Semantics of Classical Higher-Order Logic

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Own Research in HOL

Automated Theorem Proving

Own Research in HOL

Automated Theorem Proving

- Extensional Resolution, Equality Reasoning

Own Research in HOL

Semantics

Automated Theorem Proving

- Extensional Resolution, Equality Reasoning

Own Research in HOL

Semantics

- Model Classes (different extensionality properties)

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Own Research in HOL

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- Model Classes (different extensionality properties)
- Abstract Consistency Proof Method

Automated Theorem Proving

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Own Research in HOL

Semantics

- Model Classes (different extensionality properties)
- Abstract Consistency Proof Method
- Test Problems

Automated Theorem Proving

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Own Research in HOL

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- Model Classes (different extensionality properties)
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- Test Problems

Automated Theorem Proving

- Extensional Resolution, Equality Reasoning
- Combination with FO-ATP

Own Research in HOL

Semantics

- Model Classes (different extensionality properties)
- Abstract Consistency Proof Method
- Test Problems

Proof Theory

Automated Theorem Proving

- Extensional Resolution, Equality Reasoning
- Combination with FO-ATP

Own Research in HOL

Semantics

- Model Classes (different extensionality properties)
- Abstract Consistency Proof Method
- Test Problems

Proof Theory

- Cut-simulation

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Own Research in HOL

Semantics — ESSLLI-06, WS-05/06

- Model Classes (different extensionality properties) [JSL'04]
- Abstract Consistency Proof Method [JSL'04]
- Test Problems [TPHOLs'05]

Proof Theory

- Cut-simulation [IJCAR'06]

Automated Theorem Proving — SS-06 (DA), WS-04/05

- Extensional Resolution, Paramod. [CADE'98/99,Synthese'02]
- Combination with FO-ATP [LPAR'04]

Syntax

Syntax

HOL-Syntax: Simple Types

Simple Types $\mathcal{T}$:	o	(truth values)
	ι	(individuals)
	$(\alpha \longrightarrow \beta)$	(functions from α to β)

$(\alpha \longrightarrow \beta)$ is sometimes written $(\beta\alpha)$

$(\alpha \longrightarrow \beta \longrightarrow \gamma)$ abbreviates $(\alpha \longrightarrow (\beta \longrightarrow \gamma))$

HOL-Syntax: Simple Types

Simple Types $\mathcal{T}$:	o	(truth values)
	ι	(individuals)
	$(\alpha \rightarrow \beta)$	(functions from α to β)

$\mathcal{T}$ is a freely generated, inductive set.

Induction on Types: We can prove a property $\varphi(\alpha)$ holds for all types α by proving

- $\varphi(\mathsf{o})$
- $\varphi(\iota)$
- If $\varphi(\alpha)$ and $\varphi(\beta)$, then $\varphi(\alpha \rightarrow \beta)$.

HOL-Syntax: Simple Types

Simple Types $\mathcal{T}$:	o	(truth values)
	ι	(individuals)
	$(\alpha \rightarrow \beta)$	(functions from α to β)

Recursion on Types: We can uniquely define a family $\mathcal{D}_\alpha$ for $\alpha \in \mathcal{T}$ by specifying:

- $\mathcal{D}_{\mathsf{o}}$
- $\mathcal{D}_\iota$
- A rule for forming $\mathcal{D}_{\alpha \rightarrow \beta}$ given $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$.

HOL-Syntax: Simply Typed λ-Terms

Typed Terms:

X_α	Variables ($\mathcal{V}$)
c_α	Constants & Parameters (Σ & $\mathcal{P}$)
$(\mathbf{F}_{\alpha\longrightarrow\beta}\,\mathbf{B}_\alpha)_\beta$	Application
$(\lambda \mathsf{Y}_\alpha\,\mathbf{A}_\beta)_{\alpha\longrightarrow\beta}$	λ-abstraction

HOL-Syntax: Simply Typed λ-Terms

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Equality of Terms:

α-conversion	Changing bound variables	
β-reduction	$((\lambda Y_\beta\,\mathbf{A}_\alpha)\,\mathbf{B}_\beta) \xrightarrow{\beta} [\mathbf{B}/Y]\mathbf{A}$	
η-reduction	$(\lambda Y_\alpha\,(\mathbf{F}_{\alpha\longrightarrow\beta}\,Y)) \xrightarrow{\eta} \mathbf{F}$	$(Y_\beta \notin \mathbf{Free}(\mathbf{F}))$

HOL-Syntax: Simply Typed λ-Terms

Typed Terms:

X_α	Variables ($\mathcal{V}$)
c_α	Constants & Parameters (Σ & $\mathcal{P}$)
$(\mathbf{F}_{\alpha\longrightarrow\beta}\,\mathbf{B}_\alpha)_\beta$	Application
$(\lambda \mathsf{Y}_\alpha\, \mathbf{A}_\beta)_{\alpha\longrightarrow\beta}$	λ-abstraction

Equality of Terms:

Every term has a unique $\beta\eta$-normal form (up to α-conversion).

HOL: Adding Logical Connectives

- $\top_o$ – true
- $\bot_o$ – false
- $\neg_{o\to o}$ – negation
- $\vee_{o\to o\to o}$ – disjunction
- $\wedge_{o\to o\to o}$ – conjunction
- $\supset_{o\to o\to o}$ – implication
- $\Leftrightarrow_{o\to o\to o}$ – equivalence
- $\forall X_\alpha \ldots$ – universal quantification over type α (∀ types α)
- $\exists X_\alpha \ldots$ – existential quantification over type α (∀ types α)
- $=_{\alpha\to\alpha\to o}$ – equality at type α (∀ types α)

HOL: Adding Logical Constants to Σ

Our choice for signature Σ:

- $\neg_{o \to o}$ – negation
- $\vee_{o \to o \to o}$ – disjunction
- $\Pi_{(\alpha \to o) \to o}$ – universal quantification over type α ($\forall$ types α)
- $=_{\alpha \to \alpha \to o}$ – equality at type α ($\forall$ types α)

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Use abbreviations for other logical operators

$\mathbf{A} \vee \mathbf{B}$	means	$(\vee\, \mathbf{A}\, \mathbf{B})$
$\mathbf{A} \wedge \mathbf{B}$	means	$\neg(\neg \mathbf{A} \vee \neg \mathbf{B})$
$\mathbf{A} \supset \mathbf{B}$	means	$\neg \mathbf{A} \vee \mathbf{B}$
$\mathbf{A} \Leftrightarrow \mathbf{B}$	means	$(\mathbf{A} \supset \mathbf{B}) \wedge (\mathbf{B} \supset \mathbf{A})$
$\forall X\, \mathbf{A}$	means	$\Pi(\lambda X\, \mathbf{A})$
$\exists X\, \mathbf{A}$	means	$\neg(\forall X\, \neg \mathbf{A})$

HOL: Adding Logical Constants to Σ

Our choice for signature Σ:

- $\neg_{o \to o}$ – negation
- $\vee_{o \to o \to o}$ – disjunction
- $\Pi_{(\alpha \to o) \to o}$ – universal quantification over type α ($\forall$ types α)

Use Leibniz-equality to encode equality

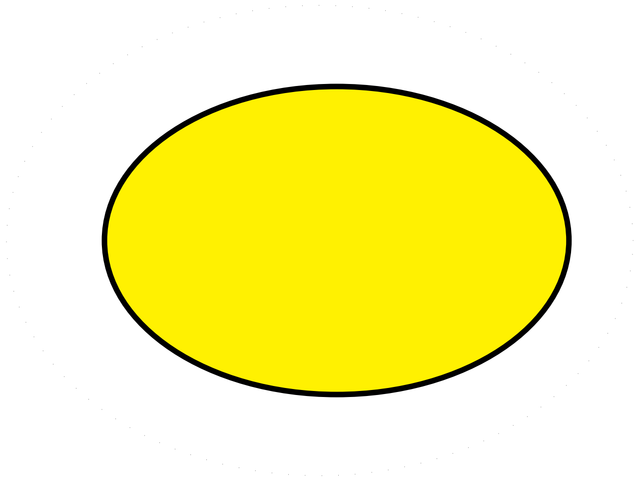
$\mathbf{A}_\alpha \doteq \mathbf{B}_\alpha$ means $\forall P_{\alpha \to o}(P\,\mathbf{A} \supset P\,\mathbf{B})$

resp. $\Pi(\lambda P_{\alpha \to o}(\neg P \mathbf{A} \vee P \mathbf{B}))$

Semantics

Model Classes
(different extensionality
properties)

Model Classes (Extensionality)



Standard Models $\mathfrak{ST}(\Sigma)$

- Idea of Standard Semantics:

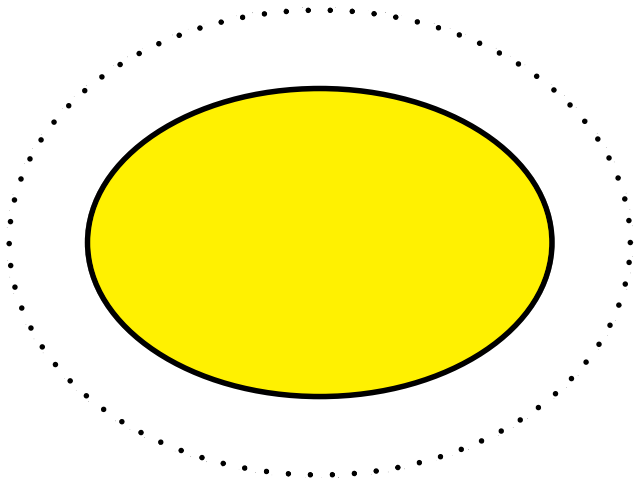
$\iota \longrightarrow \mathcal{D}_\iota$ (choose)

$o \longrightarrow \mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$ (fixed)

$(\alpha \to \beta) \longrightarrow$

$\mathcal{D}_{\alpha\to\beta} = \mathcal{F}(\mathcal{D}_\alpha, \mathcal{D}_\beta)$ (fixed)

Model Classes (Extensionality)



Standard Models $\mathfrak{ST}(\Sigma)$

- Idea of Standard Semantics:

$$\iota \longrightarrow \mathcal{D}_\iota \qquad \text{(choose)}$$
$$o \longrightarrow \mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\} \qquad \text{(fixed)}$$
$$(\alpha \to \beta) \longrightarrow$$
$$\mathcal{D}_{\alpha\to\beta} = \mathcal{F}(\mathcal{D}_\alpha, \mathcal{D}_\beta) \qquad \text{(fixed)}$$

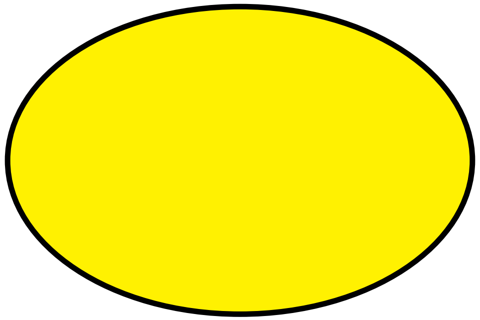
- Henkin's Generalization:

$$\mathcal{D}_{\alpha\to\beta} \subseteq \mathcal{F}(\mathcal{D}_\alpha, \mathcal{D}_\beta) \qquad \text{(choose)}$$

but elements are still functions!

[Henkin-50]

Model Classes (Extensionality)

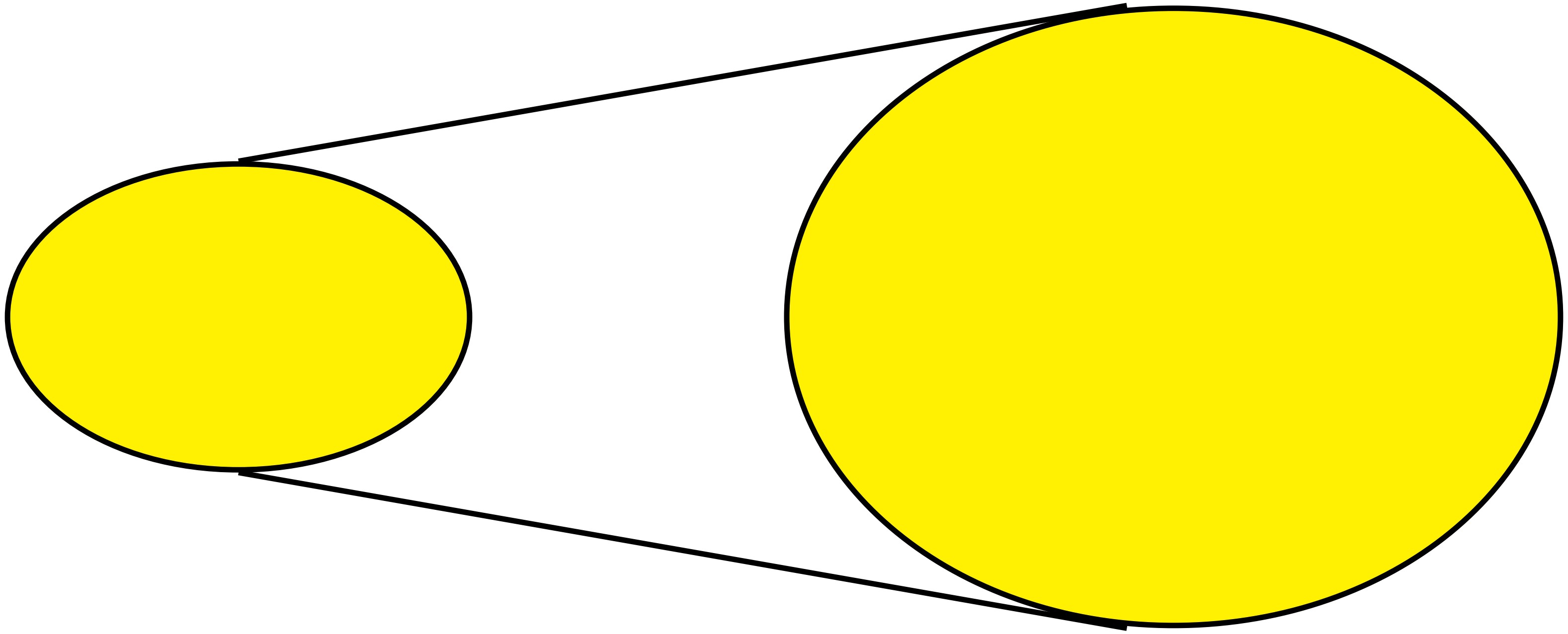


Standard Models $\mathfrak{ST}(\Sigma)$

choose: $\mathcal{D}_\iota$

fixed: $\mathcal{D}_o, \mathcal{D}_{\alpha\to\beta}$, functions

Model Classes (Extensionality)



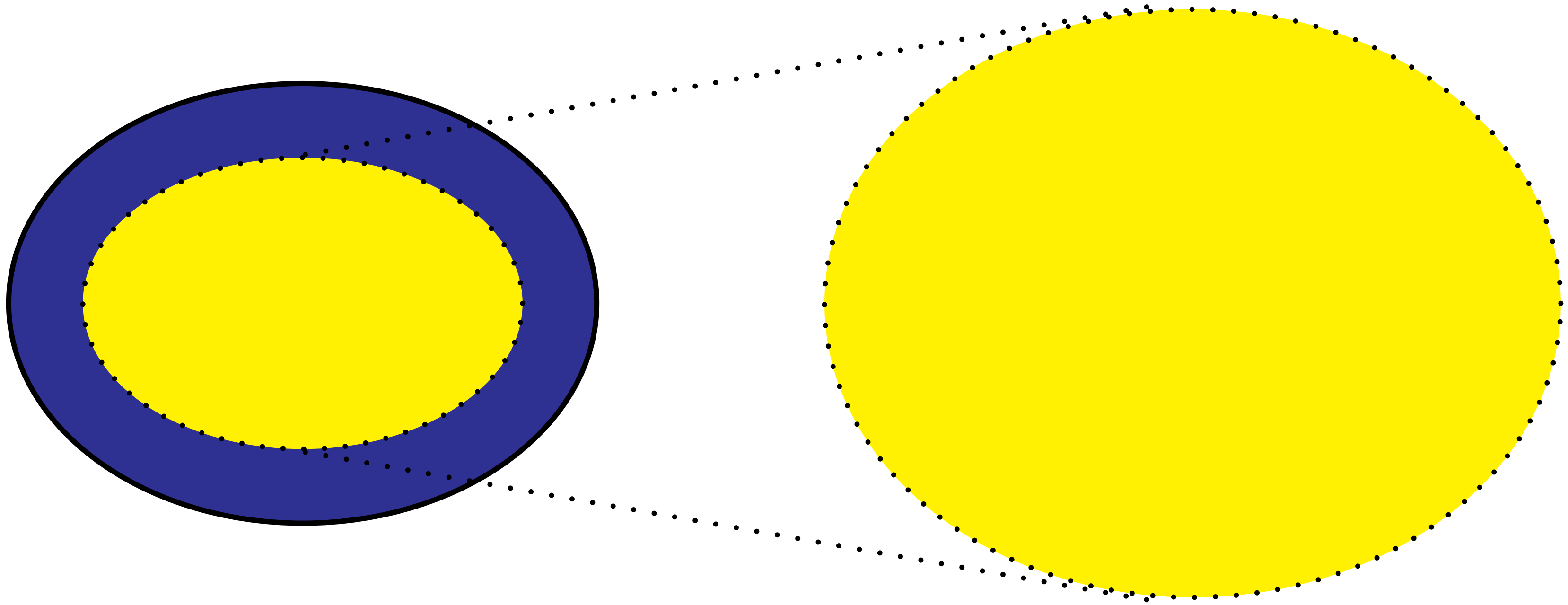
Standard Models $\mathfrak{ST}(\Sigma)$

Formulas valid in $\mathfrak{ST}(\Sigma)$

choose: $\mathcal{D}_\iota$

fixed: $\mathcal{D}_o, \mathcal{D}_{\alpha\to\beta}$, functions

Model Classes (Extensionality)



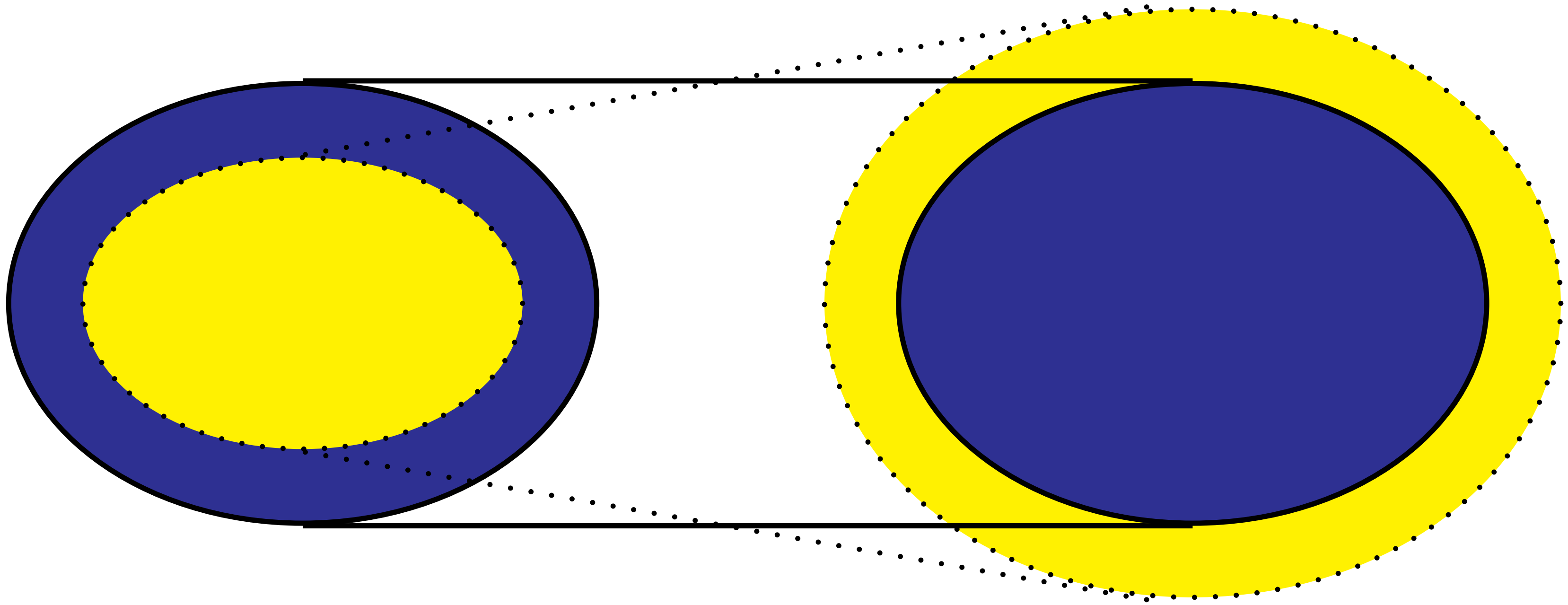
Henkin Models $\mathfrak{H}(\Sigma) = \mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma)$

choose: $\mathcal{D}_\iota, \mathcal{D}_{\alpha\rightarrow\beta}$

fixed: $\mathcal{D}_o$, functions

Formulas valid in $\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma)$?

Model Classes (Extensionality)



Henkin Models $\mathfrak{H}(\Sigma) = \mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma)$

Formulas valid in $\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma)$

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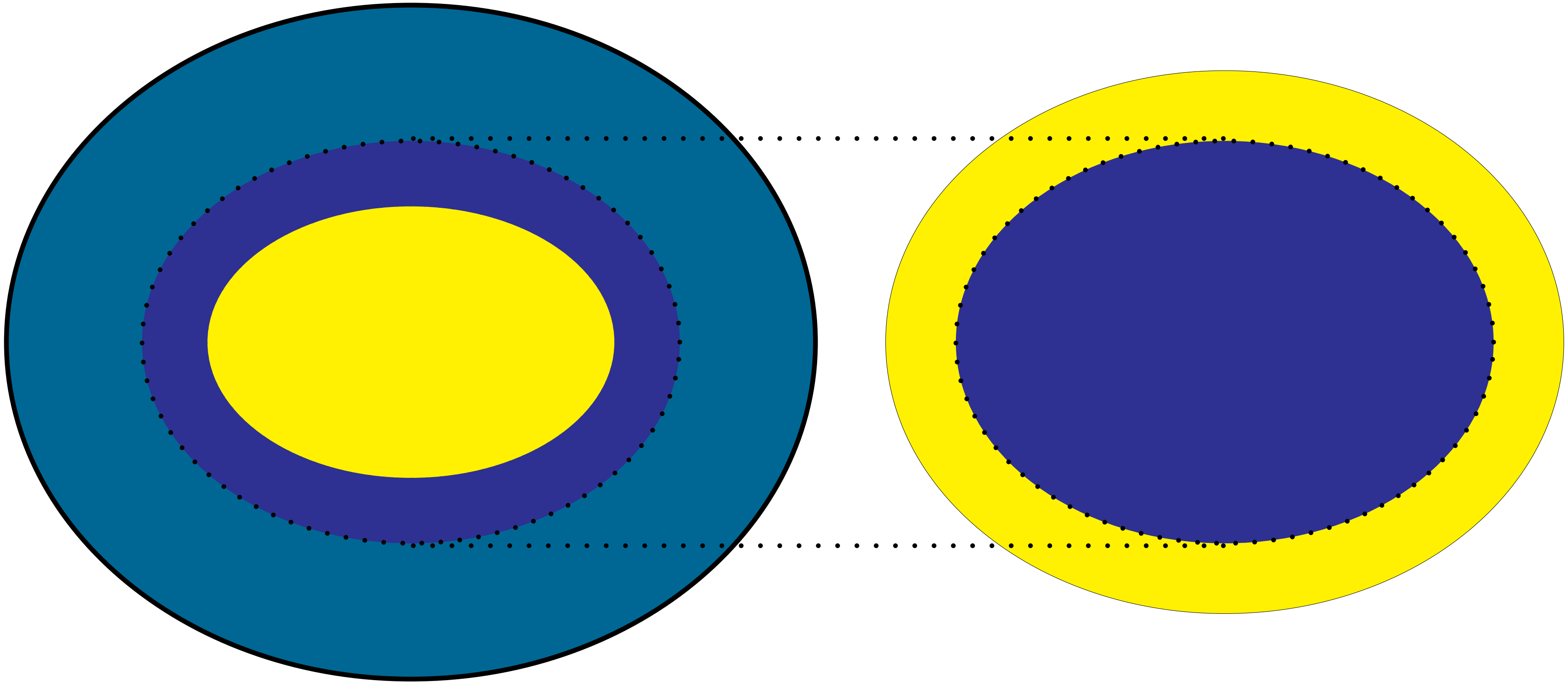
Model Classes (Extensionality)



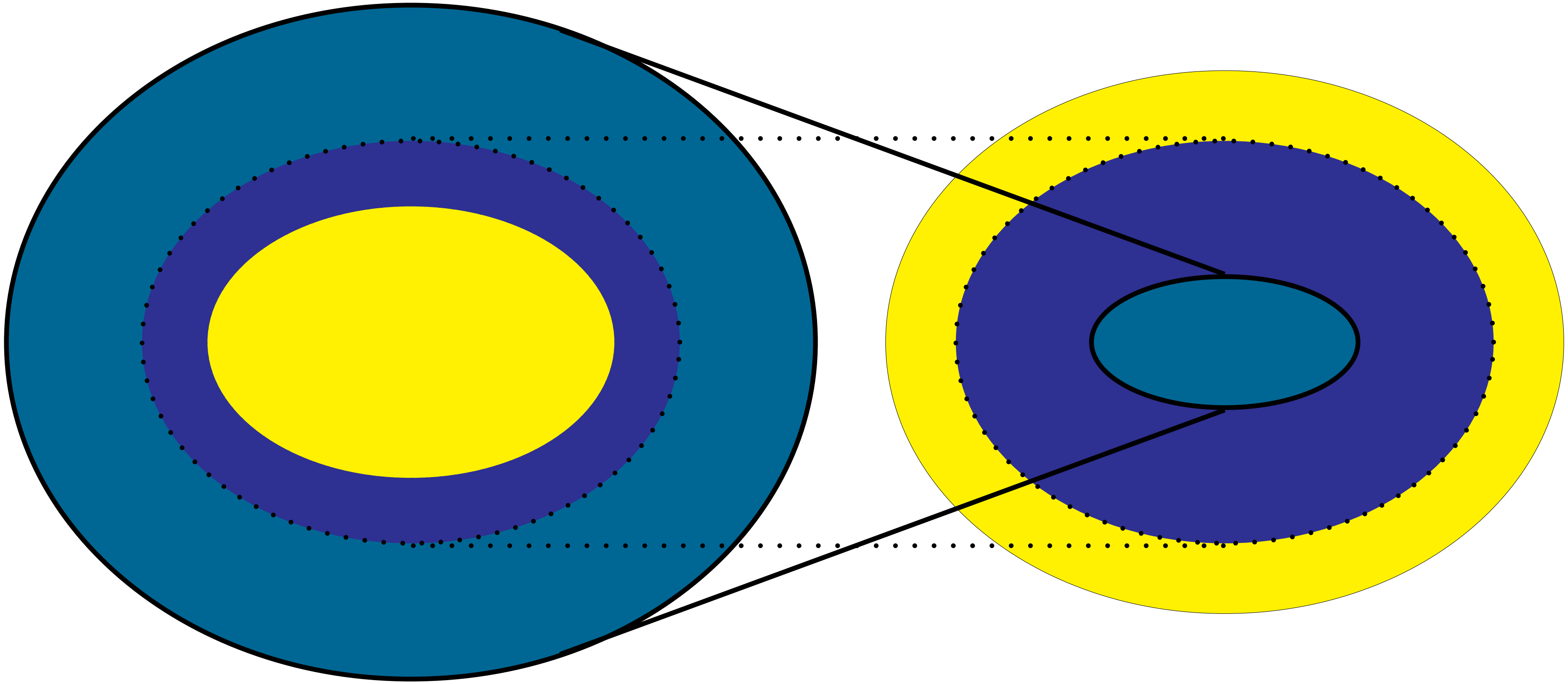
Non-Extensional Models $\mathfrak{M}_\beta(\Sigma)$

Formulas valid in $\mathfrak{M}_\beta(\Sigma)$?

choose: $\mathcal{D}_\iota, \mathcal{D}_{\alpha\rightarrow\beta}$, also non−functions, $\mathcal{D}_o$

fixed:

Model Classes (Extensionality)



Non-Extensional Models $\mathfrak{M}_\beta(\Sigma)$

Formulas valid in $\mathfrak{M}_\beta(\Sigma)$?

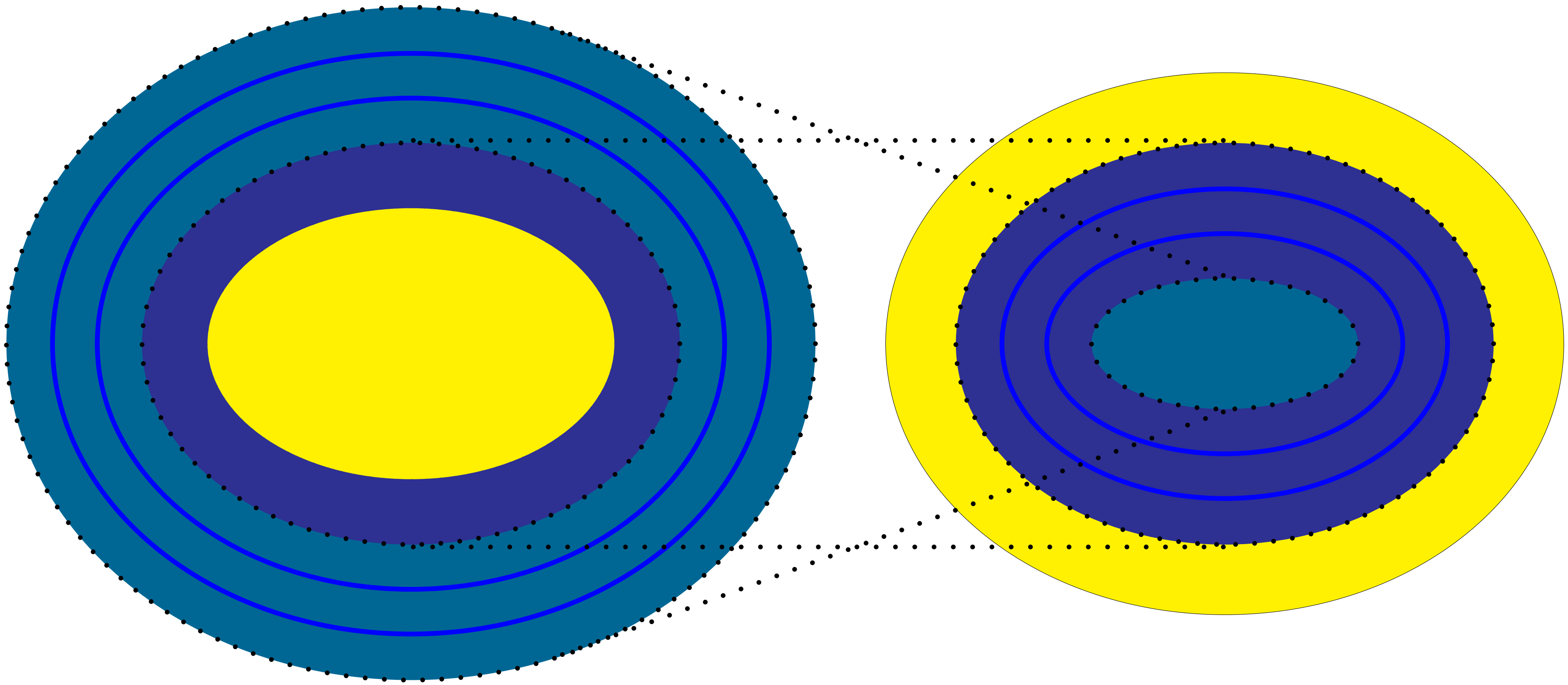
choose: $\mathcal{D}_\iota, \mathcal{D}_{\alpha\to\beta}$, also non−functions, $\mathcal{D}_o$

fixed:

Ex.: $\forall X. \forall Y. X \vee Y \Leftrightarrow Y \vee X$

vs. $\vee \doteq \lambda X. \lambda Y. Y \vee X$

Model Classes (Extensionality)



We additionally studied different model classes with 'varying degrees of extensionality'

$\forall X. \forall Y. X \vee Y \Leftrightarrow Y \vee X$ $\qquad$ $\forall X. \forall Y. X \vee Y \doteq Y \vee X$

$\lambda X. \lambda Y. X \vee Y \doteq \lambda X. \lambda Y. Y \vee X$ $\qquad$ $\vee \doteq \lambda X. \lambda Y. Y \vee X$

Model Classes (Extensionality)

$\mathfrak{M}_{\beta}(\Sigma)$

$\mathfrak{M}_{\beta}(\Sigma)$ non-extensional Σ-models

$\mathfrak{b}$: Boolean extensionality, $\mathcal{D}_{o} = \{\mathtt{T}, \mathtt{F}\}$

$\mathfrak{f}(= \eta + \xi)$: functional extensionality

η: η-functional

ξ: ξ-functionality

$\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma)$

$\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma)$ Henkin models

full

$\mathfrak{ST}(\Sigma)$

Model Classes (Extensionality)

$$
\begin{tikzcd}
\mathfrak{M}_{\beta}(\Sigma) \arrow[dr, "\mathfrak{b}"] \arrow[dd, dotted, "{\mathfrak{b},\, \mathfrak{f}(= \eta + \xi)}"'] & \\
& \mathfrak{M}_{\beta\mathfrak{b}}(\Sigma) \arrow[dl, "\mathfrak{f}"] \\
\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma) \arrow[d, dotted, "\text{full}"] & \\
\mathfrak{ST}(\Sigma) &
\end{tikzcd}
$$

Motivation for
Models without Functional Extensionality

- modeling programs:
  $p_1 \neq p_2$ even if $p_1@a = p_2@a$ for every $a \in \mathcal{D}_\alpha$
- consider, e.g., run-time complexity:
  $p_1 \leftarrow \lambda X.1$
  and
  $p_2 \leftarrow \lambda X.1 + (X+1)^2 - (X^2 + 2X + 1)$

Model Classes (Extensionality)

$$
\begin{tikzcd}
\mathfrak{M}_{\beta}(\Sigma) \arrow[dr, "\mathfrak{b}"] \arrow[dd, dotted, "{\mathfrak{b}, \mathfrak{f}(= \eta + \xi)}"'] & \\
& \mathfrak{M}_{\beta\mathfrak{b}}(\Sigma) \arrow[dl, "\mathfrak{f}"] \\
\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma) \arrow[d, dotted, "\text{full}"] & \\
\mathfrak{S}\mathfrak{T}(\Sigma) &
\end{tikzcd}
$$

Motivation for
Models without Boolean Extensionality?

- modeling of intensional concepts like 'knowledge', 'believe', etc.
- example:
  $\mathbf{O} := 2 + 2 = 4$
  $\mathbf{F} := \forall x, y, z, n > 2.x^n + y^n = z^n \Rightarrow x = y = z = 0$
  We want to model:
  $\mathbf{O} \Leftrightarrow \mathbf{F}$ but
  $\mathsf{john\_knows}(\mathbf{F}) \not\Leftrightarrow \mathsf{john\_knows}(\mathbf{O})$
- if we have $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$ then
  $\mathbf{O} \Leftrightarrow \mathbf{F}$ implies $\mathbf{O} = \mathbf{F}$
  which also enforces
  $\mathsf{john\_knows}(\mathbf{F}) \Leftrightarrow \mathsf{john\_knows}(\mathbf{O})$

Model Classes (Extensionality)

$$\begin{tikzcd}
 & \mathfrak{M}_{\beta}(\Sigma) \arrow[dl, "\mathfrak{f}"'] \arrow[dr, "\mathfrak{b}"] \arrow[dd, dotted, "{\mathfrak{b}, \mathfrak{f}(= \eta + \xi)}"] & \\
\mathfrak{M}_{\beta\mathfrak{f}}(\Sigma) \arrow[dr, "\mathfrak{b}"'] & & \mathfrak{M}_{\beta\mathfrak{b}}(\Sigma) \arrow[dl, "\mathfrak{f}"] \\
 & \mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma) \arrow[d, dotted, "\text{full}"] & \\
 & \mathfrak{S}\mathfrak{T}(\Sigma) &
\end{tikzcd}$$

Models without η

$$\mathcal{E}_{\varphi}(\mathsf{A}) = \mathcal{E}_{\varphi}(\mathsf{A}\downarrow_{\eta})$$

Model Classes (Extensionality)

$$
\begin{tikzcd}
 & \mathfrak{M}_{\beta}(\Sigma) \arrow[dl, "\mathfrak{f}"'] \arrow[dr, "\mathfrak{b}"] \arrow[dd, dotted, "{\mathfrak{b}, \mathfrak{f}(= \eta + \xi)}" description] & \\
\mathfrak{M}_{\beta\mathfrak{f}}(\Sigma) \arrow[dr, "\mathfrak{b}"'] & & \mathfrak{M}_{\beta\mathfrak{b}}(\Sigma) \arrow[dl, "\mathfrak{f}"] \\
 & \mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma) \arrow[d, dotted, "\text{full}"] & \\
 & \mathfrak{ST}(\Sigma) &
\end{tikzcd}
$$

Models without η

$$\mathcal{E}_{\varphi}(\mathsf{A}) = \mathcal{E}_{\varphi}(\mathsf{A}\downarrow_{\eta})$$

Model Classes (Extensionality)

$$
\begin{tikzcd}
& \mathfrak{M}_{\beta}(\Sigma) \arrow[dl, "\xi"'] \arrow[dr, "\mathfrak{b}"] \arrow[ddl, dotted, "\mathfrak{f}"] & \\
\mathfrak{M}_{\beta\xi}(\Sigma) \arrow[d, "\eta"] & & \mathfrak{M}_{\beta\mathfrak{b}}(\Sigma) \arrow[ddl, "\mathfrak{f}"] \\
\mathfrak{M}_{\beta\mathfrak{f}}(\Sigma) \arrow[dr, "\mathfrak{b}"] & & \\
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& \mathfrak{S}\mathfrak{T}(\Sigma) &
\end{tikzcd}
$$

Models without ξ

$$\mathcal{E}_{\varphi}(\lambda X_{\alpha}\text{.}M_{\beta}) = \mathcal{E}_{\varphi}(\lambda X_{\alpha}\text{.}N_{\beta}) \text{ iff}$$
$$\mathcal{E}_{\varphi,[a/X]}(M) = \mathcal{E}_{\varphi,[a/X]}(N)\ (\forall a \in \mathcal{D}_{\alpha})$$

Model Classes (Extensionality)

$$
\begin{tikzcd}
 & \mathfrak{M}_{\beta}(\Sigma) \arrow[dl, "\xi"'] \arrow[dr, "\mathfrak{b}"] \arrow[ddl, dotted, "\mathfrak{f}"] & \\
\mathfrak{M}_{\beta\xi}(\Sigma) \arrow[d, "\eta"] & & \mathfrak{M}_{\beta\mathfrak{b}}(\Sigma) \arrow[dl, "\xi"'] \arrow[ddl, dotted, "\mathfrak{f}"] \\
\mathfrak{M}_{\beta\mathfrak{f}}(\Sigma) \arrow[dr, "\mathfrak{b}"] & \mathfrak{M}_{\beta\xi\mathfrak{b}}(\Sigma) \arrow[d, "\eta"] & \\
 & \mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma) \arrow[d, dotted, "\text{full}"] & \\
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\end{tikzcd}
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Model Classes (Extensionality)

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\mathfrak{M}_{\beta\xi}(\Sigma) \arrow[d, "\eta"] & \mathfrak{M}_{\beta\eta}(\Sigma) \arrow[dl, "\xi"'] & \mathfrak{M}_{\beta\mathfrak{b}}(\Sigma) \arrow[dl, "\xi"'] \arrow[d, "\eta"] \arrow[ddl, dotted, "\mathfrak{f}"] \\
\mathfrak{M}_{\beta\mathfrak{f}}(\Sigma) \arrow[dr, "\mathfrak{b}"] & \mathfrak{M}_{\beta\xi\mathfrak{b}}(\Sigma) \arrow[d, "\eta"] & \mathfrak{M}_{\beta\eta\mathfrak{b}}(\Sigma) \arrow[dl, "\xi"'] \\
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Model Classes (Extensionality)

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\begin{tikzcd}
 & & \mathfrak{M}_{\beta}(\Sigma) \arrow[dll, "\xi"'] \arrow[d, "\eta"] \arrow[drr, "\mathfrak{b}"] \arrow[ddll, dotted, "\mathfrak{f}"] & & \\
\mathfrak{M}_{\beta\xi}(\Sigma) \arrow[d, "\eta"'] \arrow[drr, "\mathfrak{b}"] & & \mathfrak{M}_{\beta\eta}(\Sigma) \arrow[dll, "\xi"'] \arrow[drr, "\mathfrak{b}"] & & \mathfrak{M}_{\beta\mathfrak{b}}(\Sigma) \arrow[dll, "\xi"'] \arrow[d, "\eta"] \arrow[ddll, dotted, "\mathfrak{f}"] \\
\mathfrak{M}_{\beta\mathfrak{f}}(\Sigma) \arrow[drr, "\mathfrak{b}"'] & & \mathfrak{M}_{\beta\xi\mathfrak{b}}(\Sigma) \arrow[d, "\eta"] & & \mathfrak{M}_{\beta\eta\mathfrak{b}}(\Sigma) \arrow[dll, "\xi"] \\
 & & \mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \arrow[d, dotted, "\text{full}"] & & \\
 & & \mathfrak{S}\mathfrak{T}(\Sigma) & &
\end{tikzcd}
$$

Model Classes (Extensionality)

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\begin{tikzcd}
 & & \mathfrak{M}_{\beta}(\Sigma) \arrow[dll, "\xi"'] \arrow[d, "\eta"] \arrow[drr, "\mathfrak{b}"] \arrow[ddll, "\mathfrak{f}"] & & \\
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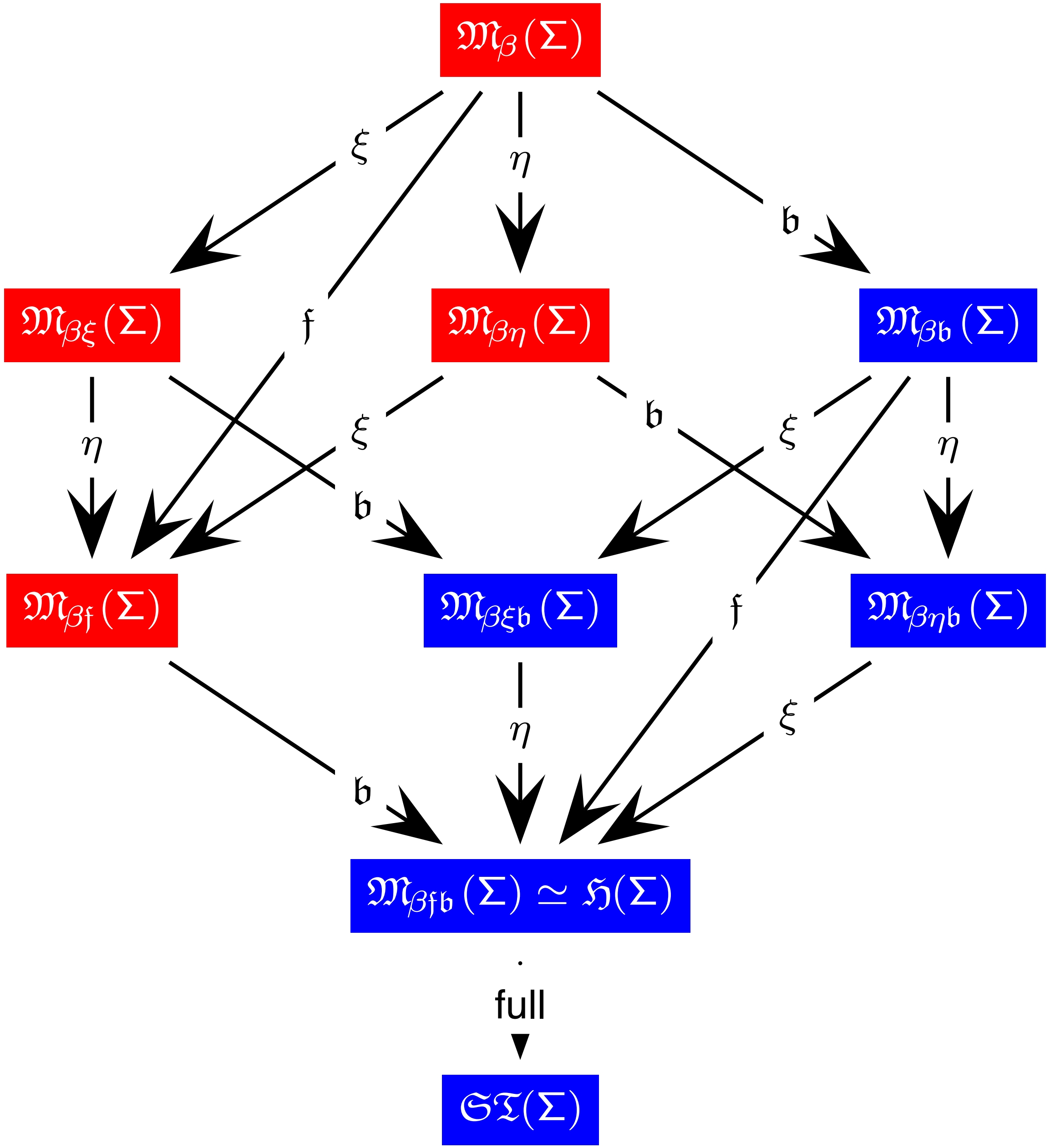
 & & \mathfrak{ST}(\Sigma) & &
\end{tikzcd}
$$

- $\forall X.\forall Y.X \vee Y \Leftrightarrow Y \vee X$

valid for all model classes

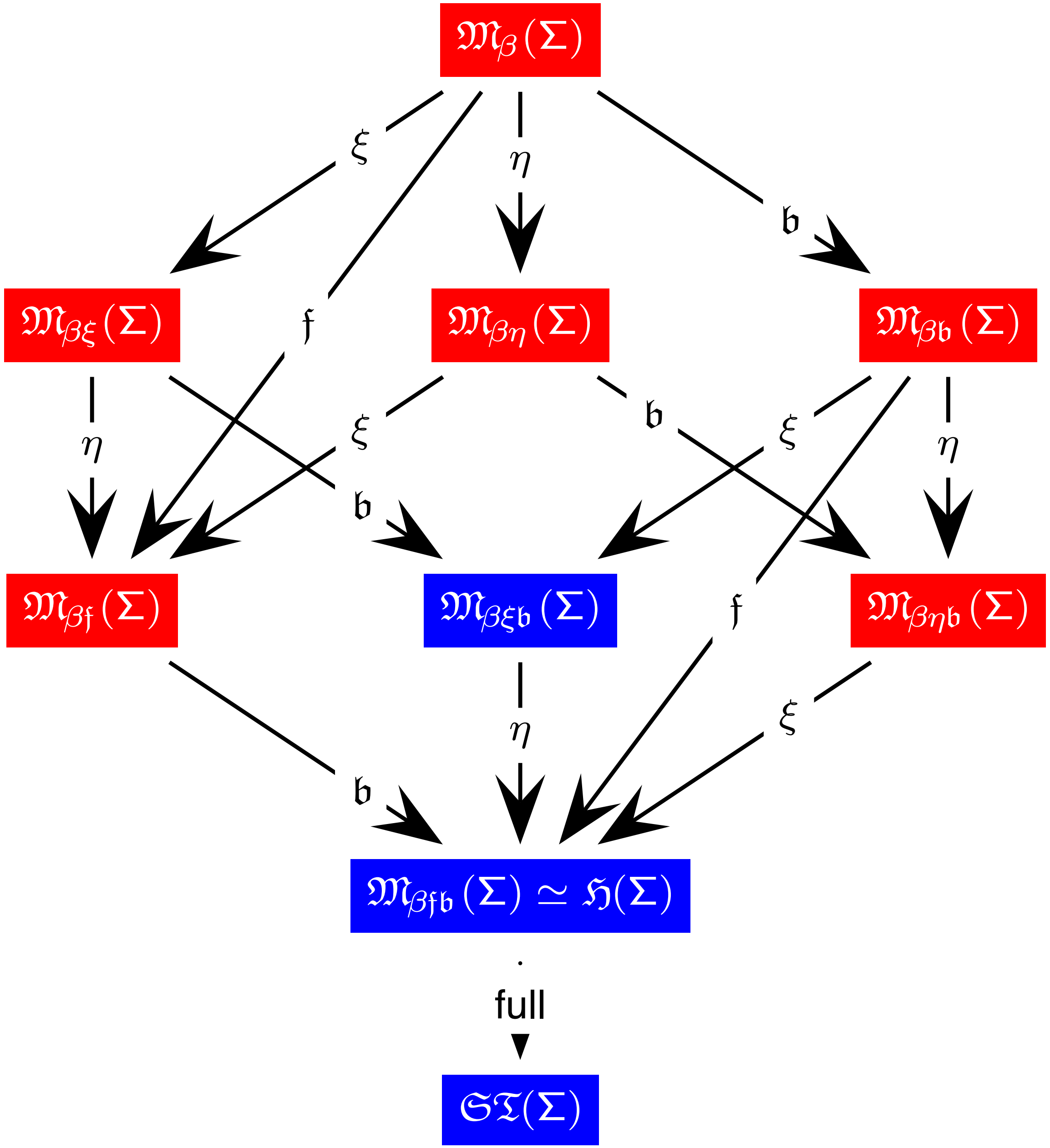
Model Classes (Extensionality)



- $\forall X.\forall Y.X \vee Y \Leftrightarrow Y \vee X$
- $\forall X.\forall Y.X \vee Y \doteq Y \vee X$

validity requires $\mathfrak{b}$

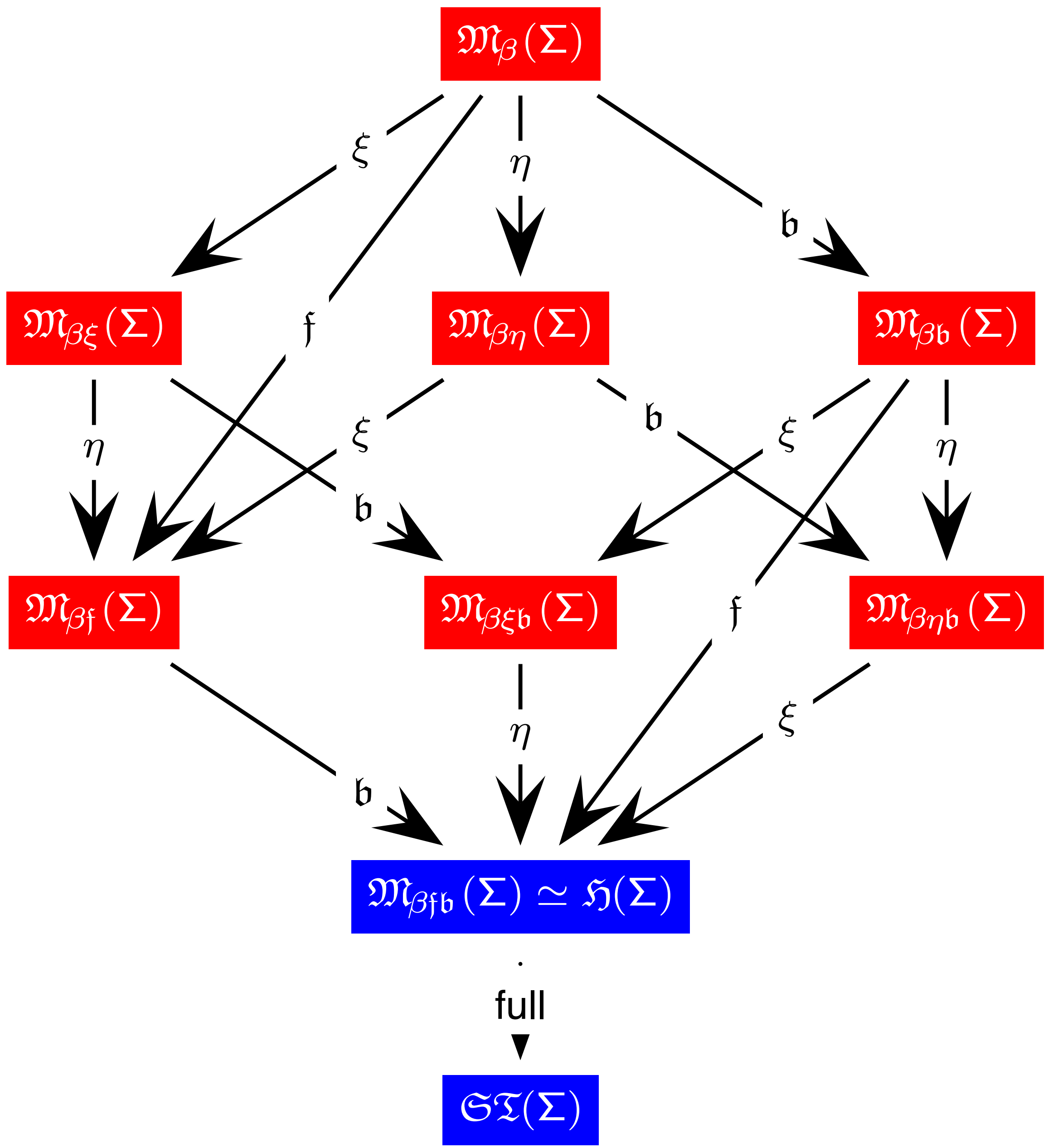
Model Classes (Extensionality)



- $\forall X. \forall Y. X \vee Y \Leftrightarrow Y \vee X$
- $\forall X. \forall Y. X \vee Y \doteq Y \vee X$
- $\lambda X. \lambda Y. X \vee Y \doteq \lambda X. \lambda Y. Y \vee X$

validity requires $\mathfrak{b}$ and ξ

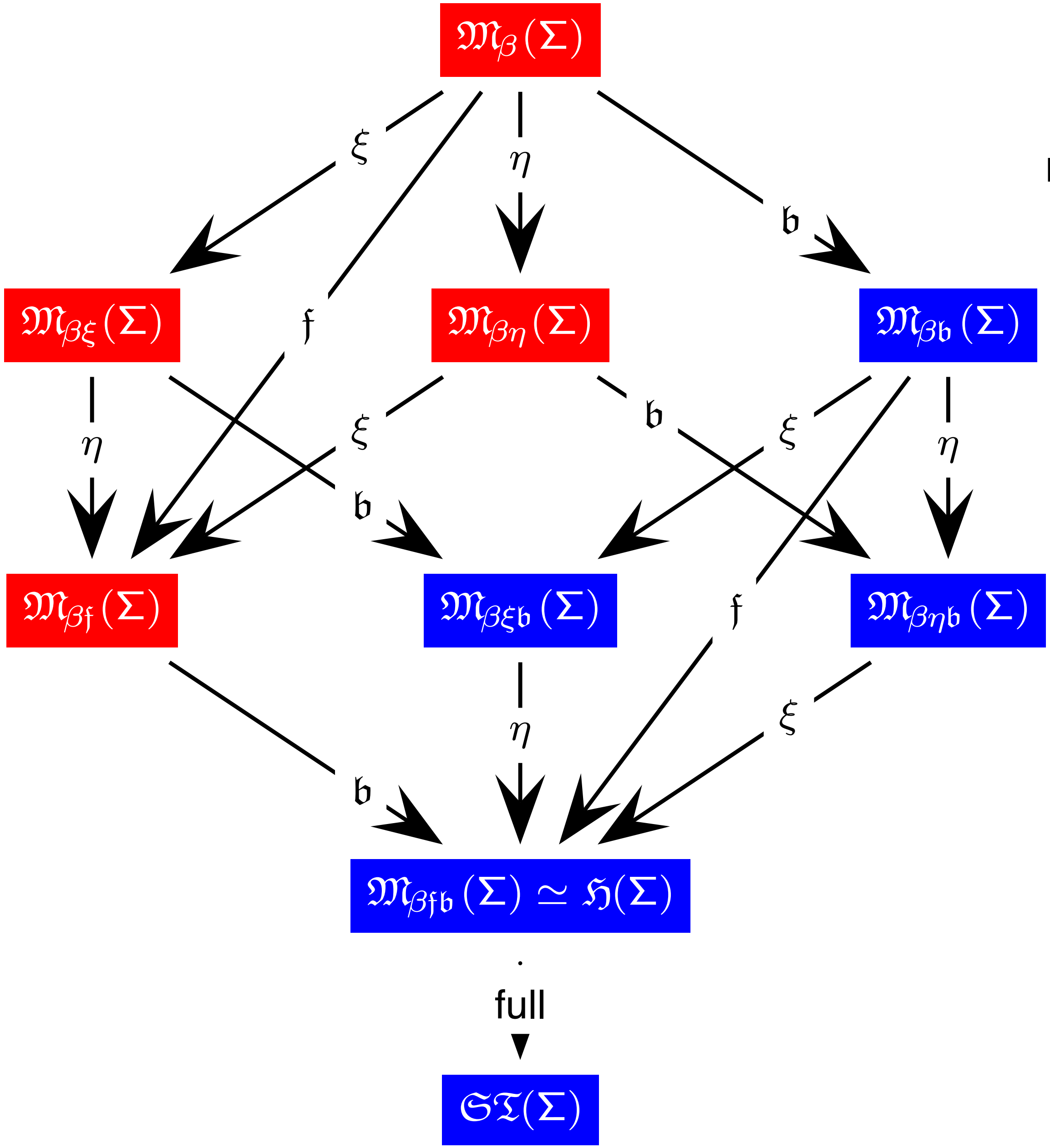
Model Classes (Extensionality)



- $\forall X. \forall Y. X \vee Y \Leftrightarrow Y \vee X$
- $\forall X. \forall Y. X \vee Y \doteq Y \vee X$
- $\lambda X. \lambda Y. X \vee Y \doteq \lambda X. \lambda Y. Y \vee X$
- $\vee \doteq \lambda X. \lambda Y. Y \vee X$

validity requires $\mathfrak{b}$ and $\mathfrak{f}$

Useful: Test Problems for TPs



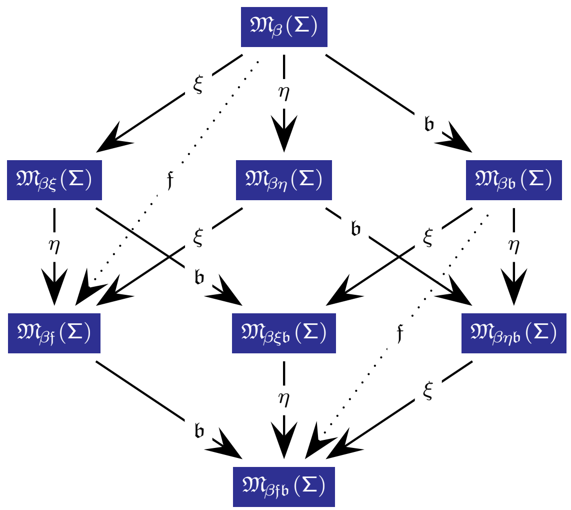
Examples requiring property $\mathfrak{b}$

- $(p\ a_o) \land (p\ b_o) \Rightarrow (p\ (a \land b))$
- $(h_{o\to\iota}((h\top) \doteq (h\bot))) \doteq (h\bot)$

Semantics - Calculi - Abstract Consistency

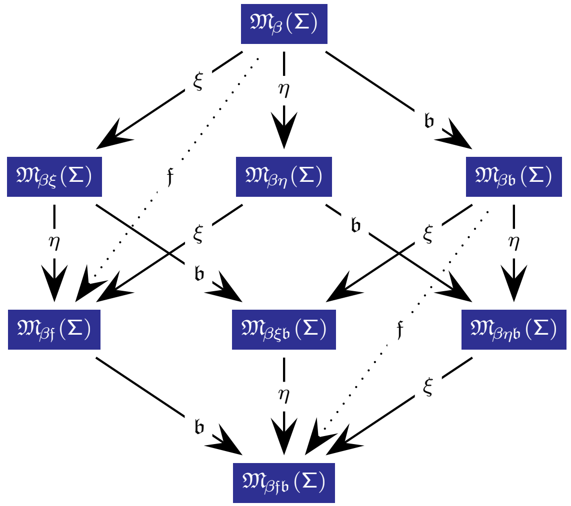
Semantics:

Model Classes (Extensionality)

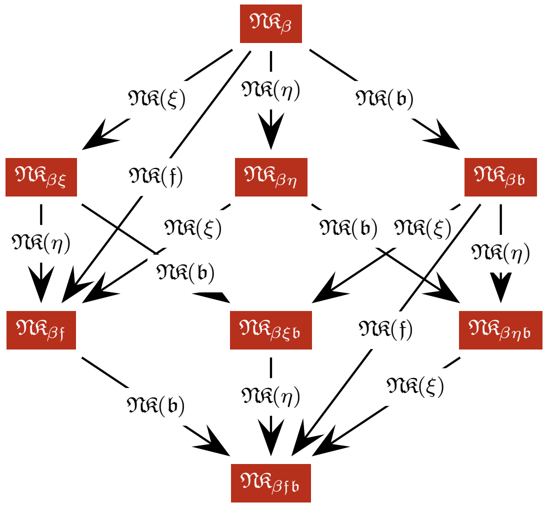


Semantics - Calculi - Abstract Consistency

Semantics:
Model Classes (Extensionality)



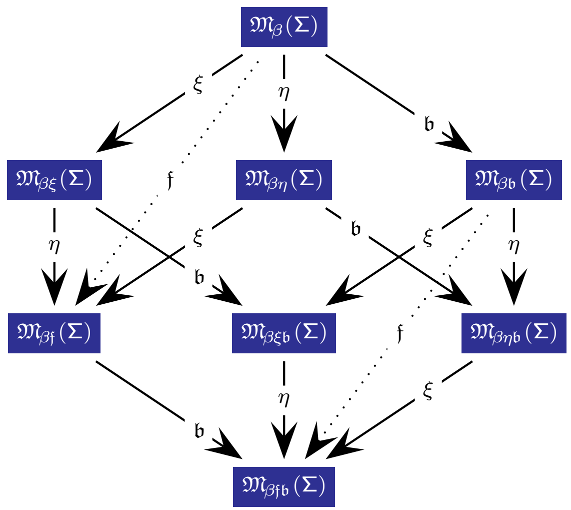
Reference Calculi:
ND (and others)



Semantics - Calculi - Abstract Consistency

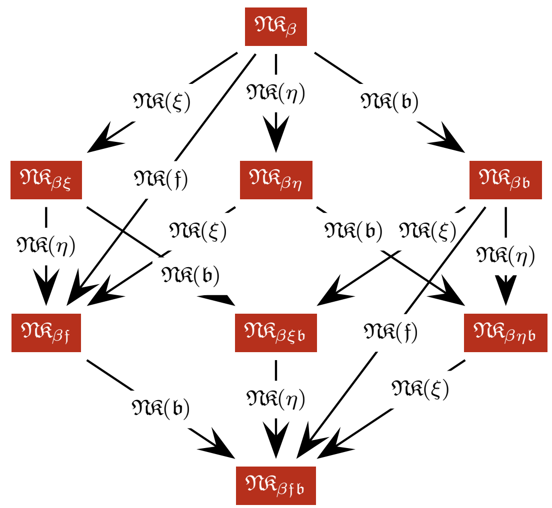
Semantics:

Model Classes (Extensionality)



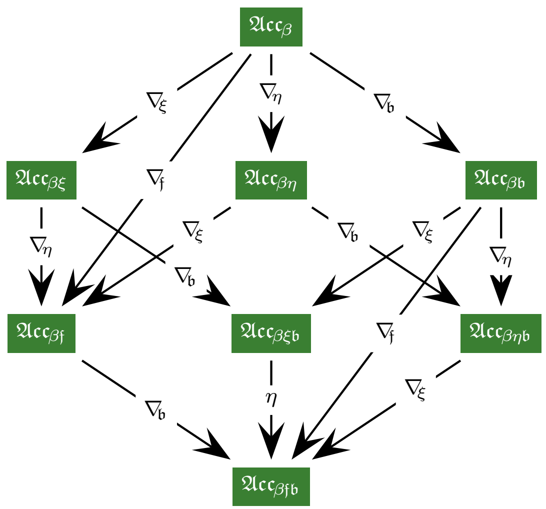
Reference Calculi:

ND (and others)



Abstract Consistency / Unifying Principle:

Extensions of Smullyan-63 and Andrews-71



Introduction

History

- Cantor’s Set Theory – late 1800’s
- Frege’s Logic – late 1800’s
- Russell’s Paradox – 1902
- Zermelo’s Axiomatic Set Theory – 1908
- Russell’s Type Theory – 1908
- Church’s Untyped λ-Calculus (Computation) – 1930’s
- Church’s Type Theory – HOL (Mathematics) – 1940
- Henkin Models and Completeness – 1950
- Cut-Elimination (Takahashi, Prawitz, Andrews) – 1967-1972
- Theorem Proving – 1980’s - today
- More Semantics and Cut-Elimination – mid 1990’s - today

A Standard Frame

$$\mathcal{D}_{o} = \{\mathtt{T}, \mathtt{F}\}.$$

$\mathcal{D}_{\iota}$ = $\mathbf{N}$ (natural numbers).

$\mathcal{D}_{\alpha\to\beta}$ = $\mathcal{D}_{\beta}^{\mathcal{D}_{\alpha}}$, all functions from $\mathcal{D}_{\alpha}$ to $\mathcal{D}_{\beta}$.

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$\mathcal{D}_{\iota\to o} \cong \mathcal{P}(\mathbf{N})$:

$X \subseteq \mathbf{N}$ induces $\chi_X \in \mathcal{D}_{\iota\to o}$ (characteristic function)

$$\chi_X(x) := \begin{cases} \mathtt{T} & \text{if } x \in X \\ \mathtt{F} & \text{if } x \notin X \end{cases}$$

Every $f \in \mathcal{D}_{\iota\to o}$ is χ_X where

$$X := \{x \in \mathcal{D}_\iota | f(x) = \mathtt{T}\}$$

A Standard Frame

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$\mathcal{D}_{\alpha\to\beta} = \mathcal{D}_\beta^{\mathcal{D}_\alpha}$, all functions from $\mathcal{D}_\alpha$ to $\mathcal{D}_\beta$.

$\mathcal{D}_{\iota\to\iota\to o} \cong \mathcal{P}(\mathbf{N} \times \mathbf{N})$: Binary relations on $\mathbf{N}$

$\mathcal{D}_{(\iota\to o)\to o} \cong \mathcal{P}(\mathcal{P}(\mathbf{N}))$

Standard Frames

$$\mathcal{D}_o = \text{any nonempty set}$$

$$\mathcal{D}_\iota = \text{any nonempty set}$$

$$\mathcal{D}_{\alpha\to\beta} = (\mathcal{D}_\beta)^{\mathcal{D}_\alpha}, \text{ all functions from } \mathcal{D}_\alpha \text{ to } \mathcal{D}_\beta.$$

Standard Frames are Determined by Domains of Base Type: If $\mathcal{D}$ and $\mathcal{E}$ are standard frames, $\mathcal{D}_o = \mathcal{E}_o$, and $\mathcal{D}_\iota = \mathcal{E}_\iota$, then $\mathcal{D} = \mathcal{E}$.

Proof: Induction on types.

Peano Arithmetic

Easy to Encode Peano's Axioms with ι as $\mathbf{N}$,
0_ι a parameter and $S_{\iota\to\iota}$ a parameter

1. Zero is a natural number.
   0 has type ι
2. n natural number $\Rightarrow$ successor of n is a natural number
   $[S\,\mathbf{N}]$ has type ι for any term $\mathbf{N}_\iota$
3. No successor is zero.
   $\forall n_\iota \neg[[S\,n] = 0]$
4. The successor function is injective.
5. Induction:

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3. No successor is zero.
   $[\Pi^{\iota}_{(\iota\to o)\to o}\,[\lambda n_\iota [\neg_{o\to o}\,[=^{\iota}_{\iota\to\iota\to o}\,[S_{\iota\to\iota}\,n]\,0_\iota]]]]_o$
4. The successor function is injective.

5. Induction:

Peano Arithmetic

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   $\forall n_\iota \neg[[S\,n] = 0]$
4. The successor function is injective.
   $\forall n_\iota \forall m_\iota [[[S\,n] = [S\,m]] \supset n = m]$
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Peano Arithmetic

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   $\forall n_\iota \forall m_\iota [[[S\,n] = [S\,m]] \supset n = m]$
5. Induction: $\forall p_{\iota\to o}[[p\,0] \wedge [\forall n_\iota [[p\,n] \supset [p\,[S\,n]]]] \supset [\forall n_\iota\,[p\,n]]]$

Incompleteness wrt Standard Frames

Only ONE standard frame with $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$ satisfies Peano: $\mathcal{D}_\iota = \mathbf{N}$

Suppose we have a recursively axiomatizable deduction system $\vdash$ for HOL sound and complete for standard models with $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$.

Gödel construction gives: $\mathbf{G}_o$

$\mathbf{G}$ evaluates to $\mathtt{T}$ in standard frame $\mathcal{D}$ above $\Leftrightarrow\ \not\vdash [\mathsf{PA} \supset \mathbf{G}]$

$\vdash [\mathsf{PA} \supset \mathbf{G}] \Rightarrow_{\text{Soundness}} \mathbf{G}$ evaluates to $\mathtt{T}$ in $\mathcal{D} \Rightarrow\ \not\vdash [\mathsf{PA} \supset \mathbf{G}]$

$\not\vdash [\mathsf{PA} \supset \mathbf{G}] \Rightarrow \mathbf{G}$ evaluates to $\mathtt{T}$ in $\mathcal{D} \Rightarrow_{\text{Completeness}} \vdash [\mathsf{PA} \supset \mathbf{G}]$

Incompleteness wrt Standard Frames

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Suppose we have a recursively axiomatizable deduction system $\vdash$ for HOL sound and complete for standard models with $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$.

Gödel construction gives: $\mathbf{G}_o$

$\mathbf{G}$ evaluates to $\mathtt{T}$ in standard frame $\mathcal{D}$ above $\Leftrightarrow \not\vdash [\mathrm{PA} \supset \mathbf{G}]$

There is no recursively axiomatizable deduction system for HOL sound and complete wrt standard models.

Frames in General

$$\mathcal{D}_o = \text{any nonempty set}$$

$$\mathcal{D}_\iota = \text{any nonempty set}$$

$$\mathcal{D}_{\alpha\to\beta} \subseteq (\mathcal{D}_\beta)^{\mathcal{D}_\alpha} \text{ (maybe not all functions)}$$

Frames are NOT Determined by Domains of Base Type.

Henkin Completeness (1950): Church's Deductive System is Complete wrt a Class of General Frames ("Henkin Models")

Theorem Proving in HOL

Interactive systems for constructing formal theories (these use extensions of Church's Type Theory):

- Isabelle-HOL
- HOL-light
- HOL4

Systems performing automated search for proofs in (fragments of) Church's Type Theory:

- TPS
- LEO

Theorem Proving: Extensionality

Consider $[A_o \wedge B_o \wedge [Q_{o \to o}\, A]] \supset [Q\, B]$.

Theorem? Yes, assuming Boolean extensionality.
Idea: A and B true implies A and B are equal.

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Automatic Search? Clauses to Refute:

A

B

$[Q A]$

$\neg [Q B]$

What to resolve?

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None Unify Syntactically.

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Automatic Search? Clauses to Refute:

A

B

$[Q\,A]$

$\neg[Q\,B]$

What to resolve?

None Unify Syntactically.

Idea: Resolve $[Q\,A]$ and $\neg[Q\,B]$, then prove $A = B$

Theorem Proving: Extensionality

There are similar examples for functional extensionality

TPS traditionally searches without extensionality.

TPS could not prove such examples

TPS was not “Henkin Complete” (but maybe wrt other model classes)?

LEO (1999) introduced search with extensionality

Coming Attractions

- Semantics without all Logical Constants
- Semantics without full Extensionality

Generalizing the Semantics

More Syntax

- α-conversion: We consider terms "identical" if they are the same up renaming of bound variables.

Example: $[\lambda \mathsf{x}_\iota \lambda \mathsf{y}_{\iota\to\mathsf{o}} [\mathsf{y}\,\mathsf{x}]]$ is identical to $[\lambda \mathsf{y}_\iota \lambda \mathsf{z}_{\iota\to\mathsf{o}} [\mathsf{z}\,\mathsf{y}]]$

- $[\mathbf{A}/\mathsf{x}]\mathbf{B}$ denotes substitution of $\mathbf{A}$ for free occurrences of x in $\mathbf{B}$. We rename bound variables to ensure no capture.

Example: $[\mathsf{y}/\mathsf{x}][\lambda \mathsf{y}_\iota\, [\mathsf{p}_{\iota\to\iota\to\mathsf{o}}\, \mathsf{x}\,\mathsf{y}]]$ is $[\lambda \mathsf{z}_\iota\, [\mathsf{p}\,\mathsf{y}\,\mathsf{z}]]$.

- We may also consider simultaneous substitutions θ for a finite number of variables.

More Syntax

- We will consider β and η reduction and conversion.

β: $[[\lambda x_\alpha\, \mathbf{B}_\beta]\, \mathbf{A}]$ β-reduces to $[\mathbf{A}/x]\mathbf{B}$

η: $[\lambda x_\alpha\, [\mathbf{F}_{\alpha\rightarrow\beta}\, x]]$ η-reduces to $\mathbf{F}$ if $x \notin \mathbf{Free}(\mathbf{F})$

- We write $\mathbf{A} \xrightarrow{\beta}_1 \mathbf{B}$ if $\mathbf{B}$ is obtained by β-reducing in some position in $\mathbf{A}$.

- We write $\mathbf{A} \xrightarrow{\eta}_1 \mathbf{B}$ if $\mathbf{B}$ is obtained by η-reducing in some position in $\mathbf{A}$.

- We write $\xrightarrow{\beta}$ to denote the reflexive, transitive closure of $\xrightarrow{\beta}_1$.

- We write $\xrightarrow{\beta\eta}$ to denote the reflexive, transitive closure of $\xrightarrow{\beta}_1 \cup \xrightarrow{\eta}_1$.

More Syntax

- We will consider β and η reduction and conversion.

β: $[[\lambda \mathsf{x}_\alpha\, \mathbf{B}_\beta]\, \mathbf{A}]$ β-reduces to $[\mathbf{A}/\mathsf{x}]\mathbf{B}$

η: $[\lambda \mathsf{x}_\alpha\, [\mathbf{F}_{\alpha\rightarrow\beta}\, \mathsf{x}]]$ η-reduces to $\mathbf{F}$ if $\mathsf{x} \notin \mathbf{Free}(\mathbf{F})$

Facts: $\xrightarrow{\beta}$ and $\xrightarrow{\beta\eta}$ satisfy the strong Church-Rosser property: Every term $\mathbf{A}$ has a unique normal form.

- $\mathbf{A}{\downarrow_\beta}$ denotes the β-normal (i.e., $\xrightarrow{\beta}$ normal) form of $\mathbf{A}$.

- $\mathbf{A}{\downarrow_{\beta\eta}}$ denotes the $\beta\eta$-normal (i.e., $\xrightarrow{\beta\eta}$ normal) form of $\mathbf{A}$.

Generalized Semantics

There are two key steps to generalize combinatory frames with evaluations to give nonextensional models.

To obtain non-functional semantics, we allow $\mathcal{D}_{\alpha\longrightarrow\beta}$ to be any nonempty set and include an "application operator" $@ : \mathcal{D}_{\alpha\longrightarrow\beta} \times \mathcal{D}_{\alpha} \longrightarrow \mathcal{D}_{\beta}$.

To generalize from two truth values, we allow $\mathcal{D}_{o}$ to be any nonempty set and include a "valuation" $v : \mathcal{D}_{o} \rightarrow \{\mathtt{T}, \mathtt{F}\}$.

Coming Attractions

1. Definition of applicative structure generalizing frames
2. Definition of logical properties relative to $\upsilon : \mathcal{D}_o \longrightarrow \{\mathtt{T}, \mathtt{F}\}$.
3. Definition of evaluations for interpreting terms in applicative structures
4. Definition of model for determining which terms of type o are true
5. Definition of model classes varying extensionality

Applicative Structures

Def.: A (typed) applicative structure is a pair $(\mathcal{D}, @)$ where $\mathcal{D}$ is a typed family of nonempty sets and $@^{\alpha\to\beta} : \mathcal{D}_{\alpha\to\beta} \times \mathcal{D}_\alpha \longrightarrow \mathcal{D}_\beta$ for each function type $(\alpha \to \beta)$.

Write $\mathsf{f}@\mathsf{a}$ for $\mathsf{f}@^{\alpha\to\beta}\mathsf{a}$ when $\mathsf{f} \in \mathcal{D}_{\alpha\to\beta}$ and $\mathsf{a} \in \mathcal{D}_\alpha$ are clear in context.

Def.: Let $\mathcal{A} := (\mathcal{D}, @)$ be an applicative structure. We say $\mathcal{A}$ is functional if for all types α and β and objects $\mathsf{f}, \mathsf{g} \in \mathcal{D}_{\alpha\to\beta}$, $\mathsf{f} = \mathsf{g}$ whenever $\mathsf{f}@\mathsf{a} = \mathsf{g}@\mathsf{a}$ for every $\mathsf{a} \in \mathcal{D}_\alpha$.

Logical Properties

Suppose $\upsilon : \mathcal{D}_o \to \{\mathtt{T}, \mathtt{F}\}$ is a function.

Def.: Let $\mathcal{A} := (\mathcal{D}, @)$ be an applicative structure and $\upsilon : \mathcal{D}_o \longrightarrow \{\mathtt{T}, \mathtt{F}\}$ be a function.
For each logical constant c_α and element $\mathsf{a} \in \mathcal{D}_\alpha$, we define the properties $\mathfrak{L}_{\mathsf{c}}(\mathsf{a})$ with respect to υ given in the following table...

Logical Properties

prop.	where	holds when			for all
$\mathfrak{L}_{\neg}(\mathsf{n})$	$\mathsf{n} \in \mathcal{D}_{o\to o}$	$\upsilon(\mathsf{n}@\mathsf{a}) = \mathtt{T}$	iff	$\upsilon(\mathsf{a}) = \mathtt{F}$	$\mathsf{a} \in \mathcal{D}_o$
$\mathfrak{L}_{\vee}(\mathsf{d})$	$\mathsf{d} \in \mathcal{D}_{o\to o\to o}$	$\upsilon(\mathsf{d}@\mathsf{a}@\mathsf{b}) = \mathtt{T}$	iff	$\upsilon(\mathsf{a}) = \mathtt{T}$ or $\upsilon(\mathsf{b}) = \mathtt{T}$	$\mathsf{a}, \mathsf{b} \in \mathcal{D}_o$
$\mathfrak{L}_{\wedge}(\mathsf{c})$	$\mathsf{c} \in \mathcal{D}_{o\to o\to o}$	$\upsilon(\mathsf{c}@\mathsf{a}@\mathsf{b}) = \mathtt{T}$	iff	$\upsilon(\mathsf{a}) = \mathtt{T}$ and $\upsilon(\mathsf{b}) = \mathtt{T}$	$\mathsf{a}, \mathsf{b} \in \mathcal{D}_o$
$\mathfrak{L}_{\Pi^\alpha}(\pi)$	$\pi \in \mathcal{D}_{(\alpha\to o)\to o}$	$\upsilon(\pi@\mathsf{f}) = \mathtt{T}$	iff	$\forall \mathsf{a} \in \mathcal{D}_\alpha\ \upsilon(\mathsf{f}@\mathsf{a}) = \mathtt{T}$	$\mathsf{f} \in \mathcal{D}_{\alpha\to o}$
$\mathfrak{L}_{\Sigma^\alpha}(\sigma)$	$\sigma \in \mathcal{D}_{(\alpha\to o)\to o}$	$\upsilon(\sigma@\mathsf{f}) = \mathtt{T}$	iff	$\exists \mathsf{a} \in \mathcal{D}_\alpha\ \upsilon(\mathsf{f}@\mathsf{a}) = \mathtt{T}$	$\mathsf{f} \in \mathcal{D}_{\alpha\to o}$
$\mathfrak{L}_{=^\alpha}(\mathsf{q})$	$\mathsf{q} \in \mathcal{D}_{\alpha\to\alpha\to o}$	$\upsilon(\mathsf{q}@\mathsf{a}@\mathsf{b}) = \mathtt{T}$	iff	$\mathsf{a} = \mathsf{b}$	$\mathsf{a}, \mathsf{b} \in \mathcal{D}_\alpha$

Logical Properties

Def.: Suppose $(\mathcal{D}, @)$ is an applicative structure and $\upsilon : \mathcal{D}_{o} \longrightarrow \{\mathtt{T}, \mathtt{F}\}$ is a function.

We say $(\mathcal{D}, @, \upsilon)$ realizes a logical constant c_α if there is some $a \in \mathcal{D}_\alpha$ such that $\mathfrak{L}_c(a)$ holds with respect to this υ. We say $(\mathcal{D}, @, \upsilon)$ realizes a signature Σ if it realizes every $c \in \Sigma$.

Variable Assignment

Def.: Let $\mathcal{A} := (\mathcal{D}, @)$ be an applicative structure.
A typed function $\varphi\colon \mathcal{V} \longrightarrow \mathcal{D}$ is called a variable assignment into $\mathcal{D}$.

Given a variable assignment φ, variable x_α, and value $\mathsf{a} \in \mathcal{D}_\alpha$, we use $\varphi, [\mathsf{a}/\mathsf{x}]$ to denote the variable assignment with
$(\varphi, [\mathsf{a}/\mathsf{x}])(\mathsf{x}) = \mathsf{a}$ and
$(\varphi, [\mathsf{a}/\mathsf{x}])(\mathsf{y}) = \varphi(\mathsf{y})$ for variables y other than x.

Evaluations

Def.: Let $\mathcal{A} = (\mathcal{D}, @)$ be an applicative structure.

An Σ-evaluation function $\mathcal{E}$ for $\mathcal{A}$ is a function taking assignments φ and terms $\mathbf{A}_\alpha$ to $\mathcal{E}_\varphi(\mathbf{A}) \in \mathcal{D}_\alpha$ satisfying the following properties:

1. $\mathcal{E}_\varphi(\mathsf{x}) = \varphi(\mathsf{x})$ for $\mathsf{x} \in \mathcal{V}$.
2. $\mathcal{E}_\varphi([\mathbf{F}\,\mathbf{A}]) = \mathcal{E}_\varphi(\mathbf{F})@\mathcal{E}_\varphi(\mathbf{A})$ for any $\mathbf{F}_{\alpha\to\beta}$ and $\mathbf{A}_\alpha$ and types α and β.
3. $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\psi(\mathbf{A})$ for any type α and $\mathbf{A}_\alpha$, whenever φ and ψ coincide on $\mathbf{Free}(\mathbf{A})$.
4. $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{A}\!\downarrow_\beta)$ for all $\mathbf{A}_\alpha$.

Evaluations

If $\mathbf{A}$ is a closed formula, then $\mathcal{E}_\varphi(\mathbf{A})$ is independent of φ.

Then we write $\mathcal{E}(\mathbf{A})$.

Evaluations

Def.: We call $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$ an Σ-evaluation if
$(\mathcal{D}, @)$ is an applicative structure and
$\mathcal{E}$ is an evaluation function for $(\mathcal{D}, @)$.

We call an Σ-evaluation $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$ functional if the applicative structure $(\mathcal{D}, @)$ is functional.

We say $\mathcal{J}$ is a Σ-evaluation over a frame if $(\mathcal{D}, @)$ is a frame.

Evaluations Respect β

If A β-converts to B, then they have the same normal form.

Hence

$$\mathcal{E}_\varphi(\mathsf{A}) = \mathcal{E}_\varphi(\mathsf{A}{\downarrow_\beta}) = \mathcal{E}_\varphi(\mathsf{B}{\downarrow_\beta}) = \mathcal{E}_\varphi(\mathsf{B})$$

Substitution-Value Lemma

Lemma: Substitution-Value Lemma

$$\mathcal{E}_{\varphi,[\mathcal{E}_\varphi(\mathbf{B}_\beta)/\mathsf{x}_\beta]}(\mathbf{A}_\alpha) = \mathcal{E}_\varphi([\mathbf{B}/\mathsf{x}]\mathbf{A})$$

Substitution-Value Lemma

Lemma: Substitution-Value Lemma

$$\mathcal{E}_{\varphi,[\mathcal{E}_\varphi(\mathbf{B}_\beta)/\mathsf{x}_\beta]}(\mathbf{A}_\alpha) = \mathcal{E}_\varphi([\mathbf{B}/\mathsf{x}]\mathbf{A})$$

Proof:

$$\begin{array}{rcl}
\mathcal{E}_{\varphi,[\mathcal{E}_\varphi(\mathbf{B})/\mathsf{x}]}(\mathbf{A}) & = & \mathcal{E}_{\varphi,[\mathcal{E}_\varphi(\mathbf{B})/\mathsf{x}]}([[\lambda\mathsf{x}\,\mathbf{A}]\,\mathsf{x}]) \\
 & = & \mathcal{E}_{\varphi,[\mathcal{E}_\varphi(\mathbf{B})/\mathsf{x}]}([\lambda\mathsf{x}\mathbf{A}])@\mathcal{E}_{\varphi,[\mathcal{E}_\varphi(\mathbf{B})/\mathsf{x}]}(\mathsf{x}) \\
 & = & \mathcal{E}_\varphi([\lambda\mathsf{x}\,\mathbf{A}])@\mathcal{E}_\varphi(\mathbf{B}) \\
 & = & \mathcal{E}_\varphi([[\lambda\mathsf{x}\,\mathbf{A}]\,\mathbf{B}]) \\
 & = & \mathcal{E}_\varphi([\mathbf{B}/\mathsf{x}]\mathbf{A}).
\end{array}$$

Substitution-Value Lemma

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$$\mathcal{E}_{\varphi,[\mathcal{E}_\varphi(\mathbf{B}_\beta)/\mathsf{x}_\beta]}(\mathbf{A}_\alpha) = \mathcal{E}_\varphi([\mathbf{B}/\mathsf{x}]\mathbf{A})$$

Proof:

$$\begin{array}{rcl}
\mathcal{E}_{\varphi,[\mathcal{E}_\varphi(\mathbf{B})/\mathsf{x}]}(\mathbf{A}) & = & \mathcal{E}_{\varphi,[\mathcal{E}_\varphi(\mathbf{B})/\mathsf{x}]}([[\lambda \mathsf{x}\,\mathbf{A}]\,\mathsf{x}]) \\
& = & \mathcal{E}_{\varphi,[\mathcal{E}_\varphi(\mathbf{B})/\mathsf{x}]}([\lambda \mathsf{x} \mathbf{A}])@\mathcal{E}_{\varphi,[\mathcal{E}_\varphi(\mathbf{B})/\mathsf{x}]}(\mathsf{x}) \\
& = & \mathcal{E}_\varphi([\lambda \mathsf{x}\,\mathbf{A}])@\mathcal{E}_\varphi(\mathbf{B}) \\
& = & \mathcal{E}_\varphi([[\lambda \mathsf{x}\,\mathbf{A}]\,\mathbf{B}]) \\
& = & \mathcal{E}_\varphi([\mathbf{B}/\mathsf{x}]\mathbf{A}).
\end{array}$$

Proof by Andrei Paskevich

Weak Functionality

Def.: Let $\mathcal{J} = (\mathcal{D}, @, \mathcal{E})$ be an Σ-evaluation.

We say $\mathcal{J}$ is η-functional if $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{A}\downarrow_{\beta\eta})$ for any type α, term $\mathbf{A}_\alpha$, and assignment φ.

We say $\mathcal{J}$ is ξ-functional if for all $\alpha, \beta \in \mathcal{T}$, $\mathbf{M}_\beta, \mathbf{N}_\beta$, assignments φ, and variables x_α, $\mathcal{E}_\varphi([\lambda \mathsf{x}_\alpha \mathbf{M}_\beta]) = \mathcal{E}_\varphi([\lambda \mathsf{x}_\alpha \mathbf{N}_\beta])$ whenever $\mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{x}]}(\mathbf{M}) = \mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{x}]}(\mathbf{N})$ for every $\mathsf{a} \in \mathcal{D}_\alpha$.

$\mathfrak{f} = \eta + \xi$

Lemma:
- functional $\Rightarrow$ η-functional
- functional $\Rightarrow$ ξ-functional
- η-functional and ξ-functional $\Rightarrow$ functional

Proof: Exercise.

Models

Def.: Let $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$ be an Σ-evaluation.

A function $\upsilon : \mathcal{D}_o \longrightarrow \{\mathtt{T}, \mathtt{F}\}$ is called a Σ-valuation for $\mathcal{J}$ if $\mathfrak{L}_c(\mathcal{E}(c))$ holds for every $c \in \Sigma$.

In this case, $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ is called an Σ-model.

Models

Def.: An assignment φ satisfies a formula $\mathbf{A}_o$ in $\mathcal{M}$
(we write $\mathcal{M} \models_\varphi \mathbf{A}$)
if $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \mathtt{T}$.

We say that $\mathbf{A}$ is valid in $\mathcal{M}$
(and write $\mathcal{M} \models \mathbf{A}$)
if $\mathcal{M} \models_\varphi \mathbf{A}$ for all assignments φ.

When $\mathbf{A}_o$ is closed, we drop φ and write $\mathcal{M} \models \mathbf{A}$.

Finally, we say that $\mathcal{M}$ is an Σ-model for a set Φ of closed formulas
(we write $\mathcal{M} \models \Phi$)
if $\mathcal{M} \models \mathbf{A}$ for all $\mathbf{A} \in \Phi$.

Example

Assume Σ contains $\neg$ and $\vee$

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, v)$ be a Σ-model

Claim: $\mathcal{M} \models_\varphi [\vee\, \mathsf{P}\, [\neg\, \mathsf{P}]]$ (i.e., $\mathsf{P} \vee \neg\, \mathsf{P}$) where $\mathsf{P} \in \mathit{wff}_o(\Sigma)$

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Show: $\upsilon(\mathcal{E}_{\varphi}([\vee\, \mathsf{P}\, [\neg\, \mathsf{P}]])) = \mathtt{T}$

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Assume Σ contains $\neg$ and $\vee$

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model

Claim: $\mathcal{M} \models_\varphi [\vee\, \mathsf{P}\, [\neg\, \mathsf{P}]]$ (i.e., $\mathsf{P} \vee \neg\, \mathsf{P}$) where $\mathsf{P} \in \mathit{wff}_{\mathsf{o}}(\Sigma)$

Show: $\upsilon(\mathcal{E}_\varphi([\vee\, \mathsf{P}\, [\neg\, \mathsf{P}]])) = \mathtt{T}$

Note: $\mathcal{E}_\varphi([\vee\, \mathsf{P}\, [\neg\, \mathsf{P}]]) = \mathcal{E}_\varphi(\vee) @ \mathcal{E}_\varphi(\mathsf{P}) @ \mathcal{E}_\varphi([\neg\, \mathsf{P}])$ (property of $\mathcal{E}$)

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Show: $\upsilon(\mathcal{E}_{\varphi}([\vee\, \mathsf{P}\, [\neg\, \mathsf{P}]])) = \mathtt{T}$

Note: $\mathcal{E}_{\varphi}([\vee\, \mathsf{P}\, [\neg\, \mathsf{P}]]) = \mathcal{E}_{\varphi}(\vee) @ \mathcal{E}_{\varphi}(\mathsf{P}) @ \mathcal{E}_{\varphi}([\neg\, \mathsf{P}])$ (property of $\mathcal{E}$)

Use $\mathfrak{L}_{\vee}(\mathcal{E}(\vee))$ – Show: Either $\upsilon(\mathcal{E}_{\varphi}(\mathsf{P})) = \mathtt{T}$ or $\upsilon(\ \underbrace{\mathcal{E}_{\varphi}([\neg\, \mathsf{P}])}_{\mathcal{E}_{\varphi}(\neg) @ \mathcal{E}_{\varphi}(\mathsf{P})}\ ) = \mathtt{T}$

Example

Assume Σ contains $\neg$ and $\vee$

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model

Claim: $\mathcal{M} \models_{\varphi} [\vee \, \mathsf{P} \, [\neg \, \mathsf{P}]]$ (i.e., $\mathsf{P} \vee \neg \mathsf{P}$) where $\mathsf{P} \in \mathit{wff}_{o}(\Sigma)$

Show: $\upsilon(\mathcal{E}_{\varphi}([\vee \, \mathsf{P} \, [\neg \, \mathsf{P}]])) = \mathtt{T}$

Note: $\mathcal{E}_{\varphi}([\vee \, \mathsf{P} \, [\neg \, \mathsf{P}]]) = \mathcal{E}_{\varphi}(\vee) @ \mathcal{E}_{\varphi}(\mathsf{P}) @ \mathcal{E}_{\varphi}([\neg \, \mathsf{P}])$ (property of $\mathcal{E}$)

Use $\mathfrak{L}_{\vee}(\mathcal{E}(\vee))$ – Show: Either $\upsilon(\mathcal{E}_{\varphi}(\mathsf{P})) = \mathtt{T}$ or $\upsilon(\underbrace{\mathcal{E}_{\varphi}([\neg \, \mathsf{P}])}_{\mathcal{E}_{\varphi}(\neg) @ \mathcal{E}_{\varphi}(\mathsf{P})}) = \mathtt{T}$

Use $\mathfrak{L}_{\neg}(\mathcal{E}(\neg))$ – Show: Either $\upsilon(\mathcal{E}_{\varphi}(\mathsf{P})) = \mathtt{T}$ or $\upsilon(\mathcal{E}_{\varphi}(\mathsf{P})) = \mathtt{F}$

Example

Assume Σ contains $\neg$ and $\vee$

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model

Claim: $\mathcal{M} \models_{\varphi} [\vee \, \mathsf{P} \, [\neg \, \mathsf{P}]]$ (i.e., $\mathsf{P} \vee \neg \mathsf{P}$) where $\mathsf{P} \in \mathit{wff}_{o}(\Sigma)$

Show: $\upsilon(\mathcal{E}_{\varphi}([\vee \, \mathsf{P} \, [\neg \, \mathsf{P}]])) = \mathtt{T}$

Note: $\mathcal{E}_{\varphi}([\vee \, \mathsf{P} \, [\neg \, \mathsf{P}]]) = \mathcal{E}_{\varphi}(\vee) @ \mathcal{E}_{\varphi}(\mathsf{P}) @ \mathcal{E}_{\varphi}([\neg \, \mathsf{P}])$ (property of $\mathcal{E}$)

Use $\mathfrak{L}_{\vee}(\mathcal{E}(\vee))$ – Show: Either $\upsilon(\mathcal{E}_{\varphi}(\mathsf{P})) = \mathtt{T}$ or $\upsilon(\underbrace{\mathcal{E}_{\varphi}([\neg \, \mathsf{P}])}_{\mathcal{E}_{\varphi}(\neg) @ \mathcal{E}_{\varphi}(\mathsf{P})}) = \mathtt{T}$

Use $\mathfrak{L}_{\neg}(\mathcal{E}(\neg))$ – Show: Either $\upsilon(\mathcal{E}_{\varphi}(\mathsf{P})) = \mathtt{T}$ or $\upsilon(\mathcal{E}_{\varphi}(\mathsf{P})) = \mathtt{F}$

OK, since $\upsilon : \mathcal{D}_{o} \longrightarrow \{\mathtt{T}, \mathtt{F}\}$.

Properties of Models

Def.: A Σ-model $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ is called functional if the applicative structure $(\mathcal{D}, @)$ is functional.

Similarly, $\mathcal{M}$ is called η-functional [ξ-functional] if the evaluation $(\mathcal{D}, @, \mathcal{E})$ is η-functional [ξ-functional].

We say $\mathcal{M}$ is an Σ-model over a frame if $(\mathcal{D}, @)$ is a frame.

Properties

Def.: Given an Σ-model $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$, we say that $\mathcal{M}$ has property

- $\mathfrak{q}$ iff for all $\alpha \in \mathcal{T}$ there is some $\mathsf{q}^\alpha \in \mathcal{D}_{\alpha\to\alpha\to o}$ such that $\mathfrak{L}_{=^\alpha}(\mathsf{q}^\alpha)$ holds.
- η iff $\mathcal{M}$ is η-functional.
- ξ iff $\mathcal{M}$ is ξ-functional.
- $\mathfrak{f}$ iff $\mathcal{M}$ is functional. (This is generally associated with functional extensionality.)
- $\mathfrak{b}$ iff υ is injective (and so $\mathcal{D}_o$ has at most two elements).

Signature Restriction

Remember: We restrict to the signature Σ being either

$\boxed{\{\neg, \vee\} \cup \{\Pi^\alpha | \alpha \in \mathcal{T}\}}$ or $\boxed{\{\neg, \vee\} \cup \{\Pi^\alpha, =^\alpha | \alpha \in \mathcal{T}\}}$.

Unless otherwise noted, other logical "constants" are abbreviations:

- $\supset$ is $[\lambda p_o \, \lambda q_o \, [\neg p \vee q]]$
- $\wedge$ is $[\lambda p_o \, \lambda q_o \, \neg[\neg p \vee \neg q]]$
- $\Leftrightarrow$ is $[\lambda p_o \, \lambda q_o \, [[p \supset q] \wedge [q \supset p]]]$
- Σ^α is $[\lambda p_{\alpha \to o} \, \neg[\Pi^\alpha \, [\lambda x_\alpha \, \neg[p\, x]]]]$

We sometimes consider "Leibniz Equality" denoted $\doteq^\alpha$:

$$[\lambda x_\alpha \lambda y_\alpha \forall p_{\alpha \to o}[[p\, x] \supset [p\, y]]]$$

Model Classes

Denote class of Σ-models that satisfy property $\mathfrak{q}$ by $\mathfrak{M}_\beta(\Sigma)$.

Specialized subclasses of depending on the validity of the properties η, ξ, $\mathfrak{f}$, and $\mathfrak{b}$ denoted by

$\mathfrak{M}_{\beta\eta}(\Sigma)$, $\mathfrak{M}_{\beta\xi}(\Sigma)$, $\mathfrak{M}_{\beta\mathfrak{f}}(\Sigma)$, $\mathfrak{M}_{\beta\mathfrak{b}}(\Sigma)$,

$\mathfrak{M}_{\beta\eta\mathfrak{b}}(\Sigma)$, $\mathfrak{M}_{\beta\xi\mathfrak{b}}(\Sigma)$, and $\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma)$.

Semantics: HOL-CUBE

Semantics: HOL-CUBE

$\mathfrak{M}_\beta(\Sigma)$

$\mathfrak{b}$: υ is injective ($|\mathcal{D}_o| = 2$)
$\mathfrak{f}(= \eta + \xi)$: $\mathcal{M}$ is functional
η: $\mathcal{M}$ is η-functional
ξ: $\mathcal{M}$ is ξ-functional

$\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma)$

full

$\mathfrak{ST}(\Sigma)$

$\mathfrak{M}_\beta(\Sigma)$ elementary type theory (Σ-models)

Assume that logical symbols are

$\{\neg, \vee\} \cup \{\Pi^\alpha\}$ or $\{\neg, \vee\} \cup \{\Pi^\alpha, =^\alpha\}$

We also require property $\mathfrak{q}$:

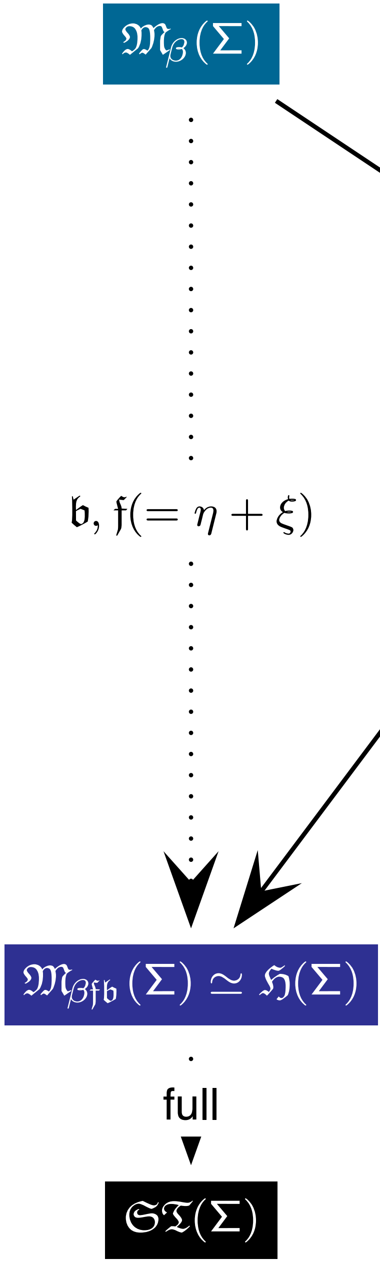
$\forall \alpha : \mathsf{id} \in \mathcal{D}_{\alpha\to\alpha\to o}$

without this equality $\doteq$ not necessarily evaluates to identity relation even in Henkin models [Andrews72]

$\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma)$ extensional type theory (Henkin models)

Standard models

Semantics: HOL-CUBE



Motivation for Models without Functional Extensionality

- modeling programs: $p_1 \neq p_2$ even if $f@a = g@a$ for every $a \in \mathcal{D}_\alpha$
- consider properties like run-time complexity:
- $\mathbf{P}_1 := \lambda X_{nat}.1$ and $\mathbf{P}_2 := \lambda X_{nat}.1 + (X + 1)^2 - (X^2 + 2X + 1)$
- $\mathbf{P}_1$ has constant complexity, $\mathbf{P}_2$ has not
- however, $\mathbf{P}_1$ behaves like $\mathbf{P}_2$ on all inputs
- a logic with a functionally extensional model theory (property $\mathfrak{f}$) necessarily conflates $\mathbf{P}_1$ and $\mathbf{P}_2$ semantically

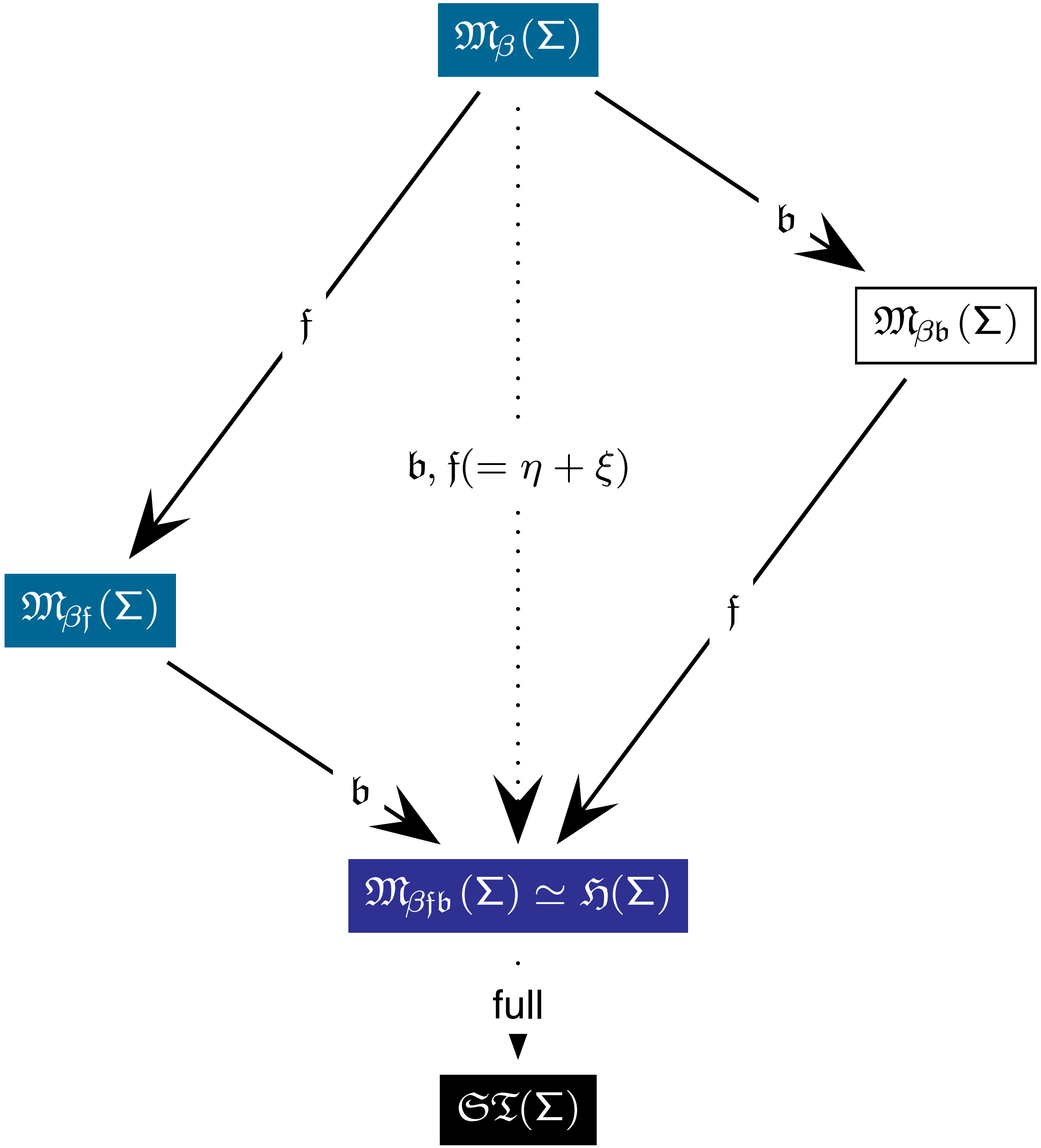
Semantics: HOL-CUBE

$$
\begin{tikzcd}
\mathfrak{M}_{\beta}(\Sigma) \arrow[dr, "\mathfrak{b}"] \arrow[dd, dotted, "{\mathfrak{b}, \mathfrak{f}(= \eta + \xi)}"] & \\
& \mathfrak{M}_{\beta\mathfrak{b}}(\Sigma) \arrow[dl, "\mathfrak{f}"] \\
\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma) \arrow[d, dotted, "\text{full}"] & \\
\mathfrak{S}\mathfrak{T}(\Sigma) &
\end{tikzcd}
$$

How do we account for Models without Functional Extensionality?

- generalized the notion of domains at function types and evaluation functions
- example: $(\mathsf{efficient}, \mathsf{K}_1) \neq (\mathsf{inefficient}, \mathsf{K}_1) \in \mathcal{D}_{\mathsf{nat} \longrightarrow \mathsf{nat}}$ where K_1 is the constant-1 function and $(*^1, *^2)@\mathsf{n}$ is defined as $*^2(\mathsf{n})$
- we build on the notion of applicative structures to define Σ-evaluations, where the evaluation function is assumed to respect application and β-conversion

Semantics: HOL-CUBE

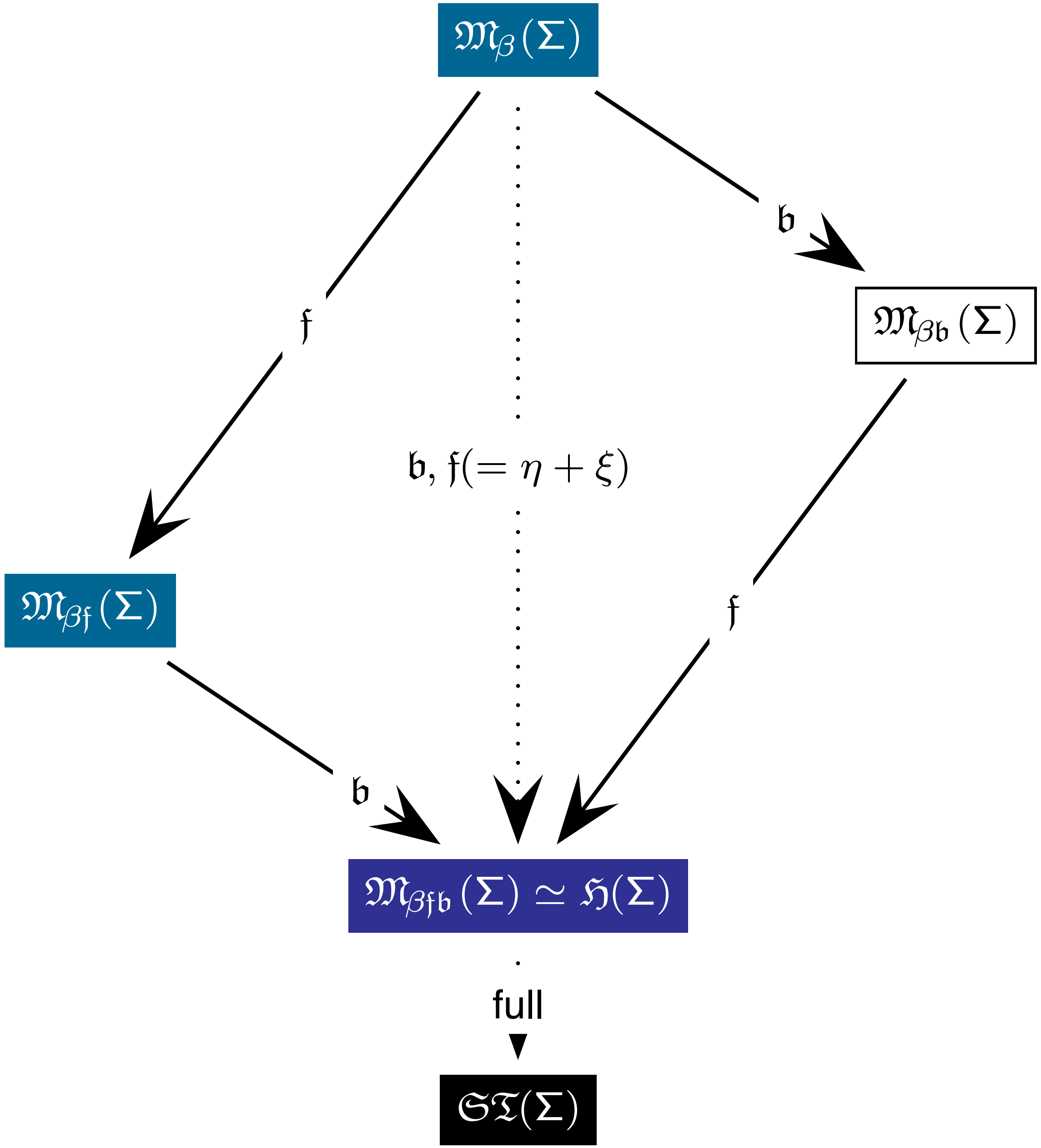


Motivation for models without Boolean Extensionality?

- modeling of intensional concepts like 'knowledge', 'believe', etc.
- example:
  $\mathbf{O} := 2 + 2 = 4$
  $\mathbf{F} := \forall x, y, z, n > 2. x^n + y^n = z^n \Rightarrow x = y = z = 0$
  We want to model:
  (1) $\mathbf{O} \Leftrightarrow \mathbf{F}$ is true
  (2)

  $\mathsf{john\_knows}(\mathbf{F}) \not\Leftrightarrow \mathsf{john\_knows}(\mathbf{O})$
- if we have $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$ then
  (1) implies $\mathbf{O} = \mathbf{F}$
  which enforces
  $\mathsf{john\_knows}(\mathbf{F}) = \mathsf{john\_knows}(\mathbf{O})$
  and
  $\mathsf{john\_knows}(\mathbf{F}) \Leftrightarrow \mathsf{john\_knows}(\mathbf{O})$

Semantics: HOL-CUBE



How do we account for models without Boolean Extensionality?

- allow that $|\mathcal{D}_o| > 2$ and use υ
- partition $|\mathcal{D}_o|$ into representatives of $\mathtt{T}$ and $\mathtt{F}$;
  e.g. $\mathcal{D}_o := \{\bot^1, \bot^2, \top^1, \top^2\}$ with $\upsilon(\bot^*) = \mathtt{F}$ and $\upsilon(\top^*) = \mathtt{T}$
- now, a predicate like john_knows may map:

  $\top^1 \longrightarrow \top^1$
  $\top^2 \longrightarrow \bot^1$
  $\bot^1 \longrightarrow \bot^1$
  $\bot^2 \longrightarrow \top^1$

  and we may choose:

  $\mathbf{O}$ evaluates to $\top^1$
  $\mathbf{F}$ evaluates to $\top^2$

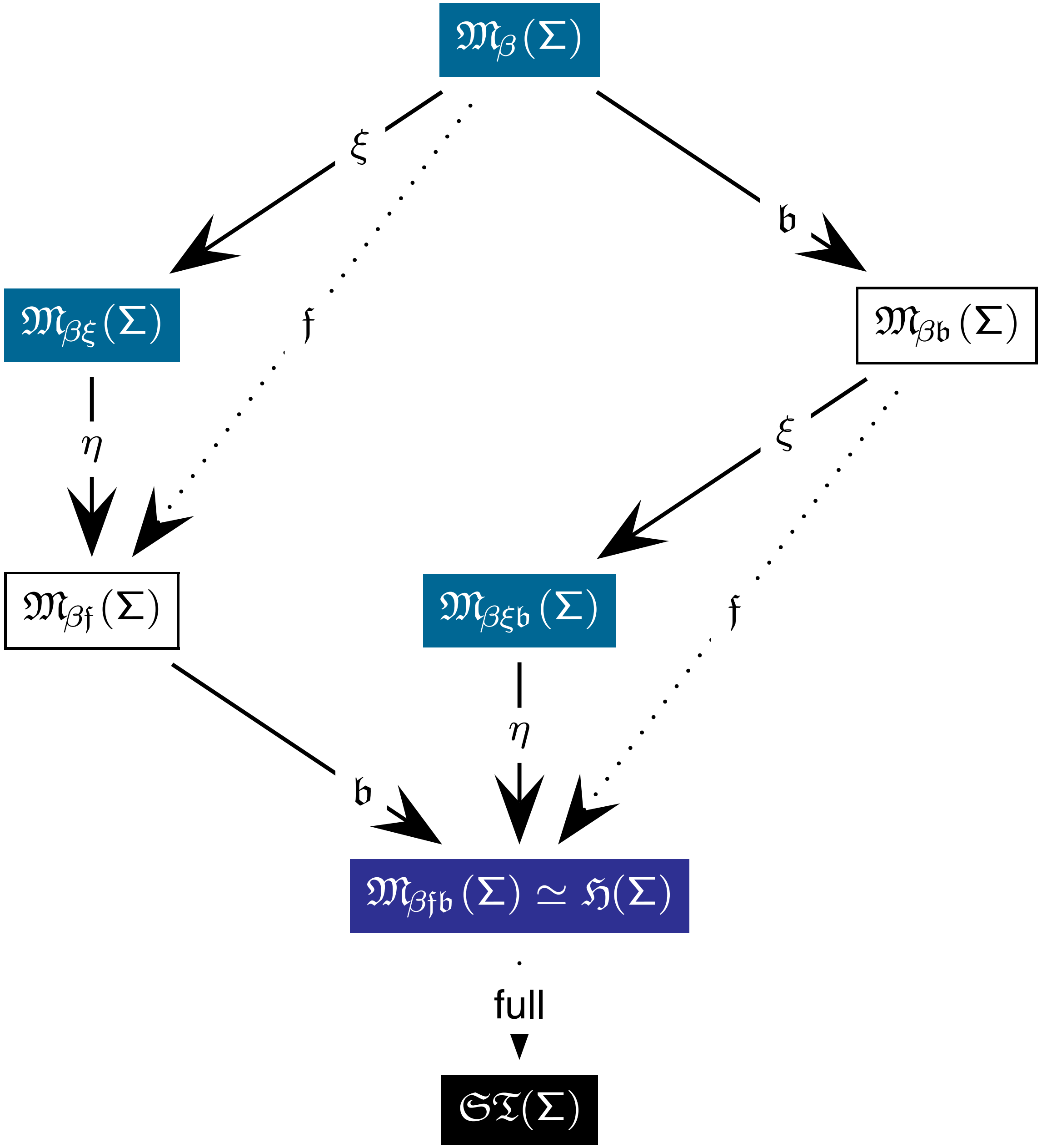
Semantics: HOL-CUBE

$$
\begin{tikzcd}
& \mathfrak{M}_{\beta}(\Sigma) \arrow[dl, "\xi"'] \arrow[dr, "\mathfrak{b}"] \arrow[ddl, dotted, "\mathfrak{f}"] & \\
\mathfrak{M}_{\beta\xi}(\Sigma) \arrow[d, "\eta"'] & & \mathfrak{M}_{\beta\mathfrak{b}}(\Sigma) \arrow[ddl, "\mathfrak{f}"] \\
\mathfrak{M}_{\beta\mathfrak{f}}(\Sigma) \arrow[dr, "\mathfrak{b}"'] & & \\
& \mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma) \arrow[d, dotted, "\text{full}"] & \\
& \mathfrak{S}\mathfrak{T}(\Sigma) &
\end{tikzcd}
$$

Models without η

$$\mathcal{E}_{\varphi}(\mathsf{A}) = \mathcal{E}_{\varphi}(\mathsf{A}\downarrow_{\beta\eta})$$

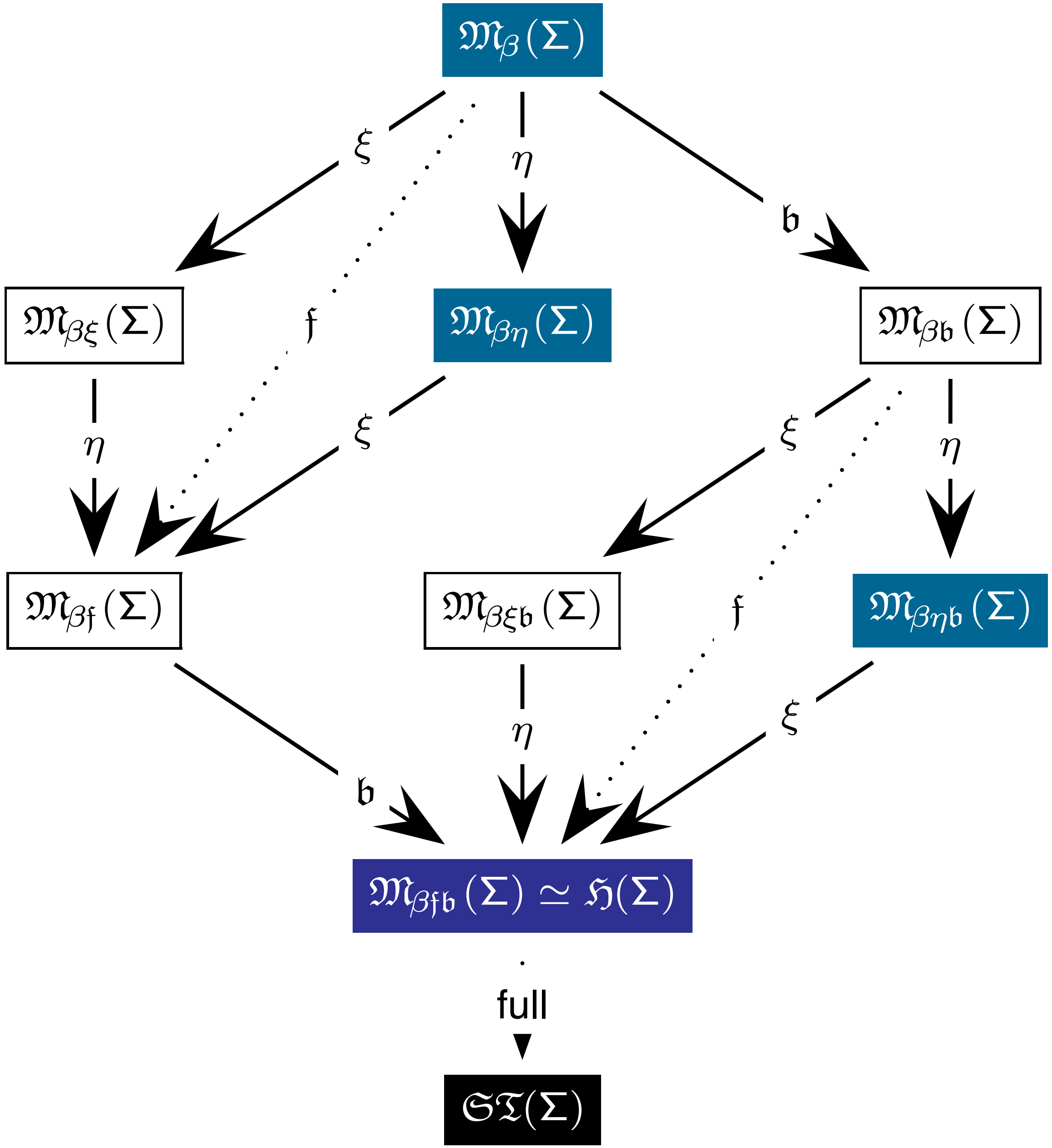
Semantics: HOL-CUBE



Models without η

$$\mathcal{E}_\varphi(\mathsf{A}) = \mathcal{E}_\varphi(\mathsf{A}\downarrow_{\beta\eta})$$

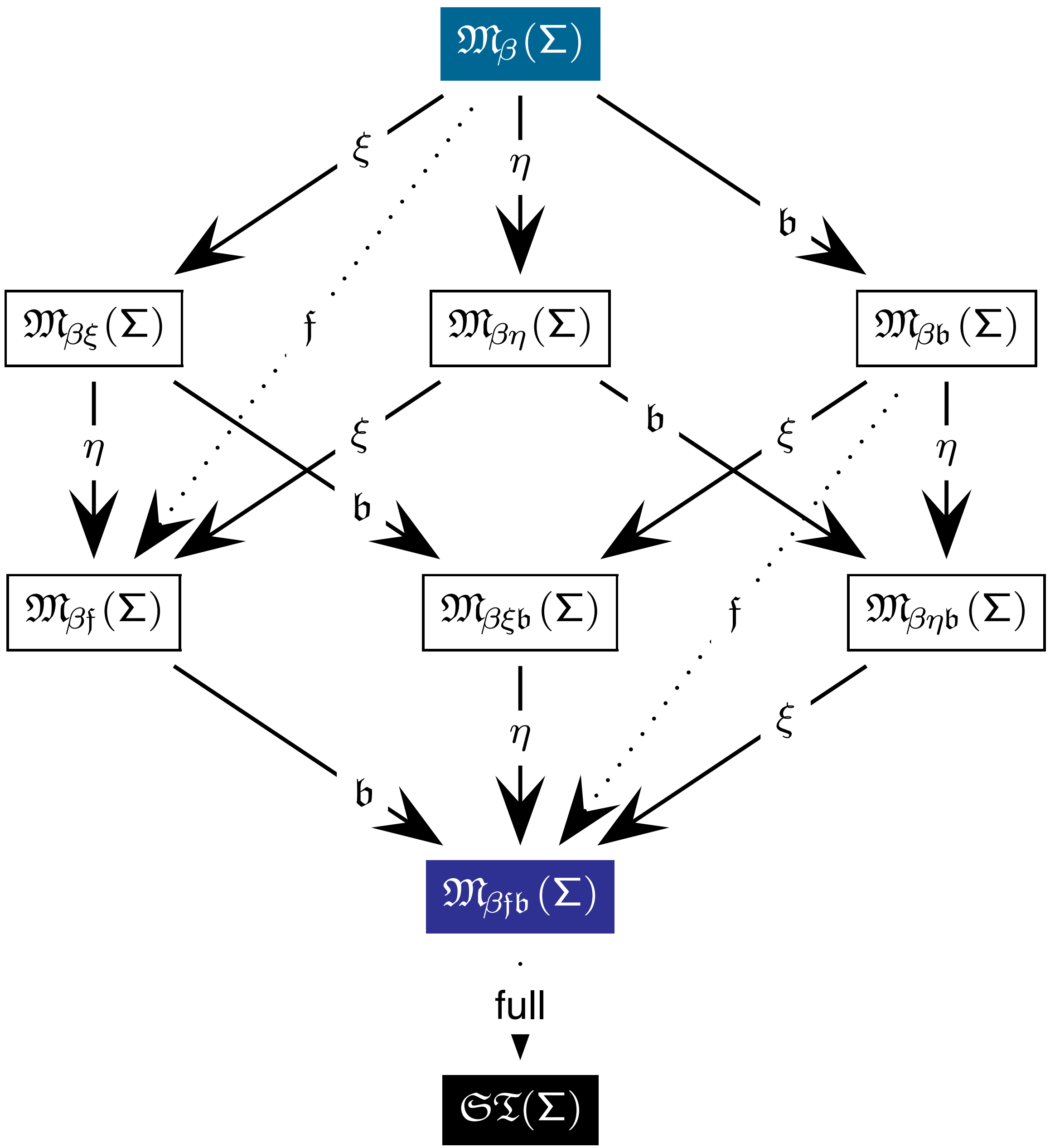
Semantics: HOL-CUBE



Models without ξ

$$\mathcal{E}_\varphi(\lambda X_\alpha . M_\beta) = \mathcal{E}_\varphi(\lambda X_\alpha . N_\beta) \text{ iff}$$
$$\mathcal{E}_{\varphi,[a/X]}(M) = \mathcal{E}_{\varphi,[a/X]}(N) \; (\forall a \in \mathcal{D}_\alpha)$$

Semantics: HOL-CUBE



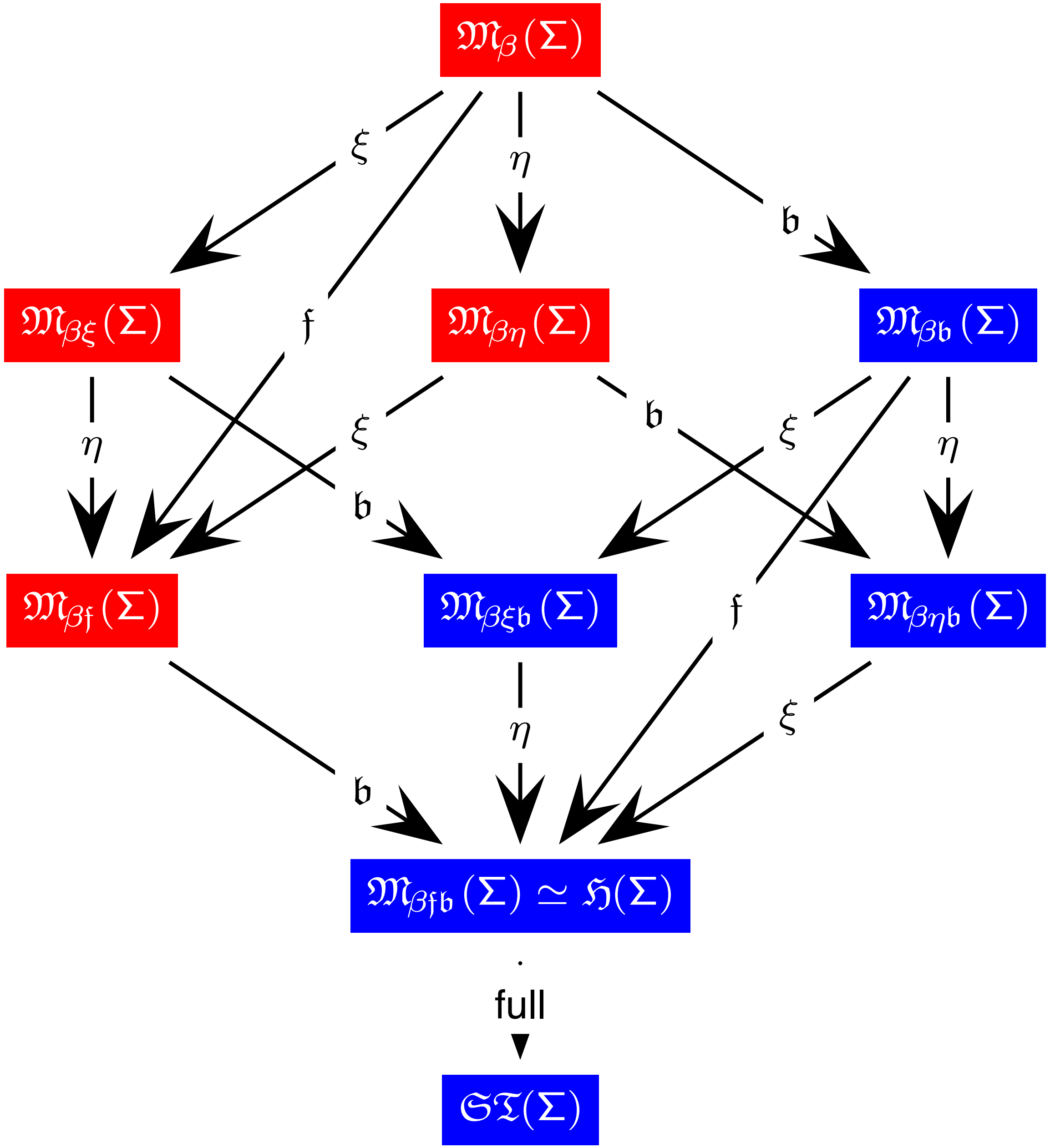
HOL Example Problems

$$
\begin{tikzcd}
 & & \mathfrak{M}_{\beta}(\Sigma) \arrow[dll, "\xi"'] \arrow[ddll, "\mathfrak{f}"] \arrow[d, "\eta"] \arrow[drr, "\mathfrak{b}"] & & \\
\mathfrak{M}_{\beta\xi}(\Sigma) \arrow[d, "\eta"] \arrow[drr, "\mathfrak{b}"] & & \mathfrak{M}_{\beta\eta}(\Sigma) \arrow[dll, "\xi"'] \arrow[drr, "\mathfrak{b}"] & & \mathfrak{M}_{\beta\mathfrak{b}}(\Sigma) \arrow[dll, "\xi"'] \arrow[ddll, "\mathfrak{f}"] \arrow[d, "\eta"] \\
\mathfrak{M}_{\beta\mathfrak{f}}(\Sigma) \arrow[drr, "\mathfrak{b}"] & & \mathfrak{M}_{\beta\xi\mathfrak{b}}(\Sigma) \arrow[d, "\eta"] & & \mathfrak{M}_{\beta\eta\mathfrak{b}}(\Sigma) \arrow[dll, "\xi"'] \\
 & & \mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma) \arrow[d, dotted, "\text{full}"] & & \\
 & & \mathfrak{S}\mathfrak{T}(\Sigma) & &
\end{tikzcd}
$$

- $\forall X. \forall Y. X \lor Y \Leftrightarrow Y \lor X$

valid for all model classes

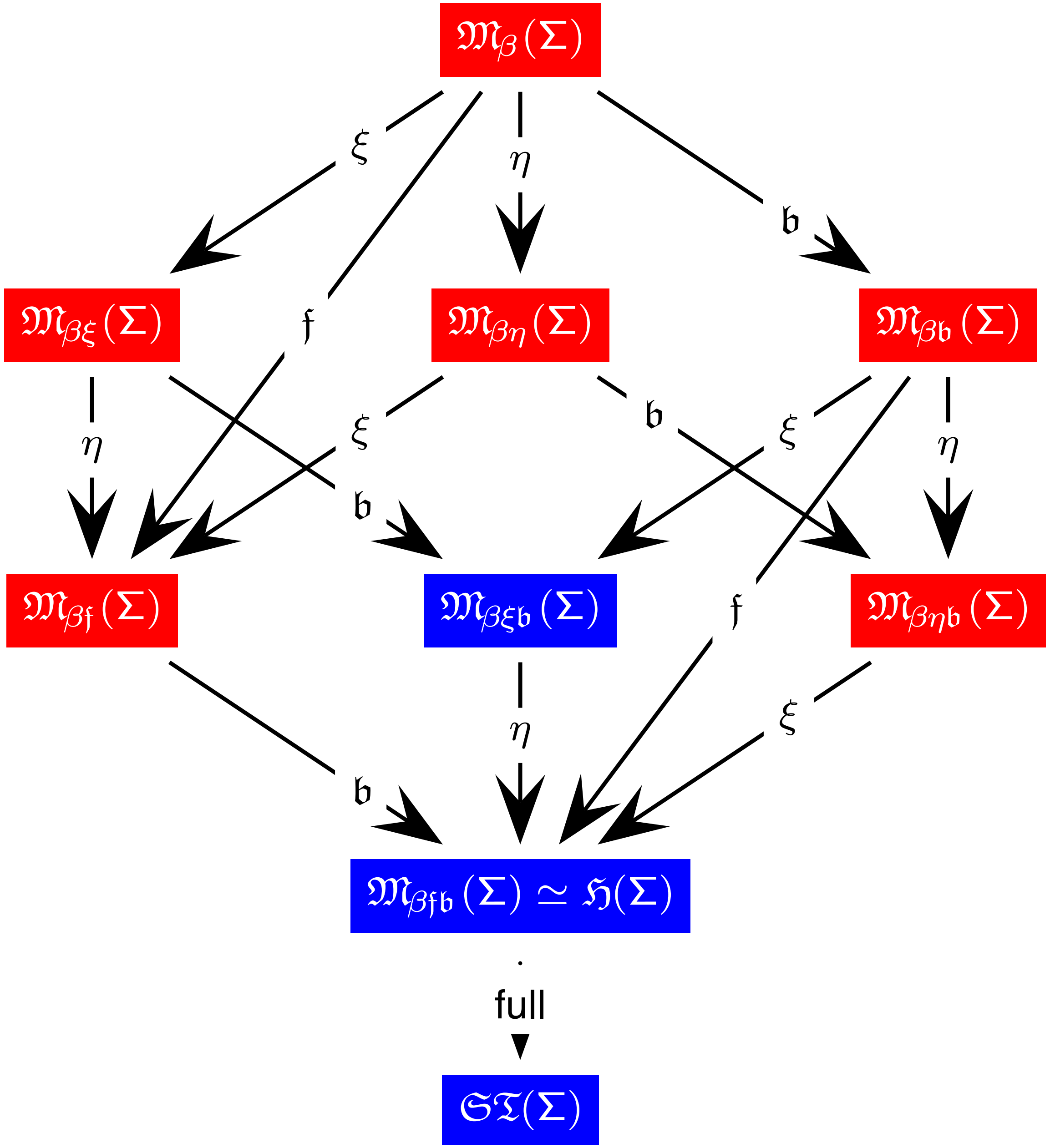
HOL Example Problems



- $\forall X. \forall Y. X \lor Y \Leftrightarrow Y \lor X$
- $\forall X. \forall Y. X \lor Y \doteq Y \lor X$

validity requires $\mathfrak{b}$

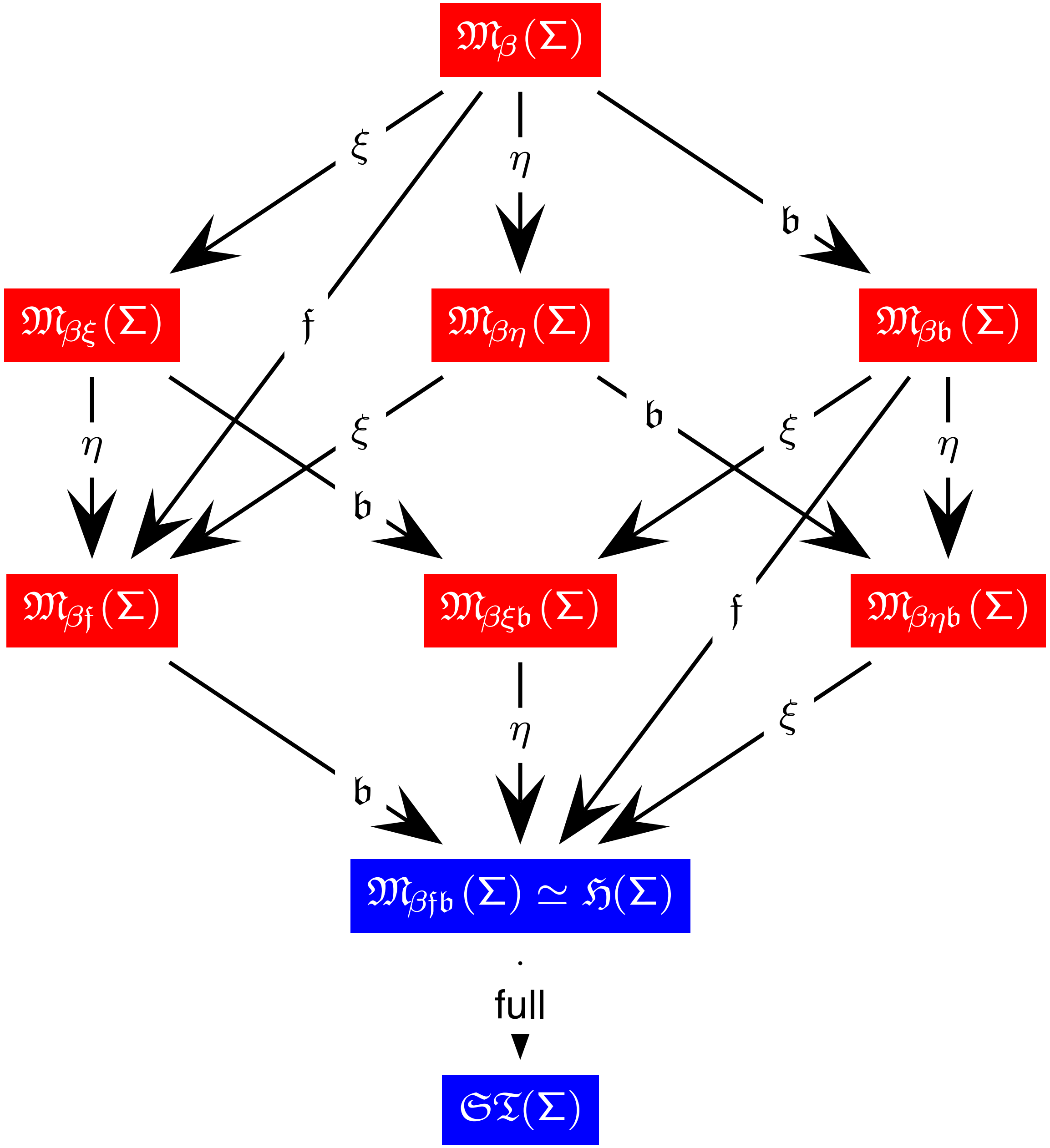
HOL Example Problems



- $\forall X. \forall Y. X \vee Y \Leftrightarrow Y \vee X$
- $\forall X. \forall Y. X \vee Y \doteq Y \vee X$
- $\lambda X. \lambda Y. X \vee Y \doteq \lambda X. \lambda Y. Y \vee X$

validity requires $\mathfrak{b}$ and ξ

HOL Example Problems



- $\forall X.\forall Y.X \vee Y \Leftrightarrow Y \vee X$
- $\forall X.\forall Y.X \vee Y \doteq Y \vee X$
- $\lambda X.\lambda Y.X \vee Y \doteq \lambda X.\lambda Y.Y \vee X$
- $\vee \doteq \lambda X.\lambda Y.Y \vee X$

validity requires $\mathfrak{b}$ and $\mathfrak{f}$

Defined Logical Connectives in Σ-Models

Defined Logical Connectives

Lemma: (Truth and Falsity in Σ-Models)

Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model and φ an assignment.

Let $\mathbf{T}_o := \forall \mathsf{P}_o.\mathsf{P} \vee \neg \mathsf{P}$ and $\mathbf{F}_o := \neg \mathbf{T}_o$.

Then $\upsilon(\mathcal{E}_\varphi(\mathbf{T}_o)) = \mathtt{T}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{F}_o)) = \mathtt{F}$.

Defined Logical Connectives

Lemma: (Truth and Falsity in Σ-Models)
Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, v)$ be a Σ-model and φ an assignment.
Let $\mathbf{T}_{\mathsf{o}} := \forall \mathsf{P}_{\mathsf{o}}. \mathsf{P} \lor \neg \mathsf{P}$ and $\mathbf{F}_{\mathsf{o}} := \neg \mathbf{T}_{\mathsf{o}}$.
Then $v(\mathcal{E}_{\varphi}(\mathbf{T}_{\mathsf{o}})) = \mathtt{T}$ and $v(\mathcal{E}_{\varphi}(\mathbf{F}_{\mathsf{o}})) = \mathtt{F}$.

Defined Logical Connectives

Lemma: (Truth and Falsity in Σ-Models)
Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model and φ an assignment.
Let $\mathbf{T}_{o} := \forall \mathsf{P}_{o} \blacksquare \mathsf{P} \vee \neg \mathsf{P}$ and $\mathbf{F}_{o} := \neg \mathbf{T}_{o}$.
Then $\upsilon(\mathcal{E}_{\varphi}(\mathbf{T}_{o})) = \mathtt{T}$ and $\upsilon(\mathcal{E}_{\varphi}(\mathbf{F}_{o})) = \mathtt{F}$.

Proof: $\upsilon(\mathcal{E}_{\varphi}(\mathbf{T}_{o})) = \mathtt{T}$

Defined Logical Connectives

Lemma: (Truth and Falsity in Σ-Models)
Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model and φ an assignment.
Let $\mathbf{T}_o := \forall P_o. P \vee \neg P$ and $\mathbf{F}_o := \neg \mathbf{T}_o$.
Then $\upsilon(\mathcal{E}_\varphi(\mathbf{T}_o)) = \mathtt{T}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{F}_o)) = \mathtt{F}$.

Proof: $\upsilon(\mathcal{E}_\varphi(\mathbf{T}_o)) = \mathtt{T}$

- iff $\upsilon(\mathcal{E}_{\varphi[p/P]}(P \vee \neg P)) = \mathtt{T}$ for all $p \in \mathcal{D}_o$

Defined Logical Connectives

Lemma: (Truth and Falsity in Σ-Models)
Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model and φ an assignment.
Let $\mathbf{T}_o := \forall \mathsf{P}_o \mathsf{.} \mathsf{P} \vee \neg \mathsf{P}$ and $\mathbf{F}_o := \neg \mathbf{T}_o$.
Then $\upsilon(\mathcal{E}_\varphi(\mathbf{T}_o)) = \mathtt{T}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{F}_o)) = \mathtt{F}$.

Proof: $\upsilon(\mathcal{E}_\varphi(\mathbf{T}_o)) = \mathtt{T}$

- iff $\upsilon(\mathcal{E}_{\varphi[\mathsf{p}/\mathsf{P}]}(\mathsf{P} \vee \neg \mathsf{P})) = \mathtt{T}$ for all $\mathsf{p} \in \mathcal{D}_o$
- This is equivalent to $\upsilon(\mathcal{E}_{\varphi[\mathsf{p}/\mathsf{P}]}(\mathsf{P})) = \mathtt{T}$ or $\upsilon(\mathcal{E}_{\varphi[\mathsf{p}/\mathsf{P}]}(\mathsf{P})) = \mathtt{F}$.

Defined Logical Connectives

Lemma: (Truth and Falsity in Σ-Models)
Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model and φ an assignment.
Let $\mathbf{T}_o := \forall \mathsf{P}_o.\mathsf{P} \vee \neg \mathsf{P}$ and $\mathbf{F}_o := \neg \mathbf{T}_o$.
Then $\upsilon(\mathcal{E}_\varphi(\mathbf{T}_o)) = \mathtt{T}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{F}_o)) = \mathtt{F}$.

Proof: $\upsilon(\mathcal{E}_\varphi(\mathbf{T}_o)) = \mathtt{T}$

- iff $\upsilon(\mathcal{E}_{\varphi[\mathsf{p}/\mathsf{P}]}(\mathsf{P} \vee \neg \mathsf{P})) = \mathtt{T}$ for all $\mathsf{p} \in \mathcal{D}_o$
- This is equivalent to $\upsilon(\mathcal{E}_{\varphi[\mathsf{p}/\mathsf{P}]}(\mathsf{P})) = \mathtt{T}$ or $\upsilon(\mathcal{E}_{\varphi[\mathsf{p}/\mathsf{P}]}(\mathsf{P})) = \mathtt{F}$.
- This is equivalent to $\upsilon(\varphi[\mathsf{p}/\mathsf{P}](\mathsf{P})) = \mathtt{T}$ or $\upsilon(\varphi[\mathsf{p}/\mathsf{P}](\mathsf{P})) = \mathtt{F}$.

Defined Logical Connectives

Lemma: (Truth and Falsity in Σ-Models)
Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model and φ an assignment.
Let $\mathbf{T}_o := \forall \mathsf{P}_o . \mathsf{P} \vee \neg \mathsf{P}$ and $\mathbf{F}_o := \neg \mathbf{T}_o$.
Then $\upsilon(\mathcal{E}_\varphi(\mathbf{T}_o)) = \mathtt{T}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{F}_o)) = \mathtt{F}$.

Proof: $\upsilon(\mathcal{E}_\varphi(\mathbf{T}_o)) = \mathtt{T}$

- iff $\upsilon(\mathcal{E}_{\varphi[\mathsf{p}/\mathsf{P}]}(\mathsf{P} \vee \neg \mathsf{P})) = \mathtt{T}$ for all $\mathsf{p} \in \mathcal{D}_o$
- This is equivalent to $\upsilon(\mathcal{E}_{\varphi[\mathsf{p}/\mathsf{P}]}(\mathsf{P})) = \mathtt{T}$ or $\upsilon(\mathcal{E}_{\varphi[\mathsf{p}/\mathsf{P}]}(\mathsf{P})) = \mathtt{F}$.
- This is equivalent to $\upsilon(\varphi[\mathsf{p}/\mathsf{P}](\mathsf{P})) = \mathtt{T}$ or $\upsilon(\varphi[\mathsf{p}/\mathsf{P}](\mathsf{P})) = \mathtt{F}$.
- Since υ maps into $\{\mathtt{T}, \mathtt{F}\}$ this must be true.

Defined Logical Connectives

Rem.: ($|\mathcal{D}_\text{o}| \geq 2$ and υ surjective)
Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model. By the previous Lemma, $\mathcal{D}_\text{o}$ must have at least the two elements $\mathcal{E}_\varphi(\mathbf{T}_\text{o})$ and $\mathcal{E}_\varphi(\mathbf{F}_\text{o})$, and υ must be surjective.

Defined Logical Connectives

Lemma: (Equivalence)

Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ an assignment into $\mathcal{M}$, and $\mathbf{A}, \mathbf{B} \in \textit{wff}_o(\Sigma)$.

$\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$ iff $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Defined Logical Connectives

Lemma: (Equivalence)
Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ an assignment into $\mathcal{M}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_o(\Sigma)$.
$\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$ iff $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Proof:

Defined Logical Connectives

Lemma: (Equivalence)

Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ an assignment into $\mathcal{M}$, and $\mathbf{A}, \mathbf{B} \in$ *wff*$_o(\Sigma)$.

$\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$ iff $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Proof: Suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$.

Defined Logical Connectives

Lemma: (Equivalence)
Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ an assignment into $\mathcal{M}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_o(\Sigma)$.
$\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$ iff $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Proof: Suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$.

- This implies $\upsilon(\mathcal{E}_\varphi(\neg(\neg(\neg\mathbf{A} \vee \mathbf{B}) \vee \neg(\neg\mathbf{B} \vee \mathbf{A})))) = \mathtt{T}$

Defined Logical Connectives

Lemma: (Equivalence)
Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ an assignment into $\mathcal{M}$, and $\mathbf{A}, \mathbf{B} \in$ *wff*$_o(\Sigma)$.
$\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$ iff $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Proof: Suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$.

- This implies $\upsilon(\mathcal{E}_\varphi(\neg(\neg(\neg\mathbf{A} \lor \mathbf{B}) \lor \neg(\neg\mathbf{B} \lor \mathbf{A})))) = \mathtt{T}$
- This implies $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{A} \lor \mathbf{B})) = \mathtt{T}$ and $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{B} \lor \mathbf{A})) = \mathtt{T}$.

Defined Logical Connectives

Lemma: (Equivalence)
Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ an assignment into $\mathcal{M}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_o(\Sigma)$.
$\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$ iff $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Proof: Suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$.

- This implies $\upsilon(\mathcal{E}_\varphi(\neg(\neg(\neg\mathbf{A} \vee \mathbf{B}) \vee \neg(\neg\mathbf{B} \vee \mathbf{A})))) = \mathtt{T}$
- This implies $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{A} \vee \mathbf{B})) = \mathtt{T}$ and $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{B} \vee \mathbf{A})) = \mathtt{T}$.
- If $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \mathtt{T}$, then $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{A} \vee \mathbf{B})) = \mathtt{T}$ implies $\upsilon(\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{T}$, so $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \mathtt{T} = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Defined Logical Connectives

Lemma: (Equivalence)
Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ an assignment into $\mathcal{M}$, and $\mathbf{A}, \mathbf{B} \in wff_o(\Sigma)$.
$\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$ iff $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Proof: Suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$.

- This implies $\upsilon(\mathcal{E}_\varphi(\neg(\neg(\neg\mathbf{A} \vee \mathbf{B}) \vee \neg(\neg\mathbf{B} \vee \mathbf{A})))) = \mathtt{T}$
- This implies $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{A} \vee \mathbf{B})) = \mathtt{T}$ and $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{B} \vee \mathbf{A})) = \mathtt{T}$.
- If $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \mathtt{T}$, then $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{A} \vee \mathbf{B})) = \mathtt{T}$ implies $\upsilon(\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{T}$, so $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \mathtt{T} = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.
- If $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \mathtt{F}$, then $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{B} \vee \mathbf{A})) = \mathtt{T}$ implies $\upsilon(\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{F}$, so $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \mathtt{F} = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Defined Logical Connectives

Lemma: (Equivalence)
Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ an assignment into $\mathcal{M}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_o(\Sigma)$.
$\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$ iff $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Proof: Suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$.

- This implies $\upsilon(\mathcal{E}_\varphi(\neg(\neg(\neg\mathbf{A} \lor \mathbf{B}) \lor \neg(\neg\mathbf{B} \lor \mathbf{A})))) = \mathtt{T}$
- This implies $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{A} \lor \mathbf{B})) = \mathtt{T}$ and $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{B} \lor \mathbf{A})) = \mathtt{T}$.
- If $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \mathtt{T}$, then $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{A} \lor \mathbf{B})) = \mathtt{T}$ implies $\upsilon(\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{T}$, so $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \mathtt{T} = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.
- If $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \mathtt{F}$, then $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{B} \lor \mathbf{A})) = \mathtt{T}$ implies $\upsilon(\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{F}$, so $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \mathtt{F} = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.
- Since these are the only two possible values for $\upsilon(\mathcal{E}_\varphi(\mathbf{A}))$, we have $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Defined Logical Connectives

Lemma: (Equivalence)

Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ an assignment into $\mathcal{M}$, and $\mathbf{A}, \mathbf{B} \in$ *wff*$_o(\Sigma)$.

$\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$ iff $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Proof:

Suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Defined Logical Connectives

Lemma: (Equivalence)
Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ an assignment into $\mathcal{M}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_o(\Sigma)$.
$\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$ iff $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Proof:

Suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

- Either $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{T}$
  or $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{F}$.

Defined Logical Connectives

Lemma: (Equivalence)
Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ an assignment into $\mathcal{M}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_o(\Sigma)$.
$\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$ iff $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Proof:

Suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

- Either $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{T}$ or $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{F}$.
- An easy consideration of both cases verifies $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{A} \vee \mathbf{B})) = \mathtt{T}$ and $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{B} \vee \mathbf{A})) = \mathtt{T}$.

Defined Logical Connectives

Lemma: (Equivalence)

Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ an assignment into $\mathcal{M}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_o(\Sigma)$.

$\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$ iff $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Proof:

Suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

- Either $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{T}$ or $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{F}$.
- An easy consideration of both cases verifies $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{A} \vee \mathbf{B})) = \mathtt{T}$ and $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{B} \vee \mathbf{A})) = \mathtt{T}$.
- Hence, $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$.

Defined Logical Connectives

Lemma: (Equivalence)
Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ an assignment into $\mathcal{M}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_o(\Sigma)$.
$\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$ iff $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Proof:

Suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

- Either $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{T}$ or $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{F}$.
- An easy consideration of both cases verifies $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{A} \vee \mathbf{B})) = \mathtt{T}$ and $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{B} \vee \mathbf{A})) = \mathtt{T}$.
- Hence, $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$.

q.e.d.

Defined Logical Connectives

Lemma: (Equivalence)
Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ an assignment into $\mathcal{M}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_{o}(\Sigma)$.
$\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$ iff $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

Proof:

Suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B}))$.

- Either $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{T}$ or $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{F}$.
- An easy consideration of both cases verifies $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{A} \vee \mathbf{B})) = \mathtt{T}$ and $\upsilon(\mathcal{E}_\varphi(\neg\mathbf{B} \vee \mathbf{A})) = \mathtt{T}$.
- Hence, $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \Leftrightarrow \mathbf{B})) = \mathtt{T}$.

q.e.d.

Extensionality for Leibniz Equality

Def.: (Extensionality for Leibniz Equality)

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We call a formula of the form

$$\mathsf{EXT}^{\alpha\to\beta}_{\doteq} \quad := \quad \forall F_{\alpha\to\beta}.\forall G_{\alpha\to\beta}.(\forall X_\alpha.FX \doteq^{\beta} GX) \Rightarrow F \doteq^{\alpha\to\beta} G$$

an axiom of (strong) functional extensionality for Leibniz equality.

Extensionality for Leibniz Equality

Def.: (Extensionality for Leibniz Equality)

We call a formula of the form

$$\mathsf{EXT}^{\alpha\to\beta}_{\dot{=}} \quad := \quad \forall \mathsf{F}_{\alpha\to\beta}. \forall \mathsf{G}_{\alpha\to\beta}. (\forall \mathsf{X}_{\alpha}. \mathsf{FX} \doteq^{\beta} \mathsf{GX}) \Rightarrow \mathsf{F} \doteq^{\alpha\to\beta} \mathsf{G}$$

an axiom of (strong) functional extensionality for Leibniz equality.

We refer to the set

$$\mathsf{EXT}^{\to}_{\dot{=}} := \{\mathsf{EXT}^{\alpha\to\beta}_{\dot{=}} \mid \alpha, \beta \in \mathcal{T}\}$$

as the axioms of (strong) functional extensionality for Leibniz equality.

Extensionality for Leibniz Equality

Def.: (Extensionality for Leibniz Equality)

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We call the formula

$$\mathrm{EXT}^{o}_{\doteq} \quad := \quad \forall A_o. \forall B_o. (A \Leftrightarrow B) \Rightarrow A \doteq^{o} B$$

the axiom of Boolean extensionality.

Extensionality for Leibniz Equality

Def.: (Extensionality for Leibniz Equality)

We call the formula

$$\mathrm{EXT}^{o}_{\dot{=}} \quad := \quad \forall A_o. \forall B_o. (A \Leftrightarrow B) \Rightarrow A \doteq^{o} B$$

the axiom of Boolean extensionality.

We call the set $\mathrm{EXT}^{\rightarrow}_{\dot{=}} \cup \{\mathrm{EXT}^{o}_{\dot{=}}\}$ the axioms of (strong) extensionality for Leibniz equality.

Extensionality and Leibniz Equality

Lemma: (Leibniz Equality in Σ-models) Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ be an assignment, $\alpha \in \mathcal{T}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_\alpha(\Sigma)$.

Extensionality and Leibniz Equality

Lemma: (Leibniz Equality in Σ-models) Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ be an assignment, $\alpha \in \mathcal{T}$, and $\mathbf{A}, \mathbf{B} \in wff_\alpha(\Sigma)$.

1. If $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.

Extensionality and Leibniz Equality

Lemma: (Leibniz Equality in Σ-models) Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ be an assignment, $\alpha \in \mathcal{T}$, and $\mathbf{A}, \mathbf{B} \in wff_\alpha(\Sigma)$.

1. If $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.
2. If $\mathcal{M}$ satisf. $\mathfrak{q}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$, then $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.

Extensionality and Leibniz Equality

Lemma: (Leibniz Equality in Σ-models) Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ be an assignment, $\alpha \in \mathcal{T}$, and $\mathbf{A}, \mathbf{B} \in$ *wff*$_\alpha(\Sigma)$.

1. If $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.
2. If $\mathcal{M}$ satisf. $\mathfrak{q}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$, then $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.

Proof:

Extensionality and Leibniz Equality

Lemma: (Leibniz Equality in Σ-models) Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ be an assignment, $\alpha \in \mathcal{T}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_\alpha(\Sigma)$.

1. If $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.
2. If $\mathcal{M}$ satisf. $\mathfrak{q}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$, then $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.

Proof:

Let φ be any assignment into $\mathcal{M}$.

Extensionality and Leibniz Equality

Lemma: (Leibniz Equality in Σ-models) Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ be an assignment, $\alpha \in \mathcal{T}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_\alpha(\Sigma)$.

1. If $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.
2. If $\mathcal{M}$ satisf. $\mathfrak{q}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$, then $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.

Proof:

Let φ be any assignment into $\mathcal{M}$.

- For the first part, suppose $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.

Extensionality and Leibniz Equality

Lemma: (Leibniz Equality in Σ-models) Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ be an assignment, $\alpha \in \mathcal{T}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_\alpha(\Sigma)$.

1. If $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.
2. If $\mathcal{M}$ satisf. $\mathfrak{q}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$, then $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.

Proof:

Let φ be any assignment into $\mathcal{M}$.

- For the first part, suppose $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.
- Given $\mathsf{r} \in \mathcal{D}_{\alpha\to o}$, we have either
  $\upsilon(\mathsf{r}@\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathsf{r}@\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{F}$ or
  $\upsilon(\mathsf{r}@\mathcal{E}_\varphi(\mathbf{B})) = \upsilon(\mathsf{r}@\mathcal{E}_\varphi(\mathbf{A})) = \mathtt{T}$.

Extensionality and Leibniz Equality

Lemma: (Leibniz Equality in Σ-models) Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ be an assignment, $\alpha \in \mathcal{T}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_\alpha(\Sigma)$.

1. If $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.
2. If $\mathcal{M}$ satisf. $\mathfrak{q}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$, then $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.

Proof:

Let φ be any assignment into $\mathcal{M}$.

- For the first part, suppose $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.
- Given $\mathsf{r} \in \mathcal{D}_{\alpha\to\mathsf{o}}$, we have either
  $\upsilon(\mathsf{r}@\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathsf{r}@\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{F}$ or
  $\upsilon(\mathsf{r}@\mathcal{E}_\varphi(\mathbf{B})) = \upsilon(\mathsf{r}@\mathcal{E}_\varphi(\mathbf{A})) = \mathtt{T}$.
- In either case, for any variable $\mathsf{P}_{\alpha\to\mathsf{o}}$ not in
  $\mathbf{Free}(\mathbf{A}) \cup \mathbf{Free}(\mathbf{B})$, we have $\upsilon(\mathcal{E}_{\varphi,[\mathsf{r}/\mathsf{P}]}(\neg(\mathsf{P}\mathbf{A}) \vee \mathsf{P}\mathbf{B})) = \mathtt{T}$.

Extensionality and Leibniz Equality

Lemma: (Leibniz Equality in Σ-models) Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ be an assignment, $\alpha \in \mathcal{T}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_\alpha(\Sigma)$.

1. If $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.
2. If $\mathcal{M}$ satisf. $\mathfrak{q}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$, then $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.

Proof:

Let φ be any assignment into $\mathcal{M}$.

- For the first part, suppose $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.
- Given $\mathsf{r} \in \mathcal{D}_{\alpha \to \mathsf{o}}$, we have either $\upsilon(\mathsf{r}@\mathcal{E}_\varphi(\mathbf{A})) = \upsilon(\mathsf{r}@\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{F}$ or $\upsilon(\mathsf{r}@\mathcal{E}_\varphi(\mathbf{B})) = \upsilon(\mathsf{r}@\mathcal{E}_\varphi(\mathbf{A})) = \mathtt{T}$.
- In either case, for any variable $\mathsf{P}_{\alpha \to \mathsf{o}}$ not in $\mathbf{Free}(\mathbf{A}) \cup \mathbf{Free}(\mathbf{B})$, we have $\upsilon(\mathcal{E}_{\varphi,[\mathsf{r}/\mathsf{P}]}(\neg(\mathsf{P}\mathbf{A}) \vee \mathsf{P}\mathbf{B})) = \mathtt{T}$.
- So, we have $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.

Extensionality and Leibniz Equality

Lemma: (Leibniz Equality in Σ-models) Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ be an assignment, $\alpha \in \mathcal{T}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_\alpha(\Sigma)$.

1. If $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.
2. If $\mathcal{M}$ satisf. $\mathfrak{q}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$, then $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.

Proof: To show the second part, suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.

Extensionality and Leibniz Equality

Lemma: (Leibniz Equality in Σ-models) Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ be an assignment, $\alpha \in \mathcal{T}$, and $\mathbf{A}, \mathbf{B} \in wff_\alpha(\Sigma)$.

1. If $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.
2. If $\mathcal{M}$ satisf. $\mathfrak{q}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$, then $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.

Proof:

To show the second part, suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.

- By property $\mathfrak{q}$, there is some $\mathsf{q}^\alpha \in \mathcal{D}_{\alpha\to\alpha\to o}$ such that for $\mathsf{a}, \mathsf{b} \in \mathcal{D}_\alpha$ we have $\upsilon(\mathsf{q}^\alpha @ \mathsf{a} @ \mathsf{b}) = \mathtt{T}$ iff $\mathsf{a} = \mathsf{b}$.

Extensionality and Leibniz Equality

Lemma: (Leibniz Equality in Σ-models) Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ be an assignment, $\alpha \in \mathcal{T}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_\alpha(\Sigma)$.

1. If $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.
2. If $\mathcal{M}$ satisf. $\mathfrak{q}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$, then $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.

Proof:

To show the second part, suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.

- By property $\mathfrak{q}$, there is some $\mathsf{q}^\alpha \in \mathcal{D}_{\alpha\to\alpha\to o}$ such that for $\mathsf{a}, \mathsf{b} \in \mathcal{D}_\alpha$ we have $\upsilon(\mathsf{q}^\alpha @\mathsf{a}@\mathsf{b}) = \mathtt{T}$ iff $\mathsf{a} = \mathsf{b}$.
- Let $\mathsf{r} = \mathsf{q}^\alpha @\mathcal{E}_\varphi(\mathbf{A})$.

Extensionality and Leibniz Equality

Lemma: (Leibniz Equality in Σ-models) Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ be an assignment, $\alpha \in \mathcal{T}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_\alpha(\Sigma)$.

1. If $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.
2. If $\mathcal{M}$ satisf. $\mathfrak{q}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$, then $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.

Proof:

To show the second part, suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.

- By property $\mathfrak{q}$, there is some $\mathsf{q}^\alpha \in \mathcal{D}_{\alpha \to \alpha \to o}$ such that for $\mathsf{a}, \mathsf{b} \in \mathcal{D}_\alpha$ we have $\upsilon(\mathsf{q}^\alpha @ \mathsf{a} @ \mathsf{b}) = \mathtt{T}$ iff $\mathsf{a} = \mathsf{b}$.
- Let $\mathsf{r} = \mathsf{q}^\alpha @ \mathcal{E}_\varphi(\mathbf{A})$.
- From $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$ we get $\upsilon(\mathcal{E}_{\varphi,[\mathsf{r}/\mathsf{P}]}(\neg \mathsf{P}\mathbf{A} \vee \mathsf{P}\mathbf{B})) = \mathtt{T}$ (where $\mathsf{P}_{\alpha \to o} \notin \mathbf{Free}(\mathbf{A}) \cup \mathbf{Free}(\mathbf{B})$).

Extensionality and Leibniz Equality

Lemma: (Leibniz Equality in Σ-models) Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ be an assignment, $\alpha \in \mathcal{T}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_\alpha(\Sigma)$.

1. If $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.
2. If $\mathcal{M}$ satisf. $\mathfrak{q}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$, then $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.

Proof:

To show the second part, suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.

- By property $\mathfrak{q}$, there is some $\mathsf{q}^\alpha \in \mathcal{D}_{\alpha\to\alpha\to o}$ such that for $\mathsf{a}, \mathsf{b} \in \mathcal{D}_\alpha$ we have $\upsilon(\mathsf{q}^\alpha @ \mathsf{a} @ \mathsf{b}) = \mathtt{T}$ iff $\mathsf{a} = \mathsf{b}$.
- Let $\mathsf{r} = \mathsf{q}^\alpha @ \mathcal{E}_\varphi(\mathbf{A})$.
- From $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$ we get $\upsilon(\mathcal{E}_{\varphi,[\mathsf{r}/\mathsf{P}]}(\neg \mathsf{P}\mathbf{A} \lor \mathsf{P}\mathbf{B})) = \mathtt{T}$ (where $\mathsf{P}_{\alpha\to o} \notin \mathbf{Free}(\mathbf{A}) \cup \mathbf{Free}(\mathbf{B})$).
- Since $\upsilon(\mathcal{E}_{\varphi,[\mathsf{r}/\mathsf{P}]}(\mathsf{P}\mathbf{A})) = \upsilon(\mathsf{q}^\alpha @ \mathcal{E}_\varphi(\mathbf{A}) @ \mathcal{E}_\varphi(\mathbf{A})) = \mathtt{T}$, we must have $\upsilon(\mathcal{E}_{\varphi,[\mathsf{r}/\mathsf{P}]}(\mathsf{P}\mathbf{B})) = \mathtt{T}$.

Extensionality and Leibniz Equality

Lemma: (Leibniz Equality in Σ-models) Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ be an assignment, $\alpha \in \mathcal{T}$, and $\mathbf{A}, \mathbf{B} \in \mathit{wff}_\alpha(\Sigma)$.

1. If $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.
2. If $\mathcal{M}$ satisf. $\mathfrak{q}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$, then $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.

Proof:

To show the second part, suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.

- By property $\mathfrak{q}$, there is some $\mathsf{q}^\alpha \in \mathcal{D}_{\alpha\to\alpha\to o}$ such that for $\mathsf{a}, \mathsf{b} \in \mathcal{D}_\alpha$ we have $\upsilon(\mathsf{q}^\alpha @ \mathsf{a} @ \mathsf{b}) = \mathtt{T}$ iff $\mathsf{a} = \mathsf{b}$.
- Let $\mathsf{r} = \mathsf{q}^\alpha @ \mathcal{E}_\varphi(\mathbf{A})$.
- From $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$ we get $\upsilon(\mathcal{E}_{\varphi,[\mathsf{r}/P]}(\neg P\mathbf{A} \lor P\mathbf{B})) = \mathtt{T}$ (where $P_{\alpha\to o} \notin \mathbf{Free}(\mathbf{A}) \cup \mathbf{Free}(\mathbf{B})$).
- Since $\upsilon(\mathcal{E}_{\varphi,[\mathsf{r}/P]}(P\mathbf{A})) = \upsilon(\mathsf{q}^\alpha @ \mathcal{E}_\varphi(\mathbf{A}) @ \mathcal{E}_\varphi(\mathbf{A})) = \mathtt{T}$, we must have $\upsilon(\mathcal{E}_{\varphi,[\mathsf{r}/P]}(P\mathbf{B})) = \mathtt{T}$.
- That is, $\upsilon(\mathsf{q}^\alpha @ \mathcal{E}_\varphi(\mathbf{A}) @ \mathcal{E}_\varphi(\mathbf{B})) = \mathtt{T}$, hence $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.

Extensionality and Leibniz Equality

Lemma: (Leibniz Equality in Σ-models) Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, φ be an assignment, $\alpha \in \mathcal{T}$, and $\mathbf{A}, \mathbf{B} \in wff_\alpha(\Sigma)$.

1. If $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.
2. If $\mathcal{M}$ satisf. $\mathfrak{q}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$, then $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.

Proof:

To show the second part, suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$.

- By property $\mathfrak{q}$, there is some $\mathsf{q}^\alpha \in \mathcal{D}_{\alpha\to\alpha\to o}$ such that for $\mathsf{a}, \mathsf{b} \in \mathcal{D}_\alpha$ we have $\upsilon(\mathsf{q}^\alpha @\mathsf{a}@\mathsf{b}) = \mathtt{T}$ iff $\mathsf{a} = \mathsf{b}$.
- Let $\mathsf{r} = \mathsf{q}^\alpha @\mathcal{E}_\varphi(\mathbf{A})$.
- From $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) = \mathtt{T}$ we get $\upsilon(\mathcal{E}_{\varphi,[\mathsf{r}/\mathsf{P}]}(\neg \mathsf{P}\mathbf{A} \vee \mathsf{P}\mathbf{B})) = \mathtt{T}$ (where $\mathsf{P}_{\alpha\to o} \notin \mathbf{Free}(\mathbf{A}) \cup \mathbf{Free}(\mathbf{B})$).
- Since $\upsilon(\mathcal{E}_{\varphi,[\mathsf{r}/\mathsf{P}]}(\mathsf{P}\mathbf{A})) = \upsilon(\mathsf{q}^\alpha @\mathcal{E}_\varphi(\mathbf{A})@\mathcal{E}_\varphi(\mathbf{A})) = \mathtt{T}$, we must have $\upsilon(\mathcal{E}_{\varphi,[\mathsf{r}/\mathsf{P}]}(\mathsf{P}\mathbf{B})) = \mathtt{T}$.
- That is, $\upsilon(\mathsf{q}^\alpha @\mathcal{E}_\varphi(\mathbf{A})@\mathcal{E}_\varphi(\mathbf{B})) = \mathtt{T}$, hence $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{B})$.

q.e.d.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{f}$, then $\mathcal{M} \not\models \mathsf{EXT}^{\rightarrow}_{\dot{\underline{=}}}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)
Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{f}$, then $\mathcal{M} \not\models \mathsf{EXT}^{\rightarrow}_{\dot{=}}$.
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{b}$, then $\mathcal{M} \not\models \mathsf{EXT}^{o}_{\dot{=}}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{f}$, then $\mathcal{M} \not\models \mathsf{EXT}^{\rightarrow}_{\doteq}$.
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{b}$, then $\mathcal{M} \not\models \mathsf{EXT}^{o}_{\doteq}$.
3. If $\mathcal{M}$ satisfies $\mathfrak{q}$ and $\mathfrak{f}$, then $\mathcal{M} \models \mathsf{EXT}^{\rightarrow}_{\doteq}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{f}$, then $\mathcal{M} \not\models \mathsf{EXT}^{\rightarrow}_{\dot{=}}$.
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{b}$, then $\mathcal{M} \not\models \mathsf{EXT}^{o}_{\dot{=}}$.
3. If $\mathcal{M}$ satisfies $\mathfrak{q}$ and $\mathfrak{f}$, then $\mathcal{M} \models \mathsf{EXT}^{\rightarrow}_{\dot{=}}$.
4. If $\mathcal{M}$ satisfies $\mathfrak{b}$, then $\mathcal{M} \models \mathsf{EXT}^{o}_{\dot{=}}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)
Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{f}$, then $\mathcal{M} \not\models \mathrm{EXT}^{\rightarrow}_{\doteq}$.
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{b}$, then $\mathcal{M} \not\models \mathrm{EXT}^{o}_{\doteq}$.
3. If $\mathcal{M}$ satisfies $\mathfrak{q}$ and $\mathfrak{f}$, then $\mathcal{M} \models \mathrm{EXT}^{\rightarrow}_{\doteq}$.
4. If $\mathcal{M}$ satisfies $\mathfrak{b}$, then $\mathcal{M} \models \mathrm{EXT}^{o}_{\doteq}$.

in	$\mathfrak{M}_{\beta}(\Sigma)$, $\mathfrak{M}_{\beta\eta}(\Sigma)$, $\mathfrak{M}_{\beta\xi}(\Sigma)$		$\mathfrak{M}_{\beta\mathfrak{f}}(\Sigma)$		$\mathfrak{M}_{\beta\mathfrak{b}}(\Sigma)$,$\mathfrak{M}_{\beta\eta\mathfrak{b}}(\Sigma)$,$\mathfrak{M}_{\beta\xi\mathfrak{b}}(\Sigma)$		$\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma)$	
formula	valid?	by	valid?	by	valid?	by	valid?	by
$\mathrm{EXT}^{\rightarrow}_{\doteq}$	–	1.	+	3.	–	1.	+	3.
$\mathrm{EXT}^{o}_{\doteq}$	–	2.	–	2.	+	4.	+	4.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)
Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{f}$, then $\mathcal{M} \not\models \mathsf{EXT}^{\rightarrow}_{\dot{=}}$.

Proof:

Suppose $\mathcal{M}$ satisfies property $\mathfrak{q}$ but does not satisfy property $\mathfrak{f}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)
Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{f}$, then $\mathcal{M} \not\models \mathsf{EXT}^{\rightarrow}_{\dot{=}}$.

Proof: Suppose $\mathcal{M}$ satisfies property $\mathfrak{q}$ but does not satisfy property $\mathfrak{f}$.

- Then there must be types α and β and objects $\mathsf{f}, \mathsf{g} \in \mathcal{D}_{\alpha\rightarrow\beta}$ such that $\mathsf{f} \neq \mathsf{g}$ but $\mathsf{f}@\mathsf{a} = \mathsf{g}@\mathsf{a}$ for every $\mathsf{a} \in \mathcal{D}_\alpha$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{f}$, then $\mathcal{M} \not\models \mathsf{EXT}^{\rightarrow}_{\doteq}$.

Proof:

Suppose $\mathcal{M}$ satisfies property $\mathfrak{q}$ but does not satisfy property $\mathfrak{f}$.

- Then there must be types α and β and objects $\mathsf{f}, \mathsf{g} \in \mathcal{D}_{\alpha\rightarrow\beta}$ such that $\mathsf{f} \neq \mathsf{g}$ but $\mathsf{f}@\mathsf{a} = \mathsf{g}@\mathsf{a}$ for every $\mathsf{a} \in \mathcal{D}_\alpha$.
- Let $\mathsf{F}_{\alpha\rightarrow\beta}, \mathsf{G}_{\alpha\rightarrow\beta} \in \mathcal{V}_{\alpha\rightarrow\beta}$ be distinct variables, $\mathsf{X}_\alpha \in \mathcal{V}_\alpha$, and φ be any assignment with $\varphi(\mathsf{F}) = \mathsf{f}$ and $\varphi(\mathsf{G}) = \mathsf{g}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{f}$, then $\mathcal{M} \not\models \mathsf{EXT}^{\rightarrow}_{\doteq}$.

Proof:

Suppose $\mathcal{M}$ satisfies property $\mathfrak{q}$ but does not satisfy property $\mathfrak{f}$.

- Then there must be types α and β and objects $\mathsf{f}, \mathsf{g} \in \mathcal{D}_{\alpha\to\beta}$ such that $\mathsf{f} \neq \mathsf{g}$ but $\mathsf{f}@\mathsf{a} = \mathsf{g}@\mathsf{a}$ for every $\mathsf{a} \in \mathcal{D}_\alpha$.
- Let $\mathsf{F}_{\alpha\to\beta}, \mathsf{G}_{\alpha\to\beta} \in \mathcal{V}_{\alpha\to\beta}$ be distinct variables, $\mathsf{X}_\alpha \in \mathcal{V}_\alpha$, and φ be any assignment with $\varphi(\mathsf{F}) = \mathsf{f}$ and $\varphi(\mathsf{G}) = \mathsf{g}$.
- For any $\mathsf{a} \in \mathcal{D}_\alpha$, $\mathsf{f}@\mathsf{a} = \mathsf{g}@\mathsf{a}$ implies $\mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{FX}) = \mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{GX})$ implies $\upsilon(\mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{FX} \doteq^{\beta} \mathsf{GX})) = \mathtt{T}$ by Lemma 'Leibniz Equality in Σ-models(1.)'.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{f}$, then $\mathcal{M} \not\models \mathsf{EXT}^{\rightarrow}_{\doteq}$.

Proof:

Suppose $\mathcal{M}$ satisfies property $\mathfrak{q}$ but does not satisfy property $\mathfrak{f}$.

- Then there must be types α and β and objects $\mathsf{f}, \mathsf{g} \in \mathcal{D}_{\alpha\rightarrow\beta}$ such that $\mathsf{f} \neq \mathsf{g}$ but $\mathsf{f}@\mathsf{a} = \mathsf{g}@\mathsf{a}$ for every $\mathsf{a} \in \mathcal{D}_\alpha$.
- Let $\mathsf{F}_{\alpha\rightarrow\beta}, \mathsf{G}_{\alpha\rightarrow\beta} \in \mathcal{V}_{\alpha\rightarrow\beta}$ be distinct variables, $\mathsf{X}_\alpha \in \mathcal{V}_\alpha$, and φ be any assignment with $\varphi(\mathsf{F}) = \mathsf{f}$ and $\varphi(\mathsf{G}) = \mathsf{g}$.
- For any $\mathsf{a} \in \mathcal{D}_\alpha$, $\mathsf{f}@\mathsf{a} = \mathsf{g}@\mathsf{a}$ implies $\mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{FX}) = \mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{GX})$ implies $\upsilon(\mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{FX} \doteq^\beta \mathsf{GX})) = \mathtt{T}$ by Lemma 'Leibniz Equality in Σ-models(1.)'.
- Hence, we have $\upsilon(\mathcal{E}_\varphi(\forall \mathsf{X}.(\mathsf{FX} \doteq^\beta \mathsf{GX}))) = \mathtt{T}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{f}$, then $\mathcal{M} \not\models \mathsf{EXT}^{\rightarrow}_{\doteq}$.

Proof:

Suppose $\mathcal{M}$ satisfies property $\mathfrak{q}$ but does not satisfy property $\mathfrak{f}$.

- Then there must be types α and β and objects $\mathsf{f}, \mathsf{g} \in \mathcal{D}_{\alpha\rightarrow\beta}$ such that $\mathsf{f} \neq \mathsf{g}$ but $\mathsf{f}@\mathsf{a} = \mathsf{g}@\mathsf{a}$ for every $\mathsf{a} \in \mathcal{D}_\alpha$.
- Let $\mathsf{F}_{\alpha\rightarrow\beta}, \mathsf{G}_{\alpha\rightarrow\beta} \in \mathcal{V}_{\alpha\rightarrow\beta}$ be distinct variables, $\mathsf{X}_\alpha \in \mathcal{V}_\alpha$, and φ be any assignment with $\varphi(\mathsf{F}) = \mathsf{f}$ and $\varphi(\mathsf{G}) = \mathsf{g}$.
- For any $\mathsf{a} \in \mathcal{D}_\alpha$, $\mathsf{f}@\mathsf{a} = \mathsf{g}@\mathsf{a}$ implies $\mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{FX}) = \mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{GX})$ implies $\upsilon(\mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{FX} \doteq^\beta \mathsf{GX})) = \mathtt{T}$ by Lemma 'Leibniz Equality in Σ-models(1.)'.
- Hence, we have $\upsilon(\mathcal{E}_\varphi(\forall \mathsf{X}_{\bullet}(\mathsf{FX} \doteq^\beta \mathsf{GX}))) = \mathtt{T}$.
- On the other hand, since $\mathsf{f} \neq \mathsf{g}$ and $\mathcal{M}$ satisfies property $\mathfrak{q}$, we have $\upsilon(\mathcal{E}_\varphi(\mathsf{F} \doteq^{\alpha\rightarrow\beta} \mathsf{G})) = \mathtt{F}$ by contraposition of Lemma 'Leibniz Equality in Σ-models(2.)'.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)
Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{f}$, then $\mathcal{M} \not\models \mathsf{EXT}^{\rightarrow}_{\doteq}$.

Proof:

Suppose $\mathcal{M}$ satisfies property $\mathfrak{q}$ but does not satisfy property $\mathfrak{f}$.

- Then there must be types α and β and objects $\mathsf{f}, \mathsf{g} \in \mathcal{D}_{\alpha\rightarrow\beta}$ such that $\mathsf{f} \neq \mathsf{g}$ but $\mathsf{f}@\mathsf{a} = \mathsf{g}@\mathsf{a}$ for every $\mathsf{a} \in \mathcal{D}_\alpha$.
- Let $\mathsf{F}_{\alpha\rightarrow\beta}, \mathsf{G}_{\alpha\rightarrow\beta} \in \mathcal{V}_{\alpha\rightarrow\beta}$ be distinct variables, $\mathsf{X}_\alpha \in \mathcal{V}_\alpha$, and φ be any assignment with $\varphi(\mathsf{F}) = \mathsf{f}$ and $\varphi(\mathsf{G}) = \mathsf{g}$.
- For any $\mathsf{a} \in \mathcal{D}_\alpha$, $\mathsf{f}@\mathsf{a} = \mathsf{g}@\mathsf{a}$ implies $\mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{FX}) = \mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{GX})$ implies $\upsilon(\mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{FX} \doteq^\beta \mathsf{GX})) = \mathtt{T}$ by Lemma 'Leibniz Equality in Σ-models(1.)'.
- Hence, we have $\upsilon(\mathcal{E}_\varphi(\forall \mathsf{X}.(\mathsf{FX} \doteq^\beta \mathsf{GX}))) = \mathtt{T}$.
- On the other hand, since $\mathsf{f} \neq \mathsf{g}$ and $\mathcal{M}$ satisfies property $\mathfrak{q}$, we have $\upsilon(\mathcal{E}_\varphi(\mathsf{F} \doteq^{\alpha\rightarrow\beta} \mathsf{G})) = \mathtt{F}$ by contraposition of Lemma 'Leibniz Equality in Σ-models(2.)'.
- This implies $\mathcal{M} \not\models \mathsf{EXT}^{\alpha\rightarrow\beta}_{\doteq}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

2. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{b}$, then $\mathcal{M} \not\models \mathsf{EXT}^{o}_{\doteq}$.

Proof: Suppose $\mathcal{M}$ satisfies property $\mathfrak{q}$ but does not satisfy property $\mathfrak{b}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

2. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{b}$, then $\mathcal{M} \not\models \mathsf{EXT}^{\mathsf{o}}_{\doteq}$.

Proof:

Suppose $\mathcal{M}$ satisfies property $\mathfrak{q}$ but does not satisfy property $\mathfrak{b}$.

- Then, there must be at least three elements in $\mathcal{D}_{\mathsf{o}}$. Since υ maps into a two element set, there must be two distinct elements $\mathsf{a}, \mathsf{b} \in \mathcal{D}_{\mathsf{o}}$ such that $\upsilon(\mathsf{a}) = \upsilon(\mathsf{b})$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

2. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{b}$, then $\mathcal{M} \not\models \mathsf{EXT}^{o}_{\doteq}$.

Proof:

Suppose $\mathcal{M}$ satisfies property $\mathfrak{q}$ but does not satisfy property $\mathfrak{b}$.

- Then, there must be at least three elements in $\mathcal{D}_o$. Since υ maps into a two element set, there must be two distinct elements $\mathsf{a}, \mathsf{b} \in \mathcal{D}_o$ such that $\upsilon(\mathsf{a}) = \upsilon(\mathsf{b})$.
- Let $\mathsf{A}_o, \mathsf{B}_o \in \mathcal{V}_o$ be distinct variables and φ be any assignment into $\mathcal{M}$ with $\varphi(\mathsf{A}) = \mathsf{a}$ and $\varphi(\mathsf{B}) = \mathsf{b}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)
Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

2. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{b}$, then $\mathcal{M} \not\models \mathsf{EXT}^{\mathsf{o}}_{\dot{=}}$.

Proof: Suppose $\mathcal{M}$ satisfies property $\mathfrak{q}$ but does not satisfy property $\mathfrak{b}$.

- Then, there must be at least three elements in $\mathcal{D}_{\mathsf{o}}$. Since υ maps into a two element set, there must be two distinct elements $\mathsf{a}, \mathsf{b} \in \mathcal{D}_{\mathsf{o}}$ such that $\upsilon(\mathsf{a}) = \upsilon(\mathsf{b})$.
- Let $\mathsf{A}_{\mathsf{o}}, \mathsf{B}_{\mathsf{o}} \in \mathcal{V}_{\mathsf{o}}$ be distinct variables and φ be any assignment into $\mathcal{M}$ with $\varphi(\mathsf{A}) = \mathsf{a}$ and $\varphi(\mathsf{B}) = \mathsf{b}$.
- By Lemma 'Equivalence', we know $\upsilon(\mathcal{E}_{\varphi}(\mathsf{A} \Leftrightarrow \mathsf{B})) = \mathtt{T}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

2. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{b}$, then $\mathcal{M} \not\models \mathrm{EXT}^{\mathrm{o}}_{\doteq}$.

Proof:

Suppose $\mathcal{M}$ satisfies property $\mathfrak{q}$ but does not satisfy property $\mathfrak{b}$.

- Then, there must be at least three elements in $\mathcal{D}_{\mathrm{o}}$. Since υ maps into a two element set, there must be two distinct elements $\mathsf{a}, \mathsf{b} \in \mathcal{D}_{\mathrm{o}}$ such that $\upsilon(\mathsf{a}) = \upsilon(\mathsf{b})$.
- Let $\mathsf{A}_{\mathrm{o}}, \mathsf{B}_{\mathrm{o}} \in \mathcal{V}_{\mathrm{o}}$ be distinct variables and φ be any assignment into $\mathcal{M}$ with $\varphi(\mathsf{A}) = \mathsf{a}$ and $\varphi(\mathsf{B}) = \mathsf{b}$.
- By Lemma 'Equivalence', we know $\upsilon(\mathcal{E}_{\varphi}(\mathsf{A} \Leftrightarrow \mathsf{B})) = \mathtt{T}$.
- Since $\mathsf{a} \neq \mathsf{b}$ and property $\mathfrak{q}$ holds, by contraposition of Lemma 'Leibniz Equality in Σ-models(2.)', we know $\upsilon(\mathcal{E}_{\varphi}(\mathsf{A} \doteq^{\mathrm{o}} \mathsf{B})) = \mathtt{F}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

2. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{b}$, then $\mathcal{M} \not\models \mathsf{EXT}^{\mathsf{o}}_{\doteq}$.

Proof:

Suppose $\mathcal{M}$ satisfies property $\mathfrak{q}$ but does not satisfy property $\mathfrak{b}$.

- Then, there must be at least three elements in $\mathcal{D}_{\mathsf{o}}$. Since υ maps into a two element set, there must be two distinct elements $\mathsf{a}, \mathsf{b} \in \mathcal{D}_{\mathsf{o}}$ such that $\upsilon(\mathsf{a}) = \upsilon(\mathsf{b})$.
- Let $\mathsf{A}_{\mathsf{o}}, \mathsf{B}_{\mathsf{o}} \in \mathcal{V}_{\mathsf{o}}$ be distinct variables and φ be any assignment into $\mathcal{M}$ with $\varphi(\mathsf{A}) = \mathsf{a}$ and $\varphi(\mathsf{B}) = \mathsf{b}$.
- By Lemma 'Equivalence', we know $\upsilon(\mathcal{E}_{\varphi}(\mathsf{A} \Leftrightarrow \mathsf{B})) = \mathtt{T}$.
- Since $\mathsf{a} \neq \mathsf{b}$ and property $\mathfrak{q}$ holds, by contraposition of Lemma 'Leibniz Equality in Σ-models(2.)', we know $\upsilon(\mathcal{E}_{\varphi}(\mathsf{A} \doteq^{\mathsf{o}} \mathsf{B})) = \mathtt{F}$.
- It follows that $\mathcal{M} \not\models \mathsf{EXT}^{\mathsf{o}}_{\doteq}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)
Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

2. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{b}$, then $\mathcal{M} \not\models \mathsf{EXT}^{\circ}_{\doteq}$.

Proof:

Suppose $\mathcal{M}$ satisfies property $\mathfrak{q}$ but does not satisfy property $\mathfrak{b}$.

- Then, there must be at least three elements in $\mathcal{D}_{o}$. Since υ maps into a two element set, there must be two distinct elements $\mathsf{a}, \mathsf{b} \in \mathcal{D}_{o}$ such that $\upsilon(\mathsf{a}) = \upsilon(\mathsf{b})$.
- Let $\mathsf{A}_{o}, \mathsf{B}_{o} \in \mathcal{V}_{o}$ be distinct variables and φ be any assignment into $\mathcal{M}$ with $\varphi(\mathsf{A}) = \mathsf{a}$ and $\varphi(\mathsf{B}) = \mathsf{b}$.
- By Lemma 'Equivalence', we know $\upsilon(\mathcal{E}_{\varphi}(\mathsf{A} \Leftrightarrow \mathsf{B})) = \mathtt{T}$.
- Since $\mathsf{a} \neq \mathsf{b}$ and property $\mathfrak{q}$ holds, by contraposition of Lemma 'Leibniz Equality in Σ-models(2.)', we know $\upsilon(\mathcal{E}_{\varphi}(\mathsf{A} \doteq^{\circ} \mathsf{B})) = \mathtt{F}$.
- It follows that $\mathcal{M} \not\models \mathsf{EXT}^{\circ}_{\doteq}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

3. If $\mathcal{M}$ satisfies $\mathfrak{q}$ and $\mathfrak{f}$, then $\mathcal{M} \models \mathsf{EXT}^{\rightarrow}_{\dot{=}}$.

Proof: Let φ be any assignment into $\mathcal{M}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)
Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

3. If $\mathcal{M}$ satisfies $\mathfrak{q}$ and $\mathfrak{f}$, then $\mathcal{M} \models \mathsf{EXT}^{\rightarrow}_{\doteq}$.

Proof:

Let φ be any assignment into $\mathcal{M}$.

- From $\upsilon(\mathcal{E}_\varphi(\forall X_\alpha . FX \doteq GX)) = \mathtt{T}$ we know $\upsilon(\mathcal{E}_{\varphi,[a/X]}(FX \doteq GX)) = \mathtt{T}$ holds for all $a \in \mathcal{D}_\alpha$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

3. If $\mathcal{M}$ satisfies $\mathfrak{q}$ and $\mathfrak{f}$, then $\mathcal{M} \models \mathsf{EXT}^{\rightarrow}_{\doteq}$.

Proof:

Let φ be any assignment into $\mathcal{M}$.

- From $\upsilon(\mathcal{E}_{\varphi}(\forall \mathsf{X}_{\alpha}.\mathsf{FX} \doteq \mathsf{GX})) = \mathtt{T}$ we know $\upsilon(\mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{FX} \doteq \mathsf{GX})) = \mathtt{T}$ holds for all $\mathsf{a} \in \mathcal{D}_{\alpha}$.
- By Lemma 'Leibniz Equality in Σ-models(2.)' we can conclude that $\mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{FX}) = \mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{GX})$ for all $\mathsf{a} \in \mathcal{D}_{\alpha}$ and hence $\mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{F})@\mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{X}) = \mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{G})@\mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathsf{X})$ for all $\mathsf{a} \in \mathcal{D}_{\alpha}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)
Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

3. If $\mathcal{M}$ satisfies $\mathfrak{q}$ and $\mathfrak{f}$, then $\mathcal{M} \models \mathsf{EXT}^{\rightarrow}_{\doteq}$.

Proof:

Let φ be any assignment into $\mathcal{M}$.

- From $\upsilon(\mathcal{E}_\varphi(\forall X_\alpha. FX \doteq GX)) = \mathtt{T}$ we know $\upsilon(\mathcal{E}_{\varphi,[a/X]}(FX \doteq GX)) = \mathtt{T}$ holds for all $a \in \mathcal{D}_\alpha$.
- By Lemma 'Leibniz Equality in Σ-models(2.)' we can conclude that $\mathcal{E}_{\varphi,[a/X]}(FX) = \mathcal{E}_{\varphi,[a/X]}(GX)$ for all $a \in \mathcal{D}_\alpha$ and hence $\mathcal{E}_{\varphi,[a/X]}(F)@\mathcal{E}_{\varphi,[a/X]}(X) = \mathcal{E}_{\varphi,[a/X]}(G)@\mathcal{E}_{\varphi,[a/X]}(X)$ for all $a \in \mathcal{D}_\alpha$.
- That is, $\mathcal{E}_{\varphi,[a/X]}(F)@a = \mathcal{E}_{\varphi,[a/X]}(G)@a$ for all $a \in \mathcal{D}_\alpha$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

3. If $\mathcal{M}$ satisfies $\mathfrak{q}$ and $\mathfrak{f}$, then $\mathcal{M} \models \mathsf{EXT}^{\rightarrow}_{\doteq}$.

Proof:

Let φ be any assignment into $\mathcal{M}$.

- From $\upsilon(\mathcal{E}_\varphi(\forall X_\alpha \mathbf{.} FX \doteq GX)) = \mathtt{T}$ we know $\upsilon(\mathcal{E}_{\varphi,[a/X]}(FX \doteq GX)) = \mathtt{T}$ holds for all $a \in \mathcal{D}_\alpha$.
- By Lemma 'Leibniz Equality in Σ-models(2.)' we can conclude that $\mathcal{E}_{\varphi,[a/X]}(FX) = \mathcal{E}_{\varphi,[a/X]}(GX)$ for all $a \in \mathcal{D}_\alpha$ and hence $\mathcal{E}_{\varphi,[a/X]}(F)@\mathcal{E}_{\varphi,[a/X]}(X) = \mathcal{E}_{\varphi,[a/X]}(G)@\mathcal{E}_{\varphi,[a/X]}(X)$ for all $a \in \mathcal{D}_\alpha$.
- That is, $\mathcal{E}_{\varphi,[a/X]}(F)@a = \mathcal{E}_{\varphi,[a/X]}(G)@a$ for all $a \in \mathcal{D}_\alpha$.
- Since X does not occur free in F or G, by property $\mathfrak{f}$ and Definition of Σ-evaluations we obtain $\mathcal{E}_\varphi(F) = \mathcal{E}_\varphi(G)$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)
Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

3. If $\mathcal{M}$ satisfies $\mathfrak{q}$ and $\mathfrak{f}$, then $\mathcal{M} \models \mathsf{EXT}_{\dot{=}}^{\rightarrow}$.

Proof: Let φ be any assignment into $\mathcal{M}$.

- From $\upsilon(\mathcal{E}_\varphi(\forall X_\alpha . FX \doteq GX)) = \mathtt{T}$ we know $\upsilon(\mathcal{E}_{\varphi,[a/X]}(FX \doteq GX)) = \mathtt{T}$ holds for all $a \in \mathcal{D}_\alpha$.
- By Lemma 'Leibniz Equality in Σ-models(2.)' we can conclude that $\mathcal{E}_{\varphi,[a/X]}(FX) = \mathcal{E}_{\varphi,[a/X]}(GX)$ for all $a \in \mathcal{D}_\alpha$ and hence $\mathcal{E}_{\varphi,[a/X]}(F)@\mathcal{E}_{\varphi,[a/X]}(X) = \mathcal{E}_{\varphi,[a/X]}(G)@\mathcal{E}_{\varphi,[a/X]}(X)$ for all $a \in \mathcal{D}_\alpha$.
- That is, $\mathcal{E}_{\varphi,[a/X]}(F)@a = \mathcal{E}_{\varphi,[a/X]}(G)@a$ for all $a \in \mathcal{D}_\alpha$.
- Since X does not occur free in F or G, by property $\mathfrak{f}$ and Definition of Σ-evaluations we obtain $\mathcal{E}_\varphi(F) = \mathcal{E}_\varphi(G)$.
- This finally gives us that $\upsilon(\mathcal{E}_\varphi(F \doteq^{\alpha\rightarrow\beta} G)) = \mathtt{T}$ with Lemma 'Leibniz Equality in Σ-models(1.)'.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

3. If $\mathcal{M}$ satisfies $\mathfrak{q}$ and $\mathfrak{f}$, then $\mathcal{M} \models \mathsf{EXT}^{\rightarrow}_{\doteq}$.

Proof:

Let φ be any assignment into $\mathcal{M}$.

- From $\upsilon(\mathcal{E}_\varphi(\forall X_\alpha . FX \doteq GX)) = T$ we know $\upsilon(\mathcal{E}_{\varphi,[a/X]}(FX \doteq GX)) = T$ holds for all $a \in \mathcal{D}_\alpha$.
- By Lemma 'Leibniz Equality in Σ-models(2.)' we can conclude that $\mathcal{E}_{\varphi,[a/X]}(FX) = \mathcal{E}_{\varphi,[a/X]}(GX)$ for all $a \in \mathcal{D}_\alpha$ and hence $\mathcal{E}_{\varphi,[a/X]}(F)@\mathcal{E}_{\varphi,[a/X]}(X) = \mathcal{E}_{\varphi,[a/X]}(G)@\mathcal{E}_{\varphi,[a/X]}(X)$ for all $a \in \mathcal{D}_\alpha$.
- That is, $\mathcal{E}_{\varphi,[a/X]}(F)@a = \mathcal{E}_{\varphi,[a/X]}(G)@a$ for all $a \in \mathcal{D}_\alpha$.
- Since X does not occur free in F or G, by property $\mathfrak{f}$ and Definition of Σ-evaluations we obtain $\mathcal{E}_\varphi(F) = \mathcal{E}_\varphi(G)$.
- This finally gives us that $\upsilon(\mathcal{E}_\varphi(F \doteq^{\alpha\rightarrow\beta} G)) = T$ with Lemma 'Leibniz Equality in Σ-models(1.)'.
- It follows that $\mathcal{M} \models \mathsf{EXT}^{\alpha\rightarrow\beta}_{\doteq}$ and $\mathcal{M} \models \mathsf{EXT}^{\rightarrow}_{\doteq}$, since α and β were chosen arbitrarily.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

4. If $\mathcal{M}$ satisfies $\mathfrak{b}$, then $\mathcal{M} \models \mathsf{EXT}^{o}_{\doteq}$.

Proof: Let $A_o, B_o \in \mathcal{V}_o$ be distinct variables and φ be any assignment into $\mathcal{M}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)
Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

4. If $\mathcal{M}$ satisfies $\mathfrak{b}$, then $\mathcal{M} \models \mathsf{EXT}^{o}_{\doteq}$.

Proof:

Let $A_o, B_o \in \mathcal{V}_o$ be distinct variables and φ be any assignment into $\mathcal{M}$.

- Since property $\mathfrak{b}$ holds, we can assume $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$ and υ is the identity function.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)
Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

4. If $\mathcal{M}$ satisfies $\mathfrak{b}$, then $\mathcal{M} \models \mathsf{EXT}^{\mathsf{o}}_{\doteq}$.

Proof:

Let $\mathsf{A_o}, \mathsf{B_o} \in \mathcal{V}_\mathsf{o}$ be distinct variables and φ be any assignment into $\mathcal{M}$.

- Since property $\mathfrak{b}$ holds, we can assume $\mathcal{D}_\mathsf{o} = \{\mathtt{T}, \mathtt{F}\}$ and υ is the identity function.
- Suppose $\upsilon(\mathcal{E}_\varphi(\mathsf{A} \Leftrightarrow \mathsf{B})) = \mathtt{T}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)
Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

4. If $\mathcal{M}$ satisfies $\mathfrak{b}$, then $\mathcal{M} \models \mathsf{EXT}^{\mathsf{o}}_{\dot{=}}$.

Proof:

Let $\mathsf{A_o}, \mathsf{B_o} \in \mathcal{V}_\mathsf{o}$ be distinct variables and φ be any assignment into $\mathcal{M}$.

- Since property $\mathfrak{b}$ holds, we can assume $\mathcal{D}_\mathsf{o} = \{\mathtt{T}, \mathtt{F}\}$ and υ is the identity function.
- Suppose $\upsilon(\mathcal{E}_\varphi(\mathsf{A} \Leftrightarrow \mathsf{B})) = \mathtt{T}$.
- By Lemma 'Equivalence', we have $\mathcal{E}_\varphi(\mathsf{A}) = \upsilon(\mathcal{E}_\varphi(\mathsf{A})) = \upsilon(\mathcal{E}_\varphi(\mathsf{B})) = \mathcal{E}_\varphi(\mathsf{B})$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

4. If $\mathcal{M}$ satisfies $\mathfrak{b}$, then $\mathcal{M} \models \mathsf{EXT}^{o}_{\doteq}$.

Proof:

Let $A_o, B_o \in \mathcal{V}_o$ be distinct variables and φ be any assignment into $\mathcal{M}$.

- Since property $\mathfrak{b}$ holds, we can assume $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$ and υ is the identity function.
- Suppose $\upsilon(\mathcal{E}_\varphi(A \Leftrightarrow B)) = \mathtt{T}$.
- By Lemma 'Equivalence', we have $\mathcal{E}_\varphi(A) = \upsilon(\mathcal{E}_\varphi(A)) = \upsilon(\mathcal{E}_\varphi(B)) = \mathcal{E}_\varphi(B)$.
- By Lemma 'Leibniz Equality in Σ-models(1.)', we have $\upsilon(\mathcal{E}_\varphi(A \doteq^{o} B)) = \mathtt{T}$.

Extensionality and Leibniz Equality

Thm.: (Extensionality in Σ-models)

Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model.

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{f}$, then $\mathcal{M} \not\models \mathsf{EXT}^{\rightarrow}_{\doteq}$.
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$ but not property $\mathfrak{b}$, then $\mathcal{M} \not\models \mathsf{EXT}^{o}_{\doteq}$.
3. If $\mathcal{M}$ satisfies $\mathfrak{q}$ and $\mathfrak{f}$, then $\mathcal{M} \models \mathsf{EXT}^{\rightarrow}_{\doteq}$.
4. If $\mathcal{M}$ satisfies $\mathfrak{b}$, then $\mathcal{M} \models \mathsf{EXT}^{o}_{\doteq}$.

Proof:

q.e.d.

Extensionality and Leibniz Equality

Thm.: (Trivial Extensionality Directions in Σ-Models)

Extensionality and Leibniz Equality

Thm.: (Trivial Extensionality Directions in Σ-Models)

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall A_o. \forall B_o. A \doteq^o B \Rightarrow (A \Leftrightarrow B)$.

Extensionality and Leibniz Equality

Thm.: (Trivial Extensionality Directions in Σ-Models)

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall A_o. \forall B_o. A \doteq^o B \Rightarrow (A \Leftrightarrow B)$.
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then
   $\mathcal{M} \models \forall F_{\alpha\rightarrow\beta}. \forall G_{\alpha\rightarrow\beta}. F \doteq^{\alpha\rightarrow\beta} G \Rightarrow (\forall X_\alpha. FX \doteq^\beta GX)$

Extensionality and Leibniz Equality

Thm.: (Trivial Extensionality Directions in Σ-Models)

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall A_o . \forall B_o . A \doteq^o B \Rightarrow (A \Leftrightarrow B)$
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then
   $\mathcal{M} \models \forall F_{\alpha\to\beta} . \forall G_{\alpha\to\beta} . F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha . FX \doteq^\beta GX)$

Proof: (1.)

$$\upsilon(\mathcal{E}_\varphi(\forall A_o . \forall B_o . A \doteq^o B \Rightarrow (A \Leftrightarrow B))) = \mathtt{T}$$

Extensionality and Leibniz Equality

Thm.: (Trivial Extensionality Directions in Σ-Models)

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall A_o. \forall B_o. A \doteq^o B \Rightarrow (A \Leftrightarrow B)$
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then
   $\mathcal{M} \models \forall F_{\alpha\to\beta}. \forall G_{\alpha\to\beta}. F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha. FX \doteq^\beta GX)$

Proof: (1.)

$\upsilon(\mathcal{E}_\varphi(\forall A_o. \forall B_o. A \doteq^o B \Rightarrow (A \Leftrightarrow B))) = \mathtt{T}$

- iff (for all $a, b \in \mathcal{D}_\alpha$)
  $\upsilon(\mathcal{E}_{\varphi[a/A][b/B]}(A \doteq^o B) = \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi[a/A][b/B]}(A \Leftrightarrow B)) = \mathtt{T}$

Extensionality and Leibniz Equality

Thm.: (Trivial Extensionality Directions in Σ-Models)

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall A_o. \forall B_o. A \doteq^o B \Rightarrow (A \Leftrightarrow B)$
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then
   $\mathcal{M} \models \forall F_{\alpha\to\beta}. \forall G_{\alpha\to\beta}. F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha. FX \doteq^\beta GX)$

Proof: (1.)

$\upsilon(\mathcal{E}_\varphi(\forall A_o. \forall B_o. A \doteq^o B \Rightarrow (A \Leftrightarrow B))) = \mathtt{T}$

- iff (for all $a, b \in \mathcal{D}_\alpha$)
  $\upsilon(\mathcal{E}_{\varphi[a/A][b/B]}(A \doteq^o B) = \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi[a/A][b/B]}(A \Leftrightarrow B)) = \mathtt{T}$
- assume $\upsilon(\mathcal{E}_{\varphi[a/A][b/B]}(A \doteq^o B)) = \mathtt{T}$

Extensionality and Leibniz Equality

Thm.: (Trivial Extensionality Directions in Σ-Models)

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall A_o . \forall B_o . A \doteq^o B \Rightarrow (A \Leftrightarrow B)$
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then
   $\mathcal{M} \models \forall F_{\alpha\to\beta} . \forall G_{\alpha\to\beta} . F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha . FX \doteq^\beta GX)$

Proof: (1.)

$\upsilon(\mathcal{E}_\varphi(\forall A_o . \forall B_o . A \doteq^o B \Rightarrow (A \Leftrightarrow B))) = \mathtt{T}$

- iff (for all $a, b \in \mathcal{D}_\alpha$)
  $\upsilon(\mathcal{E}_{\varphi[a/A][b/B]}(A \doteq^o B) = \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi[a/A][b/B]}(A \Leftrightarrow B)) = \mathtt{T}$
- assume $\upsilon(\mathcal{E}_{\varphi[a/A][b/B]}(A \doteq^o B)) = \mathtt{T}$
- then by Lemma 'Leibniz Equality in Σ-models(2.)':
  $\mathcal{E}_{\varphi[a/A][b/B]}(A) = \mathcal{E}_{\varphi[a/A][b/B]}(B)$

Extensionality and Leibniz Equality

Thm.: (Trivial Extensionality Directions in Σ-Models)

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall A_o. \forall B_o. A \doteq^o B \Rightarrow (A \Leftrightarrow B)$
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then
   $\mathcal{M} \models \forall F_{\alpha\to\beta}. \forall G_{\alpha\to\beta}. F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha. FX \doteq^\beta GX)$

Proof: (1.)

$\upsilon(\mathcal{E}_\varphi(\forall A_o. \forall B_o. A \doteq^o B \Rightarrow (A \Leftrightarrow B))) = \mathtt{T}$

- iff (for all $a, b \in \mathcal{D}_\alpha$)
  $\upsilon(\mathcal{E}_{\varphi[a/A][b/B]}(A \doteq^o B) = \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi[a/A][b/B]}(A \Leftrightarrow B)) = \mathtt{T}$
- assume $\upsilon(\mathcal{E}_{\varphi[a/A][b/B]}(A \doteq^o B)) = \mathtt{T}$
- then by Lemma 'Leibniz Equality in Σ-models(2.)':
  $\mathcal{E}_{\varphi[a/A][b/B]}(A) = \mathcal{E}_{\varphi[a/A][b/B]}(B)$
- then by Lemma 'Equivalence': $\upsilon(\mathcal{E}_{\varphi[a/A][b/B]}(A \Leftrightarrow B)) = \mathtt{T}$

Extensionality and Leibniz Equality

Thm.: (Trivial Extensionality Directions in Σ-Models)

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall A_o. \forall B_o. A \doteq^o B \Rightarrow (A \Leftrightarrow B)$
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall F_{\alpha\to\beta}. \forall G_{\alpha\to\beta}. F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha. FX \doteq^\beta GX)$

Proof: (1.)

$\upsilon(\mathcal{E}_\varphi(\forall A_o. \forall B_o. A \doteq^o B \Rightarrow (A \Leftrightarrow B))) = \mathtt{T}$

- iff (for all $a, b \in \mathcal{D}_\alpha$)
  $\upsilon(\mathcal{E}_{\varphi[a/A][b/B]}(A \doteq^o B) = \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi[a/A][b/B]}(A \Leftrightarrow B)) = \mathtt{T}$
- assume $\upsilon(\mathcal{E}_{\varphi[a/A][b/B]}(A \doteq^o B)) = \mathtt{T}$
- then by Lemma 'Leibniz Equality in Σ-models(2.)':
  $\mathcal{E}_{\varphi[a/A][b/B]}(A) = \mathcal{E}_{\varphi[a/A][b/B]}(B)$
- then by Lemma 'Equivalence': $\upsilon(\mathcal{E}_{\varphi[a/A][b/B]}(A \Leftrightarrow B)) = \mathtt{T}$

q.e.d.

Extensionality and Leibniz Equality

Thm.: (Trivial Extensionality Directions in Σ-Models)

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall A_o . \forall B_o . A \doteq^o B \Rightarrow (A \Leftrightarrow B)$
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then
   $\mathcal{M} \models \forall F_{\alpha\to\beta} . \forall G_{\alpha\to\beta} . F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha . FX \doteq^\beta GX)$

Proof: (2.) $\upsilon(\mathcal{E}_\varphi(\forall F_{\alpha\to\beta} . \forall G_{\alpha\to\beta} . F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha . FX \doteq^\beta GX))) = \mathtt{T}$

Extensionality and Leibniz Equality

Thm.: (Trivial Extensionality Directions in Σ-Models)

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall A_o.\forall B_o.A \doteq^o B \Rightarrow (A \Leftrightarrow B)$
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then
   $\mathcal{M} \models \forall F_{\alpha\to\beta}.\forall G_{\alpha\to\beta}.F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha.FX \doteq^\beta GX)$

Proof: (2.) $\upsilon(\mathcal{E}_\varphi(\forall F_{\alpha\to\beta}.\forall G_{\alpha\to\beta}.F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha.FX \doteq^\beta GX))) = \mathtt{T}$

- iff (for all $f, g \in \mathcal{D}_{\alpha\to\beta}$)
  $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(F \doteq^{\alpha\to\beta} G)) = \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(\forall X_\alpha.FX \doteq^\beta GX)) = \mathtt{T}$

Extensionality and Leibniz Equality

Thm.: (Trivial Extensionality Directions in Σ-Models)

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall A_o. \forall B_o. A \doteq^o B \Rightarrow (A \Leftrightarrow B)$
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then
$\mathcal{M} \models \forall F_{\alpha \to \beta}. \forall G_{\alpha \to \beta}. F \doteq^{\alpha \to \beta} G \Rightarrow (\forall X_\alpha. FX \doteq^\beta GX)$

Proof: (2.) $\upsilon(\mathcal{E}_\varphi(\forall F_{\alpha \to \beta}. \forall G_{\alpha \to \beta}. F \doteq^{\alpha \to \beta} G \Rightarrow (\forall X_\alpha. FX \doteq^\beta GX))) = \mathtt{T}$

- iff (for all $f, g \in \mathcal{D}_{\alpha \to \beta}$)
$\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(F \doteq^{\alpha \to \beta} G)) = \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(\forall X_\alpha. FX \doteq^\beta GX)) = \mathtt{T}$
- assume $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(F \doteq^{\alpha \to \beta} G)) = \mathtt{T}$

Extensionality and Leibniz Equality

Thm.: (Trivial Extensionality Directions in Σ-Models)

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall A_o \mathbf{.} \forall B_o \mathbf{.} A \doteq^o B \Rightarrow (A \Leftrightarrow B)$
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall F_{\alpha\to\beta} \mathbf{.} \forall G_{\alpha\to\beta} \mathbf{.} F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha \mathbf{.} FX \doteq^\beta GX)$

Proof: (2.) $\upsilon(\mathcal{E}_\varphi(\forall F_{\alpha\to\beta} \mathbf{.} \forall G_{\alpha\to\beta} \mathbf{.} F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha \mathbf{.} FX \doteq^\beta GX))) = \mathtt{T}$

- iff (for all $f, g \in \mathcal{D}_{\alpha\to\beta}$) $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(F \doteq^{\alpha\to\beta} G)) = \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(\forall X_\alpha \mathbf{.} FX \doteq^\beta GX)) = \mathtt{T}$
- assume $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(F \doteq^{\alpha\to\beta} G)) = \mathtt{T}$
- by Lemma 'Leibniz Equality in Σ-models(2.)': $\mathcal{E}_{\varphi[f/F][g/G]}(F) = \mathcal{E}_{\varphi[f/F][g/G]}(G)$

Extensionality and Leibniz Equality

Thm.: (Trivial Extensionality Directions in Σ-Models)

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall A_o. \forall B_o. A \doteq^o B \Rightarrow (A \Leftrightarrow B)$
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then
   $\mathcal{M} \models \forall F_{\alpha\to\beta}. \forall G_{\alpha\to\beta}. F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha. FX \doteq^\beta GX)$

Proof: (2.) $\upsilon(\mathcal{E}_\varphi(\forall F_{\alpha\to\beta}. \forall G_{\alpha\to\beta}. F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha. FX \doteq^\beta GX))) = \mathtt{T}$

- iff (for all $f, g \in \mathcal{D}_{\alpha\to\beta}$)
  $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(F \doteq^{\alpha\to\beta} G)) = \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(\forall X_\alpha. FX \doteq^\beta GX)) = \mathtt{T}$
- assume $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(F \doteq^{\alpha\to\beta} G)) = \mathtt{T}$
- by Lemma 'Leibniz Equality in Σ-models(2.)': $\mathcal{E}_{\varphi[f/F][g/G]}(F) = \mathcal{E}_{\varphi[f/F][g/G]}(G)$
- X not free in G or F (for any $a \in \mathcal{D}_\alpha$): $\mathcal{E}_{\varphi[f/F][g/G][a/X]}(F) = \mathcal{E}_{\varphi[f/F][g/G][a/X]}(G)$

Extensionality and Leibniz Equality

Thm.: (Trivial Extensionality Directions in Σ-Models)

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall A_o \forall B_o A \doteq^o B \Rightarrow (A \Leftrightarrow B)$
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then
$\mathcal{M} \models \forall F_{\alpha\to\beta} \forall G_{\alpha\to\beta} F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha FX \doteq^\beta GX)$

Proof: (2.) $\upsilon(\mathcal{E}_\varphi(\forall F_{\alpha\to\beta} \forall G_{\alpha\to\beta} F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha FX \doteq^\beta GX))) = \mathtt{T}$

- iff (for all $f, g \in \mathcal{D}_{\alpha\to\beta}$)
$\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(F \doteq^{\alpha\to\beta} G)) = \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(\forall X_\alpha FX \doteq^\beta GX)) = \mathtt{T}$
- assume $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(F \doteq^{\alpha\to\beta} G)) = \mathtt{T}$
- by Lemma 'Leibniz Equality in Σ-models(2.)': $\mathcal{E}_{\varphi[f/F][g/G]}(F) = \mathcal{E}_{\varphi[f/F][g/G]}(G)$
- X not free in G or F (for any $a \in \mathcal{D}_\alpha$): $\mathcal{E}_{\varphi[f/F][g/G][a/X]}(F) = \mathcal{E}_{\varphi[f/F][g/G][a/X]}(G)$
- furthermore
$\mathcal{E}_{\varphi[f/F][g/G][a/X]}(F)@\mathcal{E}_{\varphi[f/F][g/G][a/X]}(X) = \mathcal{E}_{\varphi[f/F][g/G]}(G)@\mathcal{E}_{\varphi[f/F][g/G][a/X]}(X)$

Extensionality and Leibniz Equality

Thm.: (Trivial Extensionality Directions in Σ-Models)

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall A_o. \forall B_o. A \doteq^o B \Rightarrow (A \Leftrightarrow B)$
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then
$\mathcal{M} \models \forall F_{\alpha\rightarrow\beta}. \forall G_{\alpha\rightarrow\beta}. F \doteq^{\alpha\rightarrow\beta} G \Rightarrow (\forall X_\alpha. FX \doteq^\beta GX)$

Proof: (2.) $\upsilon(\mathcal{E}_\varphi(\forall F_{\alpha\rightarrow\beta}. \forall G_{\alpha\rightarrow\beta}. F \doteq^{\alpha\rightarrow\beta} G \Rightarrow (\forall X_\alpha. FX \doteq^\beta GX))) = \mathtt{T}$

- iff (for all $f, g \in \mathcal{D}_{\alpha\rightarrow\beta}$)
$\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(F \doteq^{\alpha\rightarrow\beta} G)) = \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(\forall X_\alpha. FX \doteq^\beta GX)) = \mathtt{T}$
- assume $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(F \doteq^{\alpha\rightarrow\beta} G)) = \mathtt{T}$
- by Lemma 'Leibniz Equality in Σ-models(2.)': $\mathcal{E}_{\varphi[f/F][g/G]}(F) = \mathcal{E}_{\varphi[f/F][g/G]}(G)$
- X not free in G or F (for any $a \in \mathcal{D}_\alpha$): $\mathcal{E}_{\varphi[f/F][g/G][a/X]}(F) = \mathcal{E}_{\varphi[f/F][g/G][a/X]}(G)$
- furthermore
$\mathcal{E}_{\varphi[f/F][g/G][a/X]}(F) @ \mathcal{E}_{\varphi[f/F][g/G][a/X]}(X) = \mathcal{E}_{\varphi[f/F][g/G]}(G) @ \mathcal{E}_{\varphi[f/F][g/G][a/X]}(X)$
- thus $\mathcal{E}_{\varphi[f/F][g/G][a/X]}(FX) = \mathcal{E}_{\varphi[f/F][g/G][a/X]}(GX)$

Extensionality and Leibniz Equality

Thm.: (Trivial Extensionality Directions in Σ-Models)

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall A_o. \forall B_o. A \doteq^o B \Rightarrow (A \Leftrightarrow B)$
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then
$\mathcal{M} \models \forall F_{\alpha\to\beta}. \forall G_{\alpha\to\beta}. F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha. FX \doteq^\beta GX)$

Proof: (2.) $\upsilon(\mathcal{E}_\varphi(\forall F_{\alpha\to\beta}. \forall G_{\alpha\to\beta}. F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha. FX \doteq^\beta GX))) = \mathtt{T}$

- iff (for all $f, g \in \mathcal{D}_{\alpha\to\beta}$)
$\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(F \doteq^{\alpha\to\beta} G)) = \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(\forall X_\alpha. FX \doteq^\beta GX)) = \mathtt{T}$
- assume $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(F \doteq^{\alpha\to\beta} G)) = \mathtt{T}$
- by Lemma 'Leibniz Equality in Σ-models(2.)': $\mathcal{E}_{\varphi[f/F][g/G]}(F) = \mathcal{E}_{\varphi[f/F][g/G]}(G)$
- X not free in G or F (for any $a \in \mathcal{D}_\alpha$): $\mathcal{E}_{\varphi[f/F][g/G][a/X]}(F) = \mathcal{E}_{\varphi[f/F][g/G][a/X]}(G)$
- furthermore
$\mathcal{E}_{\varphi[f/F][g/G][a/X]}(F)@\mathcal{E}_{\varphi[f/F][g/G][a/X]}(X) = \mathcal{E}_{\varphi[f/F][g/G]}(G)@\mathcal{E}_{\varphi[f/F][g/G][a/X]}(X)$
- thus $\mathcal{E}_{\varphi[f/F][g/G][a/X]}(FX) = \mathcal{E}_{\varphi[f/F][g/G][a/X]}(GX)$
- thus $\upsilon(\mathcal{E}_{\varphi[f/F][g/G][a/X]}(FX \doteq^\beta GX)) = \mathtt{T}$ by Lemma . . .

Extensionality and Leibniz Equality

Thm.: (Trivial Extensionality Directions in Σ-Models)

1. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then $\mathcal{M} \models \forall A_o . \forall B_o . A \doteq^o B \Rightarrow (A \Leftrightarrow B)$
2. If $\mathcal{M}$ satisfies $\mathfrak{q}$, then
   $\mathcal{M} \models \forall F_{\alpha\to\beta} . \forall G_{\alpha\to\beta} . F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha . FX \doteq^\beta GX)$

Proof: (2.) $\upsilon(\mathcal{E}_\varphi(\forall F_{\alpha\to\beta} . \forall G_{\alpha\to\beta} . F \doteq^{\alpha\to\beta} G \Rightarrow (\forall X_\alpha . FX \doteq^\beta GX))) = \mathtt{T}$

- iff (for all $f, g \in \mathcal{D}_{\alpha\to\beta}$)
  $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(F \doteq^{\alpha\to\beta} G)) = \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(\forall X_\alpha . FX \doteq^\beta GX)) = \mathtt{T}$
- assume $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(F \doteq^{\alpha\to\beta} G)) = \mathtt{T}$
- by Lemma 'Leibniz Equality in Σ-models(2.)': $\mathcal{E}_{\varphi[f/F][g/G]}(F) = \mathcal{E}_{\varphi[f/F][g/G]}(G)$
- X not free in G or F (for any $a \in \mathcal{D}_\alpha$): $\mathcal{E}_{\varphi[f/F][g/G][a/X]}(F) = \mathcal{E}_{\varphi[f/F][g/G][a/X]}(G)$
- furthermore
  $\mathcal{E}_{\varphi[f/F][g/G][a/X]}(F) @ \mathcal{E}_{\varphi[f/F][g/G][a/X]}(X) = \mathcal{E}_{\varphi[f/F][g/G]}(G) @ \mathcal{E}_{\varphi[f/F][g/G][a/X]}(X)$
- thus $\mathcal{E}_{\varphi[f/F][g/G][a/X]}(FX) = \mathcal{E}_{\varphi[f/F][g/G][a/X]}(GX)$
- thus $\upsilon(\mathcal{E}_{\varphi[f/F][g/G][a/X]}(FX \doteq^\beta GX)) = \mathtt{T}$ by Lemma ...
- thus $\upsilon(\mathcal{E}_{\varphi[f/F][g/G]}(\forall X_\alpha . FX \doteq^\beta GX)) = \mathtt{T}$ q.e.d.

Leibniz Equality in Σ-Models

$$
\begin{tikzcd}[column sep=small]
 & & \mathfrak{M}_{\beta}(\Sigma) \arrow[dll, "\xi"'] \arrow[d, "\eta"] \arrow[drr, "\mathfrak{b}"] \arrow[ddll, "\mathfrak{f}"] & & \\
\mathfrak{M}_{\beta\xi}(\Sigma) \arrow[d, "\eta"'] \arrow[drr, "\mathfrak{b}"] & & \mathfrak{M}_{\beta\eta}(\Sigma) \arrow[dll, "\xi"'] \arrow[drr, "\mathfrak{b}"] & & \mathfrak{M}_{\beta\mathfrak{b}}(\Sigma) \arrow[dll, "\xi"'] \arrow[d, "\eta"] \arrow[ddll, "\mathfrak{f}"] \\
\mathfrak{M}_{\beta\mathfrak{f}}(\Sigma) \arrow[drr, "\mathfrak{b}"'] & & \mathfrak{M}_{\beta\xi\mathfrak{b}}(\Sigma) \arrow[d, "\eta"] & & \mathfrak{M}_{\beta\eta\mathfrak{b}}(\Sigma) \arrow[dll, "\xi"] \\
 & & \mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma) \arrow[d, dotted, "\text{full}"] & & \\
 & & \mathfrak{S}\mathfrak{T}(\Sigma) & &
\end{tikzcd}
$$

$\doteq$ **is equivalence relation**

- $\forall X_\alpha . X \doteq X$

Proof:

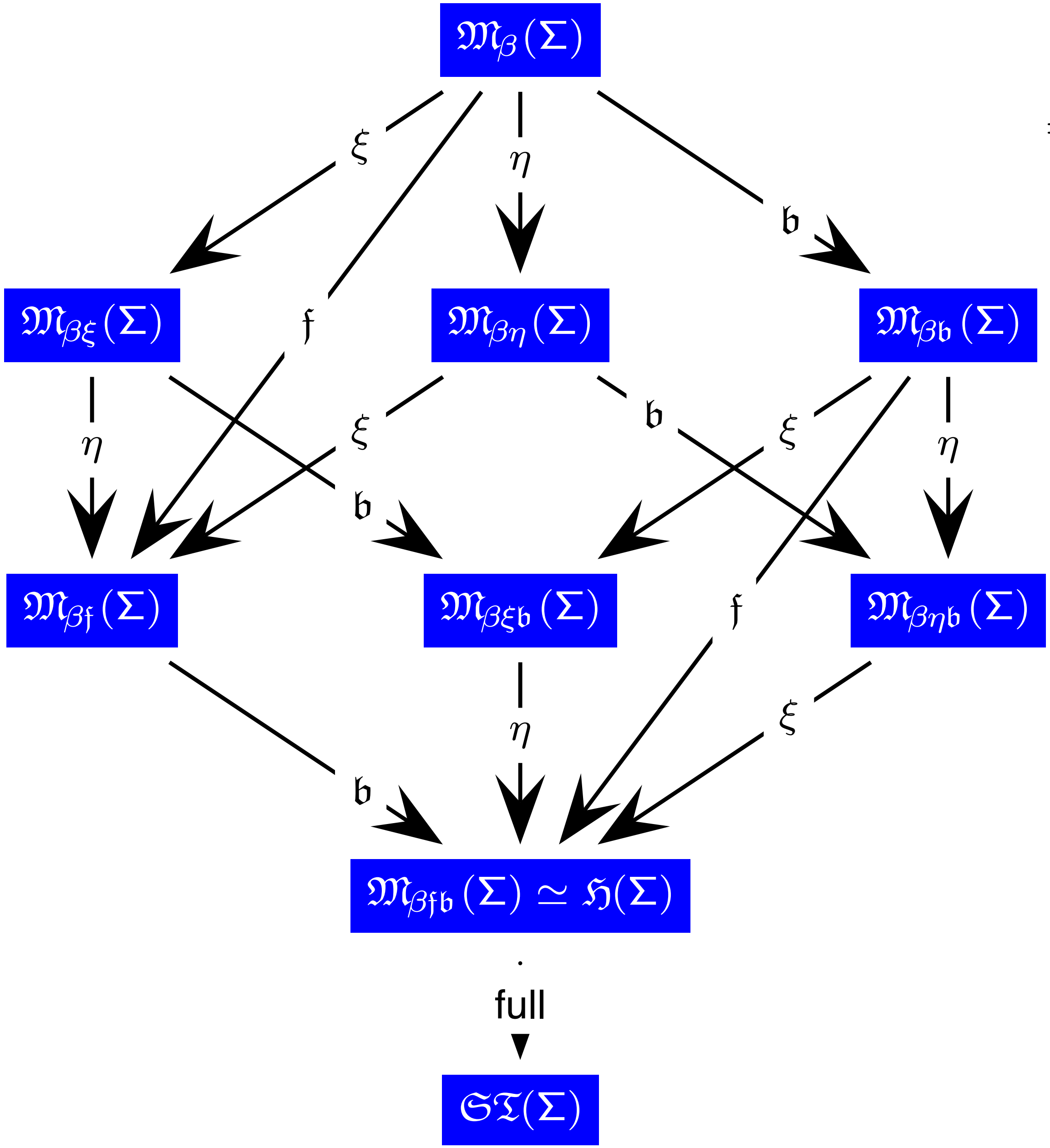
$\upsilon(\mathcal{E}_\varphi(\forall X_\alpha . X \doteq X)) = \mathtt{T}$

iff $\upsilon(\mathcal{E}_{\varphi[a/X]}(X \doteq X)) = \mathtt{T}$ **for all** $a \in \mathbf{D}_\alpha$

holds by Lemma 'Leibniz Equality in Σ-models(1.)'

since $\mathcal{E}_{\varphi[a/X]}(X) = \mathcal{E}_{\varphi[a/X]}(X)$ **for all** $a \in \mathbf{D}_\alpha$.

Leibniz Equality in Σ-Models



$\doteq$ **is equivalence relation**

- $\forall X_\alpha . X \doteq X$
- $\forall X_\alpha, Y_\alpha . X \doteq Y \Rightarrow Y \doteq X$

Proof:

$\upsilon(\mathcal{E}_\varphi(\forall X_\alpha, Y_\alpha . X \doteq Y \Leftrightarrow Y \doteq X)) = \mathtt{T}$

iff $\upsilon(\mathcal{E}_{\varphi[a/X][b/X]}(X \doteq Y)) = \mathtt{F}$ **or** $\upsilon(\mathcal{E}_{\varphi[a/X][b/X]}(Y \doteq X)) = \mathtt{T}$ **for all** $a, b \in \mathbf{D}_\alpha$

assume $\upsilon(\mathcal{E}_{\varphi[a/X][b/X]}(X \doteq Y)) = \mathtt{T}$

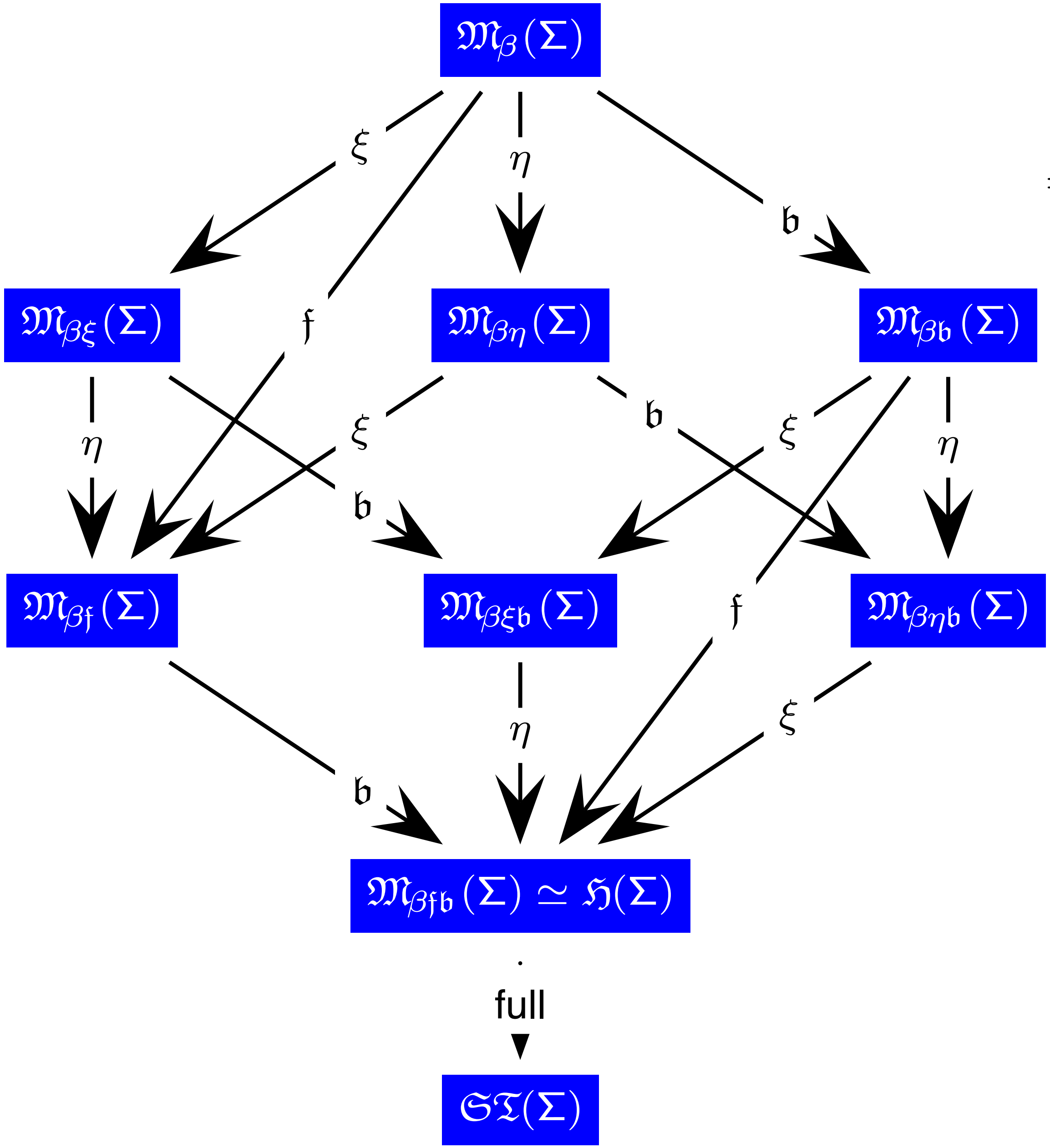
by Lemma 'Leibniz Equality in Σ-models(2.)'

$\mathcal{E}_{\varphi[a/X]}(X) = \mathcal{E}_{\varphi[a/X]}(Y)$ **for all** $a, b \in \mathbf{D}_\alpha$

hence $\mathcal{E}_{\varphi[a/X]}(Y) = \mathcal{E}_{\varphi[a/X]}(X)$ **for all** $a, b \in \mathbf{D}_\alpha$

hence $\upsilon(\mathcal{E}_{\varphi[a/X][b/X]}(X \doteq Y)) = \mathtt{T}$ **for all** $a, b \in \mathbf{D}_\alpha$ **by Lemma 'Leibniz Equality in Σ-models(2.)'**

Leibniz Equality in Σ-Models



$\doteq$ **is equivalence relation**

- $\forall X_\alpha . X \doteq X$
- $\forall X_\alpha, Y_\alpha . X \doteq Y \Rightarrow Y \doteq X$
- $\forall X_\alpha, Y_\alpha, Z_\alpha . (X \doteq Y \wedge Y \doteq Z) \Rightarrow X \doteq Z$

Proof:

analogous with

$\mathcal{E}_{\varphi[a/X][b/Y][c/Z]}(X) = \mathcal{E}_{\varphi[a/X][b/Y][c/Z]}(Y)$

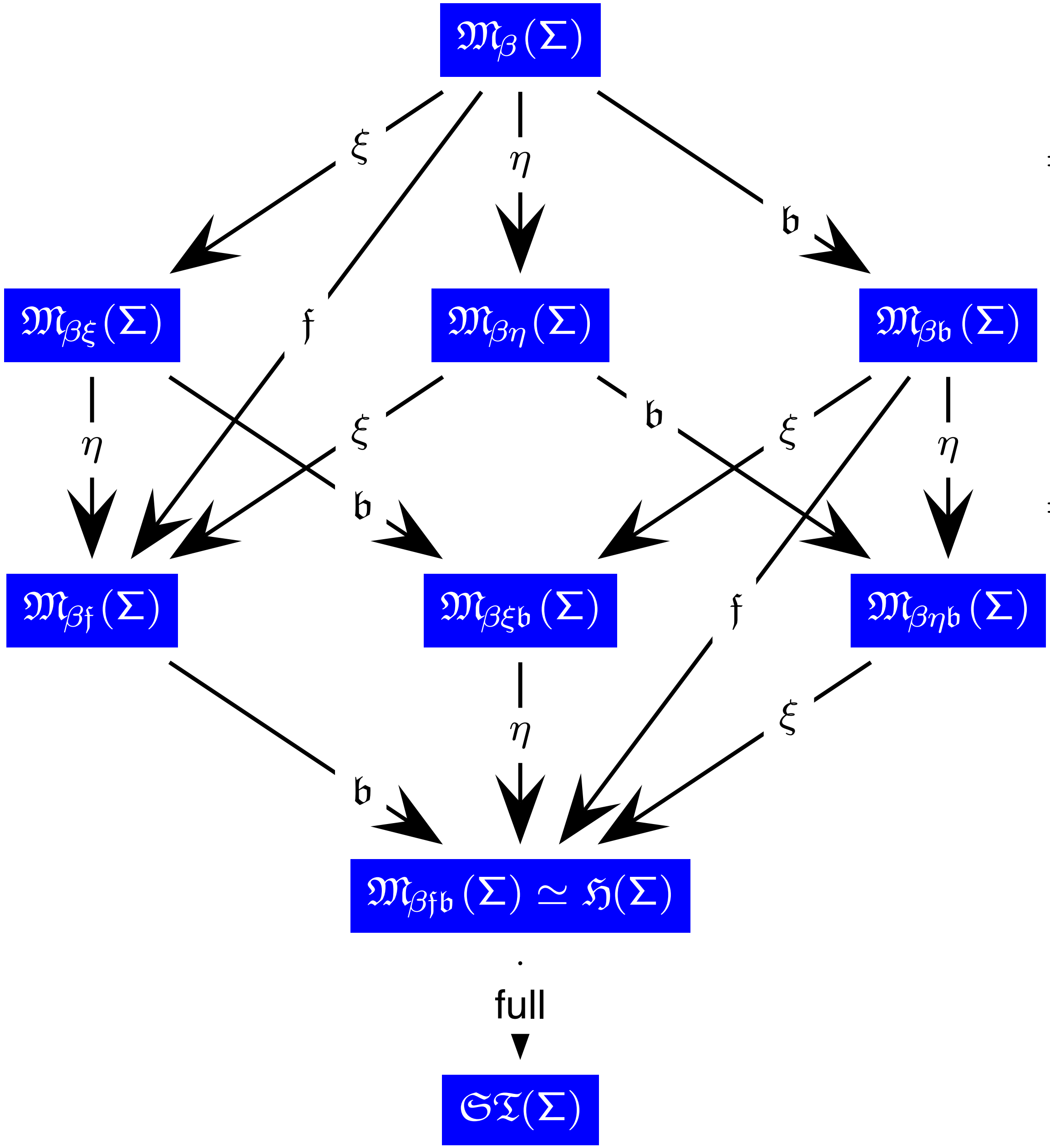
and

$\mathcal{E}_{\varphi[a/X][b/Y][c/Z]}(Y) = \mathcal{E}_{\varphi[a/X][b/Y][c/Z]}(Z)$

implies

$\mathcal{E}_{\varphi[a/X][b/Y][c/Z]}(X) = \mathcal{E}_{\varphi[a/X][b/Y][c/Z]}(Z)$

Leibniz Equality in Σ-Models



$\doteq$ is equivalence relation

- $\forall X_\alpha . X \doteq X$
- $\forall X_\alpha, Y_\alpha . X \doteq Y \Rightarrow Y \doteq X$
- $\forall X_\alpha, Y_\alpha, Z_\alpha . (X \doteq Y \wedge Y \doteq Z) \Rightarrow X \doteq Z$

$\doteq$ is congruence relation

- $\forall X_\alpha, Y_\alpha, F_{\alpha\to\beta} . X \doteq Y \Rightarrow (FX) \doteq (FY)$

Proof:

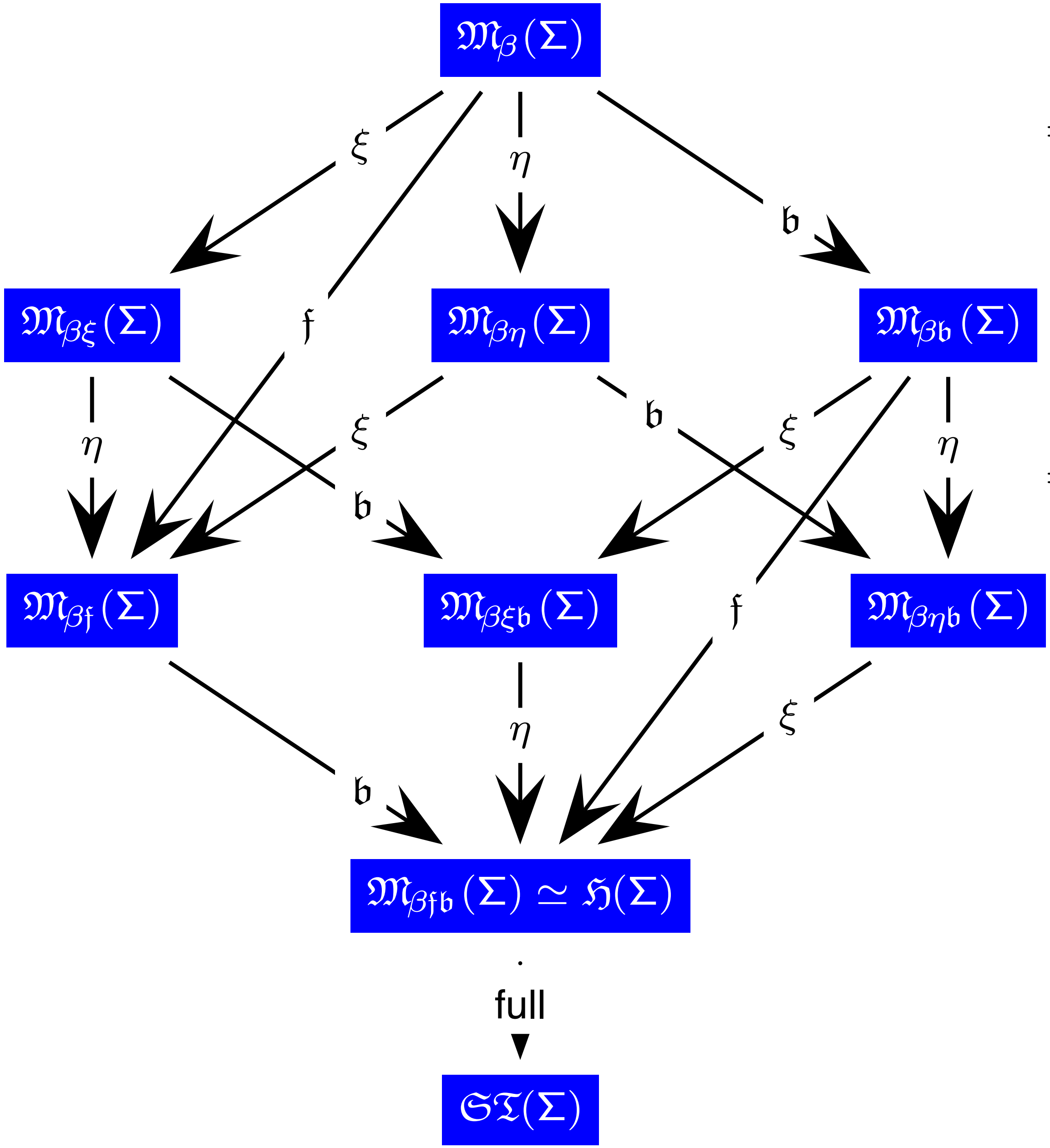
analogous with

$\mathcal{E}_{\varphi[a/X][b/Y][f/F]}(X) = \mathcal{E}_{\varphi[a/X][b/Y][f/F]}(Y)$

implies

$\mathcal{E}_{\varphi[a/X][b/Y][f/F]}(F)@\mathcal{E}_{\varphi[a/X][b/Y][f/F]}(X) = \mathcal{E}_{\varphi[a/X][b/Y][f/F]}(F)@\mathcal{E}_{\varphi[a/X][b/Y][f/F]}(Y)$

Leibniz Equality in Σ-Models



$\doteq$ **is equivalence relation**

- $\forall X_\alpha . X \doteq X$
- $\forall X_\alpha, Y_\alpha . X \doteq Y \Rightarrow Y \doteq X$
- $\forall X_\alpha, Y_\alpha, Z_\alpha . (X \doteq Y \wedge Y \doteq Z) \Rightarrow X \doteq Z$

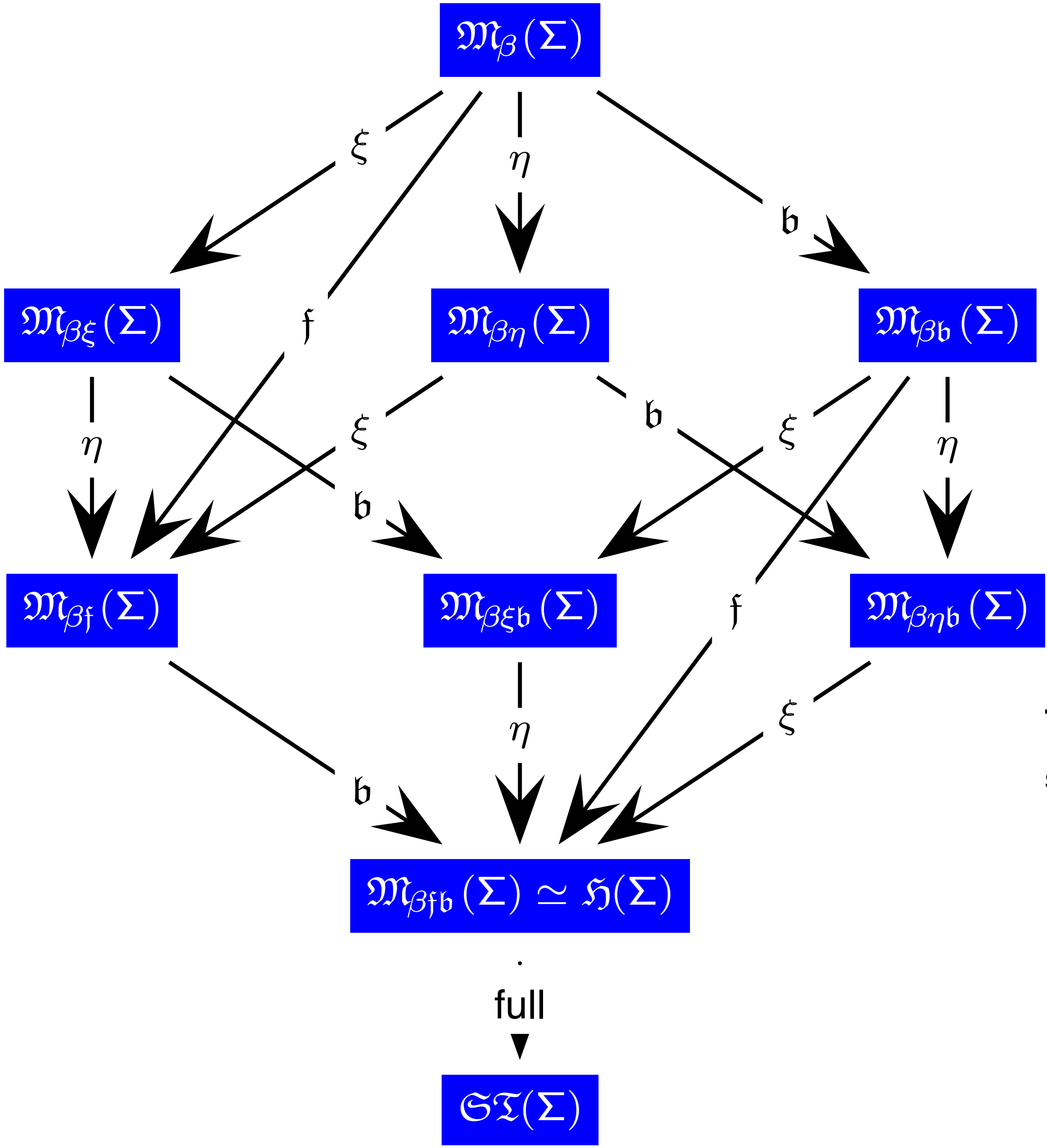
$\doteq$ **is congruence relation**

- $\forall X_\alpha, Y_\alpha, F_{\alpha\to\beta} . X \doteq Y \Rightarrow (FX) \doteq (FY)$
- $\forall X_\alpha, Y_\alpha, P_{\alpha\to o} . X \doteq Y \wedge (PX) \Rightarrow (PY)$

Proof:

analogous

Leibniz Equality in Σ-Models



$\doteq$ **is equivalence relation**

- $\forall X_\alpha . X \doteq X$
- $\forall X_\alpha, Y_\alpha . X \doteq Y \Rightarrow Y \doteq X$
- $\forall X_\alpha, Y_\alpha, Z_\alpha . (X \doteq Y \wedge Y \doteq Z) \Rightarrow X \doteq Z$

$\doteq$ **is congruence relation**

- $\forall X_\alpha, Y_\alpha, F_{\alpha\to\beta} . X \doteq Y \Rightarrow (FX) \doteq (FY)$
- $\forall X_\alpha, Y_\alpha, P_{\alpha\to o} . X \doteq Y \wedge (PX) \Rightarrow (PY)$

Trivial directions of Boolean and functional extensionality

- $\forall A_o, B_o . A \doteq B \Rightarrow (A \Leftrightarrow B)$

Proof:

by Theorem 'Trivial Extensionality Directions'

Leibniz Equality in Σ-Models

$$
\begin{tikzcd}
 & & \mathfrak{M}_{\beta}(\Sigma) \arrow[dll, "\xi"'] \arrow[ddll, "\mathfrak{f}"] \arrow[d, "\eta"] \arrow[drr, "\mathfrak{b}"] & & \\
\mathfrak{M}_{\beta\xi}(\Sigma) \arrow[d, "\eta"] \arrow[drr, "\mathfrak{b}"] & & \mathfrak{M}_{\beta\eta}(\Sigma) \arrow[dll, "\xi"'] \arrow[drr, "\mathfrak{b}"] & & \mathfrak{M}_{\beta\mathfrak{b}}(\Sigma) \arrow[dll, "\xi"'] \arrow[ddll, "\mathfrak{f}"] \arrow[d, "\eta"] \\
\mathfrak{M}_{\beta\mathfrak{f}}(\Sigma) \arrow[drr, "\mathfrak{b}"] & & \mathfrak{M}_{\beta\xi\mathfrak{b}}(\Sigma) \arrow[d, "\eta"] & & \mathfrak{M}_{\beta\eta\mathfrak{b}}(\Sigma) \arrow[dll, "\xi"'] \\
 & & \mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma) \arrow[d, dotted, "\text{full}"] & & \\
 & & \mathfrak{ST}(\Sigma) & &
\end{tikzcd}
$$

$\doteq$ **is equivalence relation**

- $\forall X_\alpha . X \doteq X$
- $\forall X_\alpha, Y_\alpha . X \doteq Y \Rightarrow Y \doteq X$
- $\forall X_\alpha, Y_\alpha, Z_\alpha . (X \doteq Y \wedge Y \doteq Z) \Rightarrow X \doteq Z$

$\doteq$ **is congruence relation**

- $\forall X_\alpha, Y_\alpha, F_{\alpha\to\beta} . X \doteq Y \Rightarrow (FX) \doteq (FY)$
- $\forall X_\alpha, Y_\alpha, P_{\alpha\to o} . X \doteq Y \wedge (PX) \Rightarrow (PY)$

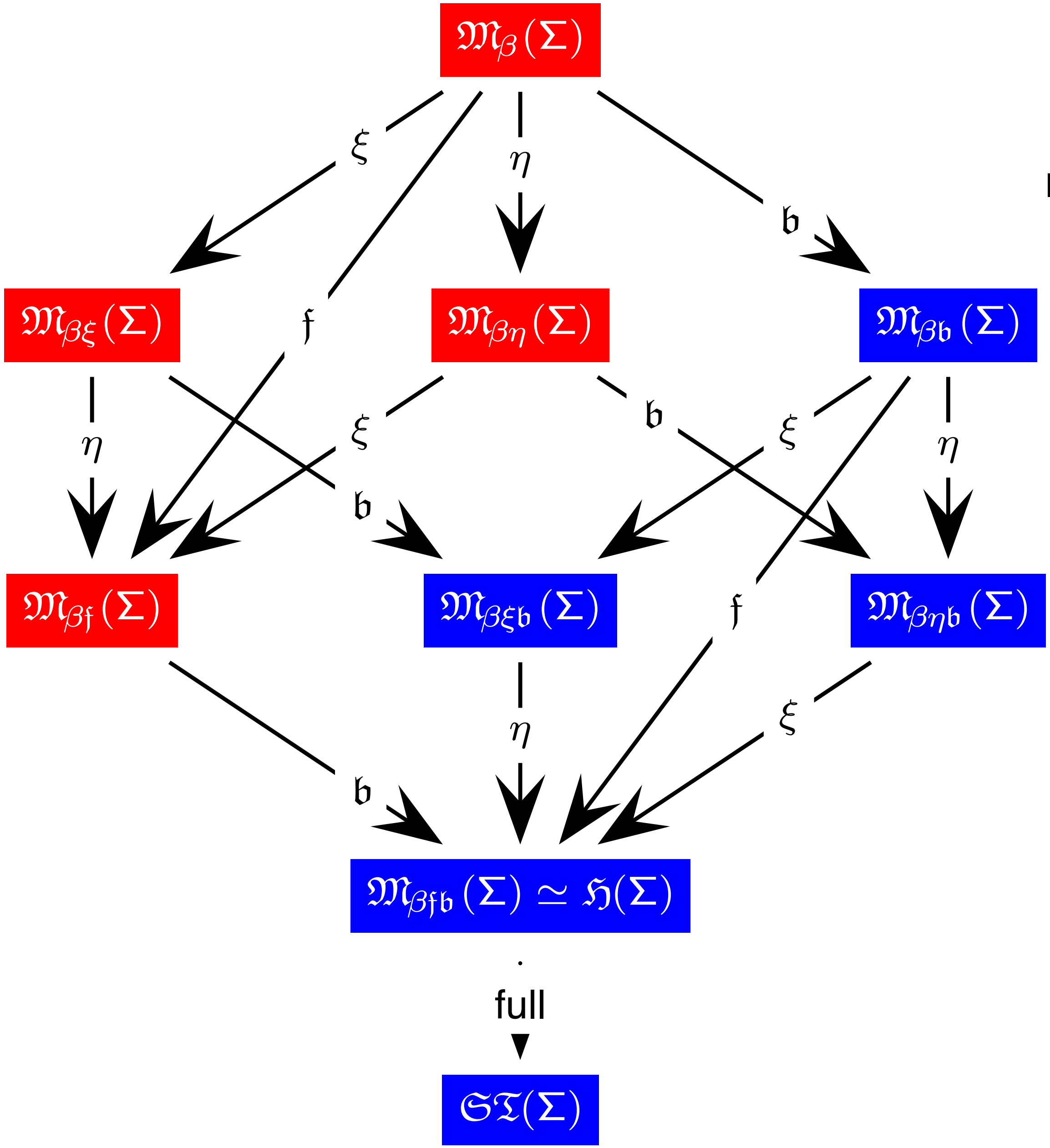
Trivial directions of Boolean and functional extensionality

- $\forall A_o, B_o . A \doteq B \Rightarrow (A \Leftrightarrow B)$
- $\forall F_{\alpha\to\beta}, G_{\alpha\to\beta} . F \doteq G \Rightarrow (\forall X_\alpha . FX \doteq GX)$

Proof:

by Theorem 'Trivial Extensionality Directions'

Leibniz Equality in Σ-Models



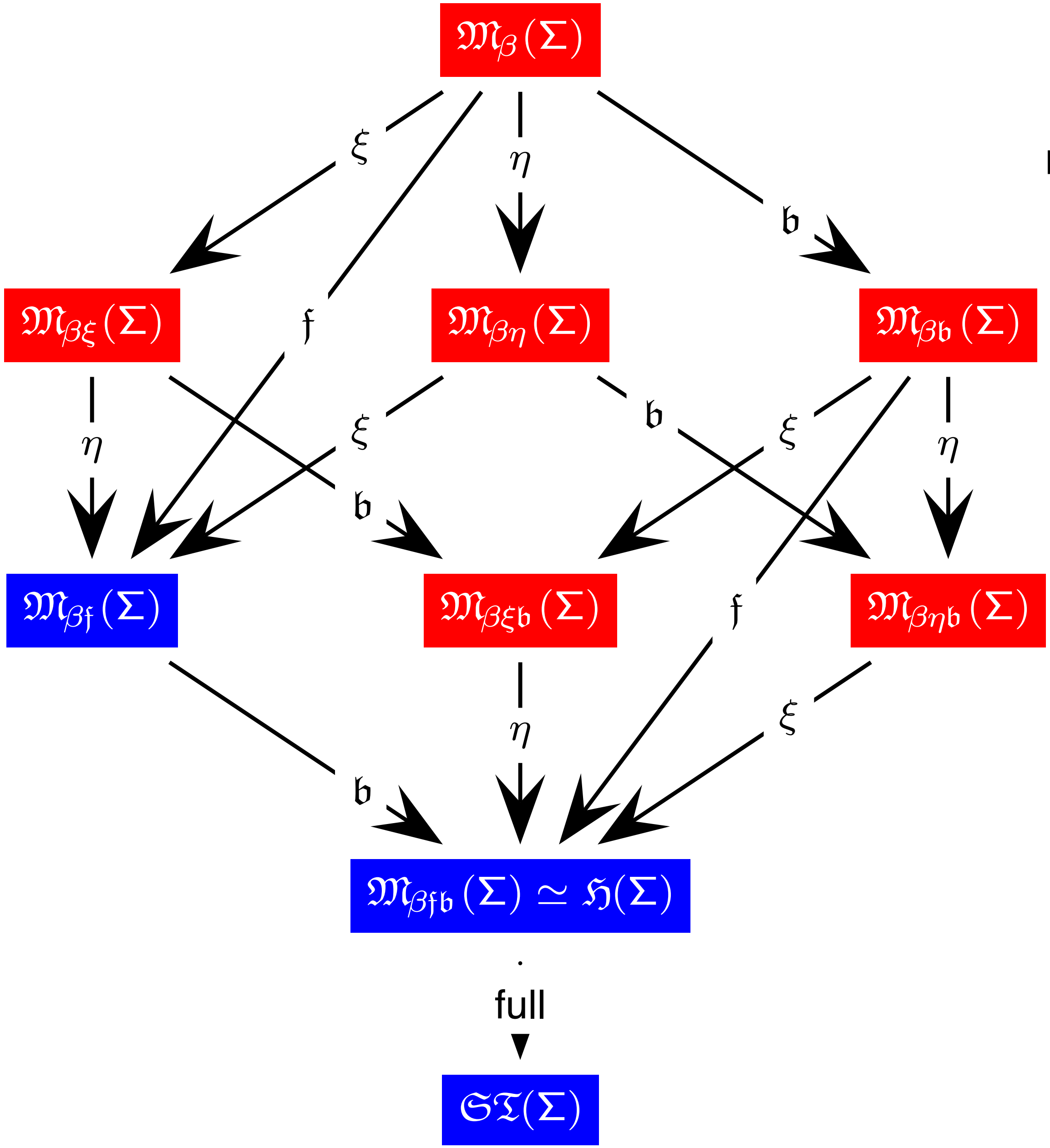
Non-trivial direction of Boolean extensionality

- $\forall A_o, B_o.(A \Leftrightarrow B) \Rightarrow A \doteq B$

Proof:

by Theorem 'Extensionality in Σ-Models'

Leibniz Equality in Σ-Models



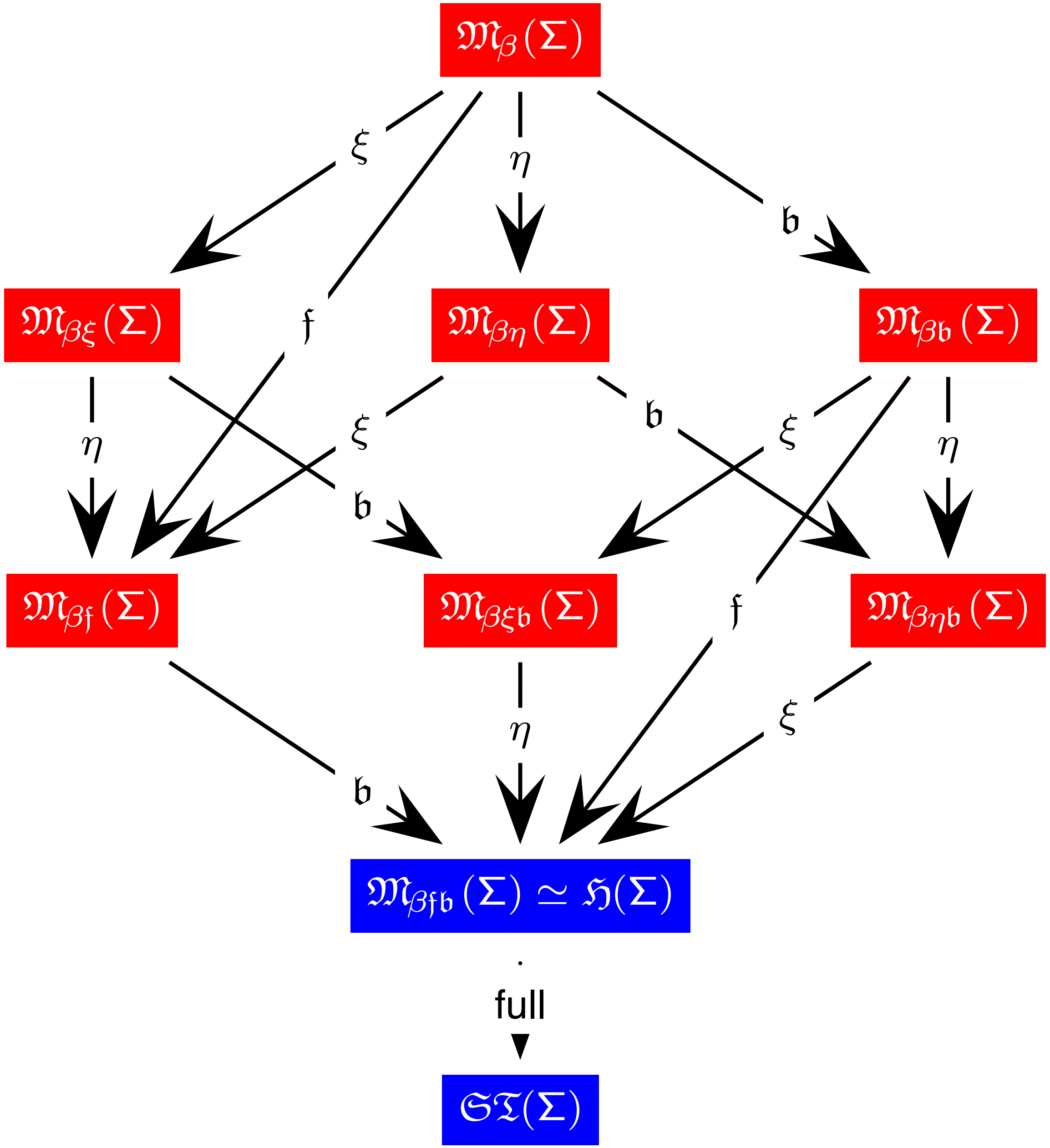
Non-trivial direct. of functional extensionality

- $\forall F_{\alpha\to\beta}, G_{\alpha\to\beta}.(\forall X_{\alpha}.FX \doteq GX) \Rightarrow F \doteq G$

Proof:

by Theorem 'Extensionality in Σ-Models'

Further Examples



- $\forall X.\forall Y.X \vee Y \Leftrightarrow Y \vee X$
- $\forall X.\forall Y.X \vee Y \doteq Y \vee X$
- $\lambda X.\lambda Y.X \vee Y \doteq \lambda X.\lambda Y.Y \vee X$
- $\vee \doteq \lambda X.\lambda Y.Y \vee X$

validity requires $\mathfrak{b}$ and $\mathfrak{f}$

Proof:

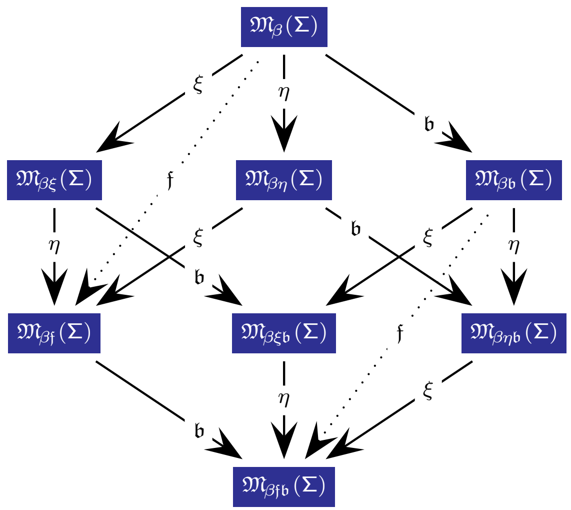
Exercise

Calculi: ND for HOL

Semantics - Calculi - Abstract Consistency

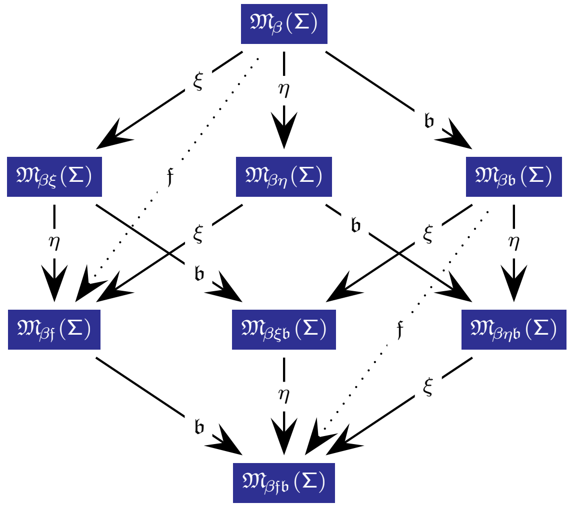
Semantics:

Model Classes (Extensionality)

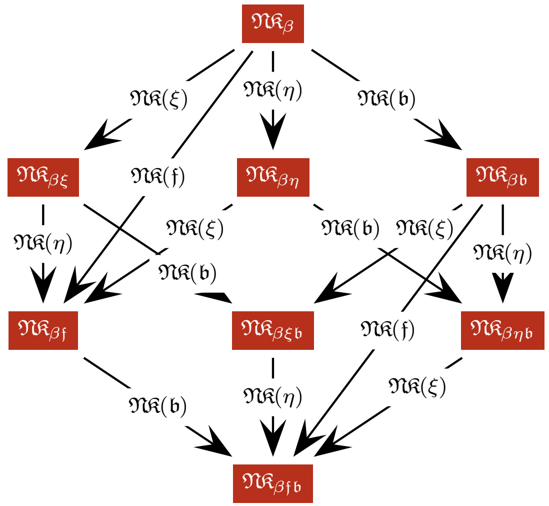


Semantics - Calculi - Abstract Consistency

Semantics:
Model Classes (Extensionality)

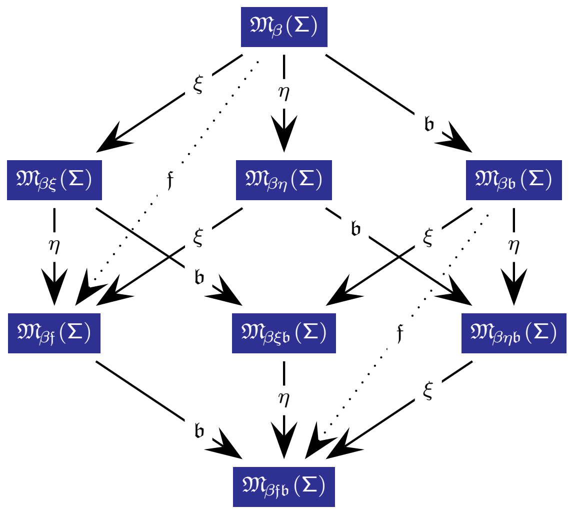


Reference Calculi:
ND (and others)

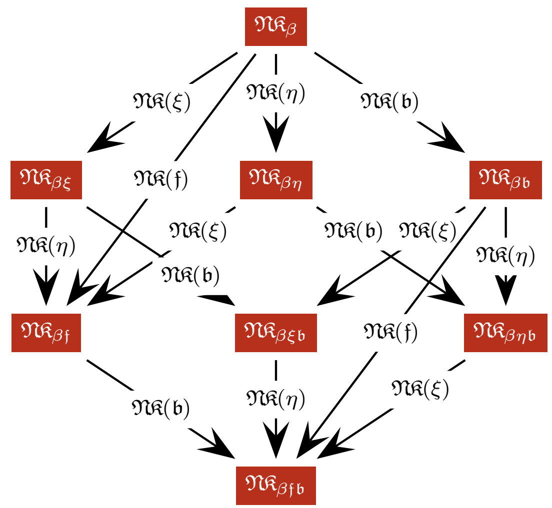


Semantics - Calculi - Abstract Consistency

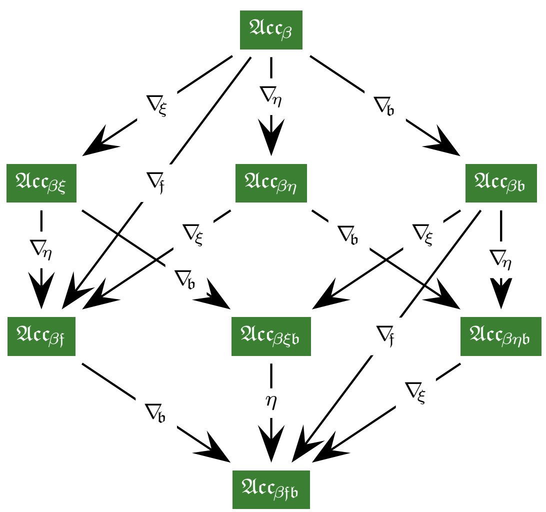
Semantics:
Model Classes (Extensionality)



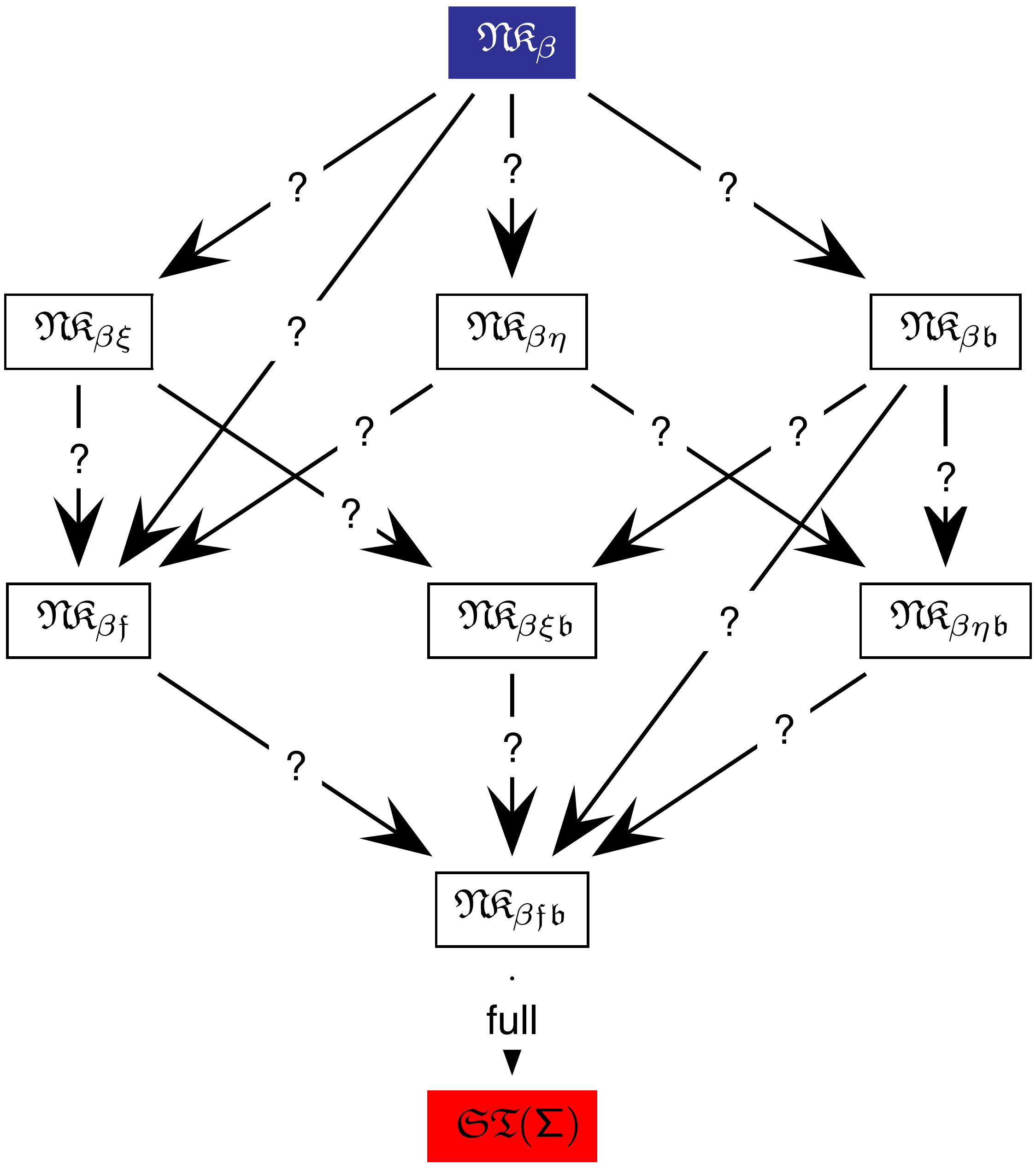
Reference Calculi:
ND (and others)



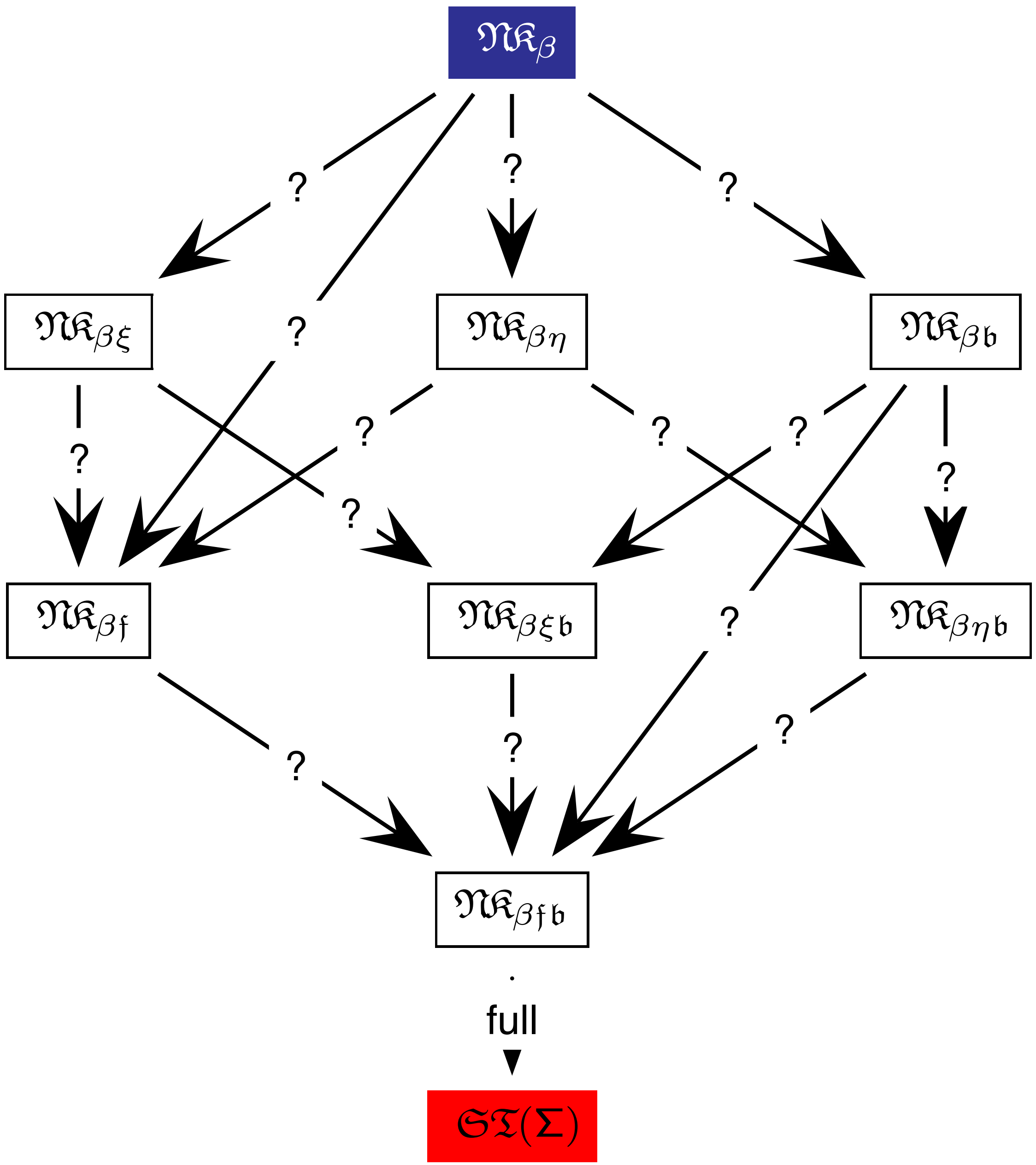
Abstract Consistency / Unifying Principle:
Extensions of Smullyan-63 and Andrews-71



ND for HOL: Base Calculus $\mathfrak{NK}_\beta$



ND for HOL: Base Calculus $\mathfrak{NK}_{\beta}$



Base Calculus $\mathfrak{NK}_{\beta}$

$\overline{}\ \mathfrak{NK}(Hyp)$ $\quad$ $\overline{}\ \mathfrak{NK}(\beta)$

$\overline{}\ \mathfrak{NK}(\neg I)$ $\quad$ $\overline{}\ \mathfrak{NK}(\neg E)$

$\overline{}\ \mathfrak{NK}(\vee I_L)$ $\quad$ $\overline{}\ \mathfrak{NK}(\vee I_R)$

$\overline{}\ \mathfrak{NK}(\vee E)$

$\overline{}\ \mathfrak{NK}(\Pi I)^{\mathrm{w}}$

$\overline{}\ \mathfrak{NK}(\Pi E)$ $\quad$ $\overline{}\ \mathfrak{NK}(Contr)$

ND for HOL: Base Calculus $\mathfrak{NK}_\beta$

$\mathfrak{NK}_\beta$:

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}} \mathfrak{NK}(Hyp)$$

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

ND for HOL: Base Calculus $\mathfrak{NK}_\beta$

$\mathfrak{NK}_\beta$:

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\beta)$$

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

ND for HOL: Base Calculus $\mathfrak{NK}_\beta$

$\mathfrak{NK}_\beta$:

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\beta)$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg \mathbf{A}}\,\mathfrak{NK}(\neg I)$$

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

ND for HOL: Base Calculus $\mathfrak{NK}_\beta$

$\mathfrak{NK}_\beta$:

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\beta)$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg\mathbf{A}}\,\mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\neg E)$$

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

ND for HOL: Base Calculus $\mathfrak{NK}_\beta$

$\mathfrak{NK}_\beta$:

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\beta)$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg\mathbf{A}}\,\mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\neg E)$$

$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\,\mathfrak{NK}(\vee I_L)$$

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

ND for HOL: Base Calculus $\mathfrak{NK}_\beta$

$\mathfrak{NK}_\beta$:

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\beta)$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg\mathbf{A}}\,\mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\neg E)$$

$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\,\mathfrak{NK}(\vee I_L) \qquad \frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\,\mathfrak{NK}(\vee I_R)$$

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

ND for HOL: Base Calculus $\mathfrak{NK}_\beta$

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$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg \mathbf{A}}\ \mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg \mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\ \mathfrak{NK}(\neg E)$$

$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\ \mathfrak{NK}(\vee I_L) \qquad \frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\ \mathfrak{NK}(\vee I_R)$$

$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}}\ \mathfrak{NK}(\vee E)$$

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$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg\mathbf{A}}\,\mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\neg E)$$

$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\,\mathfrak{NK}(\vee I_L) \qquad \frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\,\mathfrak{NK}(\vee I_R)$$

$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\vee E)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathsf{w}_\alpha \quad \mathsf{w}\text{ new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}}\,\mathfrak{NK}(\Pi I)^{\mathsf{w}}$$

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$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg \mathbf{A}}\ \mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg \mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\ \mathfrak{NK}(\neg E)$$

$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\ \mathfrak{NK}(\vee I_L) \qquad \frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\ \mathfrak{NK}(\vee I_R)$$

$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}}\ \mathfrak{NK}(\vee E)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathsf{w}_\alpha \quad \text{w new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}}\ \mathfrak{NK}(\mathit{\Pi I})^{\mathsf{w}}$$

$$\frac{\Phi \Vdash \Pi^\alpha \mathbf{G}}{\Phi \Vdash \mathbf{G}\mathbf{A}}\ \mathfrak{NK}(\mathit{\Pi E})$$

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$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg\mathbf{A}}\,\mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\neg E)$$

$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\,\mathfrak{NK}(\vee I_L) \qquad \frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\,\mathfrak{NK}(\vee I_R)$$

$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\vee E)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathsf{w}_\alpha \quad \mathsf{w}\text{ new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}}\,\mathfrak{NK}(\Pi I)^{\mathsf{w}}$$

$$\frac{\Phi \Vdash \Pi^\alpha \mathbf{G}}{\Phi \Vdash \mathbf{G}\mathbf{A}}\,\mathfrak{NK}(\Pi E) \qquad \frac{\Phi * \neg\mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(Contr)$$

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

ND for HOL: Rules for Richer Signatures

Inference rules for $\mathfrak{NK}_\beta$ (for richer signatures)

$$\frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{A}} \mathfrak{NK}(\wedge E_L)$$

ND for HOL: Rules for Richer Signatures

Inference rules for $\mathfrak{NK}_\beta$ (for richer signatures)

$$\frac{\Phi \Vdash \mathbf{A} \land \mathbf{B}}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(\land E_L) \qquad \frac{\Phi \Vdash \mathbf{A} \land \mathbf{B}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\land E_R)$$

ND for HOL: Rules for Richer Signatures

Inference rules for $\mathfrak{NK}_\beta$ (for richer signatures)

$$\frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(\wedge E_L) \qquad \frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\wedge E_R) \qquad \frac{\Phi \Vdash \mathbf{A} \quad \Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}\,\mathfrak{NK}(\wedge I)$$

ND for HOL: Rules for Richer Signatures

Inference rules for $\mathfrak{NK}_\beta$ (for richer signatures)

$$\frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{A}}\ \mathfrak{NK}(\wedge E_L) \qquad \frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{B}}\ \mathfrak{NK}(\wedge E_R) \qquad \frac{\Phi \Vdash \mathbf{A} \quad \Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}\ \mathfrak{NK}(\wedge I)$$

$$\frac{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\ \mathfrak{NK}(\Rightarrow E)$$

ND for HOL: Rules for Richer Signatures

Inference rules for $\mathfrak{NK}_\beta$ (for richer signatures)

$$\frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(\wedge E_L) \qquad \frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\wedge E_R) \qquad \frac{\Phi \Vdash \mathbf{A} \quad \Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}\,\mathfrak{NK}(\wedge I)$$

$$\frac{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\Rightarrow E) \qquad \frac{\Phi, \mathbf{A} \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B}}\,\mathfrak{NK}(\Rightarrow I)$$

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Inference rules for $\mathfrak{NK}_\beta$ (for richer signatures)

$$\frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(\wedge E_L) \qquad \frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\wedge E_R) \qquad \frac{\Phi \Vdash \mathbf{A} \quad \Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}\,\mathfrak{NK}(\wedge I)$$

$$\frac{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\Rightarrow E) \qquad \frac{\Phi, \mathbf{A} \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B}}\,\mathfrak{NK}(\Rightarrow I)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathbf{T}_\alpha}{\Phi \Vdash \Sigma^\alpha \mathbf{G}}\,\mathfrak{NK}(\Sigma I)$$

ND for HOL: Rules for Richer Signatures

Inference rules for $\mathfrak{NK}_\beta$ (for richer signatures)

$$\frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(\wedge E_L) \qquad \frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\wedge E_R) \qquad \frac{\Phi \Vdash \mathbf{A} \quad \Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}\,\mathfrak{NK}(\wedge I)$$

$$\frac{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\Rightarrow E) \qquad \frac{\Phi, \mathbf{A} \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B}}\,\mathfrak{NK}(\Rightarrow I)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathbf{T}_\alpha}{\Phi \Vdash \Sigma^\alpha \mathbf{G}}\,\mathfrak{NK}(\Sigma I) \qquad \frac{\Phi \Vdash \Sigma^\alpha \mathbf{G} \quad \Phi * \mathbf{G}\mathrm{w}_\alpha \Vdash \mathbf{C} \quad \text{w new parameter}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\Sigma E)$$

ND for HOL: Rules for Richer Signatures

Inference rules for $\mathfrak{NK}_\beta$ (for richer signatures)

$$\frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(\wedge E_L) \qquad \frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\wedge E_R) \qquad \frac{\Phi \Vdash \mathbf{A} \quad \Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}\,\mathfrak{NK}(\wedge I)$$

$$\frac{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\Rightarrow E) \qquad \frac{\Phi, \mathbf{A} \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B}}\,\mathfrak{NK}(\Rightarrow I)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathbf{T}_\alpha}{\Phi \Vdash \Sigma^\alpha \mathbf{G}}\,\mathfrak{NK}(\Sigma I) \qquad \frac{\Phi \Vdash \Sigma^\alpha \mathbf{G} \quad \Phi * \mathbf{G}\mathsf{w}_\alpha \Vdash \mathbf{C} \quad \text{w new parameter}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\Sigma E)$$

$$\frac{\Phi \Vdash \mathbf{T} =^\alpha \mathbf{W} \quad \Phi \Vdash \mathbf{A}[\mathbf{T}]}{\Phi \Vdash \mathbf{A}[\mathbf{W}]}\,\mathfrak{NK}(= Subst)$$

ND for HOL: Rules for Richer Signatures

Inference rules for $\mathfrak{NK}_\beta$ (for richer signatures)

$$\frac{\Phi \Vdash \mathbf{A} \land \mathbf{B}}{\Phi \Vdash \mathbf{A}}\ \mathfrak{NK}(\land E_L) \qquad \frac{\Phi \Vdash \mathbf{A} \land \mathbf{B}}{\Phi \Vdash \mathbf{B}}\ \mathfrak{NK}(\land E_R) \qquad \frac{\Phi \Vdash \mathbf{A} \quad \Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \land \mathbf{B}}\ \mathfrak{NK}(\land I)$$

$$\frac{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\ \mathfrak{NK}(\Rightarrow E) \qquad \frac{\Phi, \mathbf{A} \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B}}\ \mathfrak{NK}(\Rightarrow I)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathbf{T}_\alpha}{\Phi \Vdash \Sigma^\alpha \mathbf{G}}\ \mathfrak{NK}(\Sigma I) \qquad \frac{\Phi \Vdash \Sigma^\alpha \mathbf{G} \quad \Phi * \mathbf{G}\mathsf{w}_\alpha \Vdash \mathbf{C} \quad \text{w new parameter}}{\Phi \Vdash \mathbf{C}}\ \mathfrak{NK}(\Sigma E)$$

$$\frac{\Phi \Vdash \mathbf{T} =^\alpha \mathbf{W} \quad \Phi \Vdash \mathbf{A}[\mathbf{T}]}{\Phi \Vdash \mathbf{A}[\mathbf{W}]}\ \mathfrak{NK}(= Subst) \qquad \frac{}{\Phi \Vdash \mathbf{A} = \mathbf{A}}\ \mathfrak{NK}(= Refl)$$

ND for HOL: Rules for Richer Signatures

Inference rules for $\mathfrak{NK}_\beta$ (for richer signatures)

$$\frac{\Phi \Vdash \mathbf{A} \land \mathbf{B}}{\Phi \Vdash \mathbf{A}} \mathfrak{NK}(\land E_L) \quad \frac{\Phi \Vdash \mathbf{A} \land \mathbf{B}}{\Phi \Vdash \mathbf{B}} \mathfrak{NK}(\land E_R) \quad \frac{\Phi \Vdash \mathbf{A} \quad \Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \land \mathbf{B}} \mathfrak{NK}(\land I)$$

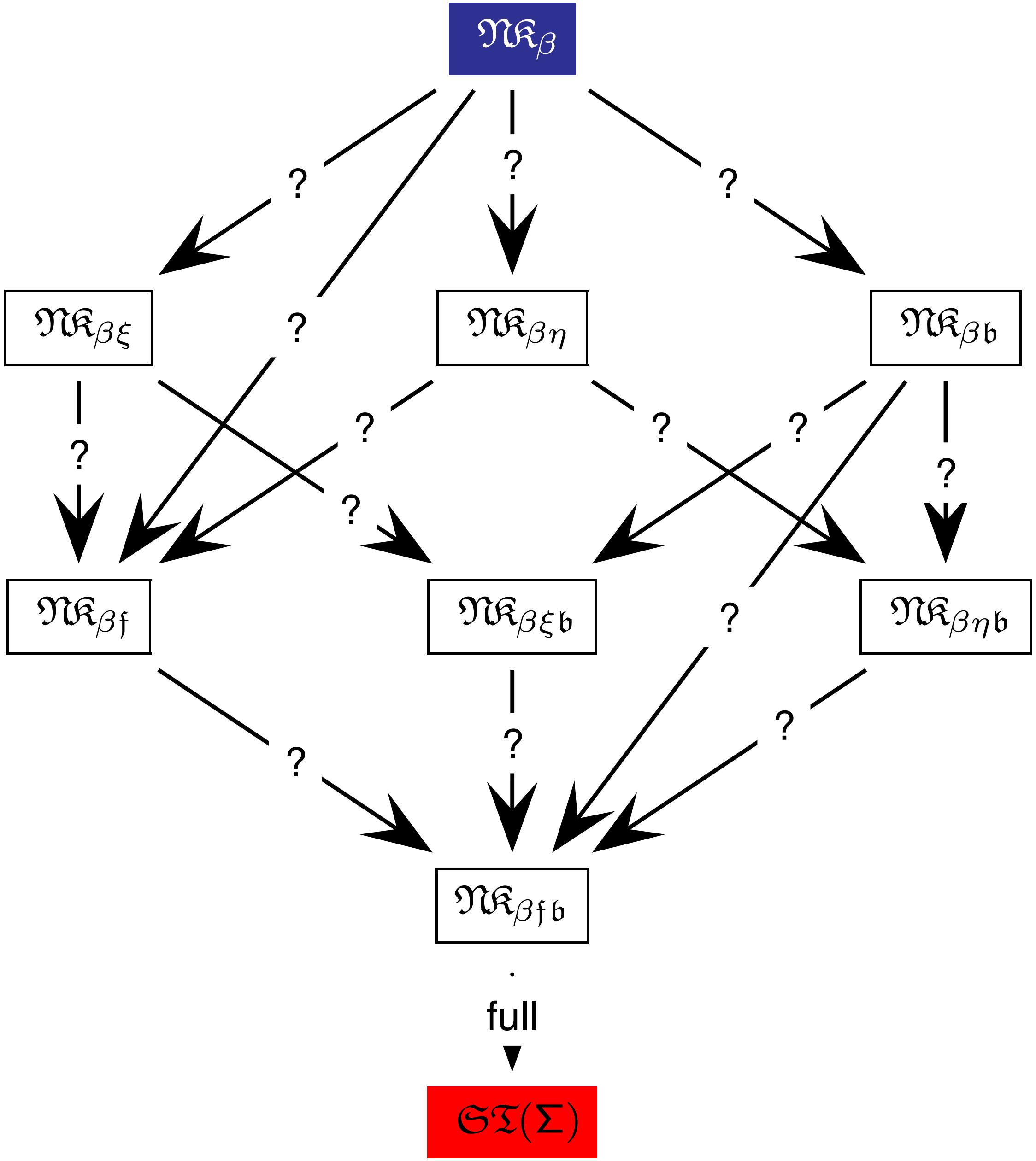
$$\frac{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}} \mathfrak{NK}(\Rightarrow E) \quad \frac{\Phi, \mathbf{A} \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B}} \mathfrak{NK}(\Rightarrow I)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathbf{T}_\alpha}{\Phi \Vdash \Sigma^\alpha \mathbf{G}} \mathfrak{NK}(\Sigma I) \quad \frac{\Phi \Vdash \Sigma^\alpha \mathbf{G} \quad \Phi * \mathbf{G}w_\alpha \Vdash \mathbf{C} \quad \text{w new parameter}}{\Phi \Vdash \mathbf{C}} \mathfrak{NK}(\Sigma E)$$

$$\frac{\Phi \Vdash \mathbf{T} =^\alpha \mathbf{W} \quad \Phi \Vdash \mathbf{A}[\mathbf{T}]}{\Phi \Vdash \mathbf{A}[\mathbf{W}]} \mathfrak{NK}(= Subst) \quad \frac{}{\Phi \Vdash \mathbf{A} = \mathbf{A}} \mathfrak{NK}(= Refl)$$

Here: we define logical constants $\land, \Rightarrow, \Sigma$, etc. in terms of $\neg, \lor, \Pi$ as usual and strictly use Leibniz equality instead of primitive equality; then the above rules are not needed

ND for HOL: Extensionality Rules



Base Calculus $\mathfrak{NK}_{\beta}$

— $\mathfrak{NK}(Hyp)$ — $\mathfrak{NK}(\beta)$

— $\mathfrak{NK}(\neg I)$ — $\mathfrak{NK}(\neg E)$

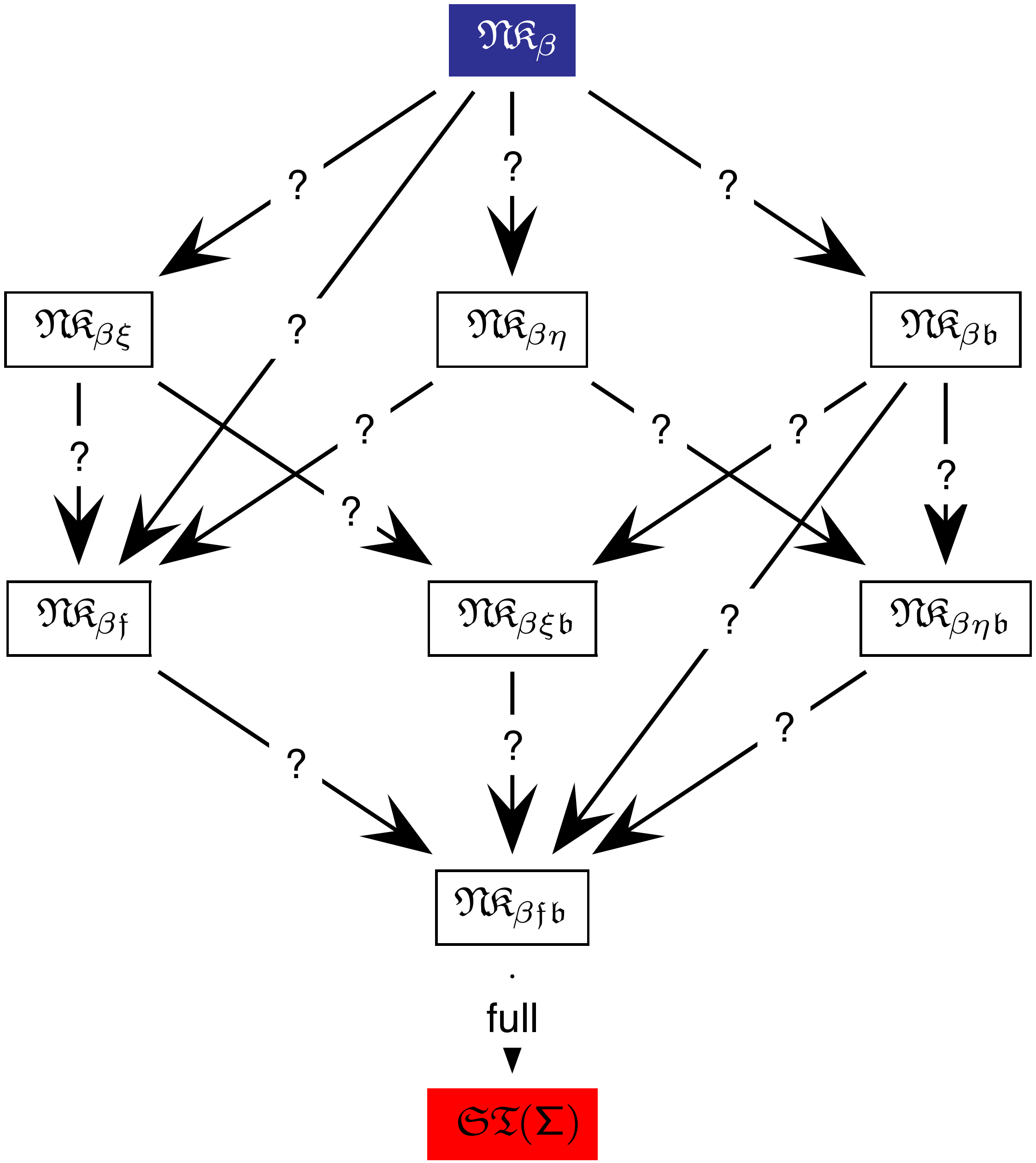
— $\mathfrak{NK}(\vee I_L)$ — $\mathfrak{NK}(\vee I_R)$

— $\mathfrak{NK}(\vee E)$

— $\mathfrak{NK}(\Pi I)^{w}$

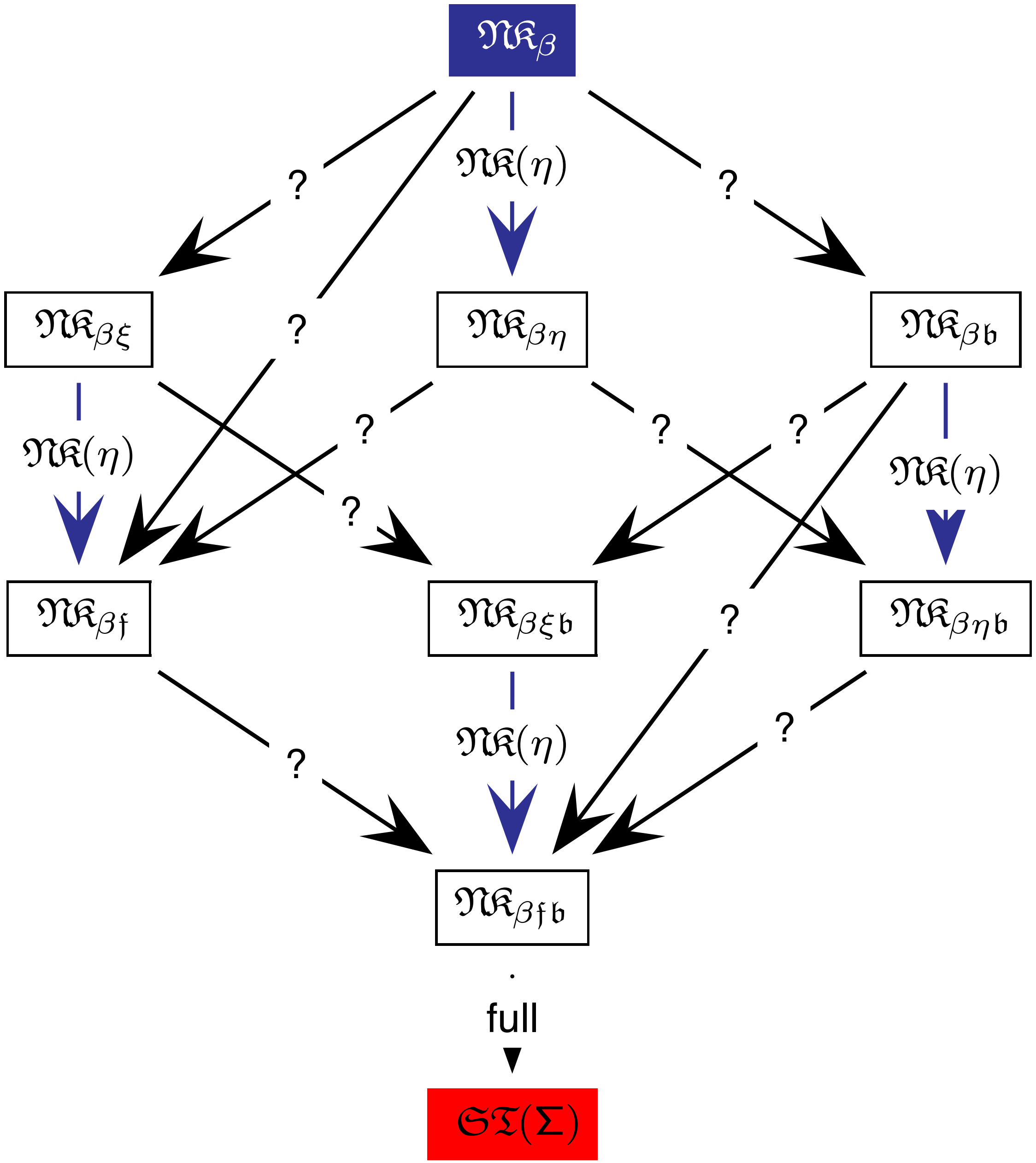
— $\mathfrak{NK}(\Pi E)$ — $\mathfrak{NK}(Contr)$

ND for HOL: Extensionality Rules



Optional Extensionality Rules

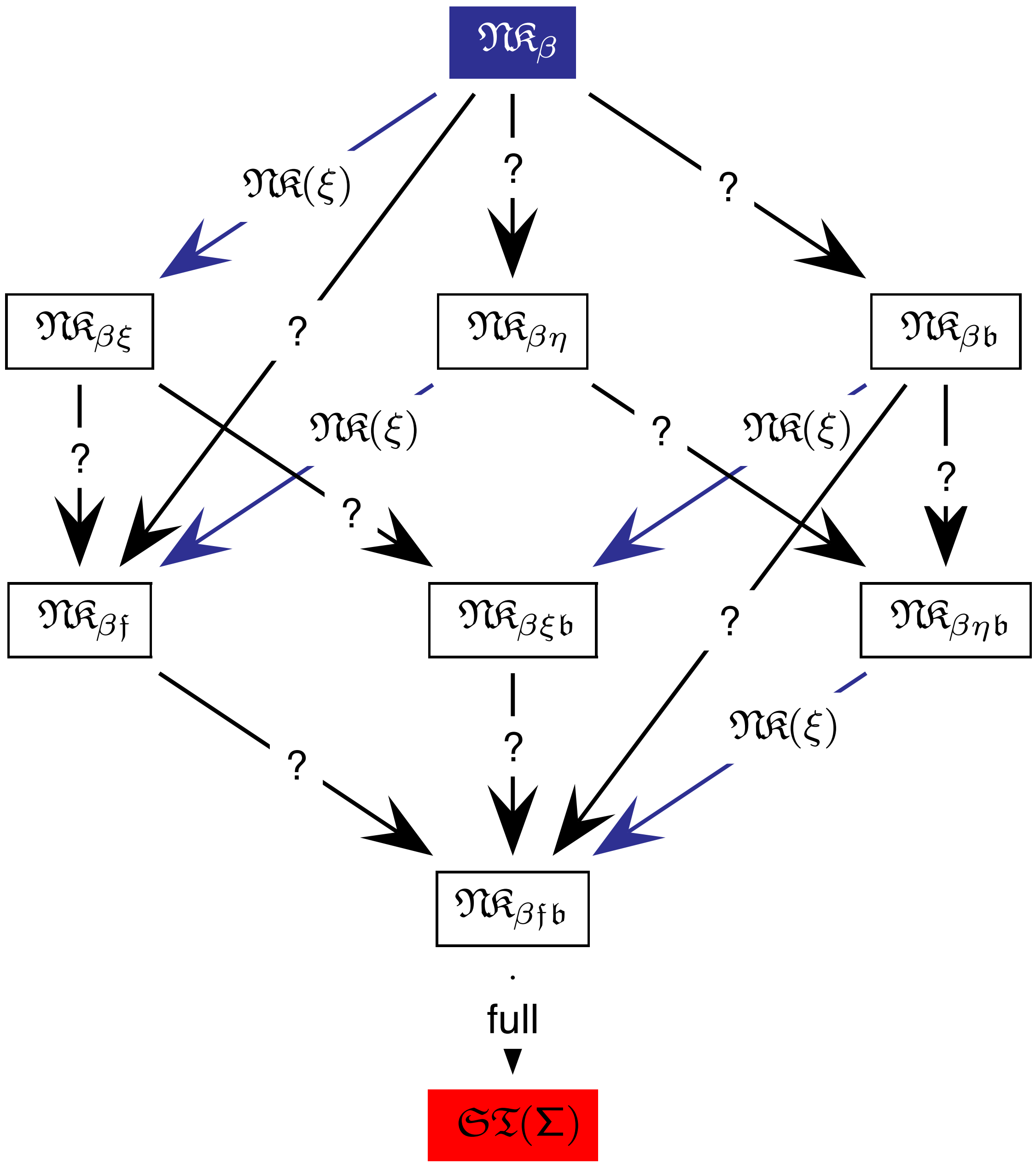
ND for HOL: Extensionality Rules



Optional Extensionality Rules

$$\frac{\mathbf{A} \overset{\beta\eta}{=} \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\ \mathfrak{NK}(\eta)$$

ND for HOL: Extensionality Rules

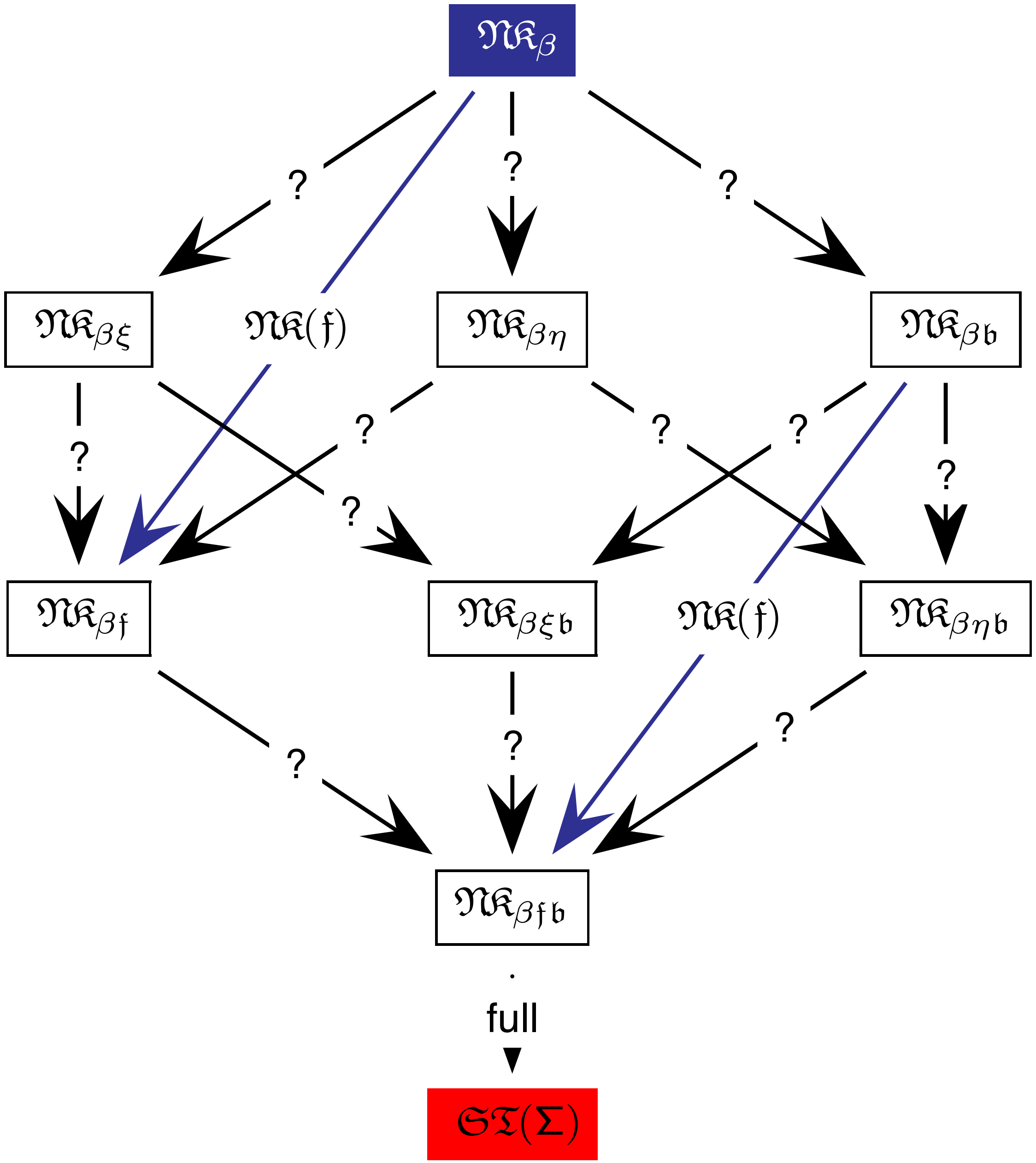


Optional Extensionality Rules

$$\frac{\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\ \mathfrak{NK}(\eta)$$

$$\frac{\Phi \Vdash \forall x_\alpha . \mathbf{M} \doteq^{\beta} \mathbf{N}}{\Phi \Vdash (\lambda x_\alpha . \mathbf{M}) \doteq^{\beta\alpha} (\lambda x_\alpha . \mathbf{N})}\ \mathfrak{NK}(\xi)$$

ND for HOL: Extensionality Rules



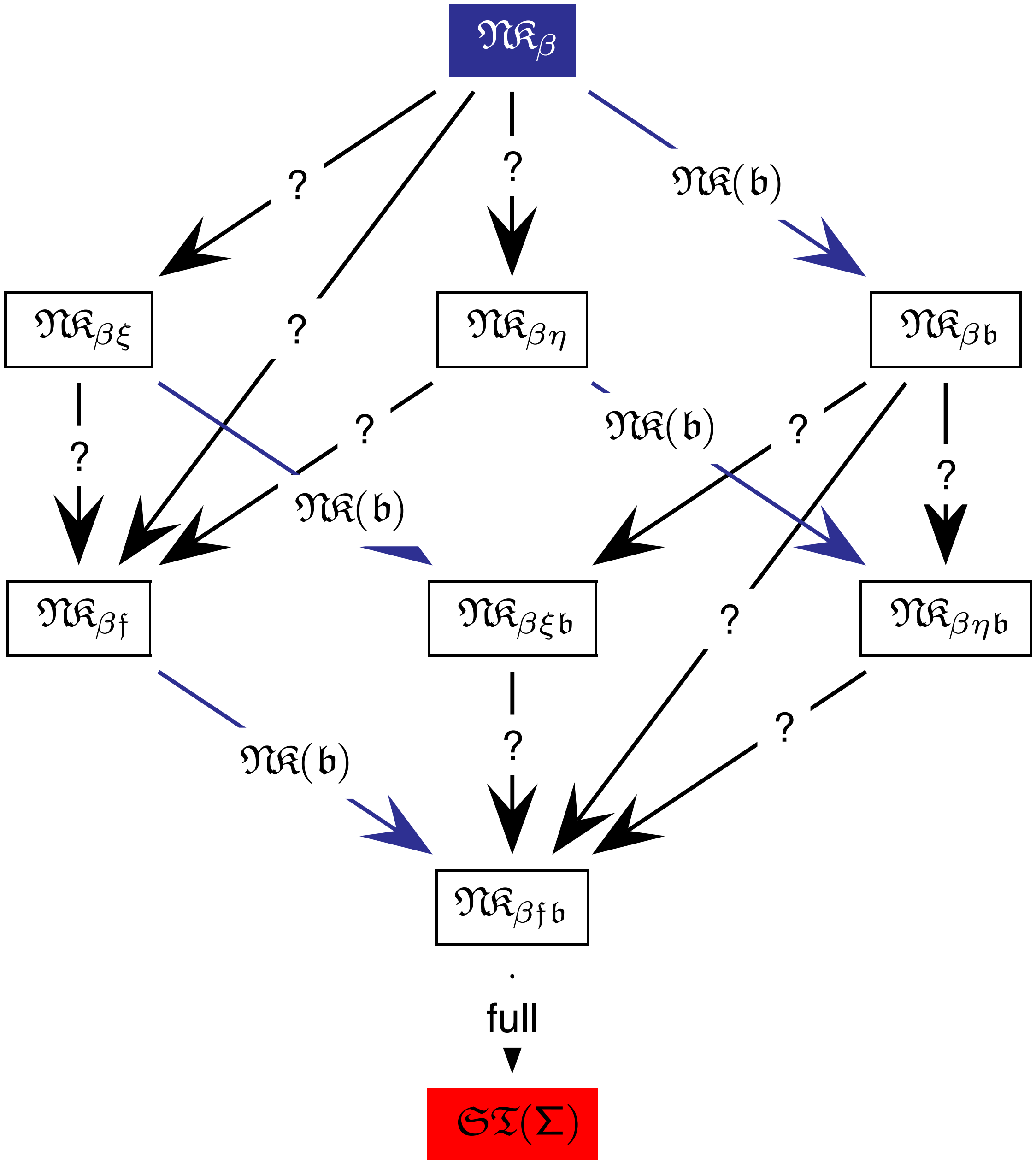
Optional Extensionality Rules

$$\frac{\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}} \; \mathfrak{NK}(\eta)$$

$$\frac{\Phi \Vdash \forall x_\alpha \mathbf{.M} \doteq^{\beta} \mathbf{N}}{\Phi \Vdash (\lambda x_\alpha \mathbf{.M}) \doteq^{\beta\alpha} (\lambda x_\alpha \mathbf{.N})} \; \mathfrak{NK}(\xi)$$

$$\frac{\Phi \Vdash \forall x_\alpha \mathbf{.G}x \doteq^{\beta} \mathbf{H}x}{\Phi \Vdash \mathbf{G} \doteq^{\beta\alpha} \mathbf{H}} \; \mathfrak{NK}(\mathfrak{f})$$

ND for HOL: Extensionality Rules



Optional Extensionality Rules

$$\frac{\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\ \mathfrak{NK}(\eta)$$

$$\frac{\Phi \Vdash \forall X_\alpha\mathbf{.M} \doteq^{\beta} \mathbf{N}}{\Phi \Vdash (\lambda X_\alpha\mathbf{.M}) \doteq^{\beta\alpha} (\lambda X_\alpha\mathbf{.N})}\ \mathfrak{NK}(\xi)$$

$$\frac{\Phi \Vdash \forall X_\alpha\mathbf{.G}X \doteq^{\beta} \mathbf{H}X}{\Phi \Vdash \mathbf{G} \doteq^{\beta\alpha} \mathbf{H}}\ \mathfrak{NK}(\mathfrak{f})$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{B} \quad \Phi * \mathbf{B} \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \doteq^{o} \mathbf{B}}\ \mathfrak{NK}(\mathfrak{b})$$

ND Calculi for HOL

Defn.: The Calculi $\mathfrak{NK}_*$

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- The calculus $\mathfrak{NK}_\beta$ consists of the inference rules for $\mathfrak{NK}_\beta$ for the provability judgment $\Vdash$ between sets of sentences Φ and sentences $\mathbf{A}$.

ND Calculi for HOL

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- The calculus $\mathfrak{NK}_\beta$ consists of the inference rules for $\mathfrak{NK}_\beta$ for the provability judgment $\Vdash$ between sets of sentences Φ and sentences $\mathbf{A}$.
- We write $\Vdash \mathbf{A}$ for $\emptyset \Vdash \mathbf{A}$.

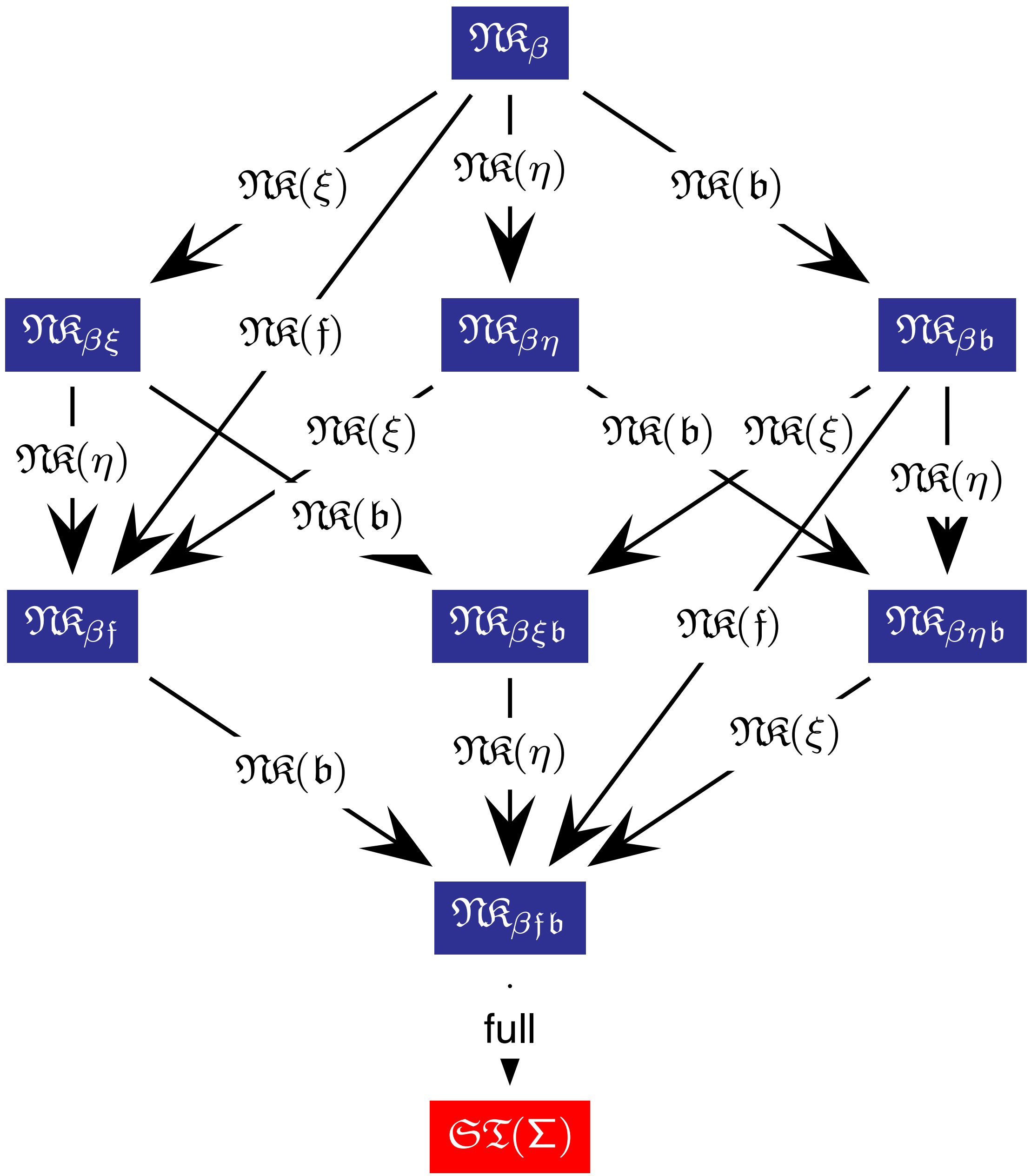
ND Calculi for HOL

Defn.: The Calculi $\mathfrak{NK}_*$

- The calculus $\mathfrak{NK}_\beta$ consists of the inference rules for $\mathfrak{NK}_\beta$ for the provability judgment $\Vdash$ between sets of sentences Φ and sentences $\mathbf{A}$.
- We write $\Vdash \mathbf{A}$ for $\emptyset \Vdash \mathbf{A}$.
- For $* \in \{\beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ we obtain the calculus $\mathfrak{NK}_*$ by adding the respective extensionality rules when specified in $*$:

$$\mathfrak{NK}_{\beta\eta}, \mathfrak{NK}_{\beta\xi}, \mathfrak{NK}_{\beta\mathfrak{f}}, \mathfrak{NK}_{\beta\mathfrak{b}}, \mathfrak{NK}_{\beta\eta\mathfrak{b}}, \mathfrak{NK}_{\beta\xi\mathfrak{b}}, \mathfrak{NK}_{\beta\mathfrak{f}\mathfrak{b}}$$

ND for HOL: The Complete Picture



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— $\mathfrak{NK}(\neg I)$ — $\mathfrak{NK}(\neg E)$

— $\mathfrak{NK}(\vee I_L)$ — $\mathfrak{NK}(\vee I_R)$

— $\mathfrak{NK}(\vee E)$

— $\mathfrak{NK}(\Pi I)^{w}$

— $\mathfrak{NK}(\Pi E)$ — $\mathfrak{NK}(Contr)$

Optional Extensionality Rules

— $\mathfrak{NK}(\eta)$ — $\mathfrak{NK}(\xi)$

— $\mathfrak{NK}(\mathfrak{f})$ — $\mathfrak{NK}(\mathfrak{b})$

ND Example Proof in $\mathfrak{NK}_\beta$

Derivation of:

$$(\mathsf{A} \doteq^{\alpha} \mathsf{A})$$

ND Example Proof in $\mathfrak{NK}_\beta$

$$\dfrac{}{\Vdash_{\mathfrak{NK}_\beta} (\mathsf{A} \doteq^\alpha \mathsf{A}) := \Pi^\alpha(\lambda \mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))}\ \mathfrak{NK}(\Pi I)$$

Derivation of:

$$(\mathsf{A} \doteq^\alpha \mathsf{A})$$

ND Example Proof in $\mathfrak{NK}_\beta$

$$\dfrac{\dfrac{}{\Vdash_{\mathfrak{NK}_\beta} ((\lambda P(\neg(PA) \vee (PA)))q)}\ \mathfrak{NK}(\beta)}{\Vdash_{\mathfrak{NK}_\beta} (A \doteq^\alpha A) := \Pi^\alpha(\lambda P(\neg(PA) \vee (PA)))}\ \mathfrak{NK}(\Pi I)$$

Derivation of:

$$(A \doteq^\alpha A)$$

ND Example Proof in $\mathfrak{NK}_\beta$

$$\dfrac{\dfrac{\dfrac{}{\Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{qA}) \vee (\mathsf{qA})}\ \mathfrak{NK}(Contr)}{\Vdash_{\mathfrak{NK}_\beta} ((\lambda \mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))\mathsf{q})}\ \mathfrak{NK}(\beta)}{\Vdash_{\mathfrak{NK}_\beta} (\mathsf{A} \doteq^\alpha \mathsf{A}) := \Pi^\alpha(\lambda \mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))}\ \mathfrak{NK}(\Pi I)$$

Derivation of:

$$(\mathsf{A} \doteq^\alpha \mathsf{A})$$

ND Example Proof in $\mathfrak{NK}_\beta$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Phi^1 := \{\neg(\neg(\mathsf{qA}) \lor (\mathsf{qA}))\} \Vdash_{\mathfrak{NK}_\beta} \mathbf{F}_o}\ \mathfrak{NK}(\neg E)}{\Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{qA}) \lor (\mathsf{qA})}\ \mathfrak{NK}(Contr)}{\Vdash_{\mathfrak{NK}_\beta} ((\lambda \mathsf{P}(\neg(\mathsf{PA}) \lor (\mathsf{PA})))\mathsf{q})}\ \mathfrak{NK}(\beta)}{\Vdash_{\mathfrak{NK}_\beta} (\mathsf{A} \doteq^\alpha \mathsf{A}) := \Pi^\alpha(\lambda \mathsf{P}(\neg(\mathsf{PA}) \lor (\mathsf{PA})))}\ \mathfrak{NK}(\Pi I)
$$

Derivation of:

$$(\mathsf{A} \doteq^\alpha \mathsf{A})$$

ND Example Proof in $\mathfrak{NK}_\beta$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Phi^1 \Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{qA}) \vee (\mathsf{qA})}{\Phi^1 := \{\neg(\neg(\mathsf{qA}) \vee (\mathsf{qA}))\} \Vdash_{\mathfrak{NK}_\beta} \mathbf{F}_o}\,\mathfrak{NK}(\neg E)}{\Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{qA}) \vee (\mathsf{qA})}\,\mathfrak{NK}(Contr)}{\Vdash_{\mathfrak{NK}_\beta} ((\lambda\mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))\mathsf{q})}\,\mathfrak{NK}(\beta)}{\Vdash_{\mathfrak{NK}_\beta} (\mathsf{A} \doteq^\alpha \mathsf{A}) := \Pi^\alpha(\lambda\mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))}\,\mathfrak{NK}(\Pi I)$$

Derivation of:

$$(\mathsf{A} \doteq^\alpha \mathsf{A})$$

ND Example Proof in $\mathfrak{NK}_\beta$

$$
\dfrac{\Phi^1 \Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{qA}) \vee (\mathsf{qA}) \qquad \dfrac{}{\Phi^1 \Vdash_{\mathfrak{NK}_\beta} \{\neg(\neg(\mathsf{qA}) \vee (\mathsf{qA}))}\,\mathfrak{NK}(Hyp)}{\dfrac{\Phi^1 := \{\neg(\neg(\mathsf{qA}) \vee (\mathsf{qA}))\} \Vdash_{\mathfrak{NK}_\beta} \mathbf{F}_o}{\dfrac{\Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{qA}) \vee (\mathsf{qA})}{\dfrac{\Vdash_{\mathfrak{NK}_\beta} ((\lambda \mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))\mathsf{q})}{\Vdash_{\mathfrak{NK}_\beta} (\mathsf{A} \doteq^\alpha \mathsf{A}) := \Pi^\alpha(\lambda \mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))}\,\mathfrak{NK}(\Pi I)}\,\mathfrak{NK}(\beta)}\,\mathfrak{NK}(Contr)}\,\mathfrak{NK}(\neg E)
$$

Derivation of:

$$(\mathsf{A} \doteq^\alpha \mathsf{A})$$

ND Example Proof in $\mathfrak{NK}_\beta$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\Phi^1 \Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{qA}) \vee (\mathsf{qA}) \qquad \cfrac{}{\Phi^1 \Vdash_{\mathfrak{NK}_\beta} \{\neg(\neg(\mathsf{qA}) \vee (\mathsf{qA}))}\;\mathfrak{NK}(Hyp)}{\Phi^1 := \{\neg(\neg(\mathsf{qA}) \vee (\mathsf{qA}))\} \Vdash_{\mathfrak{NK}_\beta} \mathbf{F}_o}\;\mathfrak{NK}(\neg E)}{\Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{qA}) \vee (\mathsf{qA})}\;\mathfrak{NK}(Contr)}{\Vdash_{\mathfrak{NK}_\beta} ((\lambda \mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))\mathsf{q})}\;\mathfrak{NK}(\beta)}{\Vdash_{\mathfrak{NK}_\beta} (\mathsf{A} \doteq^\alpha \mathsf{A}) := \Pi^\alpha(\lambda \mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))}\;\mathfrak{NK}(\Pi I)
$$

Derivation of:

$$(\mathsf{A} \doteq^\alpha \mathsf{A})$$

ND Example Proof in $\mathfrak{NK}_\beta$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{See Next Slide}}{\Phi^1 \Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{qA}) \vee (\mathsf{qA})} \qquad \dfrac{}{\Phi^1 \Vdash_{\mathfrak{NK}_\beta} \{\neg(\neg(\mathsf{qA}) \vee (\mathsf{qA}))}\,\mathfrak{NK}(Hyp)}{\Phi^1 := \{\neg(\neg(\mathsf{qA}) \vee (\mathsf{qA}))\} \Vdash_{\mathfrak{NK}_\beta} \mathbf{F}_o}\,\mathfrak{NK}(\neg E)}{\Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{qA}) \vee (\mathsf{qA})}\,\mathfrak{NK}(Contr)}{\Vdash_{\mathfrak{NK}_\beta} ((\lambda \mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))\mathsf{q})}\,\mathfrak{NK}(\beta)}{\Vdash_{\mathfrak{NK}_\beta} (\mathsf{A} \doteq^\alpha \mathsf{A}) := \Pi^\alpha(\lambda \mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))}\,\mathfrak{NK}(\Pi I)$$

Derivation of:

$$(\mathsf{A} \doteq^\alpha \mathsf{A})$$

ND Example Proof in $\mathfrak{NK}_\beta$

Derivation of:

$$\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{NK}_\beta} \neg p \vee p$$

resp. $\{\neg(\neg(qA) \vee (qA))\} \Vdash_{\mathfrak{NK}_\beta} \neg(qA) \vee (qA)$

ND Example Proof in $\mathfrak{NK}_\beta$

$$\frac{}{\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p} \vee \mathsf{p}} \; \mathfrak{NK}(\vee I_L)$$

Derivation of:

$$\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p} \vee \mathsf{p}$$

resp. $\{\neg(\neg(\mathsf{q}\mathsf{A}) \vee (\mathsf{q}\mathsf{A}))\} \Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{q}\mathsf{A}) \vee (\mathsf{q}\mathsf{A})$

ND Example Proof in $\mathfrak{NK}_\beta$

$$\dfrac{\dfrac{}{\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p}}\ \mathfrak{NK}(\neg I)}{\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p} \vee \mathsf{p}}\ \mathfrak{NK}(\vee I_L)$$

Derivation of:

$$\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p} \vee \mathsf{p}$$

resp. $\{\neg(\neg(\mathsf{qA}) \vee (\mathsf{qA}))\} \Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{qA}) \vee (\mathsf{qA})$

ND Example Proof in $\mathfrak{NK}_\beta$

$$\dfrac{\dfrac{\dfrac{}{\Phi^2 := \{\neg(\neg p \vee p), p\} \Vdash_{\mathfrak{NK}_\beta} \mathbf{F}_o}\mathfrak{NK}(\neg E)}{\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{NK}_\beta} \neg p}\mathfrak{NK}(\neg I)}{\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{NK}_\beta} \neg p \vee p}\mathfrak{NK}(\vee I_L)$$

Derivation of:

$$\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{NK}_\beta} \neg p \vee p$$

resp. $\{\neg(\neg(qA) \vee (qA))\} \Vdash_{\mathfrak{NK}_\beta} \neg(qA) \vee (qA)$

ND Example Proof in $\mathfrak{NK}_\beta$

$$
\dfrac{\dfrac{\overline{\Phi^2 \Vdash_{\mathfrak{NK}_\beta} \neg(\neg p \vee p)}\ \mathfrak{NK}(Hyp)}{\dfrac{\Phi^2 := \{\neg(\neg p \vee p), p\} \Vdash_{\mathfrak{NK}_\beta} \mathbf{F}_o}{\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{NK}_\beta} \neg p}\ \mathfrak{NK}(\neg I)}\ \mathfrak{NK}(\neg E)}{\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{NK}_\beta} \neg p \vee p}\ \mathfrak{NK}(\vee I_L)
$$

Derivation of:

$$\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{NK}_\beta} \neg p \vee p$$

resp. $\{\neg(\neg(qA) \vee (qA))\} \Vdash_{\mathfrak{NK}_\beta} \neg(qA) \vee (qA)$

ND Example Proof in $\mathfrak{NK}_\beta$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\Phi^2 \Vdash_{\mathfrak{NK}_\beta} \neg(\neg p \vee p)}\,\mathfrak{NK}(Hyp)
      \qquad
      \dfrac{}{\Phi^2 \Vdash_{\mathfrak{NK}_\beta} \neg p \vee p}\,\mathfrak{NK}(\vee I_R)
    }{\Phi^2 := \{\neg(\neg p \vee p), p\} \Vdash_{\mathfrak{NK}_\beta} \mathbf{F}_o}\,\mathfrak{NK}(\neg E)
  }{\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{NK}_\beta} \neg p}\,\mathfrak{NK}(\neg I)
}{\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{NK}_\beta} \neg p \vee p}\,\mathfrak{NK}(\vee I_L)
$$

Derivation of:

$$\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{NK}_\beta} \neg p \vee p$$

resp. $\{\neg(\neg(qA) \vee (qA))\} \Vdash_{\mathfrak{NK}_\beta} \neg(qA) \vee (qA)$

ND Example Proof in $\mathfrak{NK}_\beta$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\Phi^2 \Vdash_{\mathfrak{NK}_\beta} \neg(\neg \mathsf{p} \vee \mathsf{p})}\,\mathfrak{NK}(Hyp)
      \qquad
      \dfrac{
        \dfrac{}{\Phi^2 \Vdash_{\mathfrak{NK}_\beta} \mathsf{p}}\,\mathfrak{NK}(Hyp)
      }{\Phi^2 \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p} \vee \mathsf{p}}\,\mathfrak{NK}(\vee I_R)
    }{\Phi^2 := \{\neg(\neg \mathsf{p} \vee \mathsf{p}), \mathsf{p}\} \Vdash_{\mathfrak{NK}_\beta} \mathbf{F}_o}\,\mathfrak{NK}(\neg E)
  }{\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p}}\,\mathfrak{NK}(\neg I)
}{\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p} \vee \mathsf{p}}\,\mathfrak{NK}(\vee I_L)
$$

Derivation of:

$$\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p} \vee \mathsf{p}$$

resp. $\{\neg(\neg(\mathsf{q}\mathsf{A}) \vee (\mathsf{q}\mathsf{A}))\} \Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{q}\mathsf{A}) \vee (\mathsf{q}\mathsf{A})$

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: By induction on the derivation of $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (base case)

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}} \mathfrak{NK}(Hyp)$$

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (base case)

$$\frac{\mathbf{C} \in \Phi}{\Phi \Vdash \mathbf{C}} \mathfrak{NK}(Hyp)$$

$\mathcal{M} \models \mathbf{C}$ whenever $\mathcal{M} \models \Phi$ and $\mathbf{C} \in \Phi$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\mathbf{A} =_\beta \mathbf{C} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}} \mathfrak{NK}(\beta)$$

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\mathbf{A} =_\beta \mathbf{C} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\beta)$$

Suppose $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash \mathbf{A}$ and $\mathbf{A} =_\beta \mathbf{C}$. Let $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ be a model of Φ.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\mathbf{A} =_\beta \mathbf{C} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\ \mathfrak{NK}(\beta)$$

Suppose $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash \mathbf{A}$ and $\mathbf{A} =_\beta \mathbf{C}$. Let $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ be a model of Φ. By induction, we know $\mathcal{M} \models \mathbf{A}$ and so $\mathcal{M} \models \mathbf{C}$ since Σ-evaluations respect β-equality.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{fb}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi * \neg\mathbf{C} \Vdash \mathbf{F}_\circ}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(Contr)$$

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi * \neg\mathbf{C} \Vdash \mathbf{F}_o}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(Contr)$$

Suppose $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$, $\mathcal{M} \models \Phi$ and $\Phi \Vdash \mathbf{C}$ follows from $\Phi * \neg\mathbf{C} \Vdash \mathbf{F}_o$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi * \neg\mathbf{C} \Vdash \mathbf{F}_o}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(Contr)$$

Suppose $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$, $\mathcal{M} \models \Phi$ and $\Phi \Vdash \mathbf{C}$ follows from $\Phi * \neg\mathbf{C} \Vdash \mathbf{F}_o$. By a previous Lemma, $\mathcal{M} \not\models \mathbf{F}_o$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi * \neg\mathbf{C} \Vdash \mathbf{F}_\circ}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(Contr)$$

Suppose $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$, $\mathcal{M} \models \Phi$ and $\Phi \Vdash \mathbf{C}$ follows from $\Phi * \neg\mathbf{C} \Vdash \mathbf{F}_\circ$. By a previous Lemma, $\mathcal{M} \not\models \mathbf{F}_\circ$. So, we must have $\mathcal{M} \not\models \neg\mathbf{C}$ and, hence, $\mathcal{M} \models \mathbf{C}$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_\circ}{\Phi \Vdash \neg \mathbf{A}} \mathfrak{NK}(\neg I)$$

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_\circ}{\Phi \Vdash \neg \mathbf{A}} \mathfrak{NK}(\neg I)$$

Analogous to $\mathfrak{NK}(Contr)$

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}} \mathfrak{NK}(\neg E)$$

Suppose $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash \neg\mathbf{A}$ and $\Phi \Vdash \mathbf{A}$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\neg E)$$

Suppose $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash \neg\mathbf{A}$ and $\Phi \Vdash \mathbf{A}$. By induction, any model in $\mathfrak{M}_*(\Sigma)$ of Φ would have to model both $\mathbf{A}$ and $\neg\mathbf{A}$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \neg\mathbf{A} \qquad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\neg E)$$

Suppose $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash \neg\mathbf{A}$ and $\Phi \Vdash \mathbf{A}$. By induction, any model in $\mathfrak{M}_*(\Sigma)$ of Φ would have to model both $\mathbf{A}$ and $\neg\mathbf{A}$. So, there is no such model of Φ and we are done.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\,\mathfrak{NK}(\vee I_L)$$

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}} \mathfrak{NK}(\vee I_L)$$

Suppose $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$, $\mathcal{M} \models \Phi$, and $\Phi \Vdash (\mathbf{A} \vee \mathbf{B})$ follows from $\Phi \Vdash \mathbf{A}$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}} \mathfrak{NK}(\vee I_L)$$

Suppose $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$, $\mathcal{M} \models \Phi$, and $\Phi \Vdash (\mathbf{A} \vee \mathbf{B})$ follows from $\Phi \Vdash \mathbf{A}$. By induction, $\mathcal{M} \models \mathbf{A}$ and so $\mathcal{M} \models (\mathbf{A} \vee \mathbf{B})$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\, \mathfrak{NK}(\vee I_R)$$

Soundness of $\mathfrak{N}\mathfrak{K}_*$

Thm.: $\mathfrak{N}\mathfrak{K}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. That is, if $\Phi \Vdash_{\mathfrak{N}\mathfrak{K}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\ \mathfrak{N}\mathfrak{K}(\vee I_R)$$

Analogous to $\mathfrak{N}\mathfrak{K}(\vee I_L)$

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\vee E)$$

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}} \mathfrak{NK}(\vee E)$$

Suppose $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash (\mathbf{A} \vee \mathbf{B})$, $\Phi * \mathbf{A} \Vdash \mathbf{C}$ and $\Phi * \mathbf{B} \Vdash \mathbf{C}$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}} \mathfrak{NK}(\vee E)$$

Suppose $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash (\mathbf{A} \vee \mathbf{B})$, $\Phi * \mathbf{A} \Vdash \mathbf{C}$ and $\Phi * \mathbf{B} \Vdash \mathbf{C}$. Let $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ be a model of Φ.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\vee E)$$

Suppose $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash (\mathbf{A} \vee \mathbf{B})$, $\Phi * \mathbf{A} \Vdash \mathbf{C}$ and $\Phi * \mathbf{B} \Vdash \mathbf{C}$. Let $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ be a model of Φ. By induction, $\mathcal{M} \models \mathbf{A} \vee \mathbf{B}$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}}\, \mathfrak{NK}(\vee E)$$

Suppose $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash (\mathbf{A} \vee \mathbf{B})$, $\Phi * \mathbf{A} \Vdash \mathbf{C}$ and $\Phi * \mathbf{B} \Vdash \mathbf{C}$. Let $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ be a model of Φ. By induction, $\mathcal{M} \models \mathbf{A} \vee \mathbf{B}$. If $\mathcal{M} \models \mathbf{A}$, then by induction $\mathcal{M} \models \mathbf{C}$ since $\Phi * \mathbf{A} \Vdash \mathbf{C}$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}} \; \mathfrak{NK}(\vee E)$$

Suppose $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash (\mathbf{A} \vee \mathbf{B})$, $\Phi * \mathbf{A} \Vdash \mathbf{C}$ and $\Phi * \mathbf{B} \Vdash \mathbf{C}$. Let $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ be a model of Φ. By induction, $\mathcal{M} \models \mathbf{A} \vee \mathbf{B}$. If $\mathcal{M} \models \mathbf{A}$, then by induction $\mathcal{M} \models \mathbf{C}$ since $\Phi * \mathbf{A} \Vdash \mathbf{C}$. If $\mathcal{M} \models \mathbf{B}$, then by induction $\mathcal{M} \models \mathbf{C}$ since $\Phi * \mathbf{B} \Vdash \mathbf{C}$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}}\ \mathfrak{NK}(\vee E)$$

Suppose $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash (\mathbf{A} \vee \mathbf{B})$, $\Phi * \mathbf{A} \Vdash \mathbf{C}$ and $\Phi * \mathbf{B} \Vdash \mathbf{C}$. Let $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ be a model of Φ. By induction, $\mathcal{M} \models \mathbf{A} \vee \mathbf{B}$. If $\mathcal{M} \models \mathbf{A}$, then by induction $\mathcal{M} \models \mathbf{C}$ since $\Phi * \mathbf{A} \Vdash \mathbf{C}$. If $\mathcal{M} \models \mathbf{B}$, then by induction $\mathcal{M} \models \mathbf{C}$ since $\Phi * \mathbf{B} \Vdash \mathbf{C}$. In either case, $\Phi \Vdash \mathbf{C}$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{G}\mathsf{w}_\alpha \quad \mathsf{w}\text{ new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}}\,\mathfrak{NK}(\Pi I)^{\mathsf{w}}$$

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{G}\mathsf{w}_\alpha \quad \mathsf{w} \text{ new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}} \mathfrak{NK}(\Pi I)^{\mathsf{w}}$$

Suppose $\Phi \Vdash (\Pi^\alpha \mathbf{G})$ follows from $\Phi \Vdash \mathbf{G}\mathsf{w}$ where w_α is a fresh parameter.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{G}\mathsf{w}_\alpha \quad \mathsf{w}\text{ new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}}\ \mathfrak{NK}(\Pi I)^{\mathsf{w}}$$

Suppose $\Phi \Vdash (\Pi^\alpha \mathbf{G})$ follows from $\Phi \Vdash \mathbf{G}\mathsf{w}$ where w_α is a fresh parameter. Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_*(\Sigma)$ be a model of Φ.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{G}\mathsf{w}_\alpha \quad \mathsf{w} \text{ new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}}\ \mathfrak{NK}(\Pi I)^{\mathsf{w}}$$

Suppose $\Phi \Vdash (\Pi^\alpha \mathbf{G})$ follows from $\Phi \Vdash \mathbf{G}\mathsf{w}$ where w_α is a fresh parameter. Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_*(\Sigma)$ be a model of Φ.
Assume $\mathcal{M} \not\models \Pi^\alpha \mathbf{G}$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{G}\mathsf{w}_\alpha \quad \text{w new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}} \mathfrak{NK}(\mathit{\Pi I})^{\mathsf{w}}$$

Suppose $\Phi \Vdash (\Pi^\alpha \mathbf{G})$ follows from $\Phi \Vdash \mathbf{G}\mathsf{w}$ where w_α is a fresh parameter. Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_*(\Sigma)$ be a model of Φ. Assume $\mathcal{M} \not\models \Pi^\alpha \mathbf{G}$. Then there must be some $\mathsf{a} \in \mathcal{D}_\alpha$ such that $\upsilon(\mathcal{E}(\mathbf{G})@\mathsf{a}) = \mathtt{F}$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{G}\mathsf{w}_\alpha \quad \mathsf{w}\text{ new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}}\,\mathfrak{NK}(\Pi I)^{\mathsf{w}}$$

Suppose $\Phi \Vdash (\Pi^\alpha \mathbf{G})$ follows from $\Phi \Vdash \mathbf{G}\mathsf{w}$ where w_α is a fresh parameter. Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_*(\Sigma)$ be a model of Φ. Assume $\mathcal{M} \not\models \Pi^\alpha \mathbf{G}$. Then there must be some $\mathsf{a} \in \mathcal{D}_\alpha$ such that $\upsilon(\mathcal{E}(\mathbf{G})@\mathsf{a}) = \mathtt{F}$. From $\mathcal{E}$, one can define $\mathcal{E}'$ such that $\mathcal{E}'(\mathsf{w}) = \mathsf{a}$ and $\mathcal{E}'_\varphi(\mathbf{A}_\alpha) = \mathcal{E}_\varphi(\mathbf{A}_\alpha)$ if w does not occur in $\mathbf{A}$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{G}\mathsf{w}_\alpha \quad \mathsf{w}\text{ new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}}\ \mathfrak{NK}(\Pi I)^{\mathsf{w}}$$

Suppose $\Phi \Vdash (\Pi^\alpha \mathbf{G})$ follows from $\Phi \Vdash \mathbf{G}\mathsf{w}$ where w_α is a fresh parameter. Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_*(\Sigma)$ be a model of Φ. Assume $\mathcal{M} \not\models \Pi^\alpha \mathbf{G}$. Then there must be some $\mathsf{a} \in \mathcal{D}_\alpha$ such that $\upsilon(\mathcal{E}(\mathbf{G})@\mathsf{a}) = \mathtt{F}$. From $\mathcal{E}$, one can define $\mathcal{E}'$ such that $\mathcal{E}'(\mathsf{w}) = \mathsf{a}$ and $\mathcal{E}'_\varphi(\mathbf{A}_\alpha) = \mathcal{E}_\varphi(\mathbf{A}_\alpha)$ if w does not occur in $\mathbf{A}$. Let $\mathcal{M}' := (\mathcal{D}, @, \mathcal{E}', \upsilon)$. One can check $\mathcal{M}' \in \mathfrak{M}_*(\Sigma)$ using the fact that $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{G}\mathsf{w}_\alpha \quad \mathsf{w} \text{ new parameter}}{\Phi \Vdash \Pi^\alpha\mathbf{G}}\ \mathfrak{NK}(\Pi I)^{\mathsf{w}}$$

Suppose $\Phi \Vdash (\Pi^\alpha\mathbf{G})$ follows from $\Phi \Vdash \mathbf{G}\mathsf{w}$ where w_α is a fresh parameter. Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_*(\Sigma)$ be a model of Φ. Assume $\mathcal{M} \not\models \Pi^\alpha\mathbf{G}$. Then there must be some $\mathsf{a} \in \mathcal{D}_\alpha$ such that $\upsilon(\mathcal{E}(\mathbf{G})@\mathsf{a}) = \mathtt{F}$. From $\mathcal{E}$, one can define $\mathcal{E}'$ such that $\mathcal{E}'(\mathsf{w}) = \mathsf{a}$ and $\mathcal{E}'_\varphi(\mathbf{A}_\alpha) = \mathcal{E}_\varphi(\mathbf{A}_\alpha)$ if w does not occur in $\mathbf{A}$. Let $\mathcal{M}' := (\mathcal{D}, @, \mathcal{E}', \upsilon)$. One can check $\mathcal{M}' \in \mathfrak{M}_*(\Sigma)$ using the fact that $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$. Since $\mathcal{M}' \models \Phi$, by induction we have $\mathcal{M}' \models \mathbf{G}\mathsf{w}$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \mathbf{G}\mathsf{w}_\alpha \quad \mathsf{w} \text{ new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}}\ \mathfrak{NK}(\Pi I)^{\mathsf{w}}$$

Suppose $\Phi \Vdash (\Pi^\alpha \mathbf{G})$ follows from $\Phi \Vdash \mathbf{G}\mathsf{w}$ where w_α is a fresh parameter. Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_*(\Sigma)$ be a model of Φ. Assume $\mathcal{M} \not\models \Pi^\alpha \mathbf{G}$. Then there must be some $\mathsf{a} \in \mathcal{D}_\alpha$ such that $\upsilon(\mathcal{E}(\mathbf{G})@\mathsf{a}) = \mathtt{F}$. From $\mathcal{E}$, one can define $\mathcal{E}'$ such that $\mathcal{E}'(\mathsf{w}) = \mathsf{a}$ and $\mathcal{E}'_\varphi(\mathbf{A}_\alpha) = \mathcal{E}_\varphi(\mathbf{A}_\alpha)$ if w does not occur in $\mathbf{A}$. Let $\mathcal{M}' := (\mathcal{D}, @, \mathcal{E}', \upsilon)$. One can check $\mathcal{M}' \in \mathfrak{M}_*(\Sigma)$ using the fact that $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$. Since $\mathcal{M}' \models \Phi$, by induction we have $\mathcal{M}' \models \mathbf{G}\mathsf{w}$. This contradicts $\upsilon(\mathcal{E}'(\mathbf{G})@\mathsf{a}) = \upsilon(\mathcal{E}(\mathbf{G})@\mathsf{a}) = \mathtt{F}$. Thus, $\mathcal{M} \models \Pi^\alpha \mathbf{G}$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \Pi^\alpha \mathbf{G}}{\Phi \Vdash \mathbf{G}\mathbf{A}}\,\mathfrak{NK}(\mathit{\Pi E})$$

Soundness of $\mathfrak{N}\mathfrak{K}_*$

Thm.: $\mathfrak{N}\mathfrak{K}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{N}\mathfrak{K}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \Pi^\alpha \mathbf{G}}{\Phi \Vdash \mathbf{G}\mathbf{A}}\,\mathfrak{N}\mathfrak{K}(\mathit{\Pi E})$$

Suppose $\mathbf{C}$ is $(\mathbf{G}\mathbf{A})$ and $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash (\Pi^\alpha \mathbf{G})$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \Pi^\alpha \mathbf{G}}{\Phi \Vdash \mathbf{G}\mathbf{A}} \mathfrak{NK}(\Pi E)$$

Suppose $\mathbf{C}$ is $(\mathbf{G}\mathbf{A})$ and $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash (\Pi^\alpha \mathbf{G})$. Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_*(\Sigma)$ be a model of Φ.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \Pi^\alpha \mathbf{G}}{\Phi \Vdash \mathbf{G}\mathbf{A}}\,\mathfrak{NK}(\mathit{\Pi E})$$

Suppose $\mathbf{C}$ is $(\mathbf{G}\mathbf{A})$ and $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash (\Pi^\alpha \mathbf{G})$. Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_*(\Sigma)$ be a model of Φ. By induction, $\mathcal{M} \models (\Pi^\alpha \mathbf{G})$ and thus $\upsilon(\mathcal{E}(\mathbf{G}))@\mathsf{a} = \mathtt{T}$ for every $\mathsf{a} \in \mathcal{D}_\alpha$.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \Pi^\alpha \mathbf{G}}{\Phi \Vdash \mathbf{G}\mathbf{A}} \mathfrak{NK}(\Pi E)$$

Suppose $\mathbf{C}$ is $(\mathbf{G}\mathbf{A})$ and $\Phi \Vdash \mathbf{C}$ follows from $\Phi \Vdash (\Pi^\alpha \mathbf{G})$. Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_*(\Sigma)$ be a model of Φ. By induction, $\mathcal{M} \models (\Pi^\alpha \mathbf{G})$ and thus $\upsilon(\mathcal{E}(\mathbf{G}))@\mathsf{a} = \mathtt{T}$ for every $\mathsf{a} \in \mathcal{D}_\alpha$. In particular, $\mathcal{M} \models \mathbf{G}\mathbf{A}$.

q.e.d.

Soundness of $\mathfrak{NK}_*$

Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$.
That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\mathbf{A} \overset{\beta\eta}{=} \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}} \, \mathfrak{NK}(\eta)$$

(In this case $*$ contains property η)
Analogous to $\mathfrak{NK}(\beta)$ using property η

q.e.d.

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Thm.: $\mathfrak{NK}_*$ is sound for $\mathfrak{M}_*(\Sigma)$ for $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. That is, if $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{C}$ is derivable, then $\mathcal{M} \models \mathbf{C}$ for all models $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ in $\mathfrak{M}_*(\Sigma)$ such that $\mathcal{M} \models \Phi$.

Proof: (step cases)

$$\frac{\Phi \Vdash \forall \mathsf{x}_\alpha\mathbf{.M} \doteq^{\beta} \mathbf{N}}{\Phi \Vdash (\lambda \mathsf{x}_\alpha\mathbf{.M}) \doteq^{\beta\alpha} (\lambda \mathsf{x}_\alpha\mathbf{.N})}\ \mathfrak{NK}(\xi)$$

(In this case $*$ contains property ξ)
Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_*(\Sigma)$ be a model of Φ. By induction, we have $\mathcal{M} \models \forall \mathsf{X}_\alpha\mathbf{.M} \doteq^{\beta} \mathbf{N}$. So, for any assignment φ and $\mathsf{a} \in \mathcal{D}_\alpha$, $\mathcal{M} \models_{\varphi,[\mathsf{a}/\mathsf{X}]} \mathbf{M} \doteq^{\beta} \mathbf{N}$. Since property $\mathfrak{q}$ holds, by a previous Lemma we have $\mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathbf{M}) = \mathcal{E}_{\varphi,[\mathsf{a}/\mathsf{X}]}(\mathbf{N})$. By property ξ, $\mathcal{E}_\varphi(\lambda \mathsf{X}_\alpha\mathbf{.M}) = \mathcal{E}_\varphi(\lambda \mathsf{X}_\alpha\mathbf{.N})$ and thus $\mathcal{M} \models \mathbf{C}$ by a previous Lemma.

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Proof: (step cases)

$$\frac{\Phi \Vdash \forall \mathsf{x}_\alpha \mathbf{.} \mathbf{G}\mathsf{x} \doteq^{\beta} \mathbf{H}\mathsf{x}}{\Phi \Vdash \mathbf{G} \doteq^{\beta\alpha} \mathbf{H}} \mathfrak{NK}(\mathfrak{f})$$

(In this case $*$ contains property $\mathfrak{f}$)
Let $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ be a model of Φ. By induction, we know $\mathcal{M} \models \forall \mathsf{X}_\alpha \mathbf{.} \mathbf{G}\mathsf{X} \doteq^{\beta} \mathbf{H}\mathsf{X}$. Note that property $\mathfrak{q}$ holds for $\mathcal{M}$ since $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$. By a previous theorem, we must have $\mathcal{M} \models (\mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H})$.

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Proof: (step cases)

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{B} \qquad \Phi * \mathbf{B} \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \doteq^{o} \mathbf{B}}\,\mathfrak{NK}(\mathfrak{b})$$

(In this case $*$ contains property $\mathfrak{b}$)
Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_*(\Sigma)$ be a model of Φ. If $\mathcal{M} \models \mathbf{A}$, then $\mathcal{M} \models \mathbf{B}$ by induction. If $\mathcal{M} \models \mathbf{B}$, then $\mathcal{M} \models \mathbf{A}$ by induction. These facts imply $\upsilon(\mathcal{E}(\mathbf{A})) = \upsilon(\mathcal{E}(\mathbf{B}))$. By a previous lemma, we have $\mathcal{M} \models (\mathbf{A} \Leftrightarrow \mathbf{B})$. By a previous theorem, we must have $\mathcal{M} \models (\mathbf{A} \doteq^{o} \mathbf{B})$.

q.e.d.

Completeness of $\mathfrak{NK}_*$

Thm.: Let Φ be a sufficiently Σ-pure set of sentences, $\mathbf{A}$ be a sentence, and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. If $\mathbf{A}$ is valid in all models $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ that satisfy Φ, then $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{A}$.

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Proof:

How can we easily prove this?

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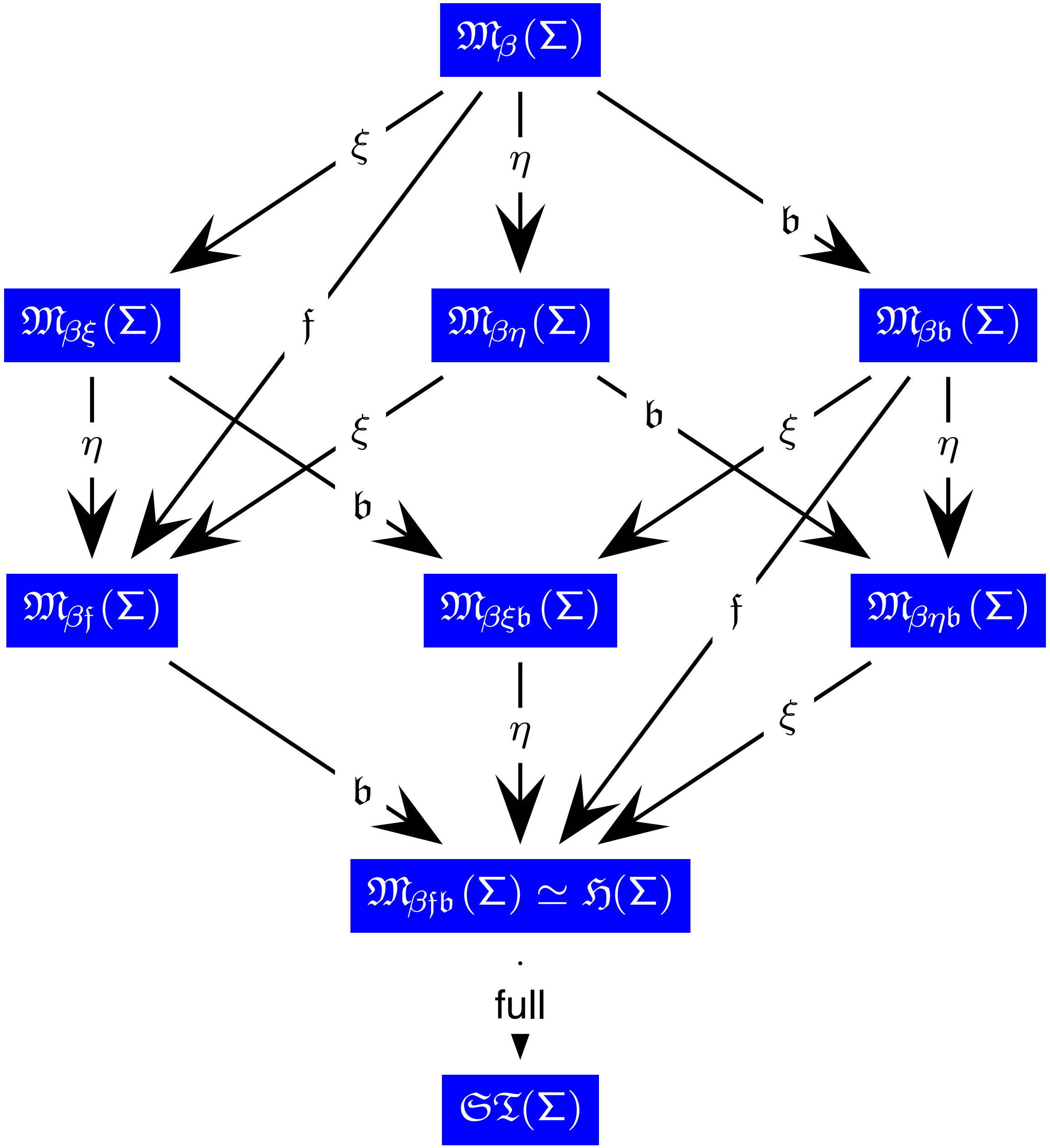
- with the help of the abstract consistency proof method (published in [JSL-04] and [Unpublished-04])

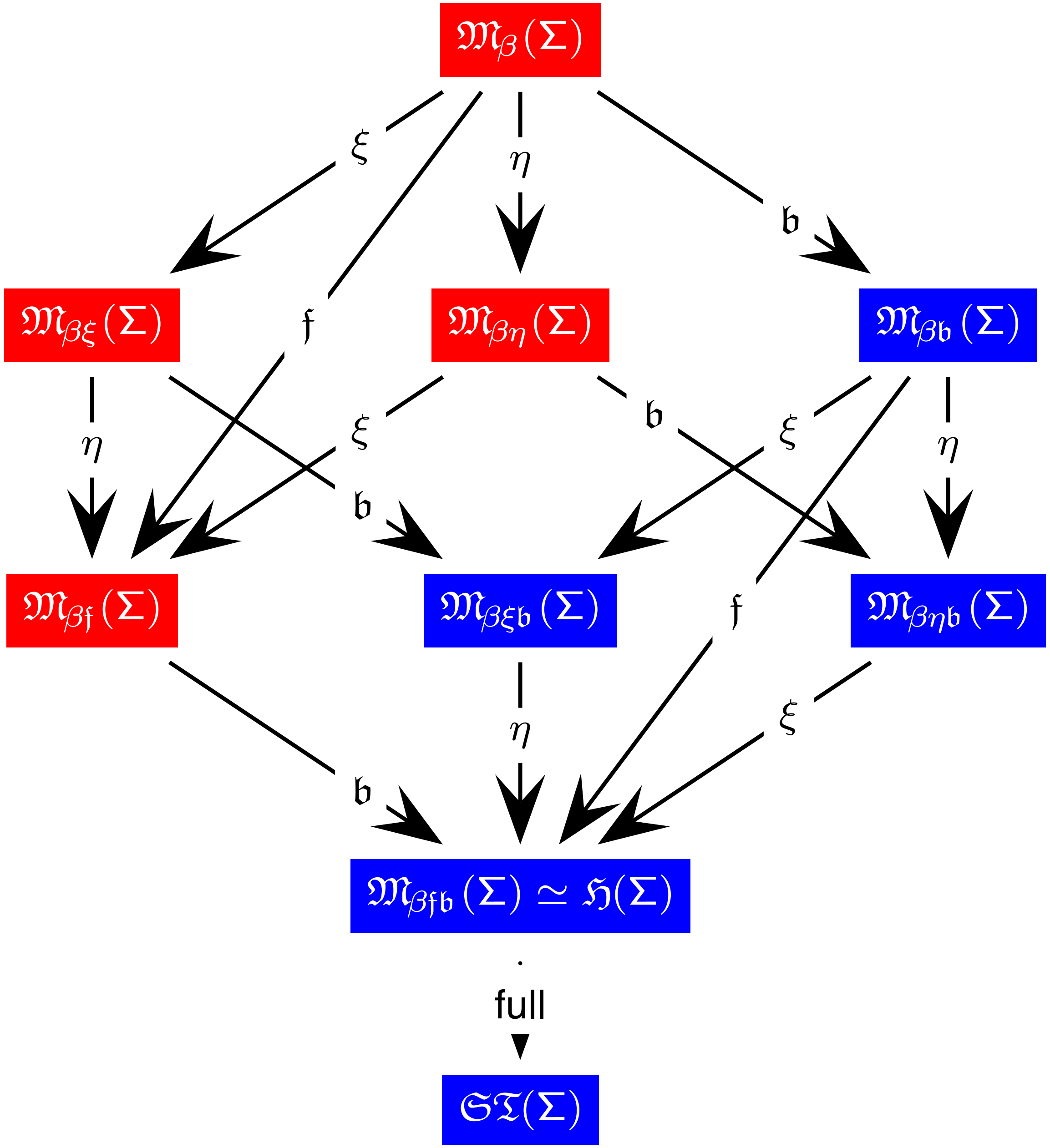
HOL Test Problems (from before)



- $\forall X. \forall y. X \vee y \Leftrightarrow y \vee X$

valid for all model classes

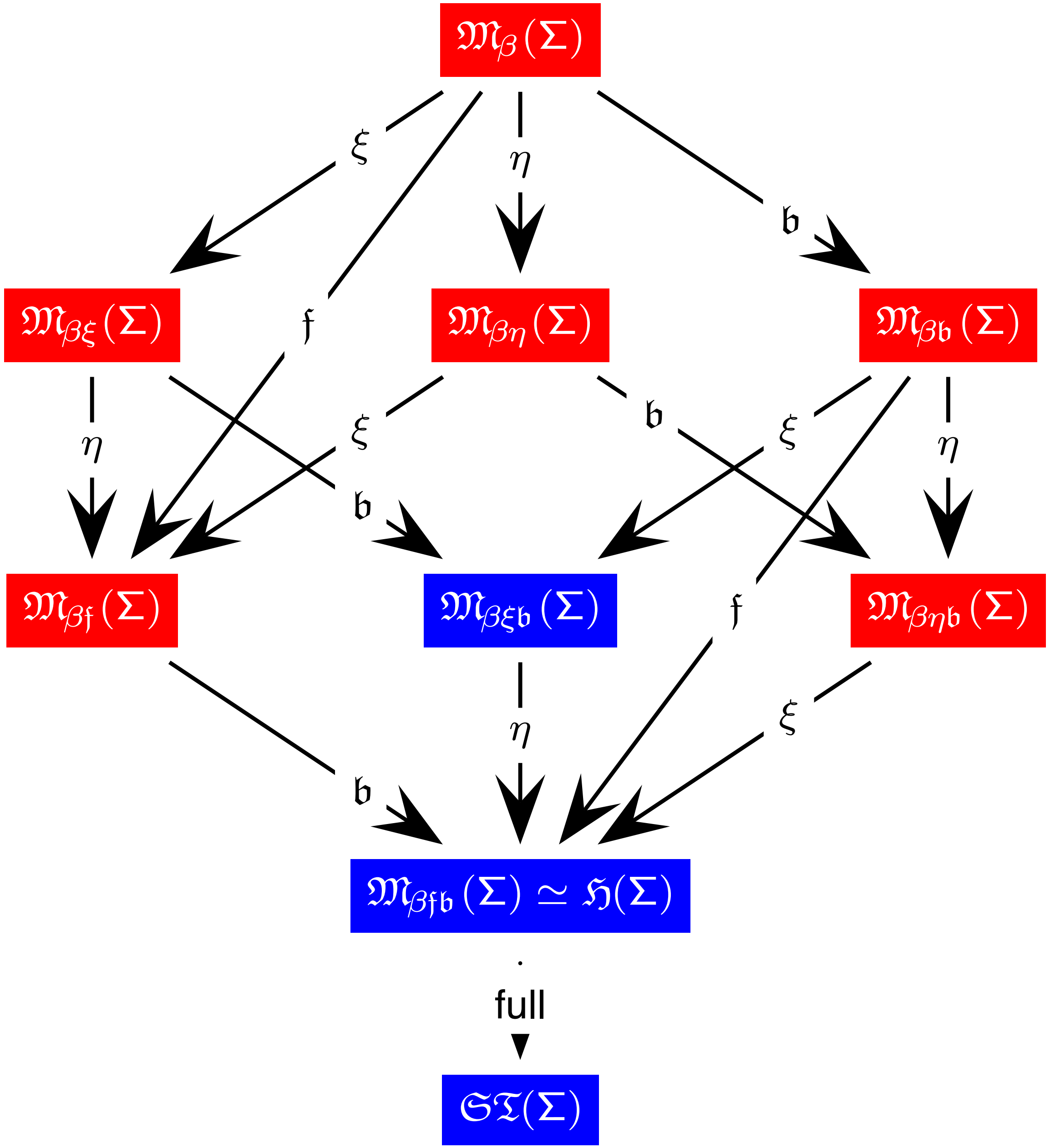
HOL Test Problems (from before)



- $\forall X.\forall Y.X \vee Y \Leftrightarrow Y \vee X$
- $\forall X.\forall Y.X \vee Y \doteq Y \vee X$

validity requires $\mathfrak{b}$

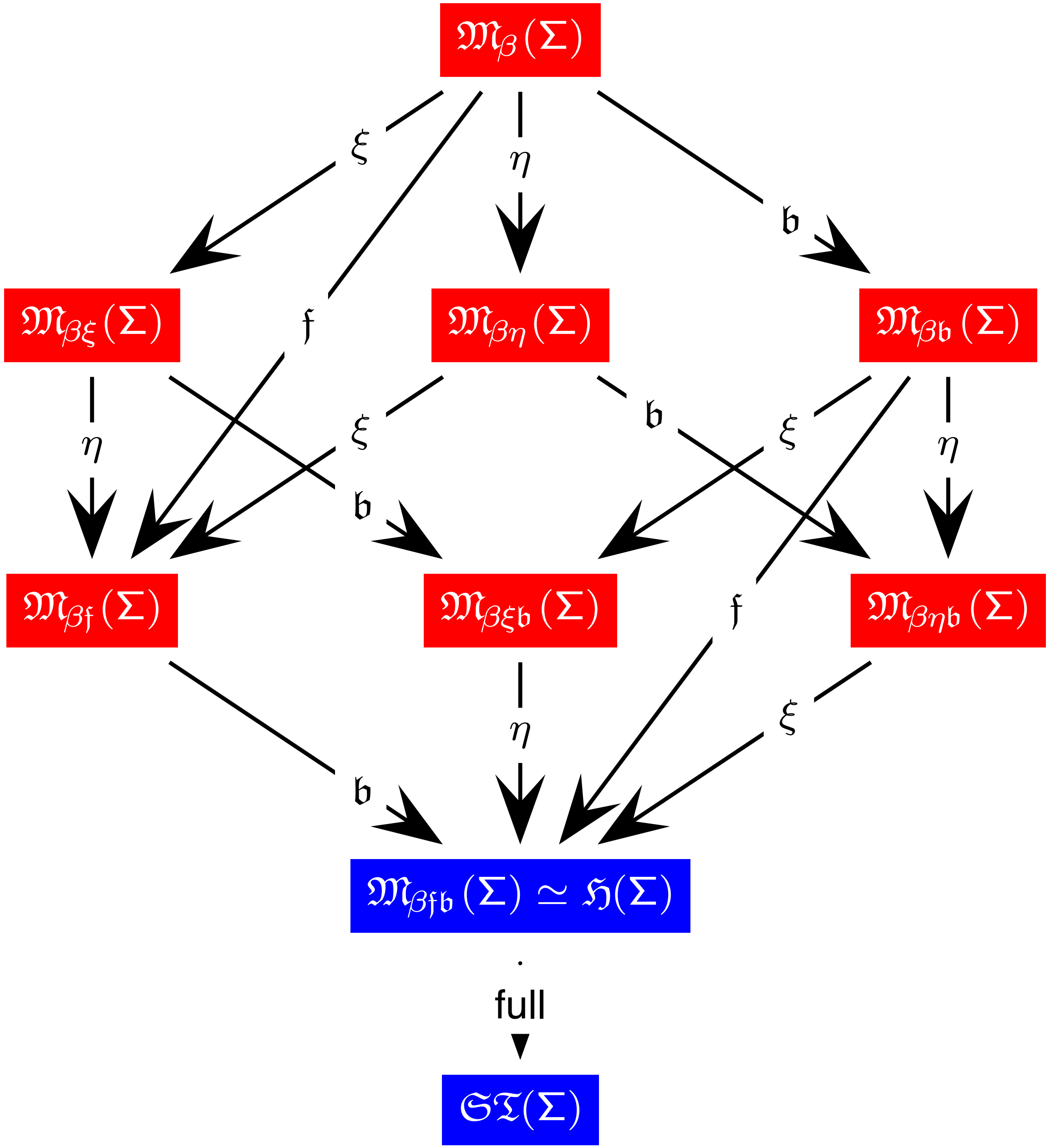
HOL Test Problems (from before)



- $\forall X.\forall Y.X \lor Y \Leftrightarrow Y \lor X$
- $\forall X.\forall Y.X \lor Y \doteq Y \lor X$
- $\lambda X.\lambda Y.X \lor Y \doteq \lambda X.\lambda Y.Y \lor X$

validity requires $\mathfrak{b}$ and ξ

HOL Test Problems (from before)



- $\forall X. \forall Y. X \lor Y \Leftrightarrow Y \lor X$
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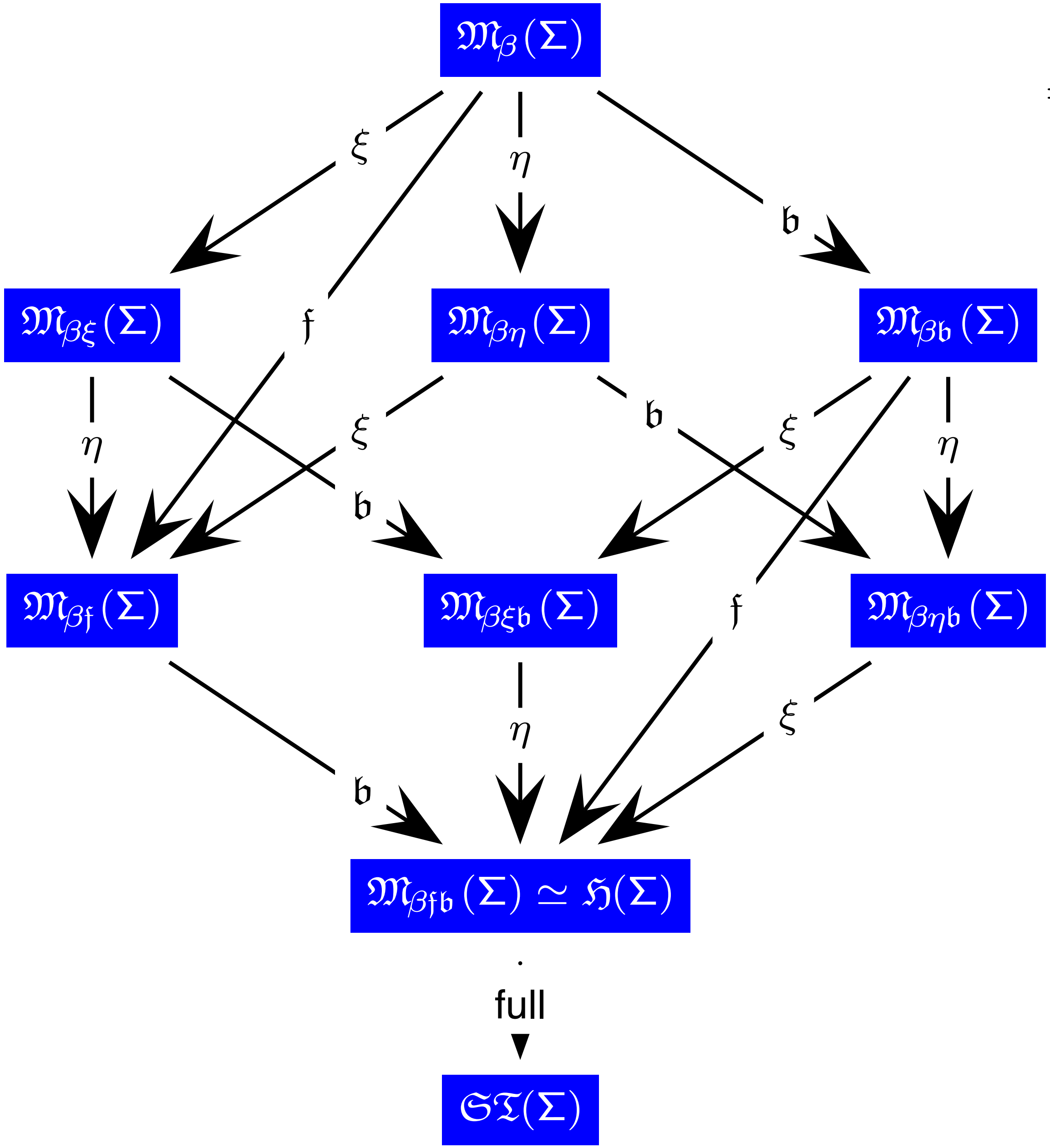
validity requires $\mathfrak{b}$ and $\mathfrak{f}$

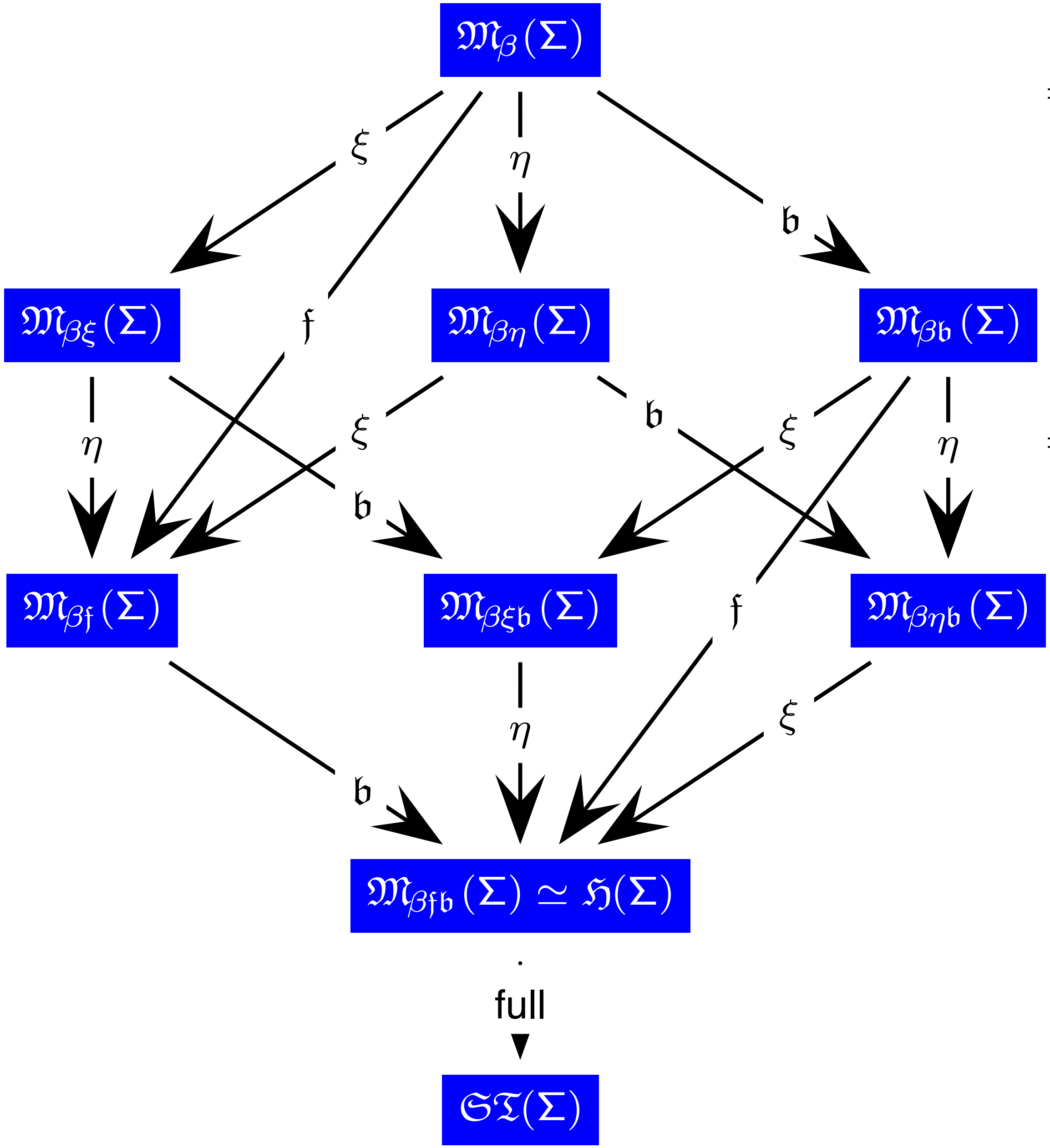
Other HOL Test Problems: β



$\stackrel{*}{=}$ **is equivalence relation**

- $\forall X_\alpha . X \stackrel{*}{=} X$
- $\forall X_\alpha, Y_\alpha . X \stackrel{*}{=} Y \supset Y \stackrel{*}{=} X$
- $\forall X_\alpha, Y_\alpha, Z_\alpha . (X \stackrel{*}{=} Y \wedge Y \stackrel{*}{=} Z) \supset X \stackrel{*}{=} Z$

Other HOL Test Problems: β



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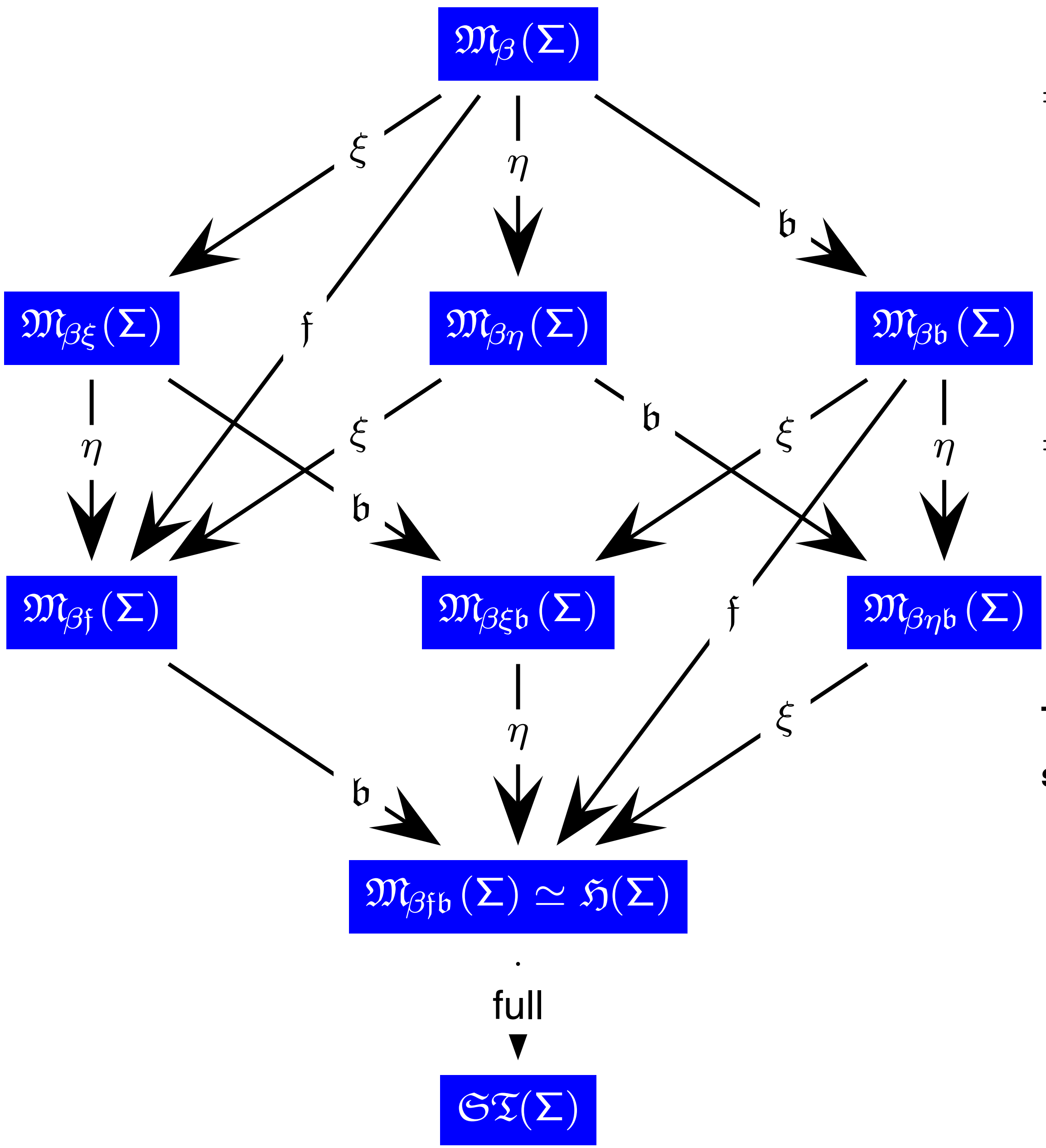
$\stackrel{*}{=}$ **is congruence relation**

- $\forall X_\alpha, Y_\alpha, F_{\alpha\alpha}. X \stackrel{*}{=} Y \supset (FX) \stackrel{*}{=} (FY)$
- $\forall X_\alpha, Y_\alpha, P_{o\alpha}. X \stackrel{*}{=} Y \wedge (PX) \supset (PY)$

Other HOL Test Problems: β



$\stackrel{*}{\doteq}$ **is equivalence relation**

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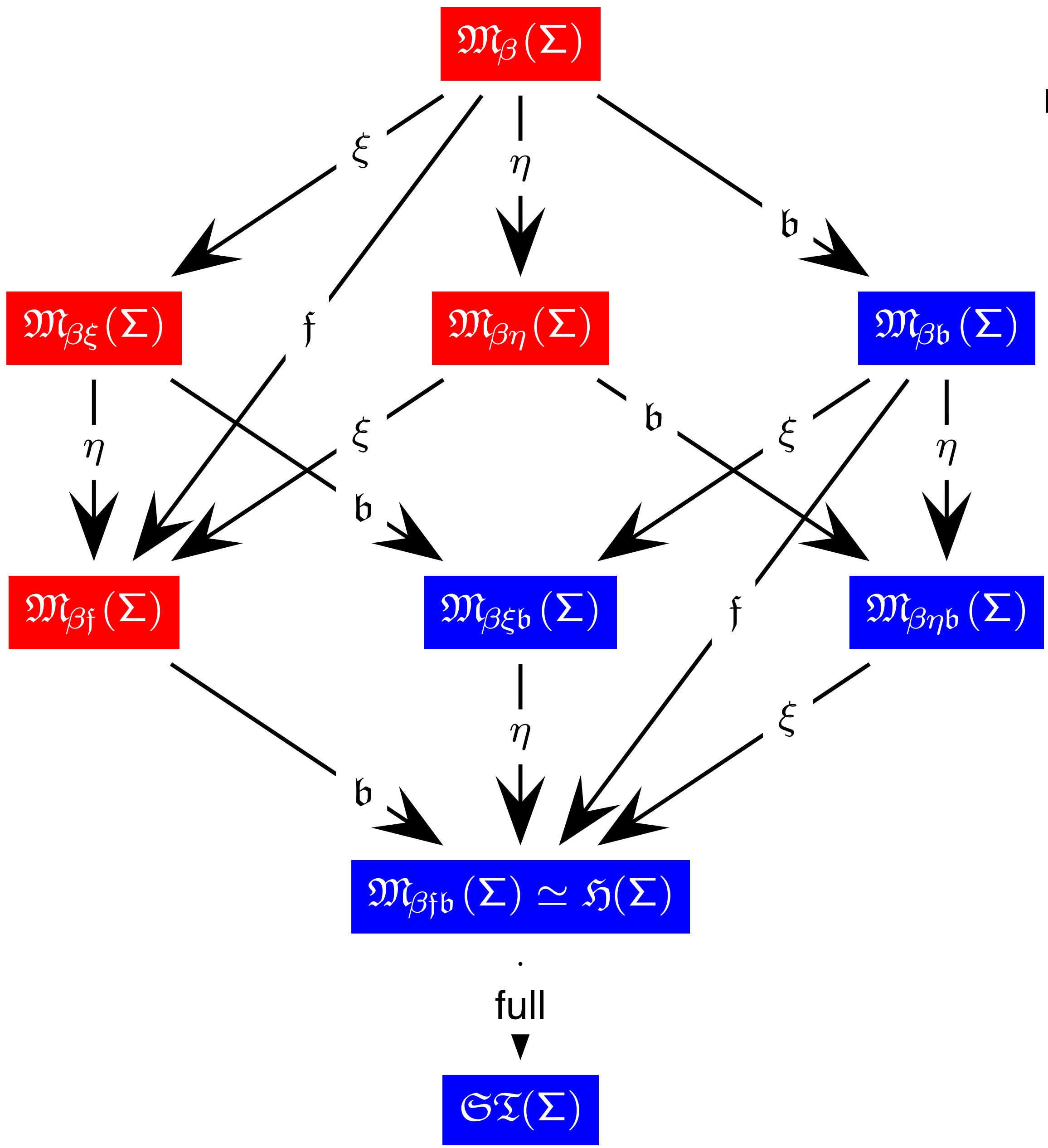
Trivial directions of Boolean and functional extensionality

- $\forall A_o, B_o . A \stackrel{*}{\doteq} B \supset (A \Leftrightarrow B)$
- $\forall F_{\beta\alpha}, G_{\beta\alpha} . F \stackrel{*}{\doteq} G \supset (\forall X_\alpha . FX \stackrel{*}{\doteq} GX)$

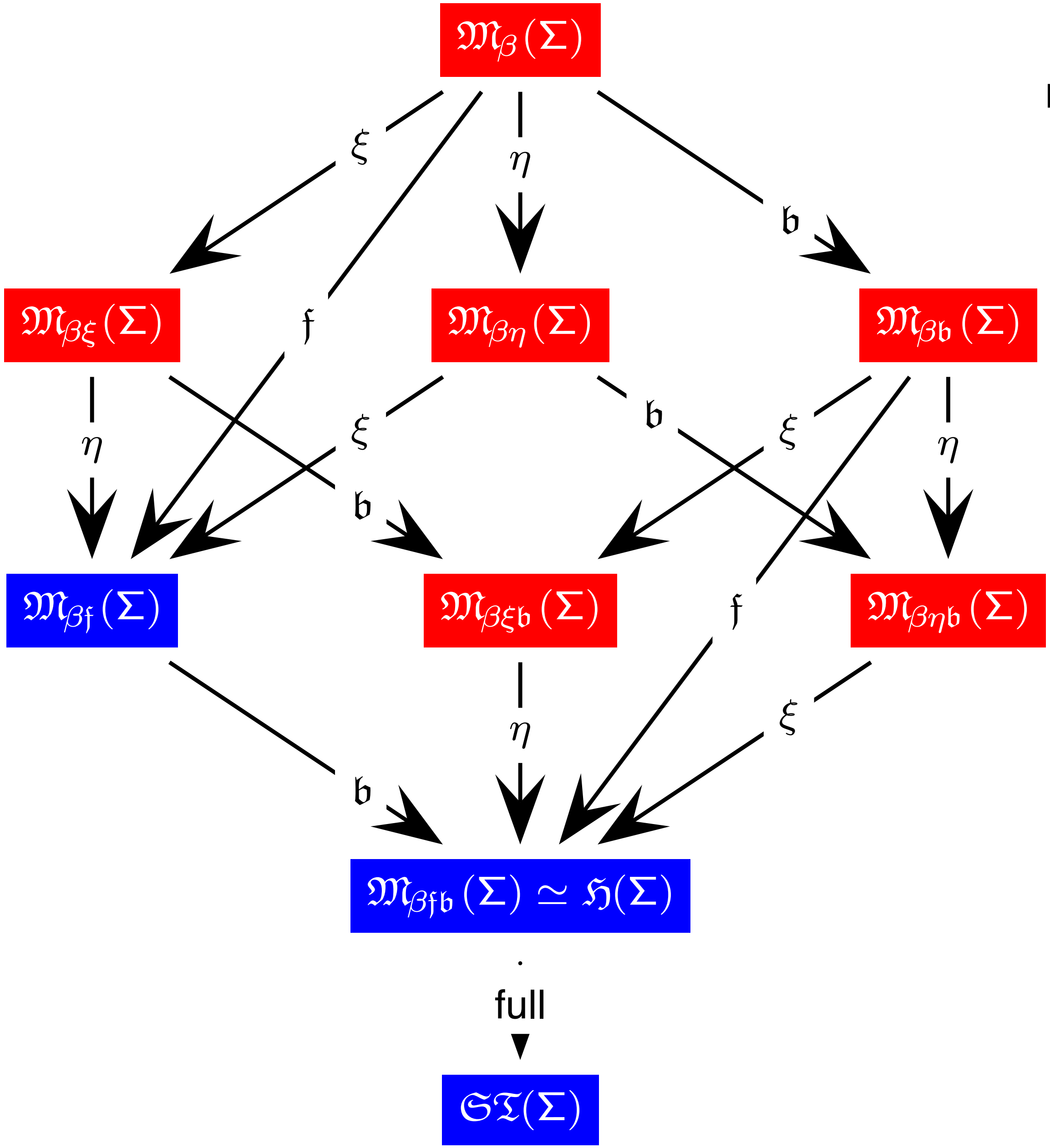
Other HOL Test Problems: $\mathfrak{b}$



Non-trivial direction of Boolean extensionality

- $\forall A_o, B_o.(A \Leftrightarrow B) \supset A \stackrel{*}{=} B$

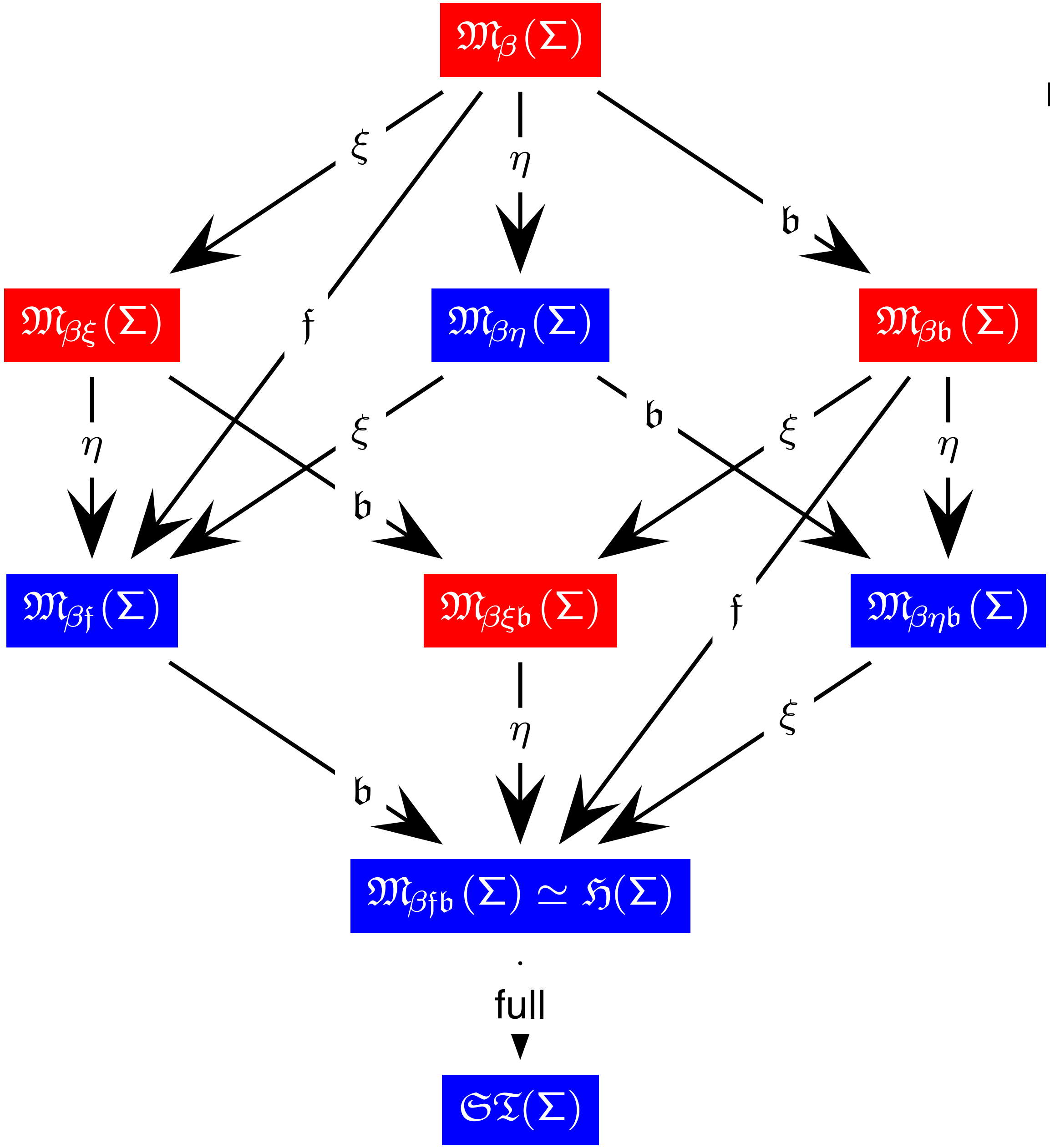
Other HOL Test Problems: $\mathfrak{f}$



Non-trivial direct. of functional extensionality

- $\forall F_{\beta\alpha}, G_{\beta\alpha}\mathbf{.}(\forall X_\alpha\mathbf{.}FX \stackrel{*}{\doteq} GX) \supset F \stackrel{*}{\doteq} G$

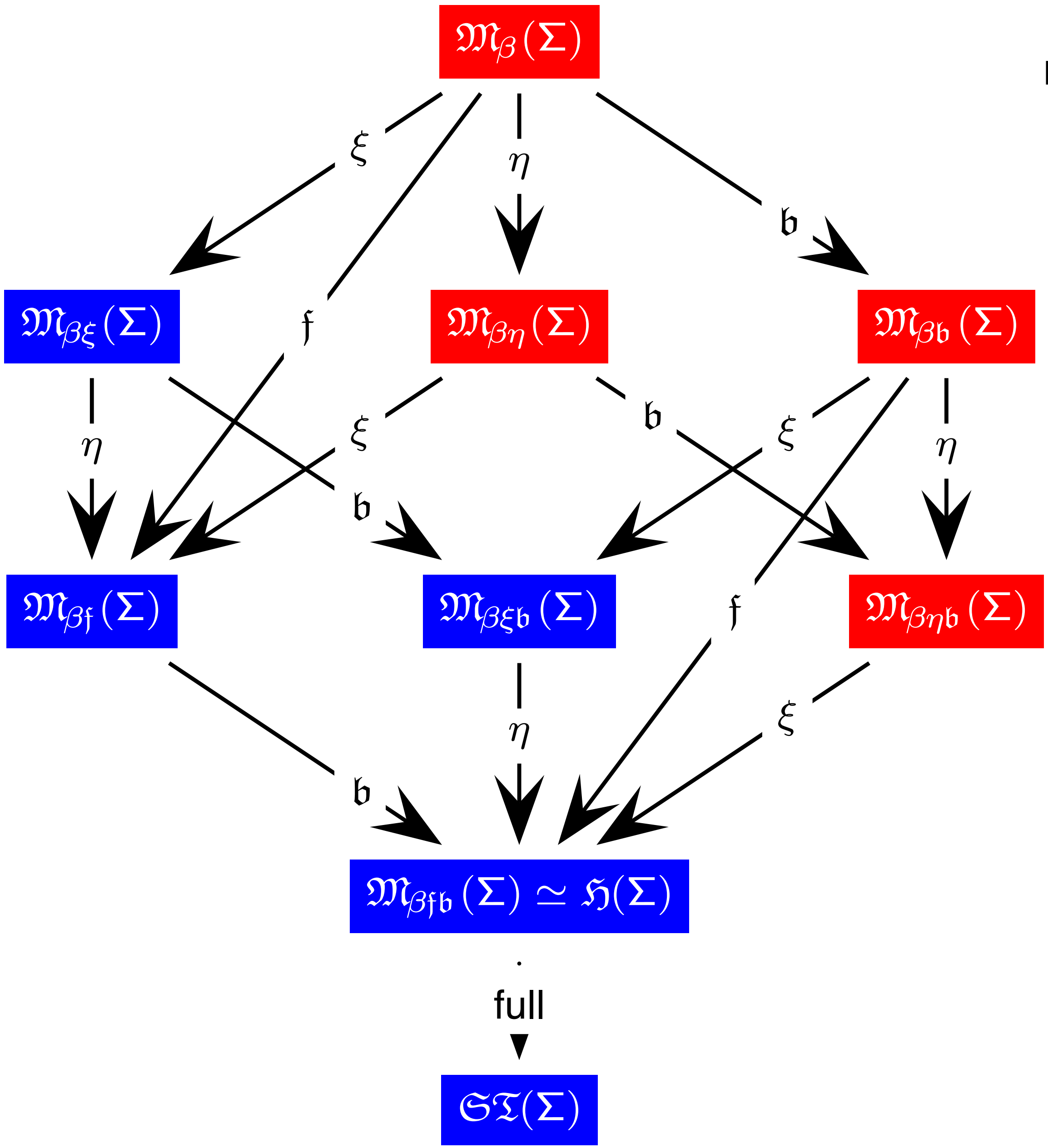
Other HOL Test Problems: η



Example requiring property η

- $(p_{o(\iota\iota)}(\lambda X_\iota . f_{\iota\iota} X)) \supset (p\ f)$

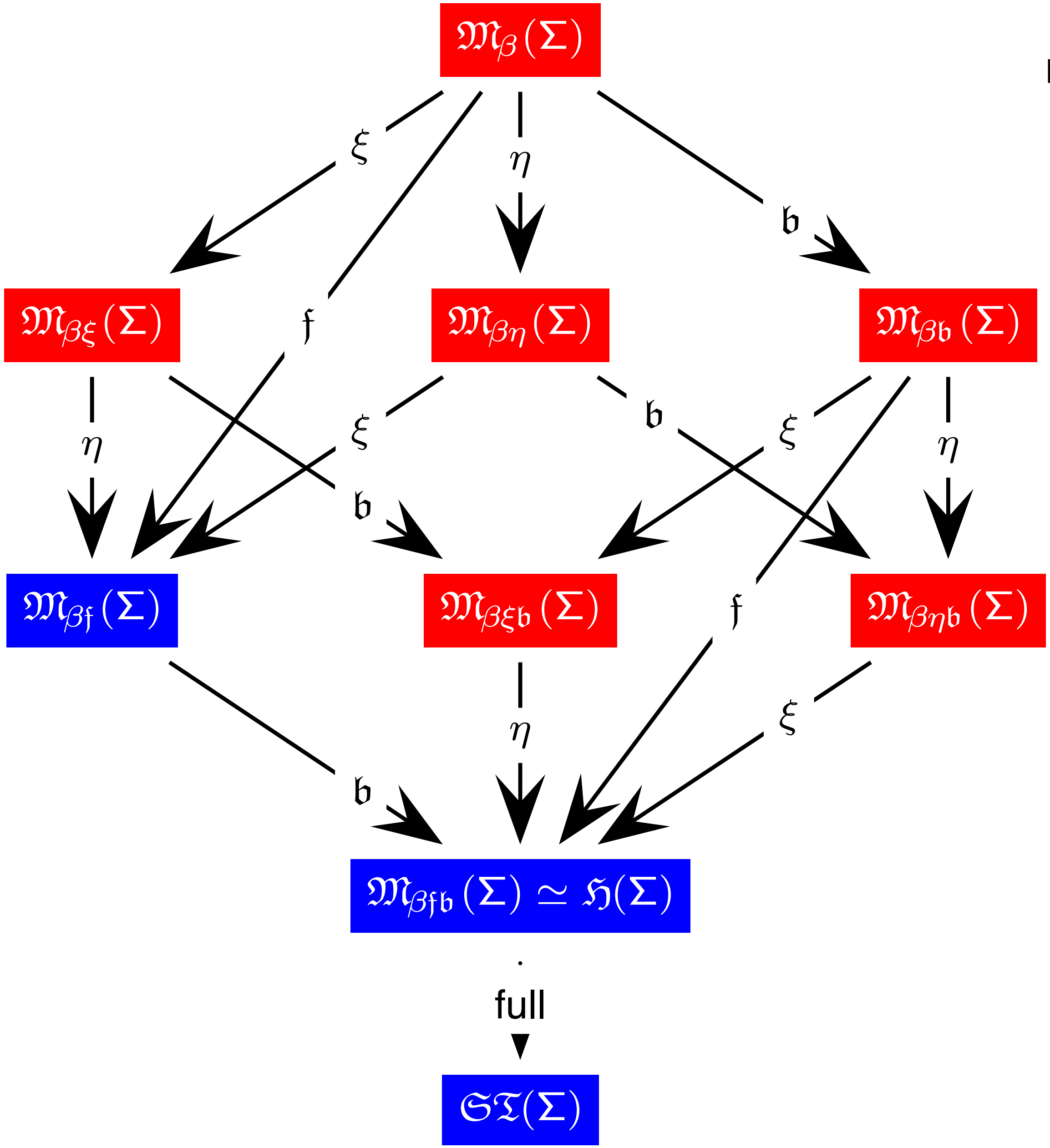
Other HOL Test Problems: ξ



Example requiring property ξ (and $\mathfrak{q}$!)

- $(\forall X_\iota . (f_{\iota\iota} X) \stackrel{*}{\doteq} X) \wedge p_{o(\iota\iota)}(\lambda X_\iota . X)$
  $\supset p(\lambda X_\iota . fX)$

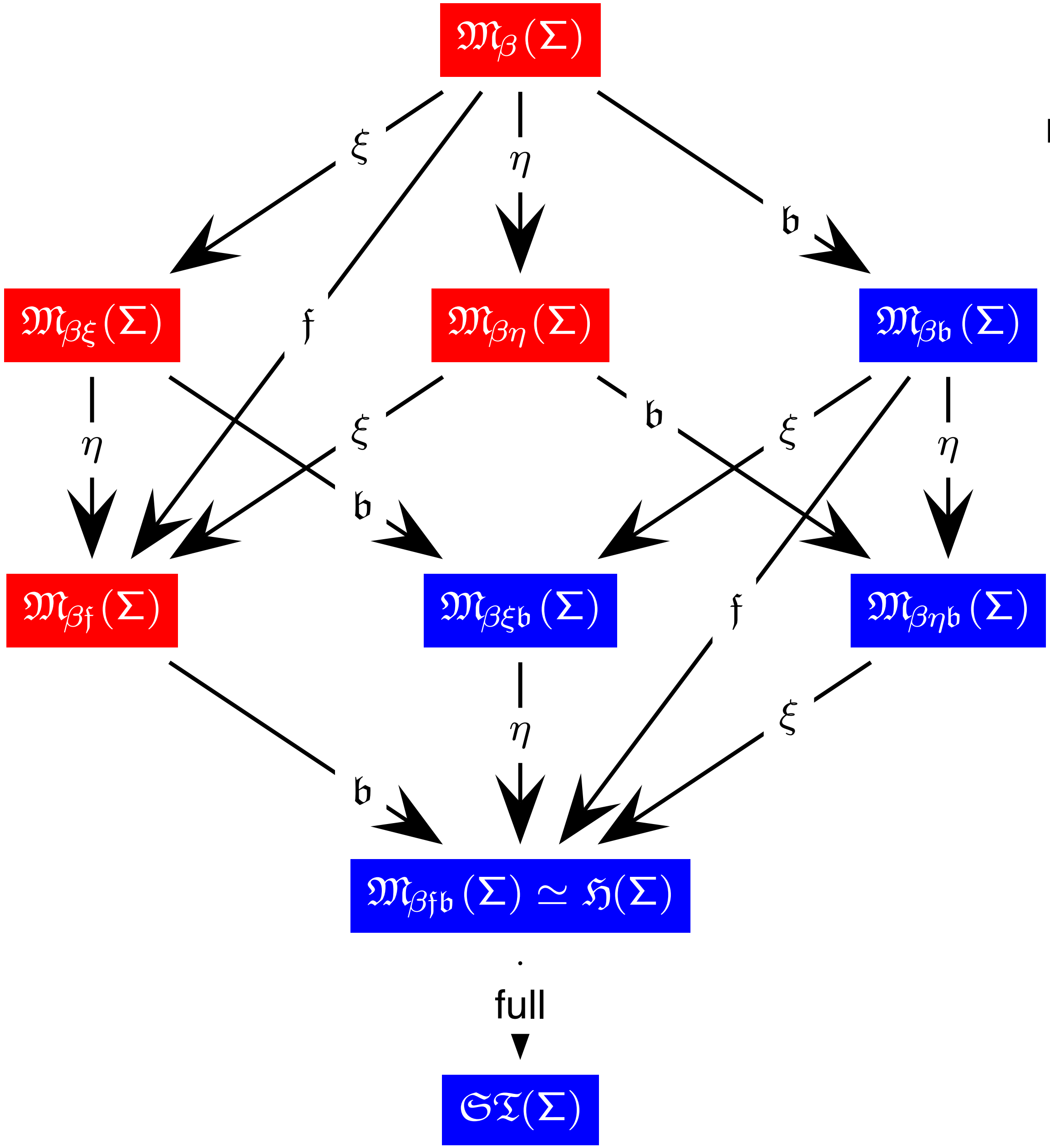
Other HOL Test Problems: $\mathfrak{f}$



Example requiring property $\mathfrak{f}$ (and $\mathfrak{q}$!)

- $(\forall X_\iota \centerdot (f_{\iota\iota} X) \stackrel{*}{\doteq} X) \land p_{o(\iota\iota)} (\lambda X_\iota \centerdot X) \supset (p\ f)$

Other HOL Test Problems: $\mathfrak{b}$



Examples requiring property $\mathfrak{b}$

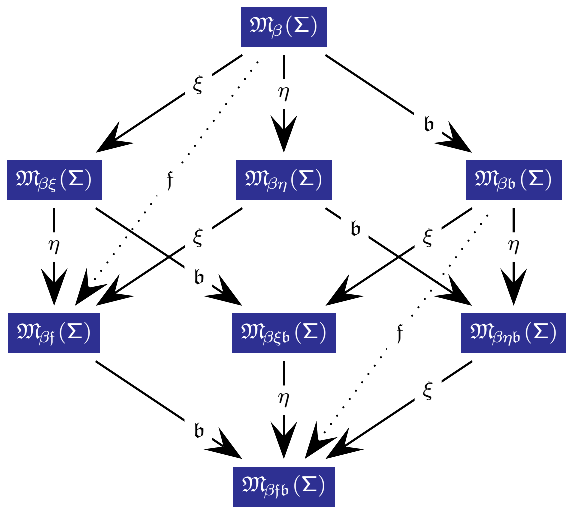
- $(p_{oo}\ a_o) \wedge (p\ b_o) \Rightarrow (p\ (a \wedge b))$
- $\neg(a \stackrel{*}{=} \neg a)$ (in particular $\neg(a = \neg a)$)
- $(h_{\iota o}((h\top) \stackrel{*}{=} (h\bot))) \stackrel{*}{=} (h\bot)$

Abstract Consistency

Semantics - Calculi - Abstract Consistency

Semantics:

Model Classes (Extensionality)



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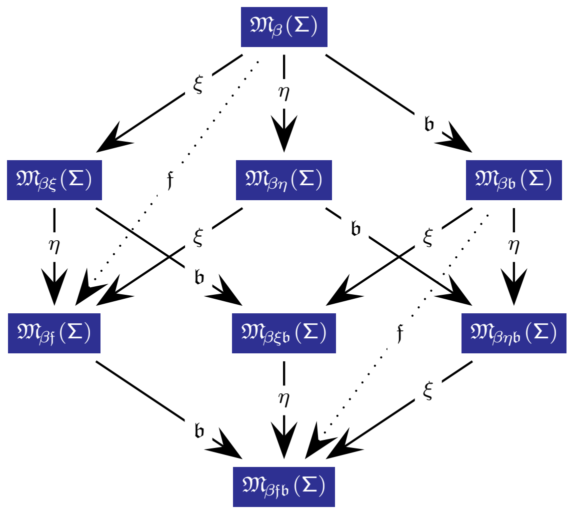
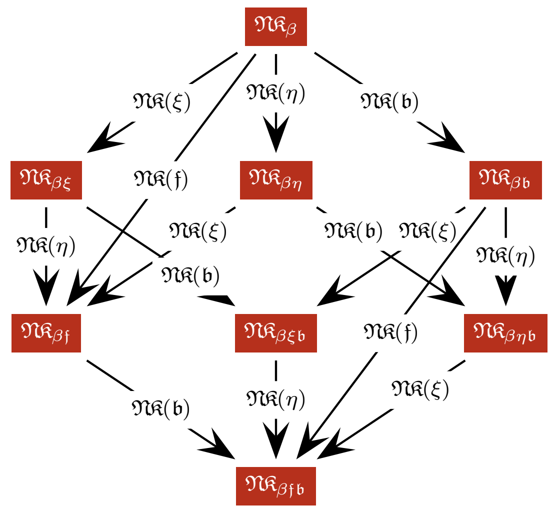


Reference Calculi:

ND (and others)



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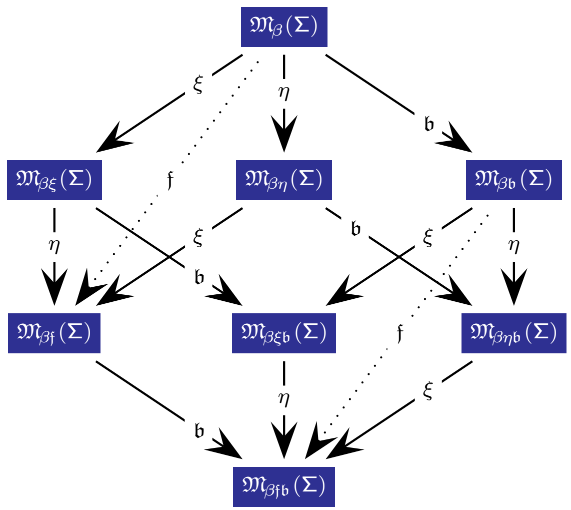

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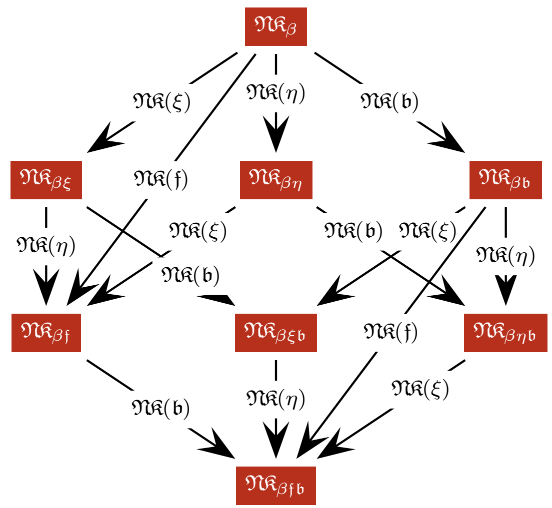
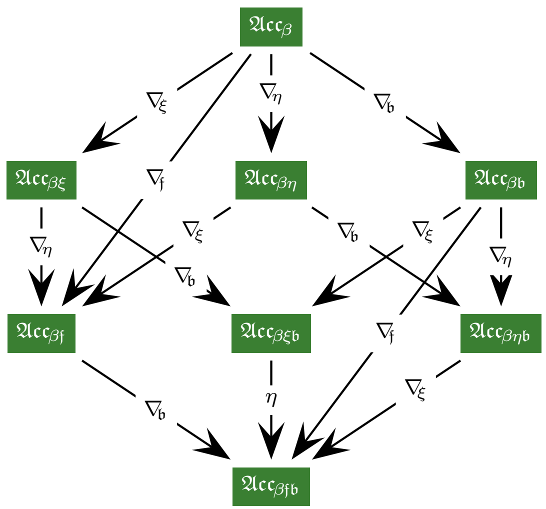


Abstract Consistency / Unifying Principle:
Extensions of Smullyan-63 and Andrews-71



Abstract Consistency: History

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- The technique has been extended to our landscape of HOL model classes in [Chris-PhD-99,Chad-PhD-04,JSL-04].

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  If a set of sentences Φ of L is a member of an (saturated) abstract consistency class Γ, then there exists a model for Φ.

Abstract Consistency: Idea

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  - For many calculi $\mathcal{C}$, this also shows A is provable, thus establishing completeness of $\mathcal{C}$.

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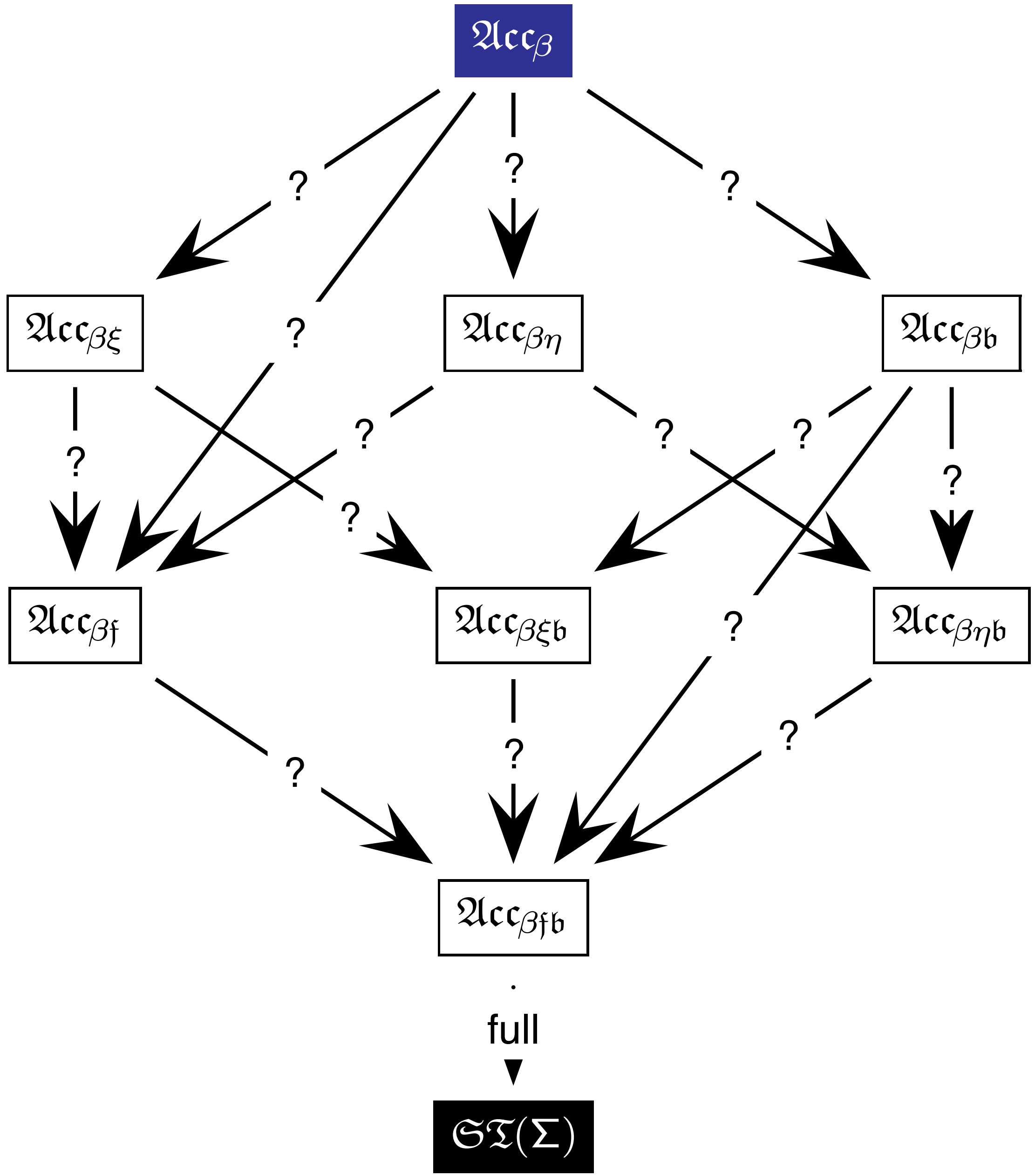

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(These properties are going back to Hintikka, Smullyan, and Andrews)

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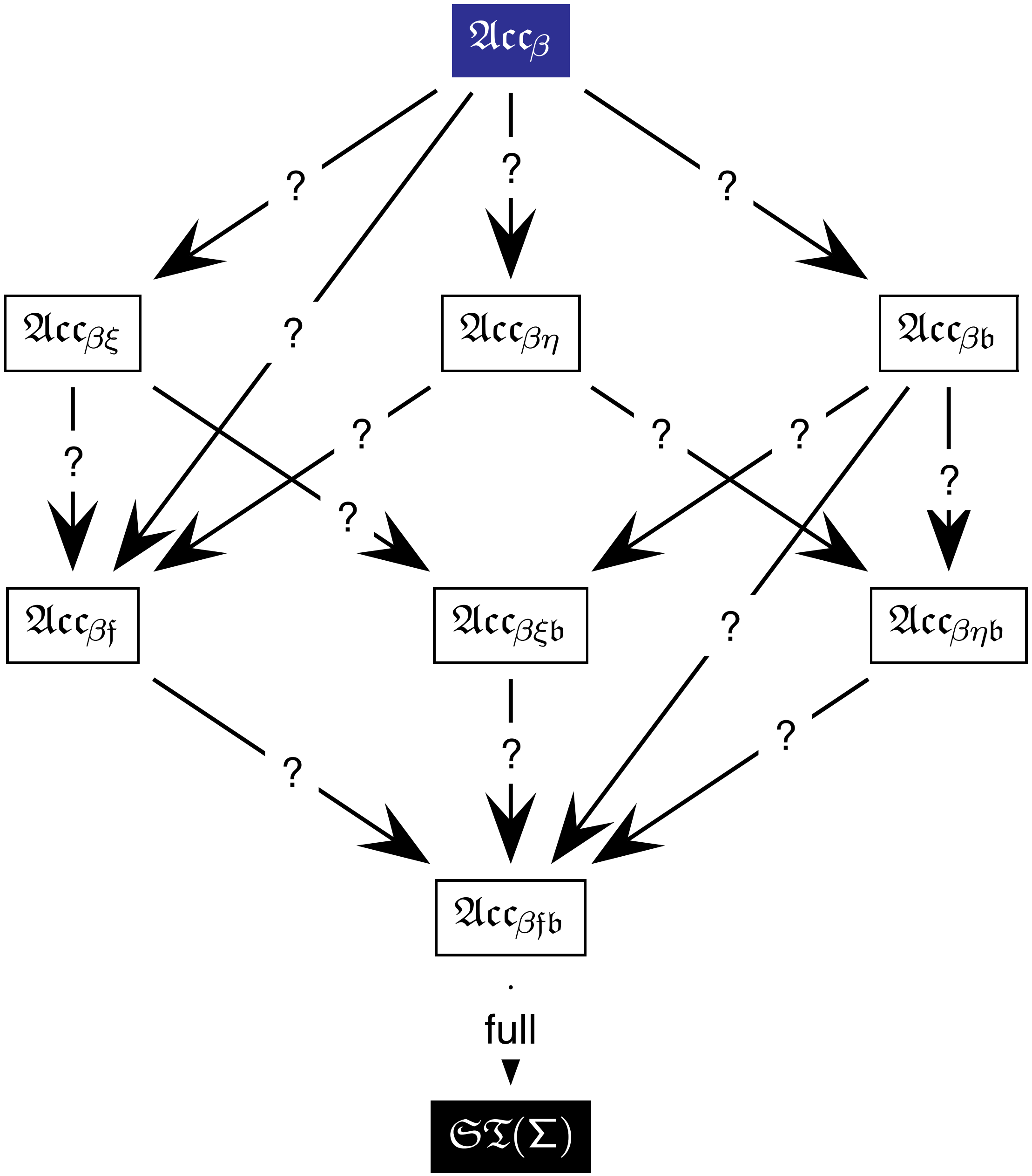

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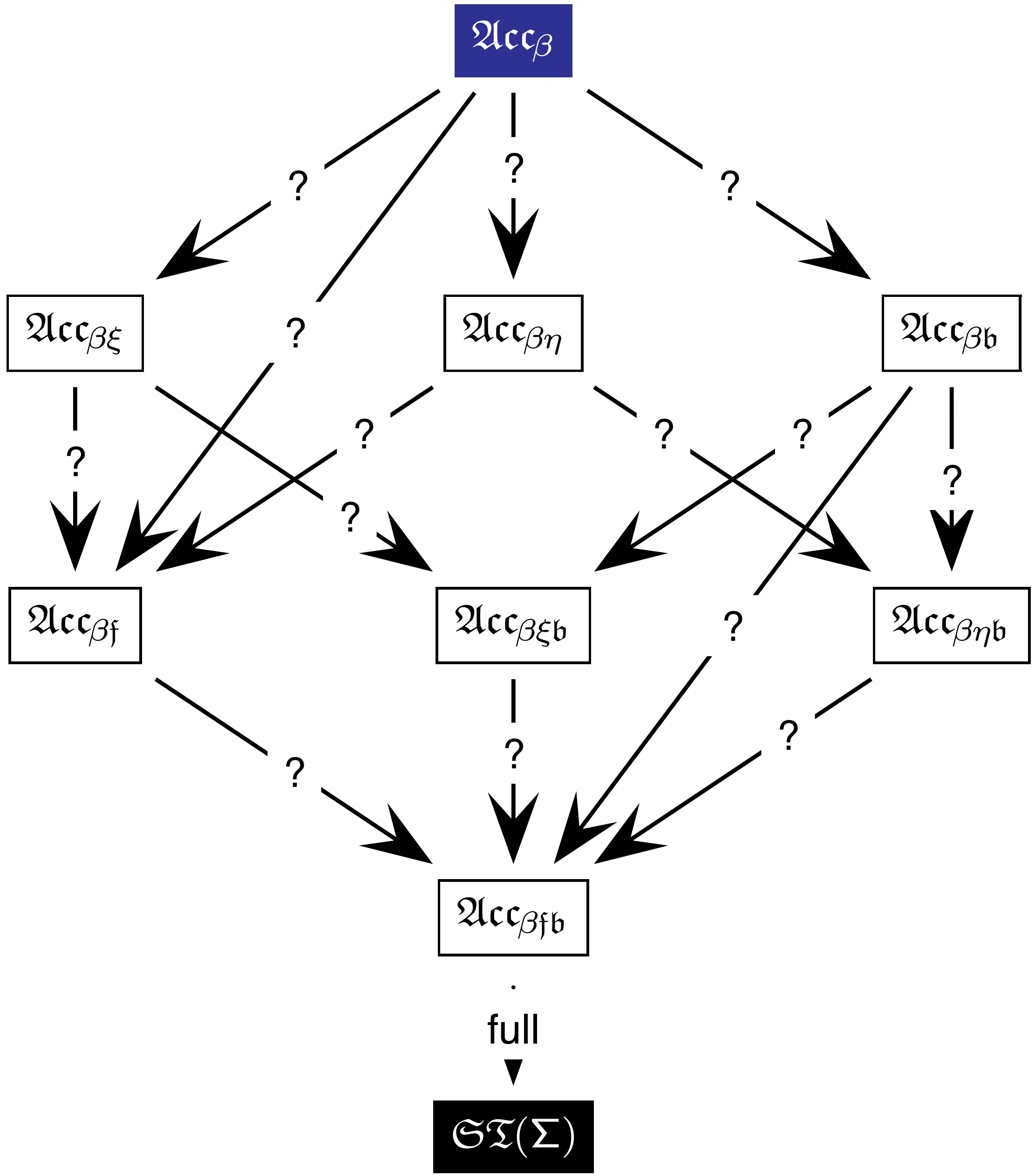

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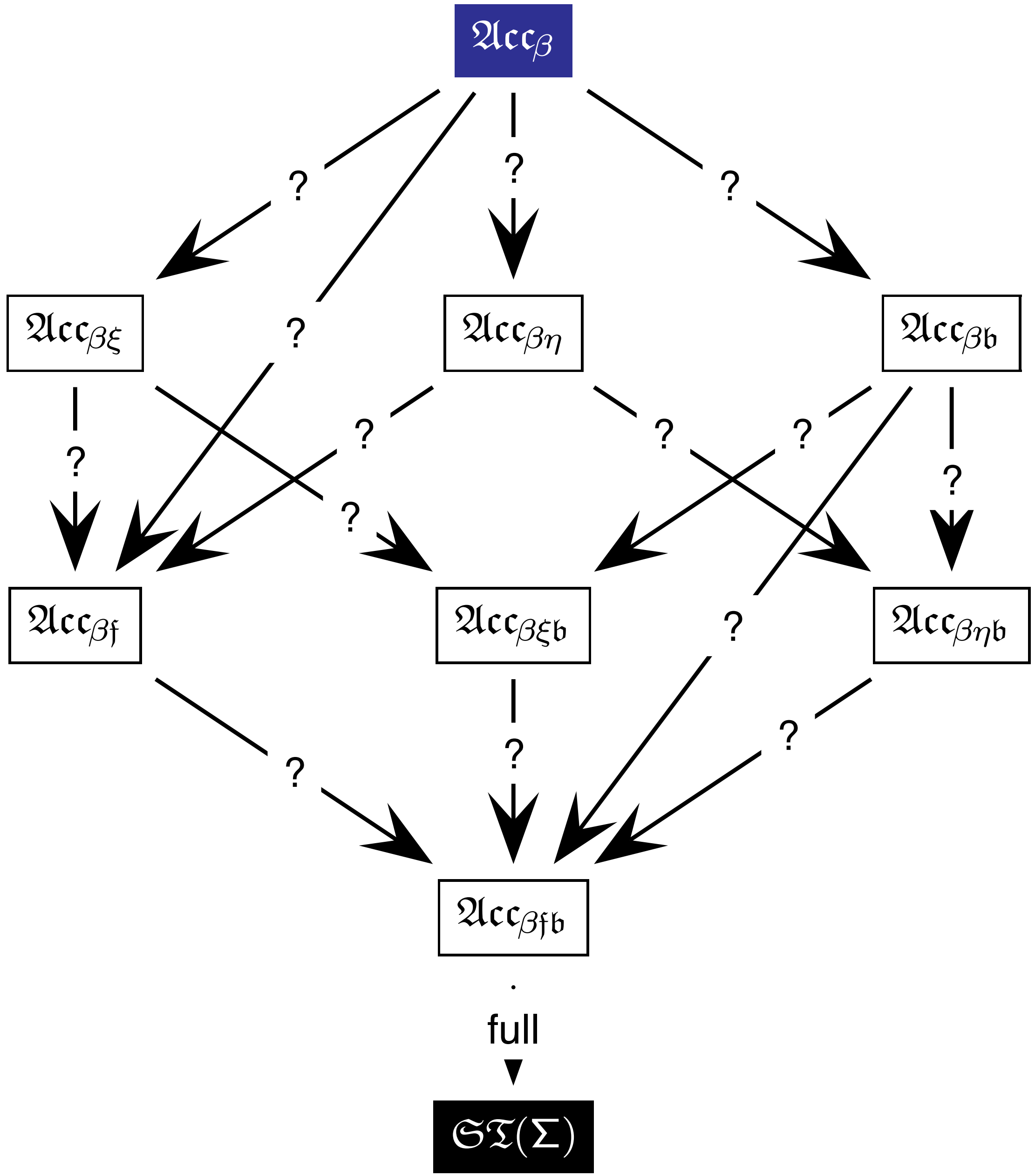

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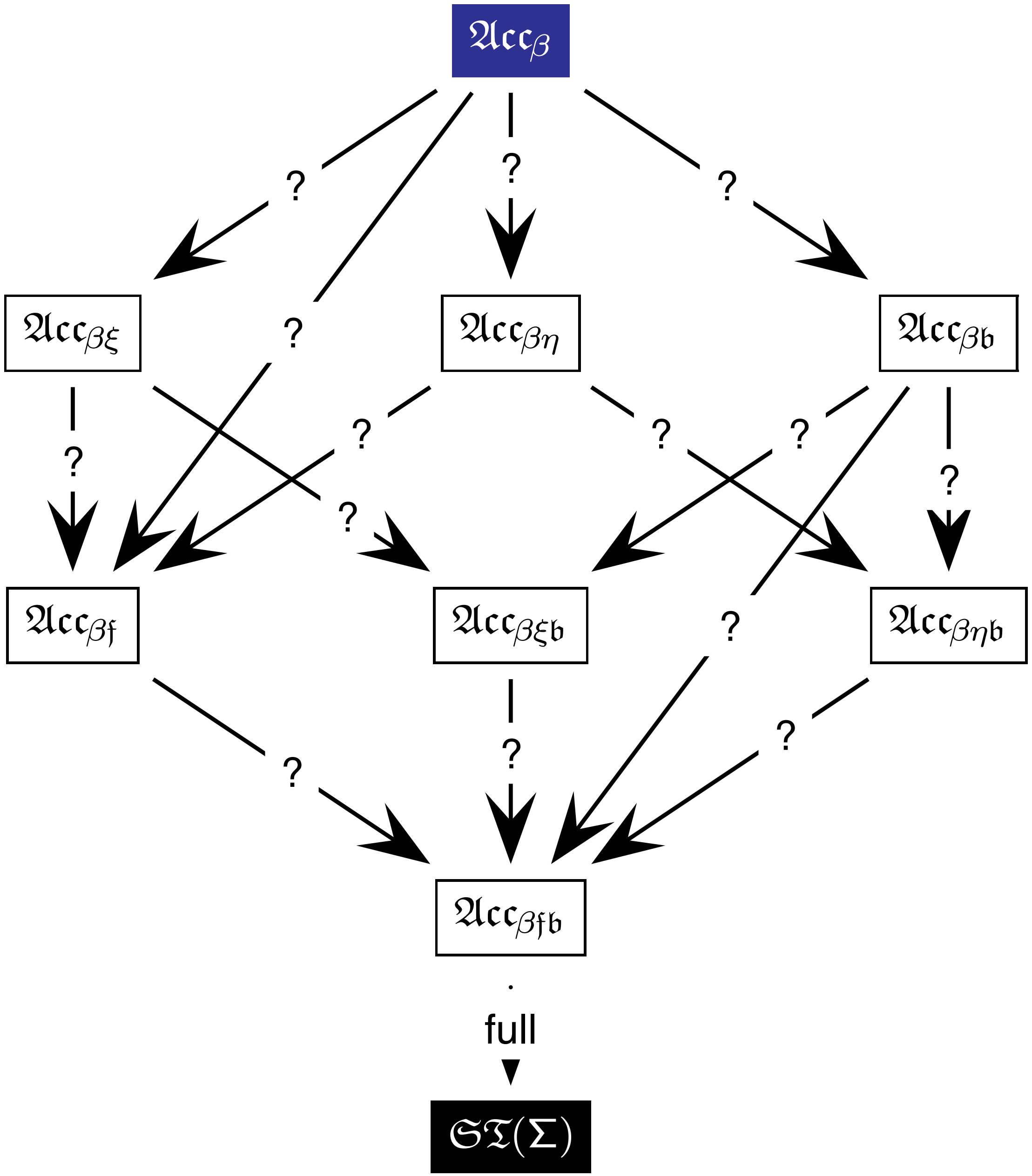

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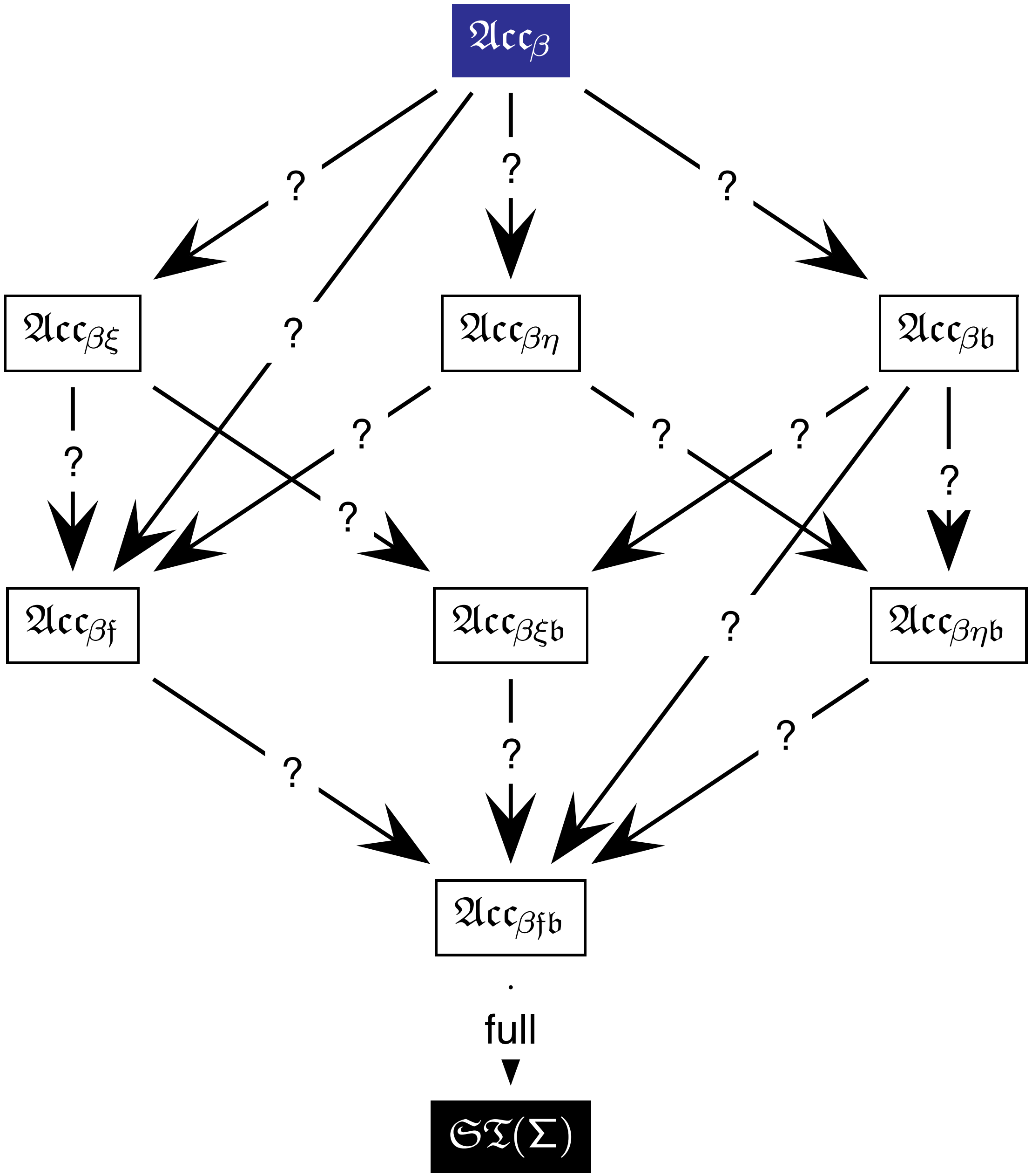

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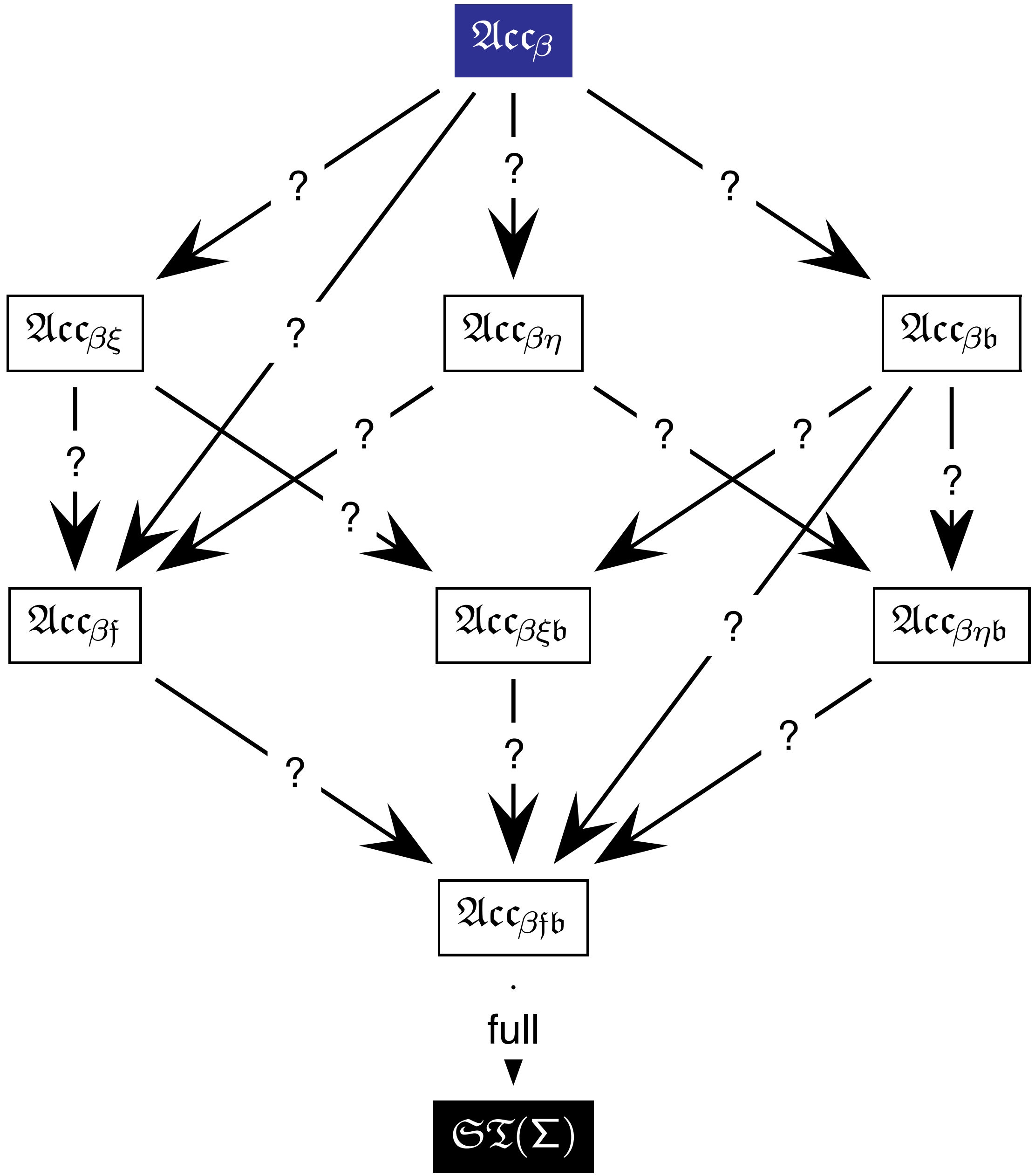

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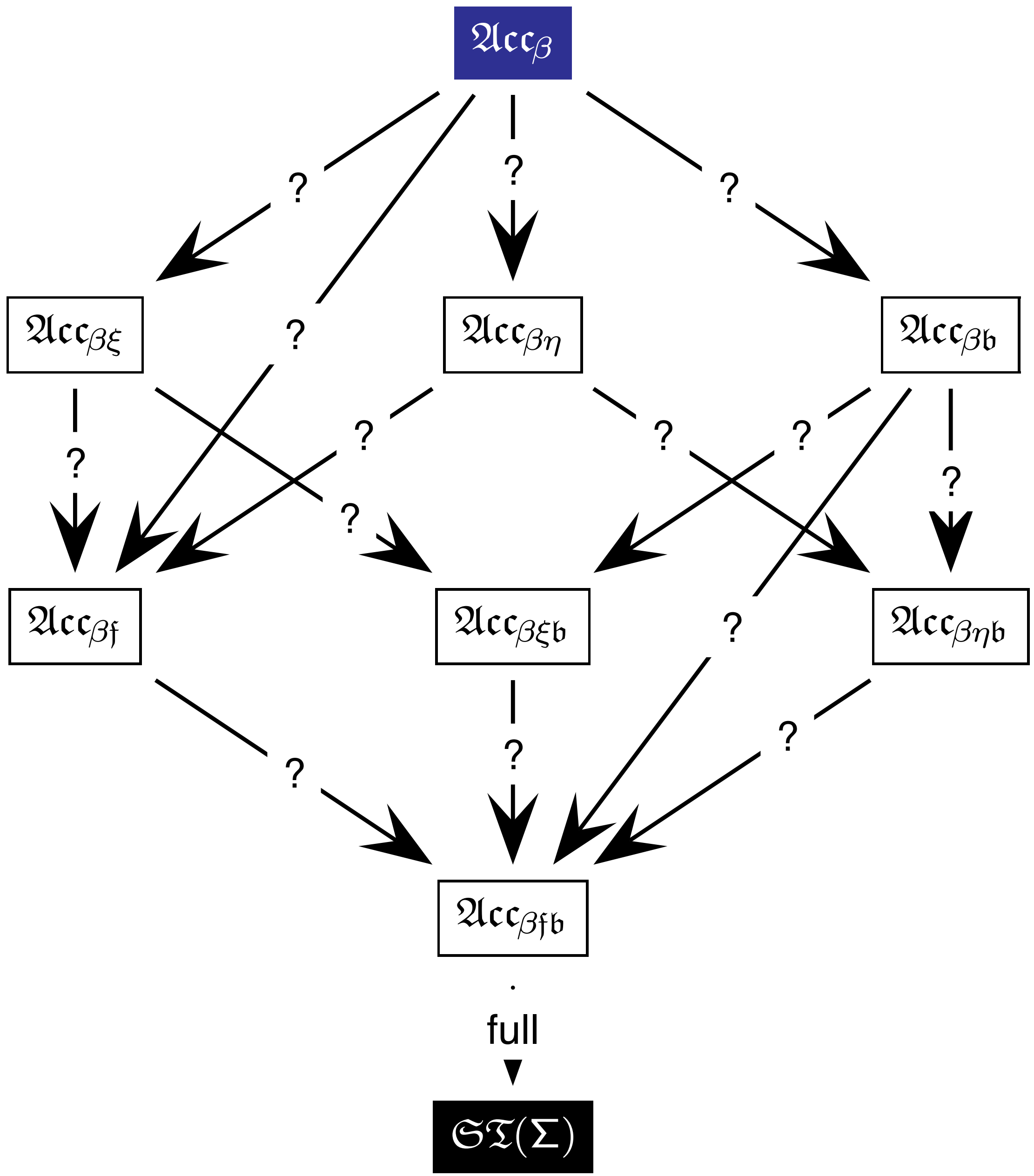

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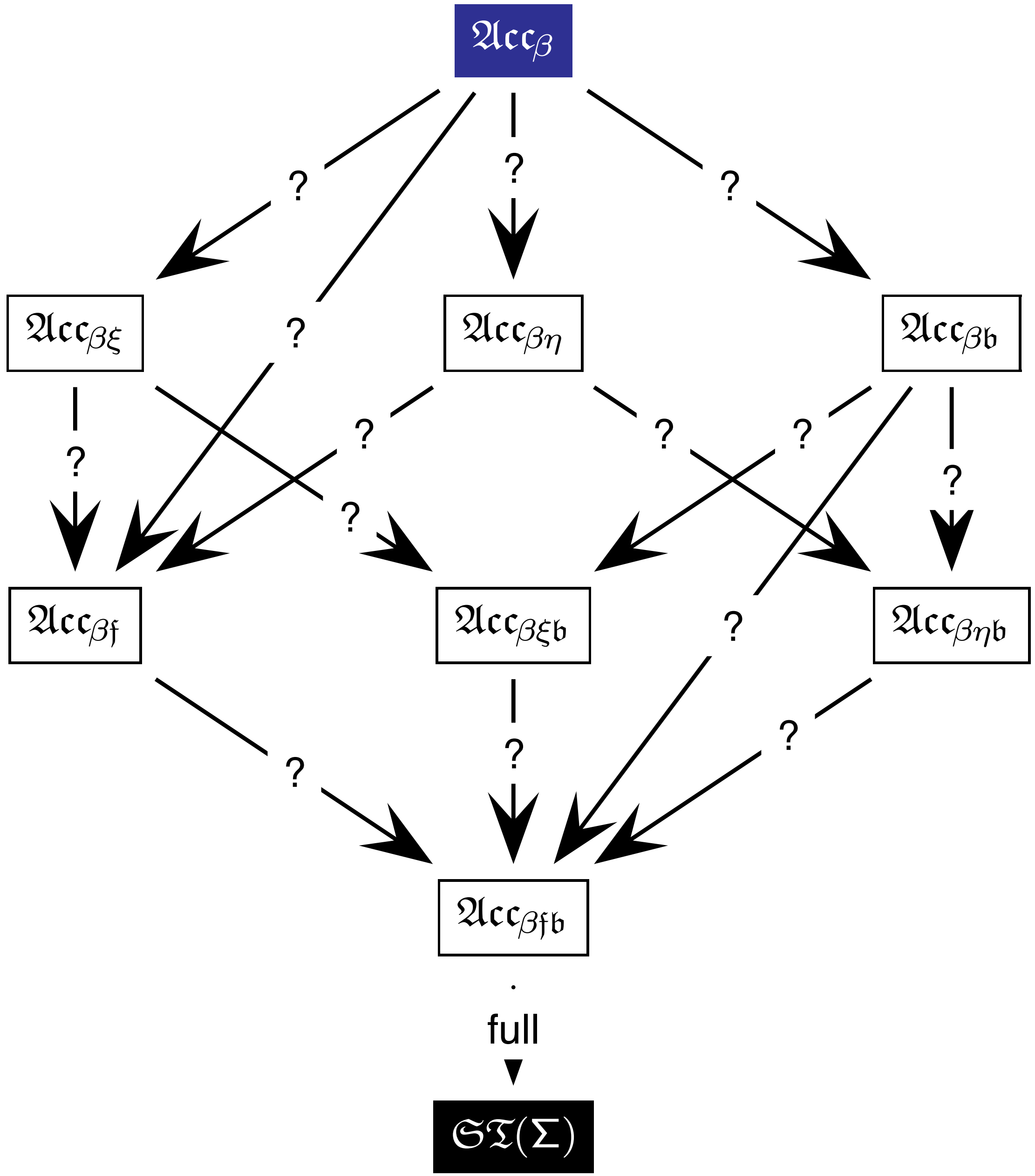

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Let Γ_Σ be a class of sets of Σ-sentences. We define (where $\Phi \in \Gamma_\Sigma$, $\alpha, \beta \in \mathcal{T}$, $\mathbf{A}, \mathbf{B} \in cwff_o(\Sigma)$, $\mathbf{G}, \mathbf{H}, (\lambda X_\alpha \mathbf{.} \mathbf{M}), (\lambda X_\alpha \mathbf{.} \mathbf{N}) \in cwff_{\alpha\to\beta}(\Sigma)$ are arbitrary):

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(These properties are new in [Chris-PhD-99,Chad-PhD-04,JSL-04])

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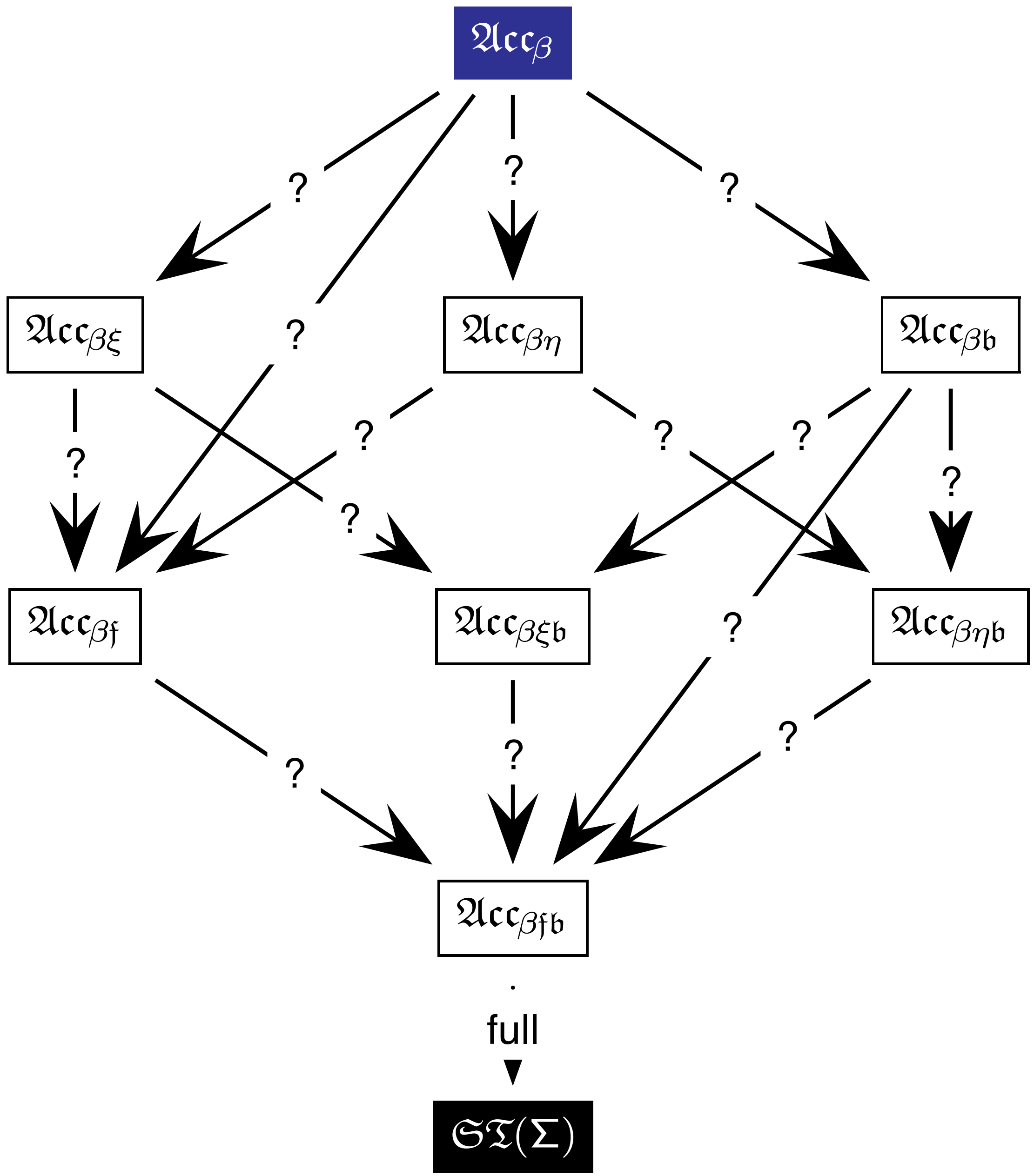

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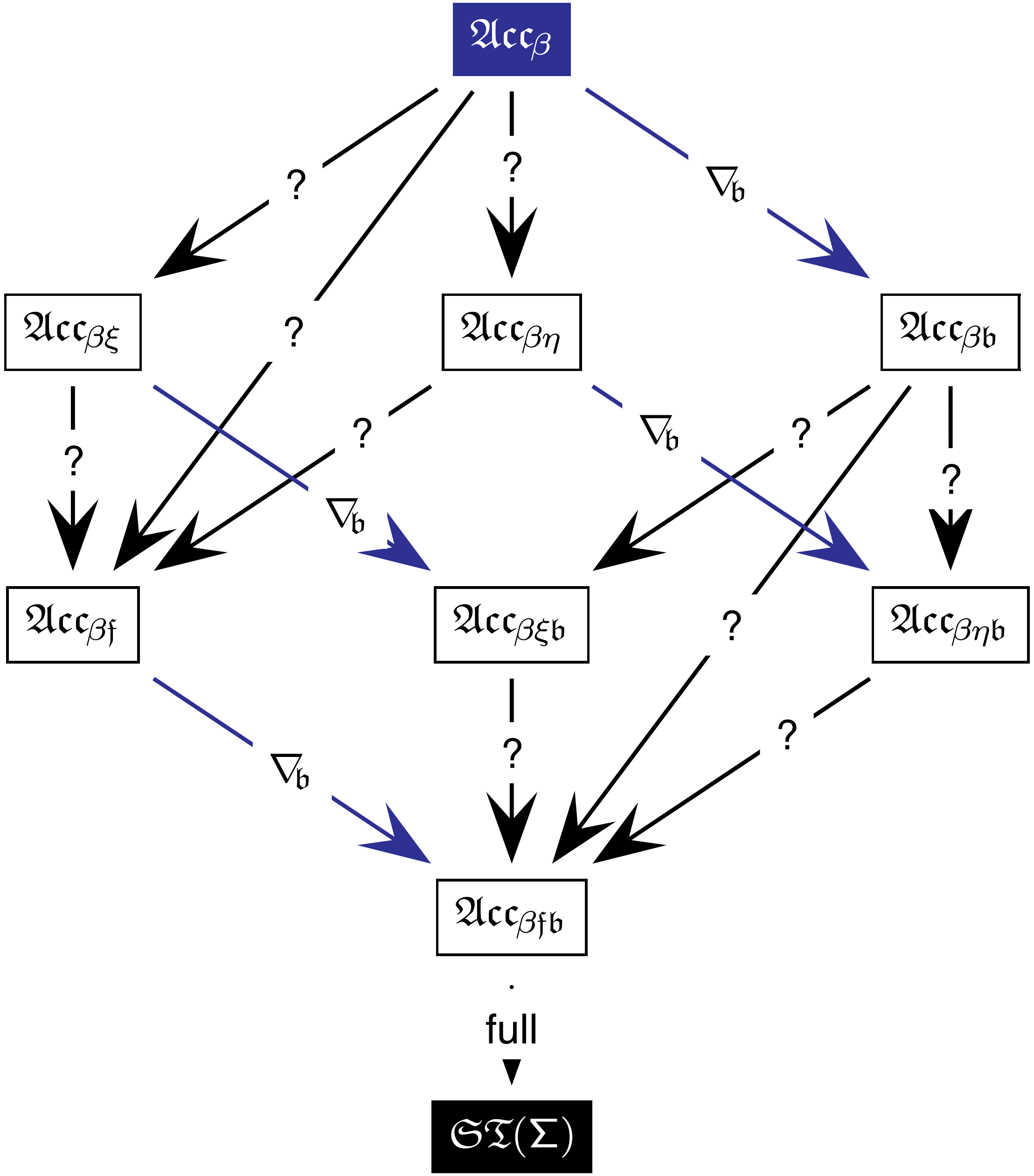

Basic AC Properties for $\mathfrak{Acc}_\beta$

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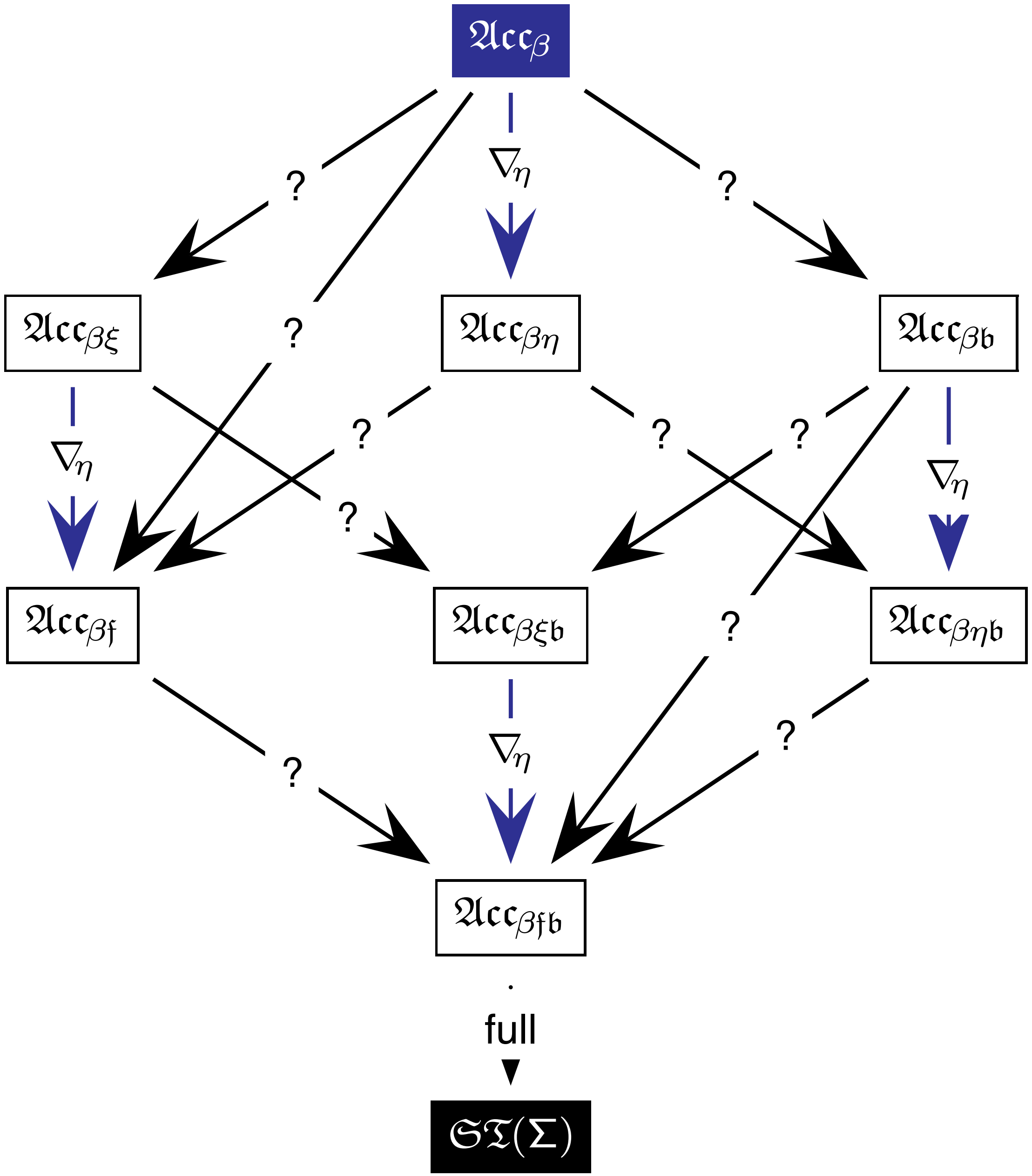

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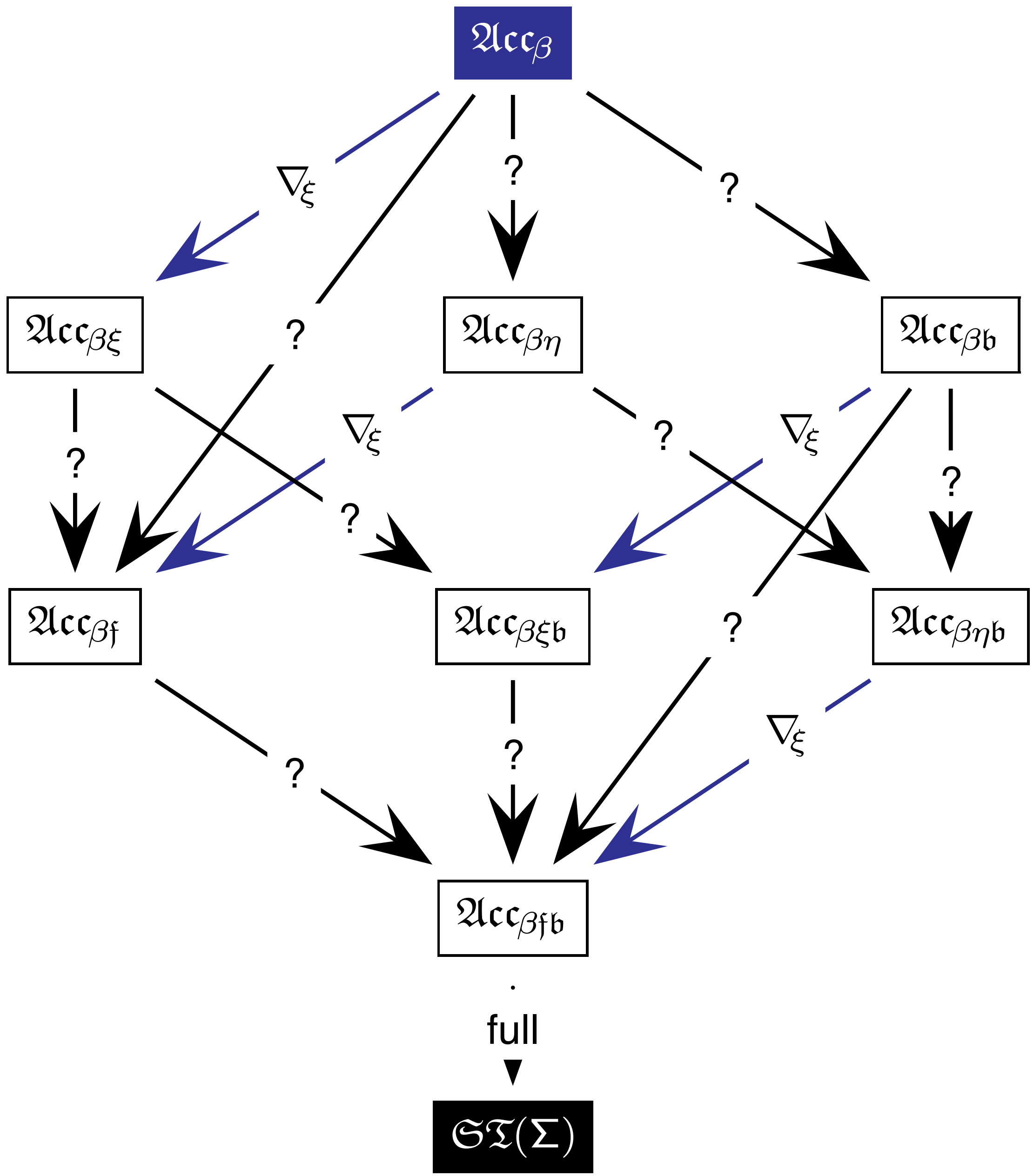

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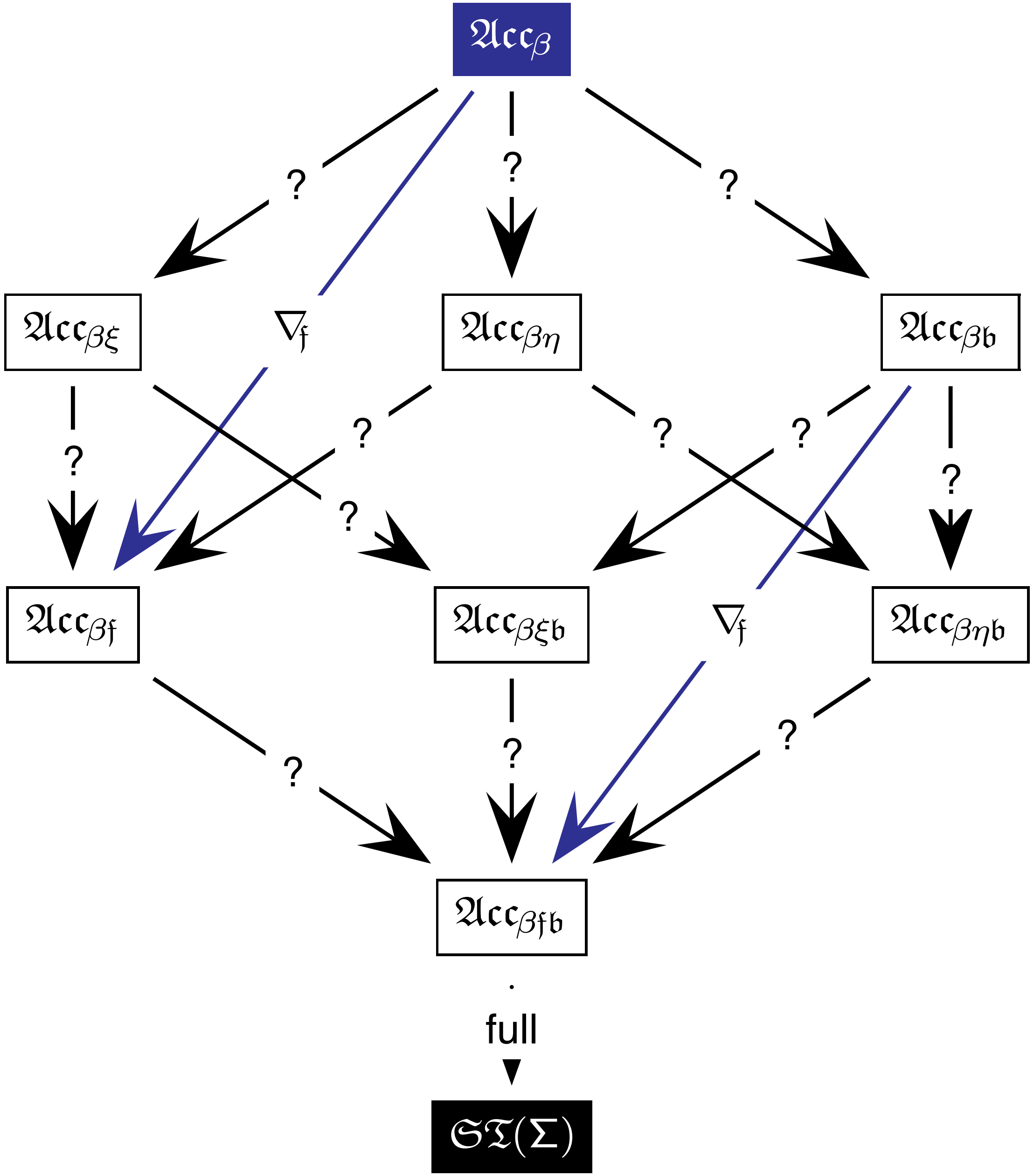

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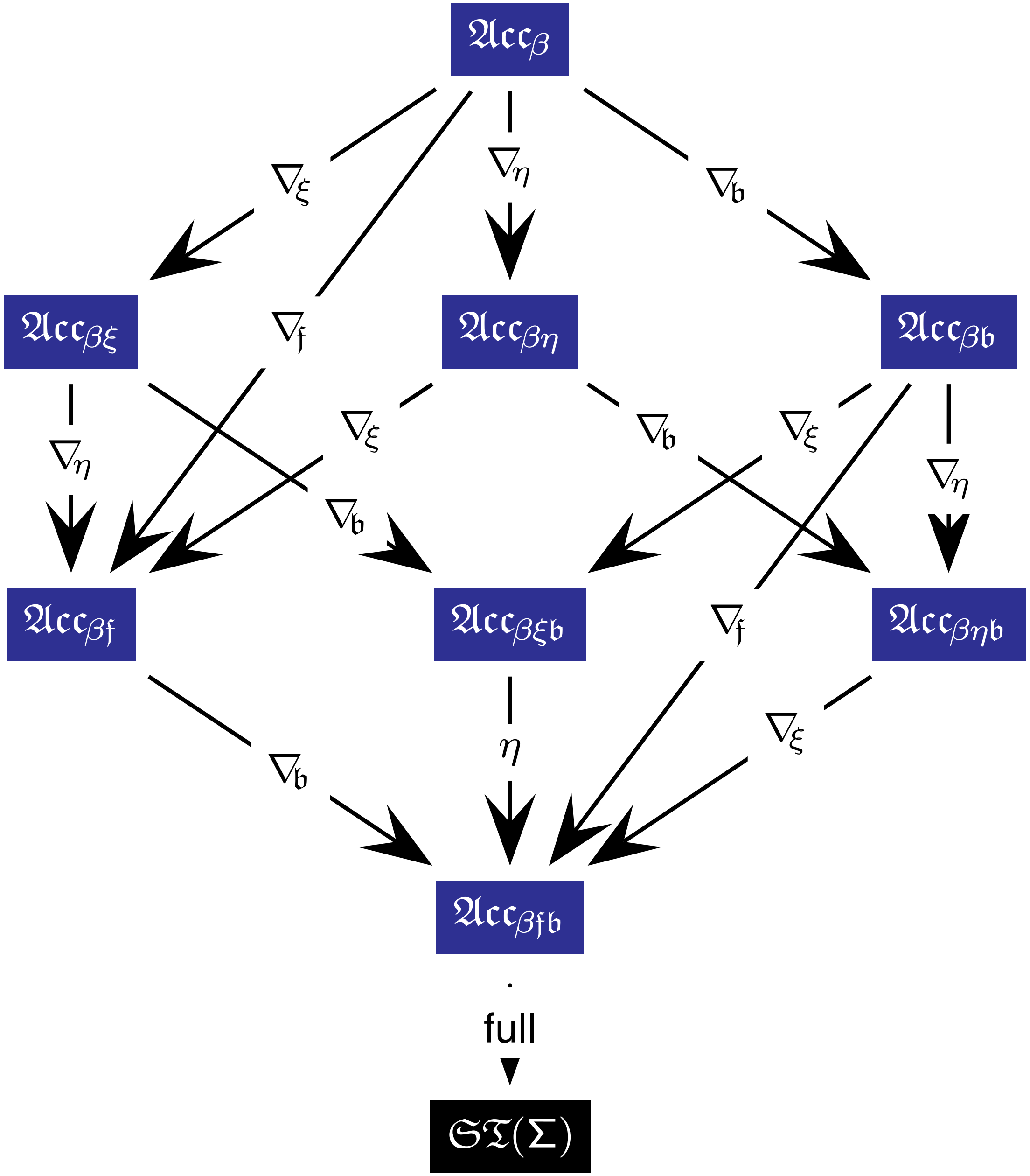

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Abstract Consistency Class $\mathfrak{Acc}_\beta$

Defn.: (Contd.) Let Σ be a signature and Γ_Σ be a class of sets of Σ-sentences that is closed under subsets.

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- Similarly, we introduce the following collections of specialized abstract consistency classes (with primitive equality): $\mathfrak{Acc}_{\beta\eta}, \mathfrak{Acc}_{\beta\xi}, \mathfrak{Acc}_{\beta\mathfrak{f}}, \mathfrak{Acc}_{\beta\mathfrak{b}}, \mathfrak{Acc}_{\beta\eta\mathfrak{b}}, \mathfrak{Acc}_{\beta\xi\mathfrak{b}}, \mathfrak{Acc}_{\beta\mathfrak{f}\mathfrak{b}}$, where we indicate by indices which additional properties from $\{\nabla_\eta, \nabla_\xi, \nabla_\mathfrak{f}, \nabla_\mathfrak{b}\}$ are required.

Ex.: Abstract Consistency Class

- not an abstract consistency class:
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- and how about this:
  $\Gamma_0 := \Gamma$
  $\Phi \in \Gamma_i \wedge A \in \Phi \wedge B =_{\beta\eta} A \wedge B \neq A \wedge (\Phi * B) \notin \Gamma_i \longrightarrow$
  $\Gamma_{i+1} :=$ close-under-subsets$(\Gamma_i * (\Phi * B))$
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We introduce a technical side-condition that ensures that we always have enough witness constants.

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> Let Σ be a signature and Φ be a set of Σ-sentences. Φ is called sufficiently Σ-pure if for each type α there is a set $\mathcal{P}_\alpha \subseteq \Sigma_\alpha$ of parameters with equal cardinality to $\mathit{wff}_\alpha(\Sigma)$, such that the elements of $\mathcal{P}_\alpha$ do not occur in the sentences of Φ.

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This can be obtained in practice by enriching the signature with spurious parameters.

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- We call an abstract consistency class Γ_Σ saturated if ∇_{sat} holds for all A.

Ex.: Saturated

- consider Γ (and Γ^*) from before: $\{\{\neg(A \lor B), \neg A, \neg B\}, \{\neg(A \lor B), \neg A\}, \{\neg(A \lor B), \neg B\}, \{\neg A, \neg B\}, \{\neg(A \lor B)\}, \{\neg A\}, \{\neg B\}, \{\}\}$

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- Γ (and Γ^*) is not saturated: for instance, it does not provide information on the formulas $(\neg A \vee B) \vee A$ and $\Pi^o(\lambda X_o.X)$

Def.: Saturated Extension

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Let $\Gamma_\Sigma, \Gamma'_\Sigma \in \mathfrak{Acc}_*$ be abstract consistency classes. We say Γ'_Σ is an extension of Γ_Σ if $\Phi \in \Gamma'_\Sigma$ for every (sufficiently Σ-pure) $\Phi \in \Gamma_\Sigma$. We say Γ'_Σ is a saturated extension of Γ_Σ if Γ'_Σ is saturated and an extension of Γ_Σ.

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Let $a_o, b_o, q_{o\to o} \in \Sigma$ and $\Phi := \{a, b, (qa), \neg(qb)\}$. We construct an abstract consistency class Γ_Σ from Φ by first building the closure Φ' of Φ under relation $=_\beta$ and then taking the power set of Φ'. It is easy to check that this Γ_Σ is in $\mathfrak{Acc}_{\beta\mathfrak{fb}}$. Suppose we have a saturated extension Γ'_Σ of Γ_Σ in $\mathfrak{Acc}_{\beta\mathfrak{fb}}$. Then $\Phi \in \Gamma'_\Sigma$ since Φ is finite (hence sufficiently pure) . By saturation, $\Phi * (a \doteq^o b) \in \Gamma'_\Sigma$ or $\Phi * \neg(a \doteq^o b) \in \Gamma'_\Sigma$. In the first case, applying $\nabla_\forall$ with the constant q, ∇_β, $\nabla_\vee$ and ∇_c contradicts $(qa), \neg(qb) \in \Phi$.

Ex.: ACC without Saturated Extension

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Model Existence Theorem

Thm.: Let Γ_Σ be a saturated abstract consistency class and let $\Phi \in \Gamma_\Sigma$ be a sufficiently Σ-pure set of sentences.
For all $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ we have:

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Proof: (Sketch)

. . . not yet . . .

Completeness of $\mathfrak{NK}_*$ via Abstract Consistency

$\mathfrak{M}\mathfrak{K}_*$-Consistent/Inconsistent

Def.: A set of sentences Φ is $\mathfrak{M}\mathfrak{K}_*$-inconsistent if $\Phi \Vdash_{\mathfrak{M}\mathfrak{K}_*} \mathbf{F}_o$, and $\mathfrak{M}\mathfrak{K}_*$-consistent otherwise.

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- We will now consider the class

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- i.e.

$$\Gamma_\Sigma^* := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \nVdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$$

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \nVdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma_\Sigma^* := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We first show: Γ_Σ^* is closed under subsets)

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We first show: Γ^*_Σ is closed under subsets)

Obviously Γ^*_Σ is closed under subsets, since any subset of an $\mathfrak{NK}_*$-consistent set is $\mathfrak{NK}_*$-consistent. (If $\Psi \subseteq \Phi$ and $\Psi \Vdash_{\mathfrak{NK}_*} \mathbf{F}_o$ then clearly $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{F}_o$)

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We now show: $\nabla_c, \nabla_\neg, \nabla_\beta, \nabla_\vee, \nabla_\wedge, \nabla_\forall, \nabla_\eta, \nabla_\xi, \nabla_\mathfrak{f}, \nabla_\mathfrak{b}, \nabla_{sat}$)

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We now check the remaining conditions. We prove all the properties by proving their contrapositive.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \nVdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: ∇_c)

Class of Sets of $\mathfrak{N}\mathfrak{K}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \nVdash_{\mathfrak{N}\mathfrak{K}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: ∇_c)

∇_c If $\mathbf{A}$ is atomic, then $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

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Suppose $\mathbf{A}, \neg\mathbf{A} \in \Phi$.

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Suppose $\mathbf{A}, \neg\mathbf{A} \in \Phi$.

$$\dfrac{\dfrac{}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(Hyp) \qquad \dfrac{}{\Phi \Vdash \neg\mathbf{A}}\,\mathfrak{NK}(Hyp)}{\Phi \Vdash \mathbf{F}_o}\,\mathfrak{NK}(\neg E)$$

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

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Hence Φ is $\mathfrak{NK}_*$-inconsistent.

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Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: ∇_β)

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Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: ∇_β)

∇_β If $\mathbf{A} =_\beta \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{B} \in \Gamma^*_\Sigma$.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma_\Sigma^* := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

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Let $\mathbf{A} \in \Phi$, $\mathbf{A} =_\beta \mathbf{B}$ and $\Phi * \mathbf{B}$ be $\mathfrak{NK}_*$-inconsistent.

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Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

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Let $\mathbf{A} \in \Phi$, $\mathbf{A}{=}_\beta\mathbf{B}$ and $\Phi * \mathbf{B}$ be $\mathfrak{NK}_*$-inconsistent. That is, $\Phi * \mathbf{B} \Vdash \mathbf{F}_o$.

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Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

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Let $\mathbf{A} \in \Phi$, $\mathbf{A} =_\beta \mathbf{B}$ and $\Phi * \mathbf{B}$ be $\mathfrak{NK}_*$-inconsistent. That is, $\Phi * \mathbf{B} \Vdash \mathbf{F}_o$.

$$\boxed{\nabla_\beta} \qquad \dfrac{\dfrac{\Phi * B \Vdash F_o}{\Phi \Vdash \neg B}\,nk(\neg I) \qquad \dfrac{\dfrac{}{\Phi \Vdash A}\,nk(Hyp)}{\Phi \Vdash B}\,nk(\beta)}{\Phi \Vdash F_o}\,nk(\neg E)$$

Class of Sets of $\mathfrak{N}\mathfrak{K}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{N}\mathfrak{K}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\neg$)

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

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Proof: (We show: $\nabla_\neg$)

$\nabla_\neg$ If $\neg\neg\mathbf{A} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma^*_\Sigma$.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma_\Sigma^* := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\neg$)

$\nabla_\neg$ If $\neg\neg\mathbf{A} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma^*$.

Suppose $\neg\neg\mathbf{A} \in \Phi$ and $\Phi * \mathbf{A}$ is $\mathfrak{NK}_*$-inconsistent.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\neg$)

$\nabla_\neg$ If $\neg\neg\mathbf{A} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma^*_\Sigma$.

Suppose $\neg\neg\mathbf{A} \in \Phi$ and $\Phi * \mathbf{A}$ is $\mathfrak{NK}_*$-inconsistent.

$$\boxed{\nabla_\neg} \qquad \dfrac{\dfrac{\Phi * A \Vdash F_o}{\Phi \Vdash \neg A}\,nk(\neg I) \qquad \dfrac{}{\Phi \Vdash \neg\neg A}\,nk(Hyp)}{\Phi \Vdash F_o}\,nk(\neg E)$$

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\vee$)

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\vee$)

$\nabla_\vee$ If $\mathbf{A} \vee \mathbf{B} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma^*_\Sigma$ or $\Phi * \mathbf{B} \in \Gamma^*_\Sigma$.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\vee$)

$\nabla_\vee$ If $\mathbf{A} \vee \mathbf{B} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma^*_\Sigma$ or $\Phi * \mathbf{B} \in \Gamma^*_\Sigma$.

Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\vee$)

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Suppose $(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and both $\Phi * \mathbf{A}$ and $\Phi * \mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent.

$\boxed{\nabla_\vee}$

$$\dfrac{\dfrac{}{\Phi \Vdash A \vee B}\,nk(Hyp) \qquad \Phi * A \Vdash F_o \qquad \Phi * B \Vdash F_o}{\Phi \Vdash F_o}\,nk(\vee E)$$

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi\Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\wedge$)

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma_\Sigma^* := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi\Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\wedge$)

$\nabla_\wedge$ If $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, then $\Phi * \neg\mathbf{A} * \neg\mathbf{B} \in \Gamma_\Sigma^*$.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi\Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\wedge$)

$\nabla_\wedge$ If $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, then $\Phi * \neg\mathbf{A} * \neg\mathbf{B} \in \Gamma^*_\Sigma$.

Suppose $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and $\Phi * \neg\mathbf{A} * \neg\mathbf{B}$ is $\mathfrak{NK}_*$-inconsistent.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi\Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

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$\nabla_\wedge$ If $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, then $\Phi * \neg\mathbf{A} * \neg\mathbf{B} \in \Gamma^*_\Sigma$.

Suppose $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ and $\Phi * \neg\mathbf{A} * \neg\mathbf{B}$ is $\mathfrak{NK}_*$-inconsistent.

$$\dfrac{\dfrac{\dfrac{\Phi * \neg A * \neg B \Vdash F_o}{\Phi * \neg A \Vdash B}\,nk(Contr)}{\Phi * \neg A \Vdash A \vee B}\,nk(\vee I_R) \qquad \dfrac{}{\Phi * \neg A \Vdash \neg(A \vee B)}\,nk(Hyp)}{\Phi * \neg A \Vdash F_o}\,nk(\neg E)$$

$$\dfrac{\dfrac{\dfrac{\Phi * \neg A \Vdash F_o}{\Phi \Vdash A}\,nk(Contr)}{\Phi \Vdash A \vee B}\,nk(\vee I_L) \qquad \dfrac{}{\Phi \Vdash \neg(A \vee B)}\,nk(Hyp)}{\Phi \Vdash F_o}\,nk(\neg E)$$

Class of Sets of $\mathfrak{N}\mathfrak{K}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{N}\mathfrak{K}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\forall$)

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\forall$)

$\nabla_\forall$ If $\Pi^\alpha \mathbf{F} \in \Phi$, then $\Phi * \mathbf{F}\mathbf{W} \in \Gamma^*_\Sigma$ for each $\mathbf{W} \in \mathit{cwff}_\alpha(\Sigma)$.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

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$\nabla_\forall$ If $\Pi^\alpha \mathbf{F} \in \Phi$, then $\Phi * \mathbf{F}\mathbf{W} \in \Gamma^*_\Sigma$ for each $\mathbf{W} \in \mathit{cwff}_\alpha(\Sigma)$.

Suppose $(\Pi^\alpha \mathbf{G}) \in \Phi$ and $\Phi * (\mathbf{G}\mathbf{A})$ is $\mathfrak{NK}_*$-inconsistent.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

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Suppose $(\Pi^\alpha \mathbf{G}) \in \Phi$ and $\Phi * (\mathbf{G}\mathbf{A})$ is $\mathfrak{NK}_*$-inconsistent.

$\boxed{\nabla_\forall}$

$$\dfrac{\dfrac{}{\Phi \Vdash \neg(\mathbf{G}\mathbf{A})}\,nk(Hyp) \qquad \dfrac{\dfrac{}{\Phi \Vdash \Pi^\alpha \mathbf{G}}\,nk(Hyp)}{\Phi \Vdash (\mathbf{G}\mathbf{A})}\,nk(\Pi E)}{\Phi \Vdash \mathbf{F}_o}\,nk(\neg E)$$

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma_\Sigma^* := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\exists$)

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \nVdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\exists$)

$\nabla_\exists$ If $\neg\Pi^\alpha\mathbf{F} \in \Phi$, then $\Phi * \neg(\mathbf{F}w) \in \Gamma^*_\Sigma$ for any parameter $w_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\exists$)

$\nabla_\exists$ If $\neg\Pi^\alpha\mathbf{F} \in \Phi$, then $\Phi * \neg(\mathbf{F}w) \in \Gamma^*_\Sigma$ for any parameter $w_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

Suppose $\neg(\Pi^\alpha\mathbf{G}) \in \Phi$, w_α is a parameter which does not occur in Φ, and $\Phi * \neg(\mathbf{G}w)$ is $\mathfrak{NK}_*$-inconsistent.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\exists$)

$\nabla_\exists$ If $\neg\Pi^\alpha\mathbf{F} \in \Phi$, then $\Phi * \neg(\mathbf{F}\mathrm{w}) \in \Gamma^*_\Sigma$ for any parameter $\mathrm{w}_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

Suppose $\neg(\Pi^\alpha\mathbf{G}) \in \Phi$, w_α is a parameter which does not occur in Φ, and $\Phi * \neg(\mathbf{G}\mathrm{w})$ is $\mathfrak{NK}_*$-inconsistent.

$$\boxed{\nabla_\exists} \qquad \dfrac{\dfrac{\dfrac{\Phi * \neg \mathbf{G}\,\mathrm{w}_\alpha \Vdash \mathbf{F}_o}{\Phi \Vdash \mathbf{G}\,\mathrm{w}_\alpha}\,\mathrm{nk}(\mathrm{Contr})}{\Phi \Vdash \Pi^\alpha \mathbf{G}}\,\mathrm{nk}(\Pi I)^{\mathrm{w}_\alpha} \qquad \dfrac{}{\Phi \Vdash \neg(\Pi^\alpha \mathbf{G})}\,\mathrm{nk}(\mathrm{Hyp})}{\Phi \Vdash \mathbf{F}_o}\,\mathrm{nk}(\neg E)$$

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma_\Sigma^* := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: ∇_{sat})

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma_\Sigma^* := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: ∇_{sat})

∇_{sat} Either $\Phi * \mathbf{A} \in \Gamma_\Sigma^*$ or $\Phi * \neg\mathbf{A} \in \Gamma_\Sigma^*$.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \nVdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: ∇_{sat})

∇_{sat} Either $\Phi * \mathbf{A} \in \Gamma^*_\Sigma$ or $\Phi * \neg\mathbf{A} \in \Gamma^*_\Sigma$.

Let $\Phi * \mathbf{A}$ and $\Phi * \neg\mathbf{A}$ be $\mathfrak{NK}_*$-inconsistent.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma_\Sigma^* := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: ∇_{sat})

∇_{sat} Either $\Phi * \mathbf{A} \in \Gamma_\Sigma^*$ or $\Phi * \neg\mathbf{A} \in \Gamma_\Sigma^*$.

Let $\Phi * \mathbf{A}$ and $\Phi * \neg\mathbf{A}$ be $\mathfrak{NK}_*$-inconsistent.

$$\boxed{\nabla_{sat}} \qquad \dfrac{\dfrac{\Phi * A \Vdash F_o}{\Phi \Vdash \neg A}\,\mathfrak{NK}(\neg I) \qquad \dfrac{\Phi * \neg A \Vdash F_o}{\Phi \Vdash \neg\neg A}\,\mathfrak{NK}(\neg I)}{\Phi \Vdash F_o}\,\mathfrak{NK}(\neg E)$$

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Thus we have shown that Γ^β_Σ is saturated and in $\mathfrak{Acc}_\beta$.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma_\Sigma^* := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \nVdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Thus we have shown that Γ_Σ^β is saturated and in $\mathfrak{Acc}_\beta$.

Now let us check the conditions for the additional properties η, ξ, $\mathfrak{f}$, and $\mathfrak{b}$.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma_\Sigma^* := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Thus we have shown that Γ_Σ^β is saturated and in $\mathfrak{Acc}_\beta$.

Now let us check the conditions for the additional properties η, ξ, $\mathfrak{f}$, and $\mathfrak{b}$.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: ∇_η)

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: ∇_η)

∇_η If $\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{B} \in \Gamma^*_\Sigma$.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \nVdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: ∇_η)

∇_η If $\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{B} \in \Gamma^*_\Sigma$.

Suppose $*$ includes η, and let $\mathbf{A} \in \Phi$, $\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B}$ and $\Phi * \mathbf{B}$ be $\mathfrak{NK}_*$-inconsistent.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: ∇_η)

∇_η If $\mathbf{A} \overset{\beta\eta}{=} \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi * \mathbf{B} \in \Gamma^*_\Sigma$.

Suppose $*$ includes η, and let $\mathbf{A} \in \Phi$, $\mathbf{A} \overset{\beta\eta}{=} \mathbf{B}$ and $\Phi * \mathbf{B}$ be $\mathfrak{NK}_*$-inconsistent.

$$\boxed{\nabla_\eta} \qquad \dfrac{\dfrac{\Phi * B \Vdash F_o}{\Phi \Vdash \neg B}\,nk(\neg I) \qquad \dfrac{\dfrac{}{\Phi \Vdash A}\,nk(Hyp)}{\Phi \Vdash B}\,nk(\eta)}{\Phi \Vdash F_o}\,nk(\neg E)$$

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: ∇_ξ)

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: ∇_ξ)

∇_ξ If $\neg(\lambda X_\alpha \mathbf{.} \mathbf{M} \doteq^{\alpha\to\beta} \lambda X_\alpha \mathbf{.} \mathbf{N}) \in \Phi$, then $\Phi * \neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N}) \in \Gamma^*_\Sigma$ for any parameter $w_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: ∇_ξ)

∇_ξ If $\neg(\lambda X_\alpha.\mathbf{M} \doteq^{\alpha\to\beta} \lambda X_\alpha.\mathbf{N}) \in \Phi$, then $\Phi * \neg([w/X]\mathbf{M} \doteq^{\beta} [w/X]\mathbf{N}) \in \Gamma^*_\Sigma$ for any parameter $w_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

Suppose $*$ includes ξ, $\neg(\lambda X.\mathbf{M} \doteq^{\alpha\to\beta} \lambda X.\mathbf{N}) \in \Phi$, and $\Phi * \neg([w/X]\mathbf{M} \doteq^{\beta} [w/X]\mathbf{N})$ is $\mathfrak{NK}_*$-inconsistent for some parameter w_α which does not occur in any sentence of Φ.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: ∇_ξ)

∇_ξ If $\neg(\lambda X_\alpha\mathbf{.}\mathbf{M} \doteq^{\alpha\to\beta} \lambda X_\alpha\mathbf{.}\mathbf{N}) \in \Phi$, then
$\Phi * \neg([w/X]\mathbf{M} \doteq^{\beta} [w/X]\mathbf{N}) \in \Gamma^*_\Sigma$ for any parameter $w_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

Suppose $*$ includes ξ, $\neg(\lambda X\mathbf{.}\mathbf{M} \doteq^{\alpha\to\beta} \lambda X\mathbf{.}\mathbf{N}) \in \Phi$, and $\Phi * \neg([w/X]\mathbf{M} \doteq^{\beta} [w/X]\mathbf{N})$ is $\mathfrak{NK}_*$-inconsistent for some parameter w_α which does not occur in any sentence of Φ.

$\Phi * \neg([w/X]M \doteq [w/X]N) \Vdash F_o$

$\overline{\Phi \Vdash ([w/X]M \doteq [w/X]N)}$ $nk(Contr)$

$\overline{\Phi \Vdash (\lambda X. M \doteq N)\, w}$ $nk(\beta)$

$\overline{\Phi \Vdash (\forall X. M \doteq N)}$ $nk(\Pi I)$

$\overline{\Phi \Vdash (\lambda X. M \doteq \lambda X. N)}$ $nk(\xi)$ $\quad$ $\overline{\Phi \Vdash \neg(\lambda X. M \doteq \lambda X. N)}$ $nk(Hyp)$

$\overline{\Phi \Vdash F_o}$

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma_\Sigma^* := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_{\mathfrak{f}}$)

Class of Sets of $\mathfrak{N}\mathfrak{K}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \nVdash_{\mathfrak{N}\mathfrak{K}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\mathfrak{f}$)

$\nabla_\mathfrak{f}$ If $\neg(\mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H}) \in \Phi$, then $\Phi * \neg(\mathbf{G}\mathrm{w} \doteq^{\beta} \mathbf{H}\mathrm{w}) \in \Gamma^*_\Sigma$ for any parameter $\mathrm{w}_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\mathfrak{f}$)

$\nabla_\mathfrak{f}$ If $\neg(\mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H}) \in \Phi$, then $\Phi * \neg(\mathbf{G}\mathsf{w} \doteq^\beta \mathbf{H}\mathsf{w}) \in \Gamma^*_\Sigma$ for any parameter $\mathsf{w}_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

Suppose $*$ includes $\mathfrak{f}$, $\neg(\mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H}) \in \Phi$, and $\Phi * \neg(\mathbf{G}\mathsf{w} \doteq^\beta \mathbf{H}\mathsf{w})$ is $\mathfrak{NK}_*$-inconsistent for some parameter w_α which does not occur in any sentence of Φ.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\mathfrak{f}$)

$\nabla_\mathfrak{f}$ If $\neg(\mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H}) \in \Phi$, then $\Phi * \neg(\mathbf{G}\mathsf{w} \doteq^\beta \mathbf{H}\mathsf{w}) \in \Gamma^*_\Sigma$ for any parameter $\mathsf{w}_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

Suppose $*$ includes $\mathfrak{f}$, $\neg(\mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H}) \in \Phi$, and $\Phi * \neg(\mathbf{G}\mathsf{w} \doteq^\beta \mathbf{H}\mathsf{w})$ is $\mathfrak{NK}_*$-inconsistent for some parameter w_α which does not occur in any sentence of Φ.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Phi * \neg(Gw \doteq Hw) \Vdash F_o}{\Phi \Vdash Gw \doteq Hw}\ nk(Contr)}{\Phi \Vdash (\lambda X. GX \doteq HX)\, w}\ nk(\beta)}{\Phi \Vdash (\forall X. GX \doteq HX)}\ nk(\Pi I)}{\Phi \Vdash G \doteq H}\ nk(f) \qquad \dfrac{}{\Phi \Vdash \neg(G \doteq H)}\ nk(Hyp)}{\Phi \Vdash F_o}\ nk(\neg E)$$

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \nVdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\mathfrak{b}$)

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_\circ(\Sigma) \mid \Phi \nVdash_{\mathfrak{NK}_*} \mathbf{F}_\circ\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\mathfrak{b}$)

$\nabla_\mathfrak{b}$ If $\neg(\mathbf{A} \doteq^\circ \mathbf{B}) \in \Phi$, then $\Phi * \mathbf{A} * \neg\mathbf{B} \in \Gamma^*_\Sigma$ or $\Phi * \neg\mathbf{A} * \mathbf{B} \in \Gamma^*_\Sigma$.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \nVdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_\mathfrak{b}$)

$\nabla_\mathfrak{b}$ If $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$, then $\Phi * \mathbf{A} * \neg\mathbf{B} \in \Gamma^*_\Sigma$ or $\Phi * \neg\mathbf{A} * \mathbf{B} \in \Gamma^*_\Sigma$.

Suppose $*$ includes $\mathfrak{b}$.

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: (We show: $\nabla_{\mathfrak{b}}$)

$\nabla_{\mathfrak{b}}$ If $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$, then $\Phi * \mathbf{A} * \neg\mathbf{B} \in \Gamma^*_\Sigma$ or $\Phi * \neg\mathbf{A} * \mathbf{B} \in \Gamma^*_\Sigma$.

Suppose $*$ includes $\mathfrak{b}$. Assume that $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$ and that both $\Phi * \neg\mathbf{A} * \mathbf{B}$ and $\Phi * \mathbf{A} * \neg\mathbf{B}$ are $\mathfrak{NK}_*$-inconsistent.

$$\boxed{\nabla_{\mathfrak{b}}} \qquad \dfrac{\dfrac{\dfrac{\Phi * A * \neg B \Vdash F_o}{\Phi * A \Vdash B}\,wk((contr)) \quad \dfrac{\Phi * B * \neg A \Vdash F_o}{\Phi * B \Vdash A}\,wk((contr))}{\Phi \Vdash A \doteq B}\,wk(b) \quad \dfrac{}{\Phi \Vdash \neg(A \doteq B)}\,wk(R_{hyp})}{\Phi \Vdash F_o}$$

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \nVdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Thus, for all $*$ we have Γ^*_Σ is a saturated $\mathfrak{Acc}_*$.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma_\Sigma^* := \{\Phi \subseteq cwff_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Thus, for all $*$ we have Γ_Σ^* is a saturated $\mathfrak{Acc}_*$.

This completes the proof of the lemma. q.e.d.

Class of Sets of $\mathfrak{NK}_*$-consistent Formulas

Lemma: $\Gamma^*_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid \Phi \not\Vdash_{\mathfrak{NK}_*} \mathbf{F}_o\}$ is a saturated $\mathfrak{Acc}_*$.

Proof: Thus, for all $*$ we have Γ^*_Σ is a saturated $\mathfrak{Acc}_*$.

This completes the proof of the lemma. q.e.d.

Henkin's Theorem for $\mathfrak{NK}_*$

Thm.: Let $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Every sufficiently Σ-pure $\mathfrak{NK}_*$-consistent set of sentences has an $\mathfrak{M}_*(\Sigma)$-model.

Henkin's Theorem for $\mathfrak{N}\mathfrak{K}_*$

Thm.: Let $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Every sufficiently Σ-pure $\mathfrak{N}\mathfrak{K}_*$-consistent set of sentences has an $\mathfrak{M}_*(\Sigma)$-model.

Proof: Let Φ be a sufficiently Σ-pure $\mathfrak{N}\mathfrak{K}_*$-consistent set of sentences.

Henkin's Theorem for $\mathfrak{NK}_*$

Thm.: Let $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Every sufficiently Σ-pure $\mathfrak{NK}_*$-consistent set of sentences has an $\mathfrak{M}_*(\Sigma)$-model.

Proof: Let Φ be a sufficiently Σ-pure $\mathfrak{NK}_*$-consistent set of sentences. By the previous lemma we know that the class of sets of $\mathfrak{NK}_*$-consistent sentences constitute a saturated $\mathfrak{Acc}_*$,

Henkin's Theorem for $\mathfrak{NK}_*$

Thm.: Let $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Every sufficiently Σ-pure $\mathfrak{NK}_*$-consistent set of sentences has an $\mathfrak{M}_*(\Sigma)$-model.

Proof: Let Φ be a sufficiently Σ-pure $\mathfrak{NK}_*$-consistent set of sentences. By the previous lemma we know that the class of sets of $\mathfrak{NK}_*$-consistent sentences constitute a saturated $\mathfrak{Acc}_*$, thus the Model Existence Theorem guarantees an $\mathfrak{M}_*(\Sigma)$ model for Φ.

Completeness Theorem for $\mathfrak{NK}_*$

Thm.: Let Φ be a sufficiently Σ-pure set of sentences, $\mathbf{A}$ be a sentence, and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. If $\mathbf{A}$ is valid in all models $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ that satisfy Φ, then $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{A}$.

Completeness Theorem for $\mathfrak{NK}_*$

Thm.: Let Φ be a sufficiently Σ-pure set of sentences, $\mathbf{A}$ be a sentence, and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. If $\mathbf{A}$ is valid in all models $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ that satisfy Φ, then $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{A}$.

Proof: Let $\mathbf{A}$ be given such that $\mathbf{A}$ is valid in all $\mathfrak{M}_*(\Sigma)$ models that satisfy Φ.

Completeness Theorem for $\mathfrak{NK}_*$

Thm.: Let Φ be a sufficiently Σ-pure set of sentences, $\mathbf{A}$ be a sentence, and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. If $\mathbf{A}$ is valid in all models $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ that satisfy Φ, then $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{A}$.

Proof: Let $\mathbf{A}$ be given such that $\mathbf{A}$ is valid in all $\mathfrak{M}_*(\Sigma)$ models that satisfy Φ. So, $\Phi * \neg\mathbf{A}$ is unsatisfiable in $\mathfrak{M}_*(\Sigma)$.

Completeness Theorem for $\mathfrak{NK}_*$

Thm.: Let Φ be a sufficiently Σ-pure set of sentences, $\mathbf{A}$ be a sentence, and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. If $\mathbf{A}$ is valid in all models $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ that satisfy Φ, then $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{A}$.

Proof: Let $\mathbf{A}$ be given such that $\mathbf{A}$ is valid in all $\mathfrak{M}_*(\Sigma)$ models that satisfy Φ. So, $\Phi * \neg\mathbf{A}$ is unsatisfiable in $\mathfrak{M}_*(\Sigma)$. Since only finitely many constants occur in $\neg\mathbf{A}$, $\Phi * \neg\mathbf{A}$ is sufficiently Σ-pure.

Completeness Theorem for $\mathfrak{N}\mathfrak{K}_*$

Thm.: Let Φ be a sufficiently Σ-pure set of sentences, $\mathbf{A}$ be a sentence, and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. If $\mathbf{A}$ is valid in all models $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ that satisfy Φ, then $\Phi \Vdash_{\mathfrak{N}\mathfrak{K}_*} \mathbf{A}$.

Proof: Let $\mathbf{A}$ be given such that $\mathbf{A}$ is valid in all $\mathfrak{M}_*(\Sigma)$ models that satisfy Φ. So, $\Phi * \neg\mathbf{A}$ is unsatisfiable in $\mathfrak{M}_*(\Sigma)$. Since only finitely many constants occur in $\neg\mathbf{A}$, $\Phi * \neg\mathbf{A}$ is sufficiently Σ-pure. So, $\Phi * \neg\mathbf{A}$ must be $\mathfrak{N}\mathfrak{K}_*$-inconsistent by Henkin's theorem above.

Completeness Theorem for $\mathfrak{NK}_*$

Thm.: Let Φ be a sufficiently Σ-pure set of sentences, $\mathbf{A}$ be a sentence, and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. If $\mathbf{A}$ is valid in all models $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$ that satisfy Φ, then $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{A}$.

Proof: Let $\mathbf{A}$ be given such that $\mathbf{A}$ is valid in all $\mathfrak{M}_*(\Sigma)$ models that satisfy Φ. So, $\Phi * \neg\mathbf{A}$ is unsatisfiable in $\mathfrak{M}_*(\Sigma)$. Since only finitely many constants occur in $\neg\mathbf{A}$, $\Phi * \neg\mathbf{A}$ is sufficiently Σ-pure. So, $\Phi * \neg\mathbf{A}$ must be $\mathfrak{NK}_*$-inconsistent by Henkin's theorem above. Thus, $\Phi \Vdash_{\mathfrak{NK}_*} \mathbf{A}$ by $\mathfrak{NK}(Contr)$.

Compactness Theorem for $\mathfrak{NK}_*$

We can use the completeness theorems obtained so far to prove a compactness theorem for $\mathfrak{NK}_*$:

Thm.: Let Φ be a sufficiently Σ-pure set of sentences and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Φ has an $\mathfrak{M}_*(\Sigma)$-model iff every finite subset of Φ has an $\mathfrak{M}_*(\Sigma)$-model.

Compactness Theorem for $\mathfrak{NK}_*$

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Proof: (interesting direction by contraposition)

Compactness Theorem for $\mathfrak{NK}_*$

We can use the completeness theorems obtained so far to prove a compactness theorem for $\mathfrak{NK}_*$:

Thm.: Let Φ be a sufficiently Σ-pure set of sentences and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Φ has an $\mathfrak{M}_*(\Sigma)$-model iff every finite subset of Φ has an $\mathfrak{M}_*(\Sigma)$-model.

Proof: (interesting direction by contraposition)
If Φ has no $\mathfrak{M}_*(\Sigma)$-model, then by the previous Henkin Theorem Φ is $\mathfrak{NK}_*$-inconsistent.

Compactness Theorem for $\mathfrak{NK}_*$

We can use the completeness theorems obtained so far to prove a compactness theorem for $\mathfrak{NK}_*$:

Thm.: Let Φ be a sufficiently Σ-pure set of sentences and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Φ has an $\mathfrak{M}_*(\Sigma)$-model iff every finite subset of Φ has an $\mathfrak{M}_*(\Sigma)$-model.

Proof: (interesting direction by contraposition)
If Φ has no $\mathfrak{M}_*(\Sigma)$-model, then by the previous Henkin Theorem Φ is $\mathfrak{NK}_*$-inconsistent. Since every $\mathfrak{NK}_*$-proof is finite, this means some finite subset Ψ of Φ is $\mathfrak{NK}_*$-inconsistent.

Compactness Theorem for $\mathfrak{NK}_*$

We can use the completeness theorems obtained so far to prove a compactness theorem for $\mathfrak{NK}_*$:

Thm.: Let Φ be a sufficiently Σ-pure set of sentences and $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$. Φ has an $\mathfrak{M}_*(\Sigma)$-model iff every finite subset of Φ has an $\mathfrak{M}_*(\Sigma)$-model.

Proof: (interesting direction by contraposition)
If Φ has no $\mathfrak{M}_*(\Sigma)$-model, then by the previous Henkin Theorem Φ is $\mathfrak{NK}_*$-inconsistent. Since every $\mathfrak{NK}_*$-proof is finite, this means some finite subset Ψ of Φ is $\mathfrak{NK}_*$-inconsistent. Hence, Ψ has no $\mathfrak{M}_*(\Sigma)$-model.

Note on the Saturation Condition

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- in fact, as we show in [Unpublished-04] and [IJCAR-06], proving ∇_{sat} is as hard as showing admissibility of cut
- if time permits, we will hear more about this later

Model Existence Theorems

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Lemma (Compactness of ACC's): For each ACC Γ_Σ there exists a compact ACC Γ'_Σ satisfying the same ∇_* properties such that $\Gamma_\Sigma \subseteq \Gamma'_\Sigma$.

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Lemma (Compactness of ACC's):

Lemma (Abstract Extension Lemma): Let Σ be a signature, Γ_Σ be a compact ACC in $\mathfrak{Acc}_*$, and let $\Phi \in \Gamma_\Sigma$ be sufficiently pure. Then there exists a Σ-Hintikka set $\mathcal{H} \in \mathfrak{Hint}_*$, such that $\Phi \subseteq \mathcal{H}$. Furthermore, if Γ_Σ is saturated, then $\mathcal{H}$ is saturated.

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Proof: The proof combines the following three ingredients:

Lemma (Compactness of ACC's):

Lemma (Abstract Extension Lemma):

Thm (Model Existence Theorem for Saturated Hintikka Sets): For all $*$ we have: If $\mathcal{H}$ is a saturated Hintikka set in $\mathfrak{Hint}_*$, then there exists a model $\mathcal{M} \in \mathfrak{M}_*$ that satisfies $\mathcal{H}$. Furthermore, each domain $\mathcal{D}_\alpha$ of $\mathcal{M}$ has cardinality at most $\aleph_s$.

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Lemma (Compactness of ACC's):

Lemma (Abstract Extension Lemma):

Thm (Model Existence Theorem for Saturated Hintikka Sets):

► ... now we sketch the proofs of these ingredients ...

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- $\Gamma'_\Sigma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma) \mid$ every finite subset of Φ is in $\Gamma_\Sigma\}$
- Show $\Gamma_\Sigma \subseteq \Gamma'_\Sigma$: Suppose $\Phi \in \Gamma_\Sigma$. Γ_Σ is closed under subsets, so every finite subset of Φ is in Γ_Σ and thus $\Phi \in \Gamma'_\Sigma$.

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- $\Gamma'_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid$ every finite subset of Φ is in $\Gamma_\Sigma\}$
- Show $\Gamma_\Sigma \subseteq \Gamma'_\Sigma$:
- Show Γ'_Σ is compact: Suppose $\Phi \in \Gamma'_\Sigma$ and Ψ is an arbitrary finite subset of Φ. By definition of Γ'_Σ all finite subsets of Φ are in Γ_Σ and therefore $\Psi \in \Gamma'_\Sigma$. Thus all finite subsets of Φ are in Γ'_Σ whenever Φ is in Γ'_Σ.
  On the other hand, suppose all finite subsets of Φ are in Γ'_Σ. Then by the definition of Γ'_Σ the finite subsets of Φ are also in Γ_Σ, so $\Phi \in \Gamma'_\Sigma$.
  Thus Γ'_Σ is compact (and closed under subsets).

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- Show that Γ'_Σ satisfies ∇_* whenever Γ_Σ satisfies ∇_*:

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- Show $\Gamma_\Sigma \subseteq \Gamma'_\Sigma$:
- Show Γ'_Σ is compact:
- Show that Γ'_Σ satisfies ∇_* whenever Γ_Σ satisfies ∇_*:

∇_c Let $\Phi \in \Gamma'_\Sigma$ and suppose there is an atom $\mathbf{A}$, such that $\{\mathbf{A}, \neg\mathbf{A}\} \subseteq \Phi$. $\{\mathbf{A}, \neg\mathbf{A}\}$ is clearly a finite subset of Φ and hence $\{\mathbf{A}, \neg\mathbf{A}\} \in \Gamma_\Sigma$ contradicting ∇_c for Γ_Σ.

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- Show $\Gamma_\Sigma \subseteq \Gamma'_\Sigma$:
- Show Γ'_Σ is compact:
- Show that Γ'_Σ satisfies ∇_* whenever Γ_Σ satisfies ∇_*:

$\nabla_\neg$ Let $\Phi \in \Gamma'_\Sigma$, $\neg\neg\mathbf{A} \in \Phi$, Ψ be any finite subset of $\Phi * \mathbf{A}$, and $\Theta := (\Psi \setminus \{\mathbf{A}\}) * \neg\neg\mathbf{A}$. Θ is a finite subset of Φ, so $\Theta \in \Gamma_\Sigma$. Since Γ_Σ is an abstract consistency class and $\neg\neg\mathbf{A} \in \Theta$, we get $\Theta * \mathbf{A} \in \Gamma_\Sigma$ by $\nabla_\neg$ for Γ_Σ. We know that $\Psi \subseteq \Theta * \mathbf{A}$ and Γ_Σ is closed under subsets, so $\Psi \in \Gamma_\Sigma$. Thus every finite subset Ψ of $\Phi * \mathbf{A}$ is in Γ_Σ and therefore by definition $\Phi * \mathbf{A} \in \Gamma'_\Sigma$.

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- Show $\Gamma_\Sigma \subseteq \Gamma'_\Sigma$:
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- Show that Γ'_Σ satisfies ∇_* whenever Γ_Σ satisfies ∇_*:

  For $\nabla_\beta, \nabla_\eta, \nabla_\vee, \nabla_\wedge, \nabla_\forall, \nabla_\exists, \nabla_\xi, \nabla_\mathfrak{f}, \nabla_\mathfrak{b}, \nabla_{\mathsf{sat}}$ see the lecture notes.

Abstract Extension Lemma

Lemma: Let Σ be a signature, Γ_Σ be a compact ACC in $\mathfrak{Acc}_*$, where $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$, and let $\Phi \in \Gamma_\Sigma$ be sufficiently pure. Then there exists a Σ-Hintikka set $\mathcal{H} \in \mathfrak{Hint}_*$, such that $\Phi \subseteq \mathcal{H}$. Furthermore, if Γ_Σ is saturated, then $\mathcal{H}$ is saturated.

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$\mathbf{E}^n$: $\mathbf{E}^n := \neg(\mathbf{B} w_\alpha^n)$ if $\mathbf{A}^n$ is of the orm $\neg(\Pi^\alpha \mathbf{B})$, and let $\mathbf{E}^n := \mathbf{A}^n$ otherwise

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$\mathbf{X}^n$: If $* \in \{\beta\mathfrak{f}, \beta\mathfrak{f}\mathfrak{b}\}$ and $\mathbf{A}^n$ is of the form $\neg(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G})$, let $\mathbf{X}^n := \neg(\mathbf{F}w^n_\alpha \doteq^\beta \mathbf{G}w^n_\alpha)$.
If $* \in \{\beta\xi, \beta\xi\mathfrak{b}\}$ and $\mathbf{A}^n$ is of the form $\neg((\lambda X_\alpha.\mathbf{M}) \doteq^{\alpha\to\beta} (\lambda X.\mathbf{N}))$, let $\mathbf{X}^n := \neg([w^n_\alpha/X]\mathbf{M} \doteq^\beta [w^n_\alpha/X]\mathbf{N})$. Otherwise, let $\mathbf{X}^n := \mathbf{A}^n$.

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Lemma: Let Σ be a signature, Γ_Σ be a compact ACC in $\mathfrak{Acc}_*$, where $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$, and let $\Phi \in \Gamma_\Sigma$ be sufficiently pure. Then there exists a Σ-Hintikka set $\mathcal{H} \in \mathfrak{Hint}_*$, such that $\Phi \subseteq \mathcal{H}$. Furthermore, if Γ_Σ is saturated, then $\mathcal{H}$ is saturated.

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params w_α^n: need to prove that always fresh parameters exists

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generalize: the above only works for the countable case; in the lecture notes we use transfinite induction for the general case

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  - Then we show by induction that $\mathcal{H}^n \in \Gamma_\Sigma$ for all n.

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  - Since Γ_Σ is compact, we also have $\mathcal{H} \in \Gamma_\Sigma$.

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  - Then we show by induction that $\mathcal{H}^n \in \Gamma_\Sigma$ for all n.
  - Since Γ_Σ is compact, we also have $\mathcal{H} \in \Gamma_\Sigma$.
  - Hence $\Phi \subseteq \mathcal{H}$ and $\mathcal{H} \in \Gamma_\Sigma$.

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  - If $\mathcal{H}^n * \mathbf{A}^n \in \Gamma_\Sigma$, then $\mathcal{H}^{n+1} := \mathcal{H}^n * \mathbf{A}^n * \mathbf{E}^n * \mathbf{X}^n$, where
  - Then we show by induction that $\mathcal{H}^n \in \Gamma_\Sigma$ for all n.
  - Since Γ_Σ is compact, we also have $\mathcal{H} \in \Gamma_\Sigma$.
  - Hence $\Phi \subseteq \mathcal{H}$ and $\mathcal{H} \in \Gamma_\Sigma$.
  - Remains to show that $\mathcal{H}$ is (subset) maximal in Γ_Σ and that $\mathcal{H}$ is indeed a Hintikka set.

Hintikka Sets

- Hintikka sets connect syntax with semantics as they provide the basis for the model constructions in the model existence theorem(s).

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Hintikka Sets

- Hintikka sets connect syntax with semantics as they provide the basis for the model constructions in the model existence theorem(s).
- We have defined eight different notions of abstract consistency classes by first defining properties ∇_*, then specifying which should hold in $\mathfrak{Acc}_*$.
- Similarly, we define Hintikka sets by first defining the desired properties.

Hintikka Properties

Defn.: Let $\mathcal{H}$ be a set of sentences. We define the following properties which $\mathcal{H}$ may satisfy, where $\mathbf{A}, \mathbf{B} \in \mathit{cwff}_{o}(\Sigma)$, $\mathbf{C}, \mathbf{D} \in \mathit{cwff}_{\alpha}(\Sigma)$, $\mathbf{F} \in \mathit{cwff}_{\alpha\to o}(\Sigma)$, and $(\lambda X_{\alpha} . \mathbf{M}), (\lambda X . \mathbf{N}), \mathbf{G}, \mathbf{H} \in \mathit{cwff}_{\alpha\to\beta}(\Sigma)$:

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$\vec{\nabla}_c$ $\mathbf{A} \notin \mathcal{H}$ or $\neg \mathbf{A} \notin \mathcal{H}$.

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$\vec{\nabla}_\forall$ If $\Pi^\alpha \mathbf{F} \in \mathcal{H}$, then $\mathbf{F}\mathbf{W} \in \mathcal{H}$ for each $\mathbf{W} \in cwff_{\alpha}(\Sigma)$.

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$\vec{\nabla}_\exists$ If $\neg\Pi^\alpha\mathbf{F} \in \mathcal{H}$, then there is a parameter $w_\alpha \in \Sigma_\alpha$ such that $\neg(\mathbf{F}w) \in \mathcal{H}$.

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$\vec{\nabla}_{\mathfrak{b}}$ If $\neg(\mathbf{A} \doteq^{o} \mathbf{B}) \in \mathcal{H}$, then $\{\mathbf{A}, \neg\mathbf{B}\} \subseteq \mathcal{H}$ or $\{\neg\mathbf{A}, \mathbf{B}\} \subseteq \mathcal{H}$.

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$\vec{\nabla}_\eta$ If $\mathbf{A} \in \mathcal{H}$ and $\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B}$, then $\mathbf{B} \in \mathcal{H}$.

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$\vec{\nabla}_{\xi}$ If $\neg(\lambda X_{\alpha} \mathbf{.} \mathbf{M} \doteq^{\alpha \to \beta} \lambda X \mathbf{.} \mathbf{N}) \in \mathcal{H}$, then there is a parameter $w_{\alpha} \in \Sigma_{\alpha}$ such that $\neg([w/X]\mathbf{M} \doteq^{\beta} [w/X]\mathbf{N}) \in \mathcal{H}$.

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$\vec{\nabla}_{\mathfrak{f}}$ If $\neg(\mathbf{G} \doteq^{\alpha \to \beta} \mathbf{H}) \in \mathcal{H}$, then there is a parameter $w_{\alpha} \in \Sigma_{\alpha}$ such that $\neg(\mathbf{G}w \doteq^{\beta} \mathbf{H}w) \in \mathcal{H}$.

Hintikka Properties

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$\vec{\nabla}_{\mathfrak{f}}$ If $\neg(\mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H}) \in \mathcal{H}$, then there is a parameter $w_\alpha \in \Sigma_\alpha$ such that $\neg(\mathbf{G}w \doteq^\beta \mathbf{H}w) \in \mathcal{H}$.

$\vec{\nabla}_{sat}$ Either $\mathbf{A} \in \mathcal{H}$ or $\neg\mathbf{A} \in \mathcal{H}$.

Σ-Hintikka Set

Defn.: A set $\mathcal{H}$ of sentences is called a Σ-Hintikka set if it satisfies $\vec{\nabla}_c$, $\vec{\nabla}_\neg$, $\vec{\nabla}_\beta$, $\vec{\nabla}_\vee$, $\vec{\nabla}_\wedge$, $\vec{\nabla}_\forall$ and $\vec{\nabla}_\exists$.

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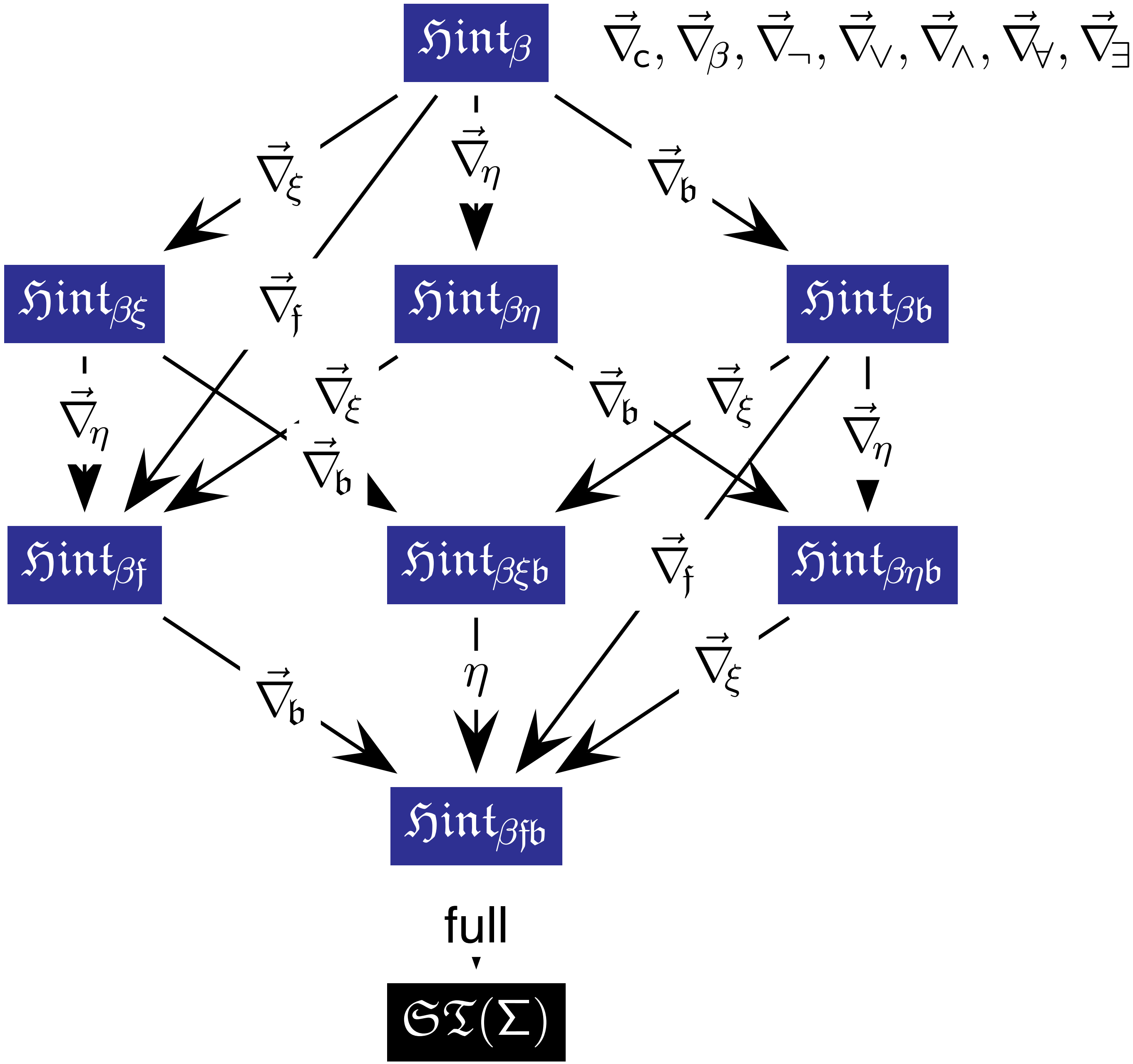
- We define the following collections of Hintikka sets: $\mathfrak{Hint}_\beta$, $\mathfrak{Hint}_{\beta\eta}$, $\mathfrak{Hint}_{\beta\xi}$, $\mathfrak{Hint}_{\beta\mathfrak{f}}$, $\mathfrak{Hint}_{\beta\mathfrak{b}}$, $\mathfrak{Hint}_{\beta\eta\mathfrak{b}}$, $\mathfrak{Hint}_{\beta\xi\mathfrak{b}}$, and $\mathfrak{Hint}_{\beta\mathfrak{f}\mathfrak{b}}$, where we indicate by indices which additional properties from $\{\vec{\nabla}_\eta, \vec{\nabla}_\xi, \vec{\nabla}_\mathfrak{f}, \vec{\nabla}_\mathfrak{b}\}$ are required.

Σ-Hintikka Sets



Model Ex. Thm for Saturated H.-Sets

Thm.: (Model Existence Theorem for Saturated Hintikka Sets)
For all $* \in \{\ldots\}$ we have: If $\mathcal{H}$ is a saturated Hintikka set in $\mathfrak{Hint}_*$, then there exists a model $\mathcal{M} \in \mathfrak{M}_*$ that satisfies $\mathcal{H}$.
Furthermore, each domain $\mathcal{D}_\alpha$ of $\mathcal{M}$ has cardinality at most $\aleph_\mathsf{s}$.

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- We construct Σ-model $\mathcal{M}_1^{\mathcal{H}} := (\mathit{cwff}(\Sigma){\downarrow_\beta}, @^\beta, \mathcal{E}^\beta, \upsilon)$ for $\mathcal{H}$

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- We construct Σ-model $\mathcal{M}_1^{\mathcal{H}} := (\mathit{cwff}(\Sigma){\downarrow}_\beta, @^\beta, \mathcal{E}^\beta, \upsilon)$ for $\mathcal{H}$
- $\mathcal{M}_1^{\mathcal{H}}$ is based on term evaluation
  $\mathcal{TE}(\Sigma)^\beta := (\mathit{cwff}(\Sigma){\downarrow}_\beta, @^\beta, \mathcal{E}^\beta)$ where
  - $\mathit{cwff}(\Sigma){\downarrow}_\beta$: closed well-formed formulae in β-normal form
  - $\mathbf{A} @^\beta \mathbf{B} := (\mathbf{AB}){\downarrow}_\beta$
  - $\mathcal{E}_\varphi^\beta(\mathbf{A}) := \sigma(\mathbf{A}){\downarrow}_\beta$

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- $\upsilon(\mathbf{A}) := \begin{cases} \mathtt{T} & \text{if } \mathbf{A} \in \mathcal{H} \\ \mathtt{F} & \text{if } \mathbf{A} \notin \mathcal{H} \end{cases}$

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- may hold: $\mathcal{M}_1^{\mathcal{H}} \notin \mathfrak{M}_*$ as it may not satisfy property $\mathfrak{q}$

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- may hold: $\mathcal{M}_1^{\mathcal{H}} \notin \mathfrak{M}_*$ as it may not satisfy property $\mathfrak{q}$
- way out: use congruence relation $\dot\sim$ on $\mathcal{M}_1^{\mathcal{H}}$

$$\mathbf{A}_\alpha \dot\sim \mathbf{B}_\alpha \text{ iff } \upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq \mathbf{B})) = \mathtt{T}$$

to construct $\mathcal{M} := \mathcal{M}_1^{\mathcal{H}}/_{\dot\sim}$

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  to construct $\mathcal{M} := \mathcal{M}_1^{\mathcal{H}}/\dot{\sim}$
- then show that $\mathcal{M}$ 'does the job'

Further Reading

- [Chris-PhD-99] C. Benzmüller: Equality and Extensionality in Higher-Order Theorem Proving. Doctoral Thesis, Computer Science, Saarland University, 1999.

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- [JSL-04] C. Benzmüller, C. Brown, and M. Kohlhase: Higher Order Semantics and Extensionality. Journal of Symbolic Logic. (2004) 69(4):1027-1088. ©JSTOR.

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Thank You!