}_{\beta\mathfrak{fb}}$ will be proven sound and complete for Henkin models, and $\mathfrak{NK}_\beta$ will be proven sound and complete for $\mathfrak{M}_\beta(\Sigma)$.
- Standard models do not admit (recursively axiomatizable) calculi that are sound and complete.

ND Calculi for HOL

- Note that $\mathfrak{NK}_\beta$ and $\mathfrak{NK}_{\beta\mathfrak{fb}}$ correspond to the extremes of the model classes in our landscape of model classes. For example, $\mathfrak{NK}_{\beta\mathfrak{fb}}$ will be proven sound and complete for Henkin models, and $\mathfrak{NK}_\beta$ will be proven sound and complete for $\mathfrak{M}_\beta(\Sigma)$.
- Standard models do not admit (recursively axiomatizable) calculi that are sound and complete.
- In the following we will develop the abstract consistency proof method for HOL (wrt all the different semantic classes $\mathfrak{M}_*(\Sigma)$ in our landscape) and we will analyse soundness and completeness of each $\mathfrak{NK}_*$ with respect to each corresponding model class $\mathfrak{M}_*(\Sigma)$ with the help of the abstract consistency method.

ND Calculi for HOL

- (Soundness for $\mathfrak{NK}_*$)
  $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

  Proof: ...exercise ...

ND Calculi for HOL

- (Soundness for $\mathfrak{NK}_*$)
  $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. That is, if $\mathbf{\Phi} \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \mathbf{\Phi}$.

  Proof: . . . exercise . . .

- (Completeness for $\mathfrak{NK}_*$)
  Let $\mathbf{\Phi}$ be a sufficiently Σ-pure set of sentences, $\mathbf{A}$ be a sentence, and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. If $\mathbf{A}$ is valid in all models $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ that satisfy $\mathbf{\Phi}$, then $\mathbf{\Phi} \Vdash_{\mathfrak{NK}_*} \mathbf{A}$.

  Proof: . . . will follow . . .

ND Calculi for HOL

- (Soundness for $\mathfrak{NK}_*$)
  $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

  Proof: . . . exercise . . .

- (Completeness for $\mathfrak{NK}_*$)
  Let Φ be a sufficiently Σ-pure set of sentences, $\mathbf{A}$ be a sentence, and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. If $\mathbf{A}$ is valid in all models $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ that satisfy Φ, then $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{A}$.

  Proof: . . . will follow . . .

ND Calculi for HOL

Derivation of $\neg(\mathsf{p} \vee \neg\mathsf{p}) \Vdash (\mathsf{p} \vee \neg\mathsf{p})$

$$\dfrac{\dfrac{\dfrac{}{\neg(\mathsf{p} \vee \neg\mathsf{p}), \mathsf{p} \Vdash \neg(\mathsf{p} \vee \neg\mathsf{p})}\,\mathfrak{NK}(Hyp) \quad \dfrac{\dfrac{}{\neg(\mathsf{p} \vee \neg\mathsf{p}), \mathsf{p} \Vdash \mathsf{p}}\,\mathfrak{NK}(Hyp)}{\neg(\mathsf{p} \vee \neg\mathsf{p}), \mathsf{p} \Vdash (\mathsf{p} \vee \neg\mathsf{p})}\,\mathfrak{NK}(\vee I_L)}{\dfrac{\neg(\mathsf{p} \vee \neg\mathsf{p}), \mathsf{p} \Vdash \mathbf{F}_o}{\neg(\mathsf{p} \vee \neg\mathsf{p}) \Vdash \neg\mathsf{p}}\,\mathfrak{NK}(\neg I)}\,\mathfrak{NK}(\neg E)}{\neg(\mathsf{p} \vee \neg\mathsf{p}) \Vdash (\mathsf{p} \vee \neg\mathsf{p})}\,\mathfrak{NK}(\vee I_R)$$