# Automating Gödel's Ontological Proof of God's Existence with Higher-order Automated Theorem Provers

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[1] Supported by DFG grant BE 2501/9-1

# Vision of Leibniz (1646–1716): *Calculemus!*

If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other . . . : Let us calculate.
(Translation by Russell)

Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo . . . dicere: calculemus. (Leibniz, 1684)

Required:
**characteristica universalis** and **calculus ratiocinator**

# Our Contribution: Towards a Computational Metaphysics

Ontological argument for the existence of God

We focused on Gödel's modern version in higher-order modal logic

Automation with provers for higher-order classical logic (HOL)

- confirmation of known results
- detection of some novel results
- systematic variation of the logic settings
- exploited HOL as a universal metalogic (characteristica universalis)

# A Long History

pros and cons

… Anselm v. C. Gaunilo … Th. Aquinas …… Descartes Spinoza Leibniz … Hume Kant … Hegel … Frege … Hartshorne Malcolm Lewis Plantinga Gödel …

Anselm's notion of God (Proslogion, 1078):

**"God is that, than which nothing greater can be conceived."**

Gödel's notion of God:

**"A God-like being possesses all 'positive' properties."**

To show by logical reasoning:

**"God exists."**

$\exists x G(x)$

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**"God is that, than which nothing greater can be conceived."**

Gödel's notion of God:

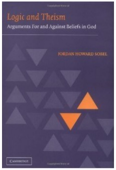
**"A God-like being possesses all 'positive' properties."**

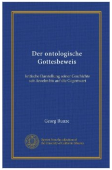
To show by logical reasoning:

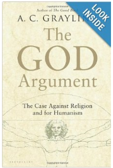
**"Necessarily God exists."**

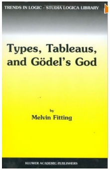
$$\Box \exists x G(x)$$

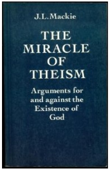
# The Ontological Proof Today

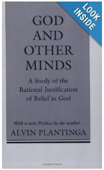


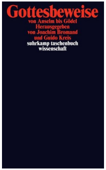



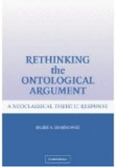



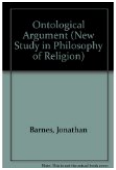



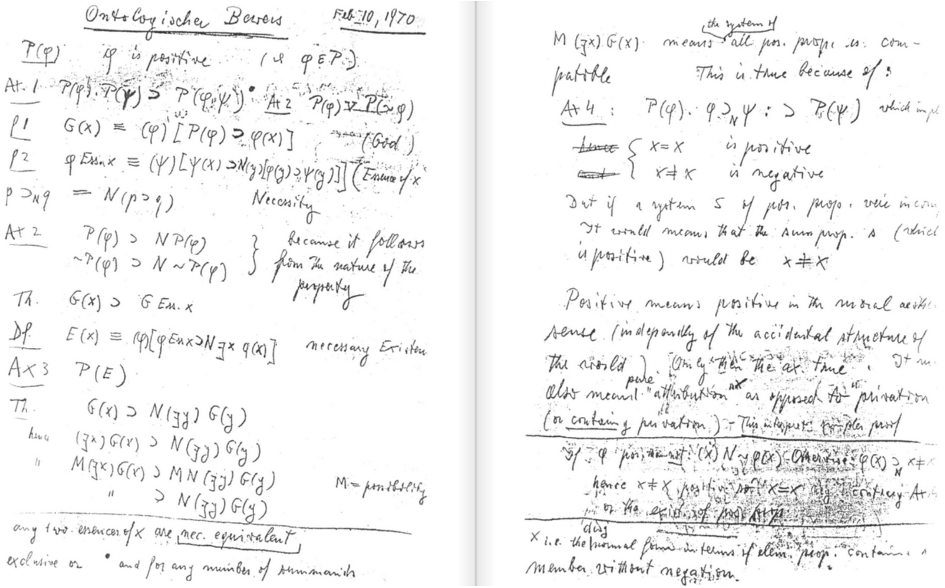







Ontologischer Beweis Feb 10, 1970

$P(\varphi)$ $\varphi$ is positive (or $\varphi \in P$)

At. 1 $P(\varphi).P(\psi) \supset P(\varphi.\psi)$ At 2 $P(\varphi) \vee P(\sim\varphi)$

P1 $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (God)

P2 $\varphi \text{ Ess } x \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Essence of x)

$p \supset_N q = N(p \supset q)$ Necessity

At 2 $P(\varphi) \supset NP(\varphi)$, $\sim P(\varphi) \supset N \sim P(\varphi)$ } because it follows from the nature of the property

Th. $G(x) \supset G \text{ Ess. } x$

Df. $E(x) \equiv (\varphi)[\varphi \text{ Ess } x \supset N \exists x\, \varphi(x)]$ necessary Existence

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y)\, G(y)$

hence $(\exists x)\, G(x) \supset N(\exists y)\, G(y)$

" $M(\exists x)\, G(x) \supset MN(\exists y)\, G(y)$ M = possibility

" $\supset N(\exists y)\, G(y)$

any two essences of x are nec. equivalent

exclusive or * and for any number of summands

$M(\exists x)\, G(x)$ means the system of all pos. props. is compatible. This is true because of:

At 4: $P(\varphi).\ \varphi \supset_N \psi : \supset P(\psi)$ which impl[ies]

{ $x = x$ is positive; $x \neq x$ is negative

But if a system S of pos. props. were incon[sistent] It would mean that the sum prop. s (which is positive) would be $x \neq x$

Positive means positive in the moral aesth[etic] sense (independently of the accidental structure of the world). Only then the ax. true. It m[ay] also mean pure "attribution"* as opposed to "privation" (or containing privation). This interpret. [illegible] proof

If φ pos. not: $(x)\, N \sim \varphi(x)$. Otherwise: $\varphi(x) \supset_N x \neq x$ hence $x \neq x$ positive so $x = x$ neg. contrary Ax. [illegible] or the exist. of pos. prop.

* i.e. the disj. normal form in terms of elem. prop. contains a member without negation.

# Scott's Version of Gödel's Axioms, Definitions and Theorems

Axiom A1 Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

Axiom A2 A property necessarily implied by a positive property is positive:
$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

Thm. T1 Positive properties are possibly exemplified: $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

Def. D1 A *God-like* being possesses all positive properties: $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$

Axiom A3 The property of being God-like is positive: $P(G)$

Cor. C Possibly, God exists: $\Diamond\exists x G(x)$

Axiom A4 Positive properties are necessarily positive: $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$

Def. D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi\ ess\ x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Thm. T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \rightarrow G\ ess\ x]$

Def. D3 *Necessary existence* of an individual is the necessary exemplification of all its essences: $E(x) \leftrightarrow \forall\phi[\phi\ ess\ x \rightarrow \Box\exists y\phi(y)]$

Axiom A5 Necessary existence is a positive property: $P(E)$

Thm. T3 Necessarily, God exists: $\Box\exists x G(x)$

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Difference to Gödel (who omits this conjunct)

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**Cor. C** Possibly, God exists: $\Diamond \exists x G(x)$

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**Def. D2** An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi \; ess \; x \leftrightarrow \phi(x) \wedge \forall \psi (\psi(x) \rightarrow \Box \forall y (\phi(y) \rightarrow \psi(y)))$

**Thm. T2** Being God-like is an essence of any God-like being: $\forall x [G(x) \rightarrow G \; ess \; x]$

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Modal operators are used

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second-order quantifiers

# Proof Overview

**D1:** $G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$

**D2:** $\varphi \; ess \; x \equiv \varphi(x) \wedge \forall\psi_{\blacksquare}.(\psi(x) \rightarrow \Box\forall x_{\blacksquare}.(\varphi(x) \rightarrow \psi(x)))$

**D3:** $NE(x) \equiv \forall\varphi_{\blacksquare}.[\varphi \; ess \; x \rightarrow \Box\exists y.\varphi(y)]$

**A3**: $P(G)$

**A2**: $\forall\varphi_{\blacksquare}.\forall\psi_{\blacksquare}.[(P(\varphi) \wedge \Box\forall x_{\blacksquare}.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

**A1a**: $\forall\varphi_{\blacksquare}.[P(\neg\varphi) \rightarrow \neg P(\varphi)]$

**T1:** $\forall\varphi_{\blacksquare}.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]$

**C:** $\Diamond\exists z.G(z)$

**A1b**: $\forall\varphi_{\blacksquare}.[\neg P(\varphi) \rightarrow P(\neg\varphi)]$

**A4**: $\forall\varphi_{\blacksquare}.[P(\varphi) \rightarrow \Box \; P(\varphi)]$

**T2:** $\forall y_{\blacksquare}.[G(y) \rightarrow G \; ess \; y]$

**A5**: $P(NE)$

**L1:** $\exists z.G(z) \rightarrow \Box\exists x.G(x)$

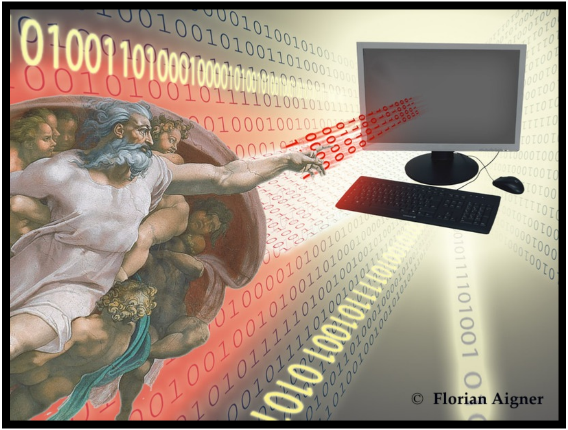
$\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)$

**S5**: $\forall\xi_{\blacksquare}.[\Diamond\Box\xi \rightarrow \Box\xi]$

**L2:** $\Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)$

**C:** $\Diamond\exists z.G(z)$ **L2:** $\Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)$

**T3:** $\Box\exists x.G(x)$




## How to automate Higher-Order Modal Logic?

Challenge: No provers for *Higher-order Modal Logic* (HOML)

Our solution: **Embedding in *Higher-order Classical Logic* (HOL)**

Then use existing HOL theorem provers for reasoning in HOML

[BenzmüllerPaulson, Logica Universalis, 2013]

Previous empirical findings:

Embedding of *First-order Modal Logic* in HOL works well

[BenzmüllerOttenRaths, ECAI, 2012]

[Benzmüller, LPAR, 2013]

HOML $\varphi, \psi \quad ::= \quad \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x\, \varphi \mid \exists x\, \varphi \mid \forall P\, \varphi$

- Kripke style semantics (possible world semantics)

HOL $s, t \quad ::= \quad C \mid x \mid \lambda x s \mid s\, t \mid \neg s \mid s \vee t \mid \forall x\, t$

- Church's simple type theory [Church, 1940], [Henkin, 1950]
- various theorem provers exist

  interactive: Isabelle/HOL, HOL4, Hol Light, Coq/HOL, PVS, . . .

  automated: TPS, LEO-II, Satallax, Nitpick, Isabelle/HOL, . . .

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HOL $s, t ::= C \mid x \mid \lambda x s \mid s\, t \mid \neg s \mid s \vee t \mid \forall x\, t$

HOML in HOL: HOML formulas $\varphi$ are mapped to HOL predicates $\varphi_{\mu\to o}$

$$
\begin{aligned}
\neg &= \lambda\varphi_{\mu\to o}\lambda w_\mu \neg\varphi w \\
\wedge &= \lambda\varphi_{\mu\to o}\lambda\psi_{\mu\to o}\lambda w_\mu(\varphi w \wedge \psi w) \\
\rightarrow &= \lambda\varphi_{\mu\to o}\lambda\psi_{\mu\to o}\lambda w_\mu(\neg\varphi w \vee \psi w) \\
\forall &= \lambda h_{\gamma\to(\mu\to o)}\lambda w_\mu \forall d_\gamma\, h d w \\
\exists &= \lambda h_{\gamma\to(\mu\to o)}\lambda w_\mu \exists d_\gamma\, h d w \\
\Box &= \lambda\varphi_{\mu\to o}\lambda w_\mu \forall u_\mu\, (\neg r w u \vee \varphi u) \\
\Diamond &= \lambda\varphi_{\mu\to o}\lambda w_\mu \exists u_\mu\, (r w u \wedge \varphi u) \\
\text{valid} &= \lambda\varphi_{\mu\to o}\forall w_\mu\, \varphi w
\end{aligned}
$$

Ax

The equations in Ax are given as axioms to the HOL provers!

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# Embedding HOML in HOL

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$$

Ax

The equations in Ax are given as axioms to the HOL provers!

Example

| | |
|---|---|
| HOML formula | $\Diamond \exists x G(x)$ |
| HOML formula in HOL | valid $(\Diamond \exists x G(x))_{\mu \to o}$ |
| expansion, $\beta\eta$-conversion | $\forall w_\mu (\Diamond \exists x G(x))_{\mu \to o}\, w$ |
| expansion, $\beta\eta$-conversion | $\forall w_\mu \exists u_\mu (rwu \wedge (\exists x G(x))_{\mu \to o} u)$ |
| expansion, $\beta\eta$-conversion | $\forall w_\mu \exists u_\mu (rwu \wedge \exists x Gxu)$ |

Expansion: user or prover may flexibly choose expansion depth

**What are we doing?**

In order to prove that $\varphi$ is valid in HOML,
–> we instead prove that valid $\varphi_{\mu \to o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

For the experts: soundness and completeness wrt Henkin semantics

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| expansion, $\beta\eta$-conversion | $\forall w_{\mu} \exists u_{\mu} (rwu \wedge \exists x Gxu)$ |

Expansion: user or prover may flexibly choose expansion depth

**What are we doing?**

In order to prove that $\varphi$ is valid in HOML,
–> we instead prove that valid $\varphi_{\mu \to o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

For the experts: soundness and completeness wrt Henkin semantics

Example

| | |
|---|---|
| HOML formula | $\Diamond \exists x G(x)$ |
| HOML formula in HOL | valid $(\Diamond \exists x G(x))_{\mu \to o}$ |
| expansion, $\beta\eta$-conversion | $\forall w_\mu (\Diamond \exists x G(x))_{\mu \to o}\, w$ |
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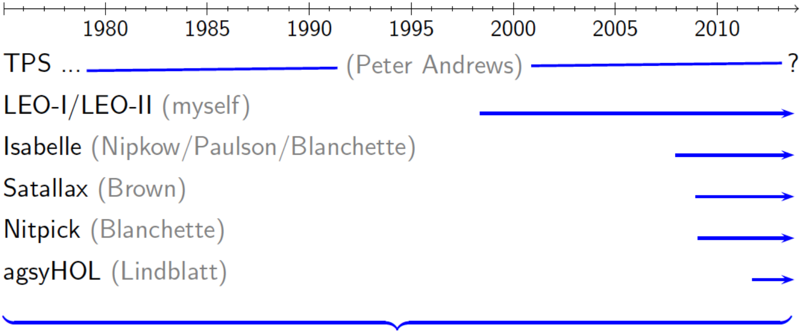
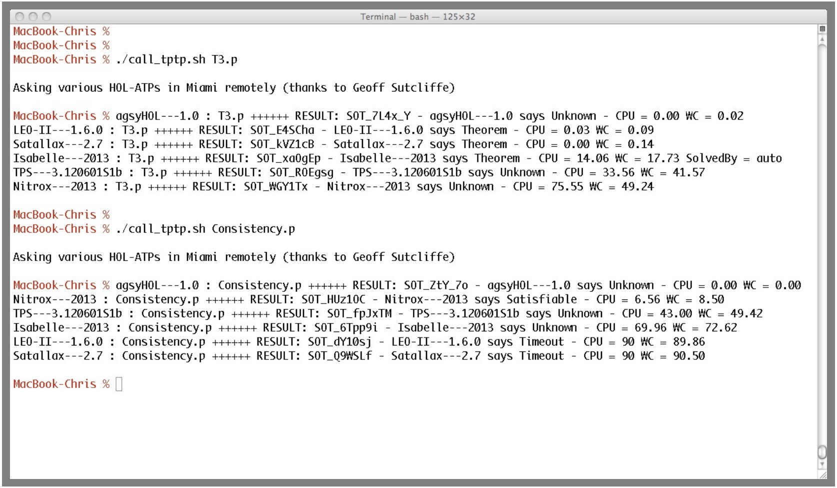
For the experts: soundness and completeness wrt Henkin semantics




- all accept TPTP THF Syntax [SutcliffeBenzmüller, J.Form.Reas, 2009]
- can be called remotely via SystemOnTPTP at Miami
- they significantly gained in strength over the last years
- they can be bundled into a combined prover HOL-P

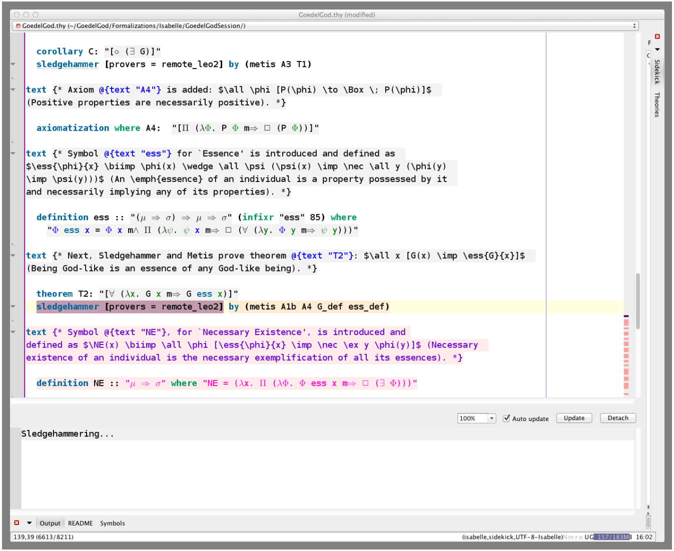
Exploit HOL with Henkin semantics as metalogic
Automate other logics (& combinations) via semantic embeddings
— HOL-P becomes a Universal Reasoner —



```
Terminal — bash — 125×32
MacBook-Chris %
MacBook-Chris %
MacBook-Chris % ./call_tptp.sh T3.p

Asking various HOL-ATPs in Miami remotely (thanks to Geoff Sutcliffe)

MacBook-Chris % agsyHOL---1.0 : T3.p ++++++ RESULT: SOT_7L4x_Y - agsyHOL---1.0 says Unknown - CPU = 0.00 WC = 0.02
LEO-II---1.6.0 : T3.p ++++++ RESULT: SOT_E4SCha - LEO-II---1.6.0 says Theorem - CPU = 0.03 WC = 0.09
Satallax---2.7 : T3.p ++++++ RESULT: SOT_kVZ1cB - Satallax---2.7 says Theorem - CPU = 0.00 WC = 0.14
Isabelle---2013 : T3.p ++++++ RESULT: SOT_xa0gEp - Isabelle---2013 says Theorem - CPU = 14.06 WC = 17.73 SolvedBy = auto
TPS---3.120601S1b : T3.p ++++++ RESULT: SOT_R0Egsg - TPS---3.120601S1b says Unknown - CPU = 33.56 WC = 41.57
Nitrox---2013 : T3.p ++++++ RESULT: SOT_WGY1Tx - Nitrox---2013 says Unknown - CPU = 75.55 WC = 49.24

MacBook-Chris %
MacBook-Chris % ./call_tptp.sh Consistency.p

Asking various HOL-ATPs in Miami remotely (thanks to Geoff Sutcliffe)

MacBook-Chris % agsyHOL---1.0 : Consistency.p ++++++ RESULT: SOT_ZtY_7o - agsyHOL---1.0 says Unknown - CPU = 0.00 WC = 0.00
Nitrox---2013 : Consistency.p ++++++ RESULT: SOT_HUz10C - Nitrox---2013 says Satisfiable - CPU = 6.56 WC = 8.50
TPS---3.120601S1b : Consistency.p ++++++ RESULT: SOT_fpJxTM - TPS---3.120601S1b says Unknown - CPU = 43.00 WC = 49.42
Isabelle---2013 : Consistency.p ++++++ RESULT: SOT_6Tpp9i - Isabelle---2013 says Unknown - CPU = 69.96 WC = 72.62
LEO-II---1.6.0 : Consistency.p ++++++ RESULT: SOT_dY10sj - LEO-II---1.6.0 says Timeout - CPU = 90 WC = 89.86
Satallax---2.7 : Consistency.p ++++++ RESULT: SOT_Q9WSLf - Satallax---2.7 says Timeout - CPU = 90 WC = 90.50

MacBook-Chris %
```

Provers are called remotely in Miami — no local installation needed!

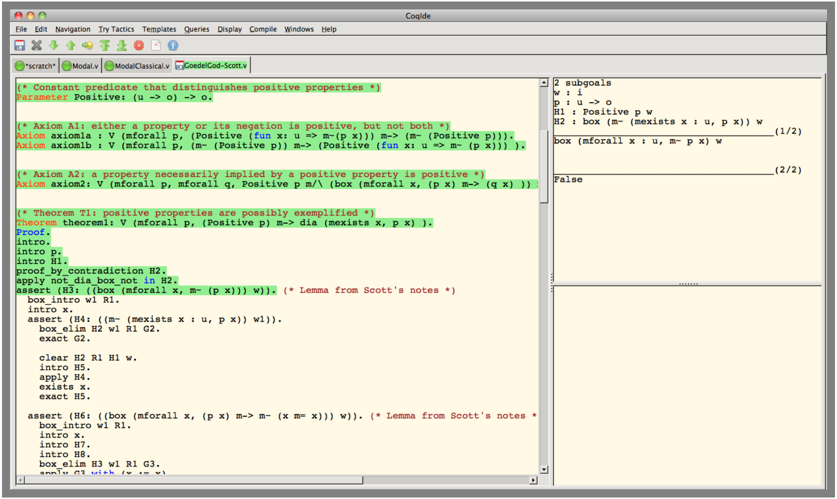
Download our experiments from `https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/THF`



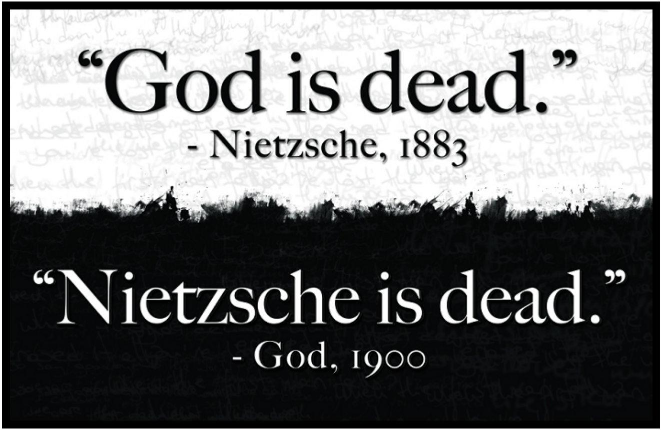

See verifiable Isabelle/HOL journal article at:

http://afp.sourceforge.net/entries/GoedelGod.shtml

# Interaction in Proof Assistant Coq




See verifiable Coq document at: `https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/Coq`




**Main Findings**

# Main Findings

| | HOL encoding | dependencies | logic | status | LEO-II const/vary | Satallax const/vary | Nitpick const/vary |
|---|---|---|---|---|---|---|---|
| A1 | $[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_{\mu}.\, \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$ | | | | | | |
| A2 | $[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\wedge}\, \dot{\Box}\dot{\forall}X_{\mu}.\, (\phi X \,\dot{\supset}\, \psi X)) \,\dot{\supset}\, p\psi]$ | | | | | | |
| T1 | $[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\supset}\, \dot{\Diamond}\dot{\exists}X_{\mu}.\, \phi X]$ | A1(⊃), A2 | K | THM | 0.1/0.1 | 0.0/0.0 | —/— |
| | | A1, A2 | K | THM | 0.1/0.1 | 0.0/5.2 | —/— |
| D1 | $g_{\mu\to\sigma} = \lambda X_{\mu}.\, \dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\supset}\, \phi X$ | | | | | | |
| A3 | $[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$ | | | | | | |
| C | $[\dot{\Diamond}\dot{\exists}X_{\mu}.\, g_{\mu\to\sigma}X]$ | T1, D1, A3 | K | THM | 0.0/0.0 | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3 | K | THM | 0.0/0.0 | 5.2/31.3 | —/— |
| A4 | $[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\supset}\, \dot{\Box}p\phi]$ | | | | | | |
| D2 | $\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_{\mu}.\, \phi X \,\dot{\wedge}\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \,\dot{\supset}\, \dot{\Box}\dot{\forall}Y_{\mu}.\, (\phi Y \,\dot{\supset}\, \psi Y))$ | | | | | | |
| T2 | $[\dot{\forall}X_{\mu}.\, g_{\mu\to\sigma}X \,\dot{\supset}\, (\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, gX)]$ | A1, D1, A4, D2 | K | THM | 19.1/18.3 | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3, A4, D2 | K | THM | 12.9/14.0 | 0.0/0.0 | —/— |
| D3 | $\mathbf{NE}_{\mu\to\sigma} = \lambda X_{\mu}.\, \dot{\forall}\phi_{\mu\to\sigma}.\, (\mathbf{ess}\, \phi X \,\dot{\supset}\, \dot{\Box}\dot{\exists}Y_{\mu}.\, \phi Y)$ | | | | | | |
| A5 | $[p_{(\mu\to\sigma)\to\sigma}\mathbf{NE}_{\mu\to\sigma}]$ | | | | | | |
| T3 | $[\dot{\Box}\dot{\exists}X_{\mu}.\, g_{\mu\to\sigma}X]$ | D1, C, T2, D3, A5 | K | CSA | —/— | —/— | 3.8/6.2 |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | K | CSA | —/— | —/— | 8.2/7.5 |
| | | D1, C, T2, D3, A5 | KB | THM | 0.0/0.1 | 0.1/5.3 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | —/— | —/— | —/— |
| MC | $[s_{\sigma} \,\dot{\supset}\, \dot{\Box}s_{\sigma}]$ | D2, T2, T3 | KB | THM | 17.9/— | 3.3/3.2 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | —/— | —/— | —/— |
| FG | $[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}X_{\mu}.\, (g_{\mu\to\sigma}X \,\dot{\supset}\, (\dot{\neg}(p_{(\mu\to\sigma)\to\sigma}\phi) \,\dot{\supset}\, \dot{\neg}(\phi X)))]$ | A1, D1 | KB | THM | 16.5/— | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | 12.8/15.1 | 0.0/5.4 | —/— |
| MT | $[\dot{\forall}X_{\mu}.\, \dot{\forall}Y_{\mu}.\, (g_{\mu\to\sigma}X \,\dot{\supset}\, (g_{\mu\to\sigma}Y \,\dot{\supset}\, X \doteq Y))]$ | D1, FG | KB | THM | —/— | 0.0/3.3 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | —/— | —/— | —/— |
| CO | $\emptyset$ (no goal, check for consistency) | A1, A2, D1, A3, A4, D2, D3, A5 | KB | SAT | —/— | —/— | 7.3/7.4 |
| D2’ | $\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_{\mu}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \,\dot{\supset}\, \dot{\Box}\dot{\forall}Y_{\mu}.\, (\phi Y \,\dot{\supset}\, \psi Y))$ | | | | | | |
| CO’ | $\emptyset$ (no goal, check for consistency) | A1(⊃), A2, D2’, D3, A5 | KB | UNS | 7.5/7.8 | —/— | —/— |
| | | A1, A2, D1, A3, A4, D2’, D3, A5 | KB | UNS | —/— | —/— | —/— |

# Main Findings

| | HOL encoding | dependencies | logic | status | LEO-II const/vary | Satallax const/vary | Nitpick const/vary |
|---|---|---|---|---|---|---|---|
| A1 | $[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$ | | | | | | |
| A2 | $[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\wedge}\, \dot{\Box}\dot{\forall}X_\mu.\, (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$ | | | | | | |
| T1 | $[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Diamond}\dot{\exists}X_\mu.\, \phi X]$ | A1(⊃), A2 | K | THM | 0.1/0.1 | 0.0/0.0 | —/— |
| | | A1, A2 | K | THM | 0.1/0.1 | 0.0/5.2 | —/— |
| D1 | $g_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \phi X$ | | | | | | |
| A3 | $[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$ | | | | | | |
| C | $[\dot{\Diamond}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$ | T1, D1, A3 | K | THM | 0.0/0.0 | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3 | K | THM | 0.0/0.0 | 5.2/31.3 | —/— |
| A4 | $[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Box}p\phi]$ | | | | | | |
| D2 | $\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \,\dot{\wedge}\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \dot{\supset} \psi Y))$ | | | | | | |
| T2 | $[\dot{\forall}X_\mu.\, g_{\mu\to\sigma}X \dot{\supset} (\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, gX)]$ | A1, D1, A4, D2 | K | THM | 19.1/18.3 | 0.0/0.0 | —/— |
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| A5 | $[p_{(\mu\to\sigma)\to\sigma}\text{NE}_{\mu\to\sigma}]$ | | | | | | |
| T3 | $[\dot{\Box}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$ | D1, C, T2, D3, A5 | K | CSA | —/— | —/— | 3.8/6.2 |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | K | CSA | —/— | —/— | 8.2/7.5 |
| | | D1, C, T2, D3, A5 | KB | THM | 0.0/0.1 | 0.1/5.3 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | —/— | —/— | —/— |
| MC | $[s_\sigma \dot{\supset} \dot{\Box}s_\sigma]$ | D2, T2, T3 | KB | THM | 17.9/— | 3.3/3.2 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | —/— | —/— | —/— |
| FG | $[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}X_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (\dot{\neg}(p_{(\mu\to\sigma)\to\sigma}\phi) \dot{\supset} \dot{\neg}(\phi X)))]$ | A1, D1 | KB | THM | 16.5/— | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | 12.8/15.1 | 0.0/5.4 | —/— |
| MT | $[\dot{\forall}X_\mu.\, \dot{\forall}Y_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (g_{\mu\to\sigma}Y \dot{\supset} X \doteq Y))]$ | D1, FG | KB | THM | —/— | 0.0/3.3 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | —/— | —/— | —/— |
| CO | $\emptyset$ (no goal, check for consistency) | A1, A2, D1, A3, A4, D2, D3, A5 | KB | SAT | —/— | —/— | 7.3/7.4 |
| D2’ | $\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \dot{\supset} \psi Y))$ | | | | | | |
| CO’ | $\emptyset$ (no goal, check for consistency) | A1(⊃), A2, D2’, D3, A5 | KB | UNS | 7.5/7.8 | —/— | —/— |
| | | A1, A2, D1, A3, A4, D2’, D3, A5 | KB | UNS | —/— | —/— | —/— |

# Main Findings

| | HOL encoding | dependencies | logic | status | LEO-II const/vary | Satallax const/vary | Nitpick const/vary |
|---|---|---|---|---|---|---|---|
| A1 | $[\dot\forall\phi_{\mu\to\sigma}\text{.}\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu\text{.}\,\dot\neg(\phi X)) \,\dot\equiv\, \dot\neg(p\phi)]$ | | | | | | |
| A2 | $[\dot\forall\phi_{\mu\to\sigma}\text{.}\,\dot\forall\psi_{\mu\to\sigma}\text{.}\,(p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\wedge\, \dot\Box\dot\forall X_\mu\text{.}\,(\phi X \,\dot\supset\, \psi X)) \,\dot\supset\, p\psi]$ | | | | | | |
| T1 | $[\dot\forall\phi_{\mu\to\sigma}\text{.}\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \dot\Diamond\dot\exists X_\mu\text{.}\,\phi X]$ | A1(⊃), A2 | K | THM | 0.1/0.1 | 0.0/0.0 | —/— |
| | | A1, A2 | K | THM | 0.1/0.1 | 0.0/5.2 | —/— |
| D1 | $g_{\mu\to\sigma} = \lambda X_\mu\text{.}\,\dot\forall\phi_{\mu\to\sigma}\text{.}\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \phi X$ | | | | | | |
| A3 | $[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$ | | | | | | |
| C | $[\dot\Diamond\dot\exists X_\mu\text{.}\, g_{\mu\to\sigma} X]$ | T1, D1, A3 | K | THM | 0.0/0.0 | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3 | K | THM | 0.0/0.0 | 5.2/31.3 | —/— |
| A4 | $[\dot\forall\phi_{\mu\to\sigma}\text{.}\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \dot\Box p\phi]$ | | | | | | |
| D2 | $\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}\text{.}\,\lambda X_\mu\text{.}\,\phi X \,\dot\wedge\, \dot\forall\psi_{\mu\to\sigma}\text{.}\,(\psi X \,\dot\supset\, \dot\Box\dot\forall Y_\mu\text{.}\,(\phi Y \,\dot\supset\, \psi Y))$ | | | | | | |
| T2 | $[\dot\forall X_\mu\text{.}\, g_{\mu\to\sigma} X \,\dot\supset\, (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$ | A1, D1, A4, D2 | K | THM | 19.1/18.3 | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3, A4, D2 | K | THM | 12.9/14.0 | 0.0/0.0 | —/— |
| D3 | $\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu\text{.}\,\dot\forall\phi_{\mu\to\sigma}\text{.}\,($e | | | | | | |
| A5 | $[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$ | | | | | | |
| T3 | $[\dot\Box\dot\exists X_\mu\text{.}\, g_{\mu\to\sigma} X]$ | | | | | | |
| MC | $[s_\sigma \,\dot\supset\, \dot\Box s_\sigma]$ | | | | | | |
| FG | $[\dot\forall\phi_{\mu\to\sigma}\text{.}\,\dot\forall X_\mu\text{.}\,(g_{\mu\to\sigma} X \,\dot\supset\, ($ | | | | | | |
| MT | $[\dot\forall X_\mu\text{.}\,\dot\forall Y_\mu\text{.}\,(g_{\mu\to\sigma} X \,\dot\supset\, (g_\mu$ | | | | | | |
| CO | ∅ (no goal, check for cons | | | | | | |
| D2' | $\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}\text{.}\,\lambda$ | | | | | | |
| CO' | ∅ (no goal, check for cons | | | | | | |

## Automating Scott's proof script

**T1: "Positive properties are possibly exemplified" proved by LEO-II and Satallax**

- **in logic: K**
- **from axioms:**
  - **A1 and A2**
- **for domain conditions:**
  - **constant domains**

| | HOL encoding | dependencies | logic | status | LEO-II const/vary | Satallax const/vary | Nitpick const/vary |
|---|---|---|---|---|---|---|---|
| A1 | $[\dot\forall\phi_{\mu\to\sigma}\cdot p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu\cdot\dot\neg(\phi X))\dot\equiv\dot\neg(p\phi)]$ | | | | | | |
| A2 | $[\dot\forall\phi_{\mu\to\sigma}\cdot\dot\forall\psi_{\mu\to\sigma}\cdot(p_{(\mu\to\sigma)\to\sigma}\phi\dot\wedge\dot\Box\dot\forall X_\mu\cdot(\phi X\dot\supset\psi X))\dot\supset p\psi]$ | | | | | | |
| T1 | $[\dot\forall\phi_{\mu\to\sigma}\cdot p_{(\mu\to\sigma)\to\sigma}\phi\dot\supset\dot\Diamond\dot\exists X_\mu\cdot\phi X]$ | A1(⊃), A2 | K | THM | 0.1/0.1 | 0.0/0.0 | —/— |
| | | A1, A2 | K | THM | 0.1/0.1 | 0.0/5.2 | —/— |
| D1 | $g_{\mu\to\sigma}=\lambda X_\mu\cdot\dot\forall\phi_{\mu\to\sigma}\cdot p_{(\mu\to\sigma)\to\sigma}\phi\dot\supset\phi X$ | | | | | | |
| A3 | $[p_{(\mu\to\sigma)\to\sigma}g_{\mu\to\sigma}]$ | | | | | | |
| C | $[\dot\Diamond\dot\exists X_\mu\cdot g_{\mu\to\sigma}X]$ | T1, D1, A3 | K | THM | 0.0/0.0 | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3 | K | THM | 0.0/0.0 | 5.2/31.3 | —/— |
| A4 | $[\dot\forall\phi_{\mu\to\sigma}\cdot p_{(\mu\to\sigma)\to\sigma}\phi\dot\supset\dot\Box p\phi]$ | | | | | | |
| D2 | $ess_{(\mu\to\sigma)\to\mu\to\sigma}=\lambda\phi_{\mu\to\sigma}\cdot\lambda X_\mu\cdot\phi X\dot\wedge\dot\forall\psi_{\mu\to\sigma}\cdot(\psi X\dot\supset\dot\Box\dot\forall Y_\mu\cdot(\phi Y\dot\supset\psi Y))$ | | | | | | |
| T2 | $[\dot\forall X_\mu\cdot g_{\mu\to\sigma}X\dot\supset(ess_{(\mu\to\sigma)\to\mu\to\sigma}gX)]$ | A1, D1, A4, D2 | K | THM | 19.1/18.3 | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3, A4, D2 | K | THM | 12.9/14.0 | 0.0/0.0 | —/— |
| D3 | $NE_{\mu\to\sigma}=\lambda X_\mu\cdot\dot\forall\phi_{\mu\to\sigma}\cdot$(e | | | | | | |
| A5 | $[p_{(\mu\to\sigma)\to\sigma}NE_{\mu\to\sigma}]$ | | | | | | |
| T3 | $[\dot\Box\dot\exists X_\mu\cdot g_{\mu\to\sigma}X]$ | | | | | | |
| MC | $[s_\sigma\dot\supset\dot\Box s_\sigma]$ | | | | | | |
| FG | $[\dot\forall\phi_{\mu\to\sigma}\cdot\dot\forall X_\mu\cdot(g_{\mu\to\sigma}X\dot\supset$ ( | | | | | | |
| MT | $[\dot\forall X_\mu\cdot\dot\forall Y_\mu\cdot(g_{\mu\to\sigma}X\dot\supset(g_\mu$ | | | | | | |
| CO | ∅ (no goal, check for cons | | | | | | |
| D2' | $ess_{(\mu\to\sigma)\to\mu\to\sigma}=\lambda\phi_{\mu\to\sigma}\cdot\lambda$ | | | | | | |
| CO' | ∅ (no goal, check for cons | | | | | | |

## Automating Scott's proof script

**T1: "Positive properties are possibly exemplified" proved by LEO-II and Satallax**

- **in logic: K**
- **from axioms:**
  - **A1 and A2**
  - **A1(⊃) and A2**
- **for domain conditions:**
  - **constant domains**

| | HOL encoding | dependencies | logic | status | LEO-II const/vary | Satallax const/vary | Nitpick const/vary |
|---|---|---|---|---|---|---|---|
| A1 | $[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$ | | | | | | |
| A2 | $[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu.\, (\phi X \dot\supset \psi X)) \dot\supset p\psi]$ | | | | | | |
| T1 | $[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\, \phi X]$ | A1(⊃), A2 | K | THM | 0.1/0.1 | 0.0/0.0 | —/— |
| | | A1, A2 | K | THM | 0.1/0.1 | 0.0/5.2 | —/— |
| D1 | $g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$ | | | | | | |
| A3 | $[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$ | | | | | | |
| C | $[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$ | T1, D1, A3 | K | THM | 0.0/0.0 | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3 | K | THM | 0.0/0.0 | 5.2/31.3 | —/— |
| A4 | $[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$ | | | | | | |
| D2 | $\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \dot\wedge \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\, (\phi Y \dot\supset \psi Y))$ | | | | | | |
| T2 | $[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$ | A1, D1, A4, D2 | K | THM | 19.1/18.3 | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3, A4, D2 | K | THM | 12.9/14.0 | 0.0/0.0 | —/— |
| D3 | $\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (\mathrm{e}$ … | | | | | | |
| A5 | $[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$ | | | | | | |
| T3 | $[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$ | | | | | | |
| MC | $[s_\sigma \dot\supset \dot\Box s_\sigma]$ | | | | | | |
| FG | $[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma}X \dot\supset ($ … | | | | | | |
| MT | $[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma}X \dot\supset (g_\mu$ … | | | | | | |
| CO | ∅ (no goal, check for cons… | | | | | | |
| D2' | $\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$ … | | | | | | |
| CO' | ∅ (no goal, check for cons… | | | | | | |

## Automating Scott's proof script

**T1: "Positive properties are possibly exemplified" proved by LEO-II and Satallax**

- **in logic: K**
- **from axioms:**
  - **A1 and A2**
  - **A1(⊃) and A2**
- **for domain conditions:**
  - **constant domains**
  - **varying domains (individuals)**

| | HOL encoding | dependencies | logic | status | LEO-II const/vary | Satallax const/vary | Nitpick const/vary |
|---|---|---|---|---|---|---|---|
| A1 | $[\dot\forall \phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$ | | | | | | |
| A2 | $[\dot\forall \phi_{\mu\to\sigma}.\, \dot\forall \psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\wedge\, \dot\Box \dot\forall X_\mu.\, (\phi X \dot\supset \psi X)) \dot\supset p\psi]$ | | | | | | |
| T1 | $[\dot\forall \phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond \dot\exists X_\mu.\, \phi X]$ | A1(⊃), A2 | K | THM | 0.1/0.1 | 0.0/0.0 | —/— |
| | | A1, A2 | K | THM | 0.1/0.1 | 0.0/5.2 | —/— |
| D1 | $g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall \phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$ | | | | | | |
| A3 | $[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$ | | | | | | |
| C | $[\dot\Diamond \dot\exists X_\mu.\, g_{\mu\to\sigma} X]$ | T1, D1, A3 | K | THM | 0.0/0.0 | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3 | K | THM | 0.0/0.0 | 5.2/31.3 | —/— |
| A4 | $[\dot\forall \phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$ | | | | | | |
| D2 | $\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \,\dot\wedge\, \dot\forall \psi_{\mu\to\sigma}.\, (\psi X \dot\supset \dot\Box \dot\forall Y_\mu.\, (\phi Y \dot\supset \psi Y))$ | | | | | | |
| T2 | $[\dot\forall X_\mu.\, g_{\mu\to\sigma} X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$ | A1, D1, A4, D2 | K | THM | 19.1/18.3 | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3, A4, D2 | K | THM | 12.9/14.0 | 0.0/0.0 | —/— |
| D3 | $\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall \phi_{\mu\to\sigma}.\, ($e | | | | | | |
| A5 | $[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$ | | | | | | |
| T3 | $[\dot\Box \dot\exists X_\mu.\, g_{\mu\to\sigma} X]$ | | | | | | |
| MC | $[s_\sigma \dot\supset \dot\Box s_\sigma]$ | | | | | | |
| FG | $[\dot\forall \phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma} X \dot\supset ($ | | | | | | |
| MT | $[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma} X \dot\supset (g_\mu$ | | | | | | |
| CO | ∅ (no goal, check for cons | | | | | | |
| D2’ | $\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$ | | | | | | |
| CO’ | ∅ (no goal, check for cons | | | | | | |

## Automating Scott’s proof script

**C: "Possibly, God exists” proved by LEO-II and Satallax**

- **in logic: K**
- **from assumptions:**
  - **T1, D1, A3**
  - **A1, A2, D1, A3**
- **for domain conditions:**
  - **constant domains**
  - **varying domains (individuals)**

| | HOL encoding | dependencies | logic | status | LEO-II const/vary | Satallax const/vary | Nitpick const/vary |
|---|---|---|---|---|---|---|---|
| A1 | $[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$ | | | | | | |
| A2 | $[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu.\, (\phi X \dot\supset \psi X)) \dot\supset p\psi]$ | | | | | | |
| T1 | $[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\, \phi X]$ | A1($\supset$), A2 | K | THM | 0.1/0.1 | 0.0/0.0 | —/— |
| | | A1, A2 | K | THM | 0.1/0.1 | 0.0/5.2 | —/— |
| D1 | $g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$ | | | | | | |
| A3 | $[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$ | | | | | | |
| C | $[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$ | T1, D1, A3 | K | THM | 0.0/0.0 | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3 | K | THM | 0.0/0.0 | 5.2/31.3 | —/— |
| A4 | $[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$ | | | | | | |
| D2 | $\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \dot\wedge \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\, (\phi Y \dot\supset \psi Y))$ | | | | | | |
| T2 | $[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$ | A1, D1, A4, D2 | K | THM | 19.1/18.3 | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3, A4, D2 | K | THM | 12.9/14.0 | 0.0/0.0 | —/— |
| D3 | $\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (\mathrm{e}$ … | | | | | | |
| A5 | $[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$ | | | | | | |
| T3 | $[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$ | | | | | | |
| MC | $[s_\sigma \dot\supset \dot\Box s_\sigma]$ | | | | | | |
| FG | $[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma}X \dot\supset ($ … | | | | | | |
| MT | $[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma}X \dot\supset (g_\mu$ … | | | | | | |
| CO | $\emptyset$ (no goal, check for cons … | | | | | | |
| D2' | $\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$ … | | | | | | |
| CO' | $\emptyset$ (no goal, check for cons … | | | | | | |

## Automating Scott's proof script

**T2: "Being God-like is an ess. of any God-like being" proved by LEO-II and Satallax**

- **in logic: K**
- **from assumptions:**
  - **A1, D1, A4, D2**
  - **A1, A2, D1, A3, A4, D2**
- **for domain conditions:**
  - **constant domains**
  - **varying domains (individuals)**

# Main Findings

| | HOL encoding | dependencies | logic | status | LEO-II const/vary | Satallax const/vary | Nitpick const/vary |
|---|---|---|---|---|---|---|---|
| A1 | $[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$ | | | | | | |
| A2 | $[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu.\, (\phi X \dot\supset \psi X)) \dot\supset p\psi]$ | | | | | | |
| T1 | $[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\, \phi X]$ | A1($\supset$), A2 | K | THM | 0.1/0.1 | 0.0/0.0 | —/— |
| | | A1, A2 | K | THM | 0.1/0.1 | 0.0/5.2 | —/— |
| D1 | $g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$ | | | | | | |
| A3 | $[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$ | | | | | | |
| C | $[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$ | T1, D1, A3 | K | THM | 0.0/0.0 | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3 | K | THM | 0.0/0.0 | 5.2/31.3 | —/— |
| A4 | $[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$ | | | | | | |
| D2 | $\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \dot\wedge \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\, (\phi Y \dot\supset \psi Y))$ | | | | | | |
| T2 | $[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \dot\supset (\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$ | A1, D1, A4, D2 | K | THM | 19.1/18.3 | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3, A4, D2 | K | THM | 12.9/14.0 | 0.0/0.0 | —/— |
| D3 | $\text{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (\text{e}$ | | | | | | |
| A5 | $[p_{(\mu\to\sigma)\to\sigma}\text{NE}_{\mu\to\sigma}]$ | | | | | | |
| T3 | $[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$ | | | | | | |
| MC | $[s_\sigma \dot\supset \dot\Box s_\sigma]$ | | | | | | |
| FG | $[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma}X \dot\supset ($ | | | | | | |
| MT | $[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma}X \dot\supset (g_\mu$ | | | | | | |
| CO | $\emptyset$ (no goal, check for cons | | | | | | |
| D2' | $\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$ | | | | | | |
| CO' | $\emptyset$ (no goal, check for cons | | | | | | |

## Automating Scott's proof script

**T3: "Necessarily, God exists" proved by LEO-II and Satallax**

- **in logic: KB**
- **from assumptions:**
  - **D1, C, T2, D3, A5**
- **for domain conditions:**
  - **constant domains**
  - **varying domains (individuals)**

**For logic K we got a countermodel by Nitpick**

| | HOL encoding | dependencies | logic | status | LEO-II const/vary | Satallax const/vary | Nitpick const/vary |
|---|---|---|---|---|---|---|---|
| A1 | $[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$ | | | | | | |
| A2 | $[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \dot{\wedge} \dot{\Box}\dot{\forall}X_\mu.\, (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$ | | | | | | |
| T1 | $[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Diamond}\dot{\exists}X_\mu.\, \phi X]$ | A1(⊃), A2 | K | THM | 0.1/0.1 | 0.0/0.0 | —/— |
| | | A1, A2 | K | THM | 0.1/0.1 | 0.0/5.2 | —/— |
| D1 | $g_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \phi X$ | | | | | | |
| A3 | $[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$ | | | | | | |
| C | $[\dot{\Diamond}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$ | T1, D1, A3 | K | THM | 0.0/0.0 | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3 | K | THM | 0.0/0.0 | 5.2/31.3 | —/— |
| A4 | $[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Box}p\phi]$ | | | | | | |
| D2 | $\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \dot{\wedge} \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \dot{\supset} \psi Y))$ | | | | | | |
| T2 | $[\dot{\forall}X_\mu.\, g_{\mu\to\sigma}X \dot{\supset} (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, gX)]$ | A1, D1, A4, D2 | K | THM | 19.1/18.3 | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3, A4, D2 | K | THM | 12.9/14.0 | 0.0/0.0 | —/— |
| D3 | $\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, (\mathrm{ess}\, \phi X \dot{\supset} \dot{\Box}\dot{\exists}Y_\mu.\, \phi Y)$ | | | | | | |
| A5 | $[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$ | | | | | | |
| T3 | $[\dot{\Box}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$ | | | | | | |
| MC | $[s_\sigma \dot{\supset} \dot{\Box}s_\sigma]$ | | | | | | |
| FG | $[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}X_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} ($ | | | | | | |
| MT | $[\dot{\forall}X_\mu.\, \dot{\forall}Y_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (g_\mu$ | | | | | | |
| CO | ∅ (no goal, check for cons | | | | | | |
| D2' | $\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$ | | | | | | |
| CO' | ∅ (no goal, check for cons | | | | | | |

## Automating Scott's proof script

### Summary

- proof verified and automated
- KB is sufficient (critisized logic S5 not needed!)
- proof works for constant and varying domains
- exact dependencies determined experimentally
- ATPs have found alternative proofs (shorter)

# Main Findings

| | HOL encoding | dependencies | logic | status | LEO-II const/vary | Satallax const/vary | Nitpick const/vary |
|---|---|---|---|---|---|---|---|
| A1 | $[\dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu. \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$ | | | | | | |
| A2 | $[\dot\forall\phi_{\mu\to\sigma}. \dot\forall\psi_{\mu\to\sigma}. (p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu. (\phi X \dot\supset \psi X)) \dot\supset p\psi]$ | | | | | | |
| T1 | $[\dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu. \phi X]$ | A1(⊃), A2 | K | THM | 0.1/0.1 | 0.0/0.0 | —/— |
| D1 | $g_{\mu\to\sigma} = \lambda X_\mu. \dot\forall\phi_{\mu\to\sigma}. p_{(\mu-}$ … | | | | | | |
| A3 | $[p_{(\mu\to\sigma)\to\sigma}g_{\mu\to\sigma}]$ | | | | | | |
| C | $[\dot\Diamond\dot\exists X_\mu. g_{\mu\to\sigma}X]$ | | | | | | |
| A4 | $[\dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p$ … | | | | | | |
| D2 | $ess_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}. \lambda$ … | | | | | | |
| T2 | $[\dot\forall X_\mu. g_{\mu\to\sigma}X \dot\supset (ess_{(\mu\to\sigma)}$ … | | | | | | |
| D3 | $NE_{\mu\to\sigma} = \lambda X_\mu. \dot\forall\phi_{\mu\to\sigma}. (e$ … | | | | | | |
| A5 | $[p_{(\mu\to\sigma)\to\sigma}NE_{\mu\to\sigma}]$ | | | | | | |
| T3 | $[\dot\Box\dot\exists X_\mu. g_{\mu\to\sigma}X]$ | | | | | | |
| | | D1, C, T2, D3, A5 | KB | THM | 0.0/0.1 | 0.1/5.3 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | —/— | —/— | —/— |
| MC | $[s_\sigma \dot\supset \dot\Box s_\sigma]$ | D2, T2, T3 | KB | THM | 17.9/— | 3.3/3.2 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | —/— | —/— | —/— |
| FG | $[\dot\forall\phi_{\mu\to\sigma}. \dot\forall X_\mu. (g_{\mu\to\sigma}X \dot\supset (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \dot\supset \dot\neg(\phi X)))]$ | A1, D1 | KB | THM | 16.5/— | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | 12.8/15.1 | 0.0/5.4 | —/— |
| MT | $[\dot\forall X_\mu. \dot\forall Y_\mu. (g_{\mu\to\sigma}X \dot\supset (g_{\mu\to\sigma}Y \dot\supset X \dot= Y))]$ | D1, FG | KB | THM | —/— | 0.0/3.3 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | —/— | —/— | —/— |
| CO | ∅ (no goal, check for consistency) | A1, A2, D1, A3, A4, D2, D3, A5 | KB | SAT | —/— | —/— | 7.3/7.4 |
| D2' | $ess_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}. \lambda X_\mu. \dot\forall\psi_{\mu\to\sigma}. (\psi X \dot\supset \dot\Box\dot\forall Y_\mu. (\phi Y \dot\supset \psi Y))$ | | | | | | |
| CO' | ∅ (no goal, check for consistency) | A1(⊃), A2, D2', D3, A5 | KB | UNS | 7.5/7.8 | —/— | —/— |
| | | A1, A2, D1, A3, A4, D2', D3, A5 | KB | UNS | —/— | —/— | —/— |

## Consistency check: Gödel vs. Scott

- **Scott's assumptions are consistent; shown by Nitpick**
- **Gödel's assumptions are inconsistent; shown by LEO-II (new philosophical result!)**

| | HOL encoding | dependencies | logic | status | LEO-II const/vary | Satallax const/vary | Nitpick const/vary |
|---|---|---|---|---|---|---|---|
| A1 | $[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$ | | | | | | |
| A2 | $[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\wedge}\, \dot{\Box}\dot{\forall}X_\mu.\,(\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$ | | | | | | |
| T1 | $[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Diamond}\dot{\exists}X_\mu.\, \phi X]$ | A1($\supset$), A2 | K | THM | 0.1/0.1 | 0.0/0.0 | —/— |
| | | A1, A2 | K | THM | 0.1/0.1 | 0.0/5.2 | —/— |
| D1 | $g_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \phi X$ | | | | | | |
| A3 | $[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$ | | | | | | |
| C | $[\dot{\Diamond}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$ | | | | | | |
| A4 | $[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Box}p$ … | | | | | | |
| D2 | $ess_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$ … | | | | | | |
| T2 | $[\dot{\forall}X_\mu.\, g_{\mu\to\sigma}X \dot{\supset} (ess_{(\mu\to\sigma)}$ … | | | | | | |
| D3 | $NE_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, (e$ … | | | | | | |
| A5 | $[p_{(\mu\to\sigma)\to\sigma}NE_{\mu\to\sigma}]$ | | | | | | |
| T3 | $[\dot{\Box}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$ | D1, C, T2, D3, A5 | K | CSA | —/— | —/— | 3.8/6.2 |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | K | CSA | —/— | —/— | 8.2/7.5 |
| | | D1, C, T2, D3, A5 | KB | THM | 0.0/0.1 | 0.1/5.3 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | —/— | —/— | —/— |
| MC | $[s_\sigma \dot{\supset} \dot{\Box}s_\sigma]$ | D2, T2, T3 | KB | THM | 17.9/— | 3.3/3.2 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | —/— | —/— | —/— |
| FG | $[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}X_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (\dot{\neg}(p_{(\mu\to\sigma)\to\sigma}\phi) \dot{\supset} \dot{\neg}(\phi X)))]$ | A1, D1 | KB | THM | 16.5/— | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | 12.8/15.1 | 0.0/5.4 | —/— |
| MT | $[\dot{\forall}X_\mu.\, \dot{\forall}Y_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (g_{\mu\to\sigma}Y \dot{\supset} X \dot{=} Y))]$ | D1, FG | KB | THM | —/— | 0.0/3.3 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | —/— | —/— | —/— |
| CO | $\emptyset$ (no goal, check for consistency) | A1, A2, D1, A3, A4, D2, D3, A5 | KB | SAT | —/— | —/— | 7.3/7.4 |
| D2’ | $ess_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \dot{\supset} \psi Y))$ | | | | | | |
| CO’ | $\emptyset$ (no goal, check for consistency) | A1($\supset$), A2, D2’, D3, A5 | KB | UNS | 7.5/7.8 | —/— | —/— |
| | | A1, A2, D1, A3, A4, D2’, D3, A5 | KB | UNS | —/— | —/— | —/— |

## Further Results

- **Monotheism holds**
- **God is flawless**

## Modal Collapse

$\forall\varphi(\varphi \supset \Box\varphi)$

- proved by LEO-II and Satallax
- for constant and varying domains

**Main critique on Gödel's ontological proof:**

- there are no contingent truths
- everything is determined / no free will
- why using modal logic in the first place?

| | HOL encoding | | | | | | |
|---|---|---|---|---|---|---|---|
| A1 | $[\dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu. \dot\neg$ … | | | | | | |
| A2 | $[\dot\forall\phi_{\mu\to\sigma}. \dot\forall\psi_{\mu\to\sigma}. (p_{(\mu\to\sigma)\to}$ … | | | | | | |
| T1 | $[\dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists$ … | | | | | | |
| D1 | $g_{\mu\to\sigma} = \lambda X_\mu. \dot\forall\phi_{\mu\to\sigma}. p_{(\mu-}$ … | | | | | | |
| A3 | $[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$ | | | | | | |
| C | $[\dot\Diamond\dot\exists X_\mu. g_{\mu\to\sigma}X]$ | | | | | | |
| A4 | $[\dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p$ … | | | | | | |
| D2 | $\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}. \lambda$ … | | | | | | |
| T2 | $[\dot\forall X_\mu. g_{\mu\to\sigma}X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)}$ … | | | | | | |
| D3 | $\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu. \dot\forall\phi_{\mu\to\sigma}. (\mathrm{e}$ … | | | | | | |
| A5 | $[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$ | | | | | | |
| T3 | $[\dot\Box\dot\exists X_\mu. g_{\mu\to\sigma}X]$ | | | | | | |
| MC | $[s_\sigma \dot\supset \dot\Box s_\sigma]$ | D2, T2, T3 | KB | THM | 17.9/— | 3.3/3.2 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | —/— | —/— | —/— |
| FG | $[\dot\forall\phi_{\mu\to\sigma}. \dot\forall X_\mu. (g_{\mu\to\sigma}X \supset (\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \supset \neg(\phi X)))]$ | A1, D1 | KB | THM | 16.5/— | 0.0/0.0 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | 12.8/15.1 | 0.0/5.4 | —/— |
| MT | $[\dot\forall X_\mu. \dot\forall Y_\mu. (g_{\mu\to\sigma}X \dot\supset (g_{\mu\to\sigma}Y \dot\supset X \doteq Y))]$ | D1, FG | KB | THM | —/— | 0.0/3.3 | —/— |
| | | A1, A2, D1, A3, A4, D2, D3, A5 | KB | THM | —/— | —/— | —/— |
| CO | ∅ (no goal, check for consistency) | A1, A2, D1, A3, A4, D2, D3, A5 | KB | SAT | —/— | —/— | 7.3/7.4 |
| D2' | $\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}. \lambda X_\mu. \dot\forall\psi_{\mu\to\sigma}. (\psi X \dot\supset \dot\Box\dot\forall Y_\mu. (\phi Y \dot\supset \psi Y))$ | | | | | | |
| CO' | ∅ (no goal, check for consistency) | A1(⊃), A2, D2', D3, A5 | KB | UNS | 7.5/7.8 | —/— | —/— |
| | | A1, A2, D1, A3, A4, D2', D3, A5 | KB | UNS | —/— | —/— | —/— |

# Avoiding the Modal Collapse: Very recent work (not yet published)

Variants of Gödel's proof that avoid the modal collapse

- [Frode Bjørdal, **Understanding Gödel's Ontological Argument**, 1998] (verified and automated)
- [Anthony Anderson, **Some emendations of Gödel's ontological proof**, 1990] (verified and automated)
- [Melvin Fitting, **Types, Tableaux and Gödel's God**, 2002] (ongoing)

Future work

- [André Fuhrmann, 2005]
- [Petr Hajek, 1996, 2001, 2002, 2008, 2011]
- [Szatkowski, 2011]
- ...

Achievements

- significant contribution towards a **Computational Metaphysics**
- **HOL** very fruitfully exploited as a **universal metalogic**
- systematic study of a **prominent philosophical argument**
- even some **novel results** were found **by HOL-ATPs**
- infrastructure can be adapted for **other logics and logic combinations**

Relevance (wrt foundations and applications)

- Theoretical Philosophy, Artificial Intelligence, Computer Science, Maths

Little related work: only for Anselm's simpler argument

- first-order ATP PROVER9 [OppenheimerZalta, 2011]
- interactive proof assistant PVS [Rushby, 2013]

Future work

- continuation of systematic study of the ontological argument
- further studies in **Computational Metaphysics**

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English Site > Germany > Science > Scientists Use Computer to Mathematically Prove Gödel God Theorem

**Holy Logic: Computer Scientists 'Prove' God Exists**

*By David Knight*

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Austrian mathematician Kurt Gödel kept his proof of God's existence a secret for decades. Now two scientists say they have proven it mathematically using a computer.

**Two scientists have formalized a theorem regarding the existence of God penned by mathematician Kurt Gödel. But the God angle is somewhat of a red herring -- the real step forward is the example it sets of how computers can make scientific progress simpler.**

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**See links at** `https://github.com/FormalTheology/GoedelGod/tree/master/Press`