Formalization, Mechanization and Automation of Gödel’s Proof of God’s Existence

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A gift to Priest Edvaldo and his church in Piracicaba, Brazil

Axiom 3
P(G')

∀φ.[P(φ) → ◇∃x.φ(x)]

∀E

P(G') → ◇∃x.G(x)

∀E

◇∃x.G(x)

Theorem 1
Germany
- Telepolis & Heise
- Spiegel Online
- FAZ
- Die Welt
- Berliner Morgenpost
- Hamburger Abendpost
- ...

Austria
- Die Presse
- Wiener Zeitung
- ORF
- ...

Italy
- Repubblica
- Ilsussidario
- ...

India
- DNA India
- Delhi Daily News
- India Today
- ...

US
- ABC News
- ...

International
- Spiegel International
- Yahoo Finance
- United Press Intl.
- ...
Introduction — Quick answers to your most pressing questions!

Are we in contact with Steve Jobs? No

Do you really need a MacBook to obtain the results? No

Is Apple sending us money? No

(but maybe they should)
Are we in contact with Steve Jobs? No

Do you really need a MacBook to obtain the results? No

Is Apple sending us money? No (but maybe they should)
Def: **Ontological Argument/Proof**

* deductive argument
* for the existence of God
* starting from premises, which are justified by pure reasoning, i.e. they do not depend on observation of the world.

Existence of God: different types of arguments/proofs

— a posteriori (use experience/observation in the world)
  — teleological
  — cosmological
  — moral
  — ...

— a priori (based on pure reasoning, independent)
  — ontological argument
    — definitional
    — modal
    — ...
  — other a priori arguments
Introduction

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  — ontological argument
    — definitional
    — modal
    — ...
  — other a priori arguments
Wohl eine jede Philosophie kreist um den ontologischen Gottesbeweis

Rich history on ontological arguments (pros and cons)

Anselm’s notion of God:
“God is that, than which nothing greater can be conceived.”

Gödel’s notion of God:
“A God-like being possesses all ‘positive’ properties.”

To show by logical reasoning:
“(Necessarily) God exists.”
Rich history on ontological arguments (pros and cons)

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Introduction

Different Interests in Ontological Arguments:

- **Philosophical**: Boundaries of Metaphysics & Epistemology
  - We talk about a metaphysical concept (God),
  - but we want to draw a conclusion for the real world.
  - Necessary Existence (NE): metaphysical NE vs. logical NE vs. modal NE

- **Theistic**: Successful argument should convince atheists

- **Ours**: Can computers (theorem provers) be used . . .
  - ... to formalize the definitions, axioms and theorems?
  - ... to verify the arguments step-by-step?
  - ... to fully automate (sub-)arguments?

“Computer-assisted Theoretical Philosophy”
Introduction

Challenge: No provers for Higher-order Quantified Modal Logic (QML)

Our solution: Embedding in Higher-order Classical Logic (HOL)

[BenzmüllerPaulson, Logica Universalis, 2013]

What we did (rough outline for remaining presentation!):

A: Pen and paper: detailed natural deduction proof
B: Formalization: in classical higher-order logic (HOL)
   Automation: theorem provers LEO-II and SATALLAX
   Consistency: model finder NITPICK (NITROX)
C: Step-by-step verification: proof assistant Coq
D: Automation & verification: proof assistant ISABELLE

Did we get any new results? Yes — let’s discuss this later!
Part A:
Informal Proof and Natural Deduction Proof
Gödel’s Manuscript: 1930’s, 1941, 1946-1955, 1970

On Gödel's Proof of God’s Existence
Scott’s Version of Gödel’s Axioms, Definitions and Theorems

A1  Either a property or its negation is positive, but not both: \( \forall \phi [P(\neg \phi) \leftrightarrow \neg P(\phi)] \)

A2  A property necessarily implied by a positive property is positive: \( \forall \phi \forall \psi [(P(\phi) \land \Box \forall x[\phi(x) \rightarrow \psi(x))] \rightarrow P(\psi)] \)

T1  Positive properties are possibly exemplified: \( \forall \varphi [P(\varphi) \rightarrow \Diamond \exists x \varphi(x)] \)

D1  A God-like being possesses all positive properties: \( G(x) \leftrightarrow \forall \phi [P(\phi) \rightarrow \phi(x)] \)

A3  The property of being God-like is positive: \( P(G) \)

C   Possibly, God exists: \( \Diamond \exists x G(x) \)

A4  Positive properties are necessarily positive: \( \forall \phi [P(\phi) \rightarrow \Box P(\phi)] \)

D2  An essence of an individual is a property possessed by it and necessarily implying any of its properties: \( \phi \text{ ess. } x \leftrightarrow \phi(x) \land \forall \psi (\psi(x) \rightarrow \Box \forall y (\phi(y) \rightarrow \psi(y))) \)

T2  Being God-like is an essence of any God-like being: \( \forall x[G(x) \rightarrow G \text{ ess. } x] \)

D3  Necessary existence of an individual is the necessary exemplification of all its essences: \( NE(x) \leftrightarrow \forall \phi [\phi \text{ ess. } x \rightarrow \Box \exists y \phi(y)] \)

A5  Necessary existence is a positive property: \( P(NE) \)

T3  Necessarily, God exists: \( \Box \exists x G(x) \)
Proof Overview

T3: □∃x. G(x)
Proof Overview

C1: \( \Diamond \exists z. G(z) \)

\[ \frac{ }{ T3: \Box \exists x. G(x) } \]
Proof Overview

C1: ∃z.G(z)       L2: ∃z.G(z) → □∃x.G(x)

T3: □∃x.G(x)
**Proof Overview**

\[ L2: \Box \exists z. G(z) \rightarrow \Box \exists x. G(x) \]

\[ C1: \Box \exists z. G(z) \]
\[ L2: \Box \exists z. G(z) \rightarrow \Box \exists x. G(x) \]

\[ T3: \Box \exists x. G(x) \]
Proof Overview

∀ξ. [◊□ξ → □ξ]

L2: ◊∃z. G(z) → □∃x. G(x)

C1: ◊∃z. G(z)  L2: ◊∃z. G(z) → □∃x. G(x)  T3: □∃x. G(x)
\begin{align*}
\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi] & \quad \text{S5} \\
\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x) & \quad \text{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)
\end{align*}

\begin{align*}
\text{C1: } \Diamond \exists z. G(z) & \quad \text{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
\text{T3: } \Box \exists x. G(x) &
\end{align*}
Proof Overview

L1: ∃z. G(z) → □∃x. G(x)

◊∃z. G(z) → ◊□∃x. G(x)

L2: ◊∃z. G(z) → □∃x. G(x)

C1: ◊∃z. G(z)

L2: ◊∃z. G(z) → □∃x. G(x)

T3: □∃x. G(x)

S5

∀ξ. [◊□ξ → □ξ]
Proof Overview

D1: \( G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)] \)

L1: \( \exists z. G(z) \rightarrow \Box \exists x. G(x) \)

\[
\begin{align*}
& \Box \exists z. G(z) \rightarrow \Box \Box \exists x. G(x) \\
& \forall \xi. [\Box \Box \xi \rightarrow \Box \xi] \quad \text{S5}
\end{align*}
\]

L2: \( \Box \exists z. G(z) \rightarrow \Box \exists x. G(x) \)

C1: \( \Box \exists z. G(z) \)

L2: \( \Box \exists z. G(z) \rightarrow \Box \exists x. G(x) \)

T3: \( \Box \exists x. G(x) \)
**Proof Overview**

**D1:** $G(x) \equiv \forall \phi. [P(\phi) \to \phi(x)]$

**D3**:* $E(x) \equiv \Box \exists y. G(y)$

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>L1:</strong></td>
<td>$\exists z. G(z) \to \Box \exists x. G(x)$</td>
</tr>
<tr>
<td><strong>S5:</strong></td>
<td>$\forall \xi. [\Diamond \xi \to \Box \xi]$</td>
</tr>
</tbody>
</table>

**C1:** $\Diamond \exists z. G(z)$

**L2:** $\Diamond \exists z. G(z) \to \Box \exists x. G(x)$

**T3:** $\Box \exists x. G(x)$
Proof Overview

**D1:** \( G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)] \)

**D3**: \( E(x) \equiv \Box \exists y. G(y) \) (cheating!)

\[ P(E) \]

\[
\frac{\exists z. G(z) \rightarrow \Box \exists x. G(x)}{\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)}
\]

**L1:** \( \exists z. G(z) \rightarrow \Box \exists x. G(x) \)

**S5**

\[ \forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi] \]

**L2:** \( \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \)

**C1:** \( \Diamond \exists z. G(z) \)

**L2:** \( \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \)

**T3:** \( \Box \exists x. G(x) \)
**D1:** \( G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)] \)

**D3\(^*\):** \( E(x) \equiv \Box \exists y. G(y) \)

**D3:** \( E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \Box \exists y. \varphi(y)] \)

**T2:** \( \forall y. [G(y) \rightarrow G \text{ ess } y] \)

\[ \begin{align*}
\text{L1:} & \quad \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
\text{L2:} & \quad \Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x) \\
\text{S5} & \quad \forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi] \\
\text{P(E)} & \quad \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
\text{C1:} & \quad \exists z. G(z) \\
\text{L2:} & \quad \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
\text{T3:} & \quad \Box \exists x. G(x)
\end{align*} \]
**Proof Overview**

**D1:** \( G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)] \)

**D3**

**D3*: \( E(x) \equiv \Box \exists y. G(y) \)

**D3:** \( E(x) \equiv \forall \varphi.[\varphi \text{ ess } x \rightarrow \Box \exists y. \varphi(y)] \)

\[
\begin{align*}
\text{T2: } & \forall y.[G(y) \rightarrow G \text{ ess } y] \\
\text{L1: } & \exists z.G(z) \rightarrow \Box \exists x.G(x) \\
& \Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x) \\
\text{S5: } & \forall \xi.[\Diamond \Box \xi \rightarrow \Box \xi]\end{align*}
\]

**A5**

\[
\text{P(E)}
\]

\[
\begin{align*}
\text{L2: } & \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x) \\
\text{C1: } & \Diamond \exists z.G(z) \\
\text{L2: } & \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x) \\
\text{T3: } & \Box \exists x.G(x)
\end{align*}
\]
Proof Overview

**D1:** \( G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)] \)

**D3**: \( E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)] \)

**D3**: \( E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)] \)

**T2**: \( \forall y. [G(y) \rightarrow G \text{ ess } y] \)

**A5**

\[
\overline{P(E)}
\]

\[
\begin{align*}
L1: & \exists z. G(z) \rightarrow \square \exists x. G(x) \\
\Rightarrow & \diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x) \\
\Rightarrow & \forall \xi. [\diamond \diamond \xi \rightarrow \square \xi] \\
L2: & \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)
\end{align*}
\]

**C1**: \( \diamond \exists z. G(z) \)

**L2**: \( \diamond \exists z. G(z) \rightarrow \square \exists x. G(x) \)

**T3**: \( \square \exists x. G(x) \)
Proof Overview

D1: \( G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)] \)

D2: \( \varphi \text{ ess } x \equiv \varphi(x) \land \forall \psi.(\psi(x) \rightarrow \Box \forall x.(\varphi(x) \rightarrow \psi(x))) \)

D3*: \( E(x) \equiv \Box \exists y.G(y) \)

D3: \( E(x) \equiv \forall \varphi.[\varphi \text{ ess } x \rightarrow \Box \exists y.\varphi(y)] \)

\[\begin{array}{c}
\forall \varphi.\lnot [P(\varphi) \rightarrow P(\lnot \varphi)] \\
\overline{\forall \varphi.[P(\varphi) \rightarrow \Box P(\varphi)]} \\
\overline{\forall y.[G(y) \rightarrow G \text{ ess } y]} \\
\lnot P(E) \\
\lnot \exists z.G(z) \rightarrow \Box \exists x.G(x) \\
\overline{\Box \exists z.G(z) \rightarrow \Box \exists x.G(x)} \\
\forall \xi.[\Box \Box \xi \rightarrow \Box \xi]
\end{array}\]

A1b

A4

A5

L1: \( \exists z.G(z) \rightarrow \Box \exists x.G(x) \)

S5

L2: \( \Box \exists z.G(z) \rightarrow \Box \exists x.G(x) \)

C1: \( \Box \exists z.G(z) \)

L2: \( \exists z.G(z) \rightarrow \Box \exists x.G(x) \)

T3: \( \Box \exists x.G(x) \)
Proof Overview

**D1:** \( G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)] \)

**D2:** \( \varphi \text{ ess } x \equiv \varphi(x) \land \forall \psi.(\psi(x) \rightarrow \Box \forall x.(\varphi(x) \rightarrow \psi(x))) \)

**D3\( ^* \): \( E(x) \equiv \Box \exists y. G(y) \)

**D3:** \( E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \Box \exists y. \varphi(y)] \)

**C1:** \( \Diamond \exists z. G(z) \)

\[
\begin{align*}
\text{A1b} & \quad \forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)] \\
\text{T2:} & \quad \forall y. [G(y) \rightarrow \text{G ess } y] \\
\text{L1:} & \quad \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
\text{L2:} & \quad \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)
\end{align*}
\]

\[
\begin{align*}
\text{A4} & \quad \forall \varphi. [P(\varphi) \rightarrow \Box P(\varphi)] \\
\text{A5} & \quad \Box \exists y. G(y) \\
\text{S5} & \quad \forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi] \\
\text{C1:} & \quad \Diamond \exists z. G(z) \\
\text{L2:} & \quad \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
\text{T3:} & \quad \Box \exists x. G(x)
\end{align*}
\]
Proof Overview

**D1:** \( G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)] \)

**D2:** \( \varphi \text{ ess } x \equiv \varphi(x) \land \forall \psi. (\psi(x) \rightarrow \Box \forall x. (\varphi(x) \rightarrow \psi(x))) \)

**D3\(^*\): \( E(x) \equiv \Box \exists y. G(y) \)  

**D3:** \( E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \Box \exists y. \varphi(y)] \)

---

\[ P(G) \]

---

**C1:** \( \Diamond \exists z. G(z) \)

**A1b**

\[ \forall \varphi. [\neg P(\varphi) \rightarrow \neg \neg P(\varphi)] \]

**A4**

\[ \forall \varphi. [P(\varphi) \rightarrow \Box \neg P(\varphi)] \]

**A5**

\[ \neg P(E) \]

**T2:** \( \forall y. [G(y) \rightarrow G \text{ ess } y] \)

**L1:** \( \exists z. G(z) \rightarrow \Box \exists x. G(x) \)

\[ \Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x) \]

**S5**

\[ \forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi] \]

**L2:** \( \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \)

**C1:** \( \Diamond \exists z. G(z) \)

**L2:** \( \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \)

**T3:** \( \Box \exists x. G(x) \)
**D1:** \( G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)] \)

**D2:** \( \varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi.(\psi(x) \rightarrow \Box \forall x.(\varphi(x) \rightarrow \psi(x))) \)

**D3**:* \( E(x) \equiv \Box \exists y.G(y) \)

**D3:** \( E(x) \equiv \forall \varphi.[\varphi \text{ ess } x \rightarrow \Box \exists y.\varphi(y)] \)

\[\begin{align*}
\text{A3} & \\
\neg P(G) & \\
\hline
\text{C1: } \Box \exists z.G(z) & \\
\end{align*}\]

\[\begin{align*}
\text{A1b} & \\
\forall \varphi.[-P(\varphi) \rightarrow P(\neg \varphi)] & \\
\forall \varphi.[P(\varphi) \rightarrow \Box P(\varphi)] & \\
\hline
\text{T2: } \forall y.[G(y) \rightarrow G \text{ ess } y] & \\
\text{A4} & \\
\text{A5} & \\
\neg P(E) & \\
\hline
\text{L1: } \exists z.G(z) \rightarrow \Box \exists x.G(x) & \\
\Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x) & \\
\Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x) & \\
\forall \xi.[\Diamond \Box \xi \rightarrow \Box \xi] & \\
\text{S5} & \\
\hline
\text{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x) & \\
\text{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x) & \\
\text{T3: } \Box \exists x.G(x) & \\
\end{align*}\]
Proof Overview

**D1**: \( G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)] \)

**D2**: \( \varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi.(\psi(x) \rightarrow \Box \forall x.(\varphi(x) \rightarrow \psi(x))) \)

**D3**: \( E(x) \equiv \Box \exists y. G(y) \)

**D3**: \( E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \Box \exists y. \varphi(y)] \)

**A3**

\[
\frac{P(G)}{P(G)}
\]

**T1**: \( \forall \varphi. [P(\varphi) \rightarrow \Box \exists x. \varphi(x)] \)

**C1**: \( \Box \exists z. G(z) \)

**A1b**

\[
\frac{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]}{\forall \varphi. [P(\varphi) \rightarrow \square P(\varphi)]}
\]

**A4**

\[
\frac{\forall \varphi. [P(\varphi) \rightarrow \square P(\varphi)]}{P(E)}
\]

**A5**

\[
\frac{P(E)}{P(E)}
\]

**T2**: \( \forall y. [G(y) \rightarrow G \text{ ess } y] \)

**L1**: \( \exists z. G(z) \rightarrow \Box \exists x. G(x) \)

\[
\frac{\Box \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\exists z. G(z) \rightarrow \Box \exists x. G(x)}
\]

**L2**: \( \Box \exists z. G(z) \rightarrow \Box \exists x. G(x) \)

**S5**

\[
\frac{\forall \xi. [\Box \exists \xi \rightarrow \Box \xi]}{\forall \xi. [\Box \exists \xi \rightarrow \Box \xi]}
\]

**T3**: \( \Box \exists x. G(x) \)
**D1:** \( G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)] \)

**D2:** \( \varphi \text{ ess } x \equiv \varphi(x) \land \forall \psi.(\psi(x) \rightarrow \Box \forall x.(\varphi(x) \rightarrow \psi(x))) \)

**D3:** \( E(x) \equiv \Box \exists y. G(y) \)

**D3** \( E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \Box \exists y. \varphi(y)] \)

---

**A3**

\[
\begin{align*}
\forall \varphi. \forall \psi & .[P(\varphi) \land \Box \forall x.[\varphi(x) \rightarrow \psi(x)]] \rightarrow P(\psi) \\
\forall \varphi. P(\varphi) \rightarrow \Box P(\varphi)
\end{align*}
\]

**T1:** \( \forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)] \)

**C1:** \( \Diamond \exists z. G(z) \)

---

**A1b**

\[
\forall \varphi . [\neg P(\varphi) \rightarrow \Box P(\neg \varphi)]
\]

**A4**

\[
\forall \varphi . [P(\varphi) \rightarrow \Box P(\varphi)]
\]

**T2:** \( \forall y. [G(y) \rightarrow G \text{ ess } y] \)

**L1:** \( \exists z. G(z) \rightarrow \Box \exists x. G(x) \)

**L2:** \( \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \)

**C1:** \( \Diamond \exists z. G(z) \)

**T3:** \( \Box \exists x. G(x) \)
Proof Overview

**D1:** $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

**D2:** $\varphi \ ess \ x \equiv \varphi(x) \land \forall \psi. (\psi(x) \rightarrow \Box \forall x. (\varphi(x) \rightarrow \psi(x)))$

**D3:** $E(x) \equiv \forall \varphi. [\varphi \ ess \ x \rightarrow \Box \exists y. \varphi(y)]$

\[\frac{\text{A3}}{P(G)} \quad \frac{\text{A2}}{\Box \forall \psi. [(P(\varphi) \land \Box \forall x. (\varphi(x) \rightarrow \psi(x))) \rightarrow P(\psi)]} \quad \frac{\text{A1a}}{\forall \varphi. [P(\neg \varphi) \rightarrow \neg P(\varphi)]} \]

\[\overline{T1: \forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)]} \]

\[\text{C1: } \Diamond \exists z. G(z) \]

\[\frac{\text{A1b}}{\Box \forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]} \quad \frac{\text{A4}}{\Box \forall \varphi. [P(\varphi) \rightarrow \Box P(\varphi)]} \]

\[\overline{T2: \forall y. [G(y) \rightarrow G \ ess \ y]} \quad \overline{P(E)} \]

\[\text{L1: } \exists z. G(z) \rightarrow \Box \exists x. G(x) \quad \text{S5} \quad \Box \forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi] \]

\[\overline{\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)} \quad \overline{\exists z. G(z) \rightarrow \Box \exists x. G(x)} \]

\[\overline{\text{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)} \]

\[\text{C1: } \Diamond \exists z. G(z) \quad \text{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \]

\[\overline{T3: \Box \exists x. G(x)} \]
Natural Deduction Calculus

\[ \frac{\neg A \quad \neg B}{A \lor B \quad \text{C} \quad \text{C}} \quad \lor_E \]

\[ \frac{A \lor B \quad C \quad C}{A \quad \text{B}} \quad \land_I \]

\[ \frac{A \quad \text{B}}{A \rightarrow B} \quad \rightarrow_I \]

\[ \frac{B}{A \rightarrow B} \quad \rightarrow_E \]

\[ \frac{A[\alpha]}{\forall x. A[x]} \quad \forall_I \]

\[ \frac{\forall x. A[x]}{A[t]} \quad \forall_E \]

\[ \frac{A[t]}{\exists x. A[x]} \quad \exists_I \]

\[ \frac{\exists x. A[x]}{A[\beta]} \quad \exists_E \]

\[ \neg A \equiv A \rightarrow \bot \]

\[ \frac{\neg \neg A}{A} \quad \neg \neg_E \]
Natural Deduction Calculus
Rules for Modalities

\[ \vdash A \quad \Box I \]

\[ \vdash \Box A \quad \Box I \]

\[ \vdash A \quad \Diamond I \]

\[ \vdash \Diamond A \equiv \neg \Box \neg A \]

\[ \vdash \Box A \quad \Box E \]

\[ \vdash \Delta A \quad \Box E \]

\[ \vdash \Delta A \quad \Diamond E \]

\[ \vdash \Diamond A \equiv \neg \Box \neg A \]
T1 and C1

\[
\begin{align*}
\textbf{A2} & \quad \forall \phi. \forall \psi. [(P(\phi) \land \Box \forall x. [\phi(x) \rightarrow \psi(x))] \rightarrow P(\psi)] \\
& \quad \forall \psi. [(P(\rho) \land \Box \forall x. [\rho(x) \rightarrow \psi(x))] \rightarrow P(\psi)] \\
& \quad (P(\rho) \land \Box \forall x. [\rho(x) \rightarrow \neg \rho(x)]) \rightarrow P(\neg \rho) \\
& \quad (P(\rho) \land \Box \forall x. [\neg \rho(x)]) \rightarrow P(\neg \rho) \\
& \quad (P(\rho) \land \Box \forall x. [\neg \rho(x)]) \rightarrow \neg P(\rho) \\
& \quad P(\rho) \rightarrow \Box \exists x. \rho(x) \\
& \quad \textbf{T1: } \forall \phi. [P(\phi) \rightarrow \Box \exists x. \phi(x)] \\
\end{align*}
\]

\[
\begin{align*}
\textbf{A1a} & \quad \forall \phi. [P(\neg \phi) \rightarrow \neg P(\phi)] \\
& \quad P(\neg \rho) \rightarrow \neg P(\rho) \\
\end{align*}
\]

\[
\begin{align*}
\textbf{T1} & \quad \forall \phi. [P(\phi) \rightarrow \Box \exists x. \phi(x)] \\
& \quad P(G) \rightarrow \Box \exists x. G(x) \\
& \quad \Box \exists x. G(x) \\
\end{align*}
\]
Natural Deduction Proofs

T2 (Partial)

\[ \psi(x) \rightarrow \Box P(\psi) \rightarrow E \]

\[ \Box P(\psi) \rightarrow E \]

\[ P(\psi) \rightarrow \forall x. (G(x) \rightarrow \psi(x)) \rightarrow E \]

\[ \Box \forall x. (G(x) \rightarrow \psi(x)) \rightarrow I \]

\[ \Box \forall x. (G(x) \rightarrow \psi(x)) \rightarrow E \]

\[ \psi(x) \rightarrow \Box \forall x. (G(x) \rightarrow \psi(x)) \rightarrow I \]
Part B:

Formalization: in classical higher-order logic (HOL)
Automation: theorem provers LEO-II and SATALLAX
Consistency: model finder NITPICK (Nitrox)
Challenge: No provers for *Higher-order Quantified Modal Logic* (QML)

Our solution: Embedding in *Higher-order Classical Logic* (HOL)
Then use existing HOL theorem provers for reasoning in QML

[BenzmüllerPaulson, Logica Universalis, 2013]

Previous empirical findings:

Embedding of *First-order Modal Logic* in HOL works well

[BenzmüllerOttenRaths, ECAI, 2012]
[Benzmüller, LPAR, 2013]
Formalization in HOL

QML \( \varphi, \psi \ ::= \ldots | \neg \varphi | \varphi \land \psi | \varphi \to \psi | \Box \varphi | \Diamond \varphi | \forall x \varphi | \exists x \varphi | \forall P \varphi \)

- Kripke style semantics (possible world semantics)

HOL \( s, t \ ::= C | x | \lambda x s | s t | \neg s | s \lor t | \forall x t \)

- meanwhile very well understood
- **Henkin semantics** vs. standard semantics
- various theorem provers do exist
  - interactive: Isabelle/HOL, HOL4, Hol Light, Coq/HOL, PVS, . . .
  - automated: TPS, LEO-II, Satallax, Nitpick, Isabelle/HOL, . . .
Formalization in HOL

QML \( \varphi, \psi \) ::= \ldots | \neg \varphi | \varphi \land \psi | \varphi \rightarrow \psi | \Box \varphi | \Diamond \varphi | \forall x \varphi | \exists x \varphi | \forall P \varphi

HOL \( s, t \) ::= C | x | \lambda xs | s t | \neg s | s \lor t | \forall x t

QML in HOL: QML formulas \( \varphi \) are mapped to HOL predicates \( \varphi_{t \rightarrow o} \)

\[
\begin{align*}
\neg &= \lambda \varphi_{t \rightarrow o} \lambda s_t \neg \varphi s \\
\land &= \lambda \varphi_{t \rightarrow o} \lambda \psi_{t \rightarrow o} \lambda s_t (\varphi s \land \psi s) \\
\rightarrow &= \lambda \varphi_{t \rightarrow o} \lambda \psi_{t \rightarrow o} \lambda s_t (\neg \varphi s \lor \psi s) \\
\Box &= \lambda \varphi_{t \rightarrow o} \lambda s_t \forall u_t (\neg rsu \lor \varphi u) \\
\Diamond &= \lambda \varphi_{t \rightarrow o} \lambda s_t \exists u_t (rsu \land \varphi u) \\
\forall &= \lambda h_{\mu \rightarrow (t \rightarrow o)} \lambda s_t \forall d_{\mu} hds \\
\exists &= \lambda h_{\mu \rightarrow (t \rightarrow o)} \lambda s_t \exists d_{\mu} hds \\
\forall &= \lambda H_{(\mu \rightarrow (t \rightarrow o)) \rightarrow (t \rightarrow o)} \lambda s_t \forall d_{\mu} Hds \\
\text{valid} &= \lambda \varphi_{t \rightarrow o} \forall w_t \varphi w
\end{align*}
\]

The equations in Ax are given as axioms to the HOL provers!

(Remark: Note that we are here dealing with constant domain quantification)

Christoph Benzmüller and Bruno Woltzenlogel Paleo

On Gödel’s Proof of God’s Existence
Formalization in HOL

QML  \( \varphi, \psi \) ::= \ldots \mid \neg \varphi \mid \varphi \land \psi \mid \varphi \to \psi \mid \Box \varphi \mid \Diamond \varphi \mid \forall x \varphi \mid \exists x \varphi \mid \forall P \varphi

HOL  \( s, t \) ::= \ C \mid x \mid \lambda xs \mid st \mid \neg s \mid s \lor t \mid \forall x t

QML in HOL: QML formulas \( \varphi \) are mapped to HOL predicates \( \varphi_{t \to o} \)

\[
\begin{align*}
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\land &= \lambda \varphi_{t \to o} \lambda \psi_{t \to o} \lambda s_t (\varphi s \land \psi s) \\
\to &= \lambda \varphi_{t \to o} \lambda \psi_{t \to o} \lambda s_t (\neg \varphi s \lor \psi s) \\
\Box &= \lambda \varphi_{t \to o} \lambda s_t (\forall u_t (\neg rsu \lor \varphi u)) \\
\Diamond &= \lambda \varphi_{t \to o} \lambda s_t (\exists u_t (rsu \lor \varphi u)) \\
\forall &= \lambda h_{(t \to o)} (\lambda s_t \forall d_{(t \to o)} hds) \\
\exists &= \lambda h_{(t \to o)} (\lambda s_t \exists d_{(t \to o)} hds) \\
\forall &= \lambda H_{(t \to o)} (\lambda s_t \forall hds)
\end{align*}
\]

valid = \( \lambda \varphi_{t \to o} (\forall w_t \varphi w) \)

The equations in Ax are given as axioms to the HOL provers!
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On Gödel’s Proof of God’s Existence
Formalization in HOL

QML  \( \varphi, \psi \) ::= \ldots \mid \neg \varphi \mid \varphi \land \psi \mid \varphi \rightarrow \psi \mid \Box \varphi \mid \Diamond \varphi \mid \forall x \varphi \mid \exists x \varphi \mid \forall P \varphi \\

HOL  \( s, t \) ::= C \mid x \mid \lambda xs \mid st \mid \neg s \mid s \lor t \mid \forall x t \\

QML in HOL: QML formulas \( \varphi \) are mapped to HOL predicates \( \varphi_{t \to o} \)

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\forall &= \lambda H_{(\mu \to (t \to o)) \to (t \to o)} \lambda s_t \forall d_{\mu} Hds \\
\text{valid} &= \lambda \varphi_{t \to o} \forall w_t \varphi w
\end{align*}
\]

The equations in \textbf{Ax} are given as axioms to the HOL provers!

(Remark: Note that we are here dealing with constant domain quantification)
Example

**QML formula**

QML formula in HOL

expansion, $\beta\eta$-conversion

expansion, $\beta\eta$-conversion

expansion, $\beta\eta$-conversion

\[ \Diamond \exists x G(x) \]

valid \( (\Diamond \exists x G(x))_{t \to o} \)

\[ \forall w_t (\Diamond \exists x G(x))_{t \to o} w \]

\[ \forall w_t \exists u_t (ruw \land (\exists x G(x))_{t \to o} u) \]

\[ \forall w_t \exists u_t (ruw \land \exists x Gxu) \]

**What are we doing?**

In order to prove that \( \varphi \) is valid in QML,

\( \rightarrow \) we instead prove that \( \text{valid} \ \varphi_{t \to o} \) can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Expansion: user or prover may flexibly choose expansion depth
Example

QML formula

QML formula in HOL

expansion, \(\beta\eta\)-conversion

expansion, \(\beta\eta\)-conversion

expansion, \(\beta\eta\)-conversion

What are we doing?

In order to prove that \(\varphi\) is valid in QML, we instead prove that valid \(\varphi_{i\rightarrow 0}\) can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Expansion: user or prover may flexibly choose expansion depth
Example

QML formula

$\exists x G(x)$

QML formula in HOL

$\exists x G(x)$

valid $\exists x G(x)$

$\forall w (\exists x G(x))$ $\to$ $w$

$\forall w \exists u (rwu \land (\exists x G(x)))$ $\to$ $u$ $\forall w \exists u (rwu \land \exists Gxu)$

What are we doing?

In order to prove that $\varphi$ is valid in QML, we instead prove that valid $\varphi$ $\to$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Expansion: user or prover may flexibly choose expansion depth
Formalization in HOL

Example

QML formula

\( \exists x G(x) \)

QML formula in HOL

valid (\( \exists x G(x) \))

expansion, \( \beta \eta \)-conversion

\( \forall w (\exists x G(x)) \)

expansion, \( \beta \eta \)-conversion

\( \forall w \exists u (r w u \land (\exists x G(x))) \)

expansion, \( \beta \eta \)-conversion

\( \forall w \exists u (r w u \land \exists G u) \)

What are we doing?

In order to prove that \( \varphi \) is valid in QML,

\( \rightarrow \) we instead prove that valid \( \varphi \) can be derived from Ax in HOL.

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Expansion: user or prover may flexibly choose expansion depth
Formalization in HOL

Example

QML formula
QML formula in HOL
expansion, $\beta\eta$-conversion
expansion, $\beta\eta$-conversion
expansion, $\beta\eta$-conversion

What are we doing?

In order to prove that $\varphi$ is valid in QML,
$\rightarrow$ we instead prove that $\text{valid } \varphi_{t\rightarrow o}$ can be derived from $Ax$ in HOL.

This can be done with interactive or automated HOL theorem provers.

Expansion: user or prover may flexibly choose expansion depth
Example

QML formula
QML formula in HOL
expansion, $\beta\eta$-conversion
expansion, $\beta\eta$-conversion
expansion, $\beta\eta$-conversion

What are we doing?

In order to prove that $\varphi$ is valid in QML,
\[ \to \] we instead prove that valid $\varphi_{i\to o}$ can be derived from $Ax$ in HOL.

This can be done with interactive or automated HOL theorem provers.

Expansion: user or prover may flexibly choose expansion depth
Formalization in HOL

Example

QML formula
QML formula in HOL
expansion, βη-conversion
expansion, βη-conversion
expansion, βη-conversion

\[ \Diamond \exists x G(x) \]
valid \( (\Diamond \exists x G(x))_{\tau \rightarrow \omega} \)
\[ \forall w (\Diamond \exists x G(x))_{\tau \rightarrow \omega} \] 
\[ \forall w \exists u (rwu \land (\exists x G(x))_{\tau \rightarrow \omega} u) \] 
\[ \forall w \exists u (rwu \land \exists Gxu) \]

What are we doing?

In order to prove that \( \varphi \) is valid in QML,
\(-\rightarrow\) we instead prove that valid \( \varphi_{\tau \rightarrow \omega} \) can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Expansion: user or prover may flexibly choose expansion depth
Automated Theorem Provers and Model Finders for HOL


TPS … (Peter Andrews) ?
LEO-I/LEO-II (myself)
Isabelle (Nipkow/Paulson/Blanchette)
Satallax (Brown)
Nitpick (Blanchette)
agsyHOL (Lindblatt)

- all accept TPTP THF Syntax [SutcliffeBenzmüller, J.Form.Reas, 2009]
  - can be called remotely via SystemOnTPTP at Miami
  - they significantly gained in strength over the last years
  - they can be bundled into a combined prover HOL-P

Exploit HOL with Henkin semantics as metalogic
Automate other logics (& combinations) via semantic embeddings

— HOL-P becomes a Universal Reasoner —
Proof Automation and Consistency Checking: Demo!

Provers are called remotely in Miami — no local installation needed!
Part C:
Formalization and Verification in Coq
Goal: verification of the natural deduction proof
  Step-by-step formalization
    Almost no automation (intentionally!)
Interesting facts:
  Embedding is transparent to the user
  Embedding gives labeled calculus for free
Goal: verification of the natural deduction proof
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Interesting facts:
  - Embedding is transparent to the user
  - Embedding gives labeled calculus for free
On Gödel’s Proof of God’s Existence

Coq Proof

(* Constant predicate that distinguishes positive properties *)
Parameter Positive: (u -> o) -> o.

(* Axiom A1: either a property or its negation is positive, but not both *)
Axiom axiomA1 : V (mforall p, (Positive (fun x: u -> m-> (p x))) m-> (m- (Positive p))).
Axiom axiomB1 : V (mforall p, (m- (Positive p))) m-> (Positive (fun x: u -> m- (p x))).

(* Axiom A2: a property necessarily implied by a positive property is positive *)
Axiom axiom2 : V (mforall p, mforall q, Positive p m\ (box (mforall x, (p x) m-> (q x)))).

(* Theorem A2: positive properties are possibly exemplified *)
Theorem theorem1: V (mforall p, (Positive p) m-> dia (mexists x, p x)).
Proof.
intro.
intro p.
intro H1.
proof_by CONTRADICTION H2.
apply not dia box not in H2.
assert (H3: (box (mforall x, m- (p x)))) w). (* Lemma from Scott’s notes *)
box_intro w1 R1.
intro x.
assert (H4: ((m- (mexists x : u, p x)) w1)).
box_elim H2 w1 R1 G2.
exact G2.
clear H2 R1 H1 w.
intro H3.
apply H4.
exists X.
exact H3.
assert (H6: ((box (mforall x, (p x) m-> m- (x m= x)))) w). (* Lemma from Scott’s notes *)
box_intro w1 R1.
intro x.
intro H7.
intro H8.
box_elim H3 w1 R1 G3.
Part D:

automation & verification: proof assistant ISABELLE
What is Isabelle?

Isabelle is a generic proof assistant. It allows mathematical formulas to be expressed in a formal language and provides tools for proving those formulas in a logical calculus. Isabelle is developed at University of Cambridge (Larry Paulson), Technische Universität München (Tobias Nipkow) and Université Paris-Sud (Markus Wenzel). See the Isabelle overview for a brief introduction.

Now available: Isabelle2013

![Download for Mac OS X](Download for Mac OS X) - ![Download for Windows](Download for Windows)

Some highlights:

- Improvements of Isabelle/Scala and Isabelle/Isar editor IDE.
- Advanced build tool based on Isabelle/Scala.
- Updated manuals: isar-ref, implementation, system.
- Pure: improved support for block-structured specification contexts.
- HOL tool enhancements: Sledgehammer, Lifting, Quickcheck.
- HOL: New SNF-based (co)datatype package.
- Improved performance thanks to Poly/ML 5.5.0.

See also the cumulative NEWS.

Distribution & Support

Isabelle is distributed for free under the BSD license. It includes source and binary packages and documentation, see the detailed installation instructions. A vast collection of Isabelle examples and applications is available from the Archive of Formal Proofs.

Support is available by ample documentation, the Isabelle community Wiki, and the following mailing lists:

- isabelle-users@cl.cam.ac.uk provides a forum for Isabelle users to discuss problems, exchange information, and make announcements. Users of official Isabelle releases should subscribe or see the archive (also available via Google groups and Archive).
- isabelle-dev@in.tum.de covers the Isabelle development process, including intermediate repository versions, and administrative issues concerning the website or testing infrastructure. Early adopters of repository versions should subscribe or see the archive (also available at mail-archive.com or gmane.org).
Isabelle/HOL (Cambridge University/TU Munich)
- HOL instance of the generic Isabelle proof assistant
- User interaction and proof automation
- Automation is supported by Sledgehammer tool
- Verification of the proofs in Isabelle/HOL’s small proof kernel

What we did?
- Proof automation of Gödel’s proof script (Scott version)
- Sledgehammer makes calls to remote THF provers in Miami
- These calls the suggest respective calls to the Metis prover
- Metis proofs are verified in Isabelle/HOL’s proof kernel

See the handout (generated from the Isabelle source file).
On Gödel's Proof of God's Existence

corollary C: 

\[ \neg (\neg G) \]

sledgehammer [provers = remote_leo2] by (metis A3 T1)

text {
* Axiom A4: is added: $\forall x \ (P(x) \to \Box \ P(x))$
(Positive properties are necessarily positive). *)

axiomatization where A4: "\[ (\forall (\lambda \phi. \ P \ \phi \Rightarrow \Box (P \ \phi)) \]

text {
* Symbol @text "ess" for 'Essence' is introduced and defined as
$\forall \phi \ (\forall x \ (P(x) \to \Box \ P(x)) \to \Box \ P(x))$
(An essence is a property possessed by it
and necessarily implying any of its properties). *)

definition ess : "(\mu \Rightarrow \sigma) \Rightarrow \mu \Rightarrow \sigma" (infixr "ess" 85) where
"\phi \ ess \ x = \phi \ x \ \equiv \ (\forall \psi \ (\lambda \ y. \ \psi \ m \Rightarrow \Box (\lambda \ y. \ \psi \ m \Rightarrow \Box \ y)))"

text {
* Next, Sledgehammer and Metis prove theorem T2: $\forall x \ (G(x) \to \Box \ ess[G][x])$
(Being God-like is an essence of any God-like being). *)

definition T2: "(\forall (\lambda \ x. \ G \ x \ m \Rightarrow \ Box \ G \ x))"

sledgehammer [provers = remote_leo2] by (metis A3b A4 G_def ess_def)

text {
* Symbol NE", for 'Necessary Existence', is introduced and
defined as $\forall \ (\exists x \ (\forall \phi \ (\exists \psi \ (\forall x \ (P(x) \to \Box \ P(x))) \to \Box \ P(x)))$
(Necessary existence of an individual is the necessary exemplification of all its essences). *)

definition NE : "(\mu \Rightarrow \sigma) \Rightarrow \mu \Rightarrow \sigma" where
"NE = (\lambda \ x. \ (\forall \phi. \ \phi \ x \ \equiv \ Box (\exists \phi))"

Sledgehammering...
“God is dead.”
- Nietzsche, 1883

“Nietzsche is dead.”
- God, 1900

Part E: Criticisms
∀P. [◊ □ P → □ P]

If something is possibly necessary, then it is necessary.

◊□(A ∨ ¬A)   □(A ∨ ¬A)

logical necessity ~ validity

for all M, M |= F → □F

logical possibility ~ satisfiability

exists M, M |= F → ◊F

What about iterations?

◊◊◊◊F

weak intuitions ⇒ dozens of modal logics

S5 is considered adequate

(But KB is sufficient!)
∀P. [◇□P → □P]

If something is possibly necessary, then it is necessary.

◇□(A ∨ ¬A) □(A ∨ ¬A)

logical necessity ≈ validity
for all M, M |= F → □F

logical possibility ≈ satisfiability
exists M, M |= F → ◇F

What about iterations?

◇□◇◇F

weak intuitions ⇒ dozens of modal logics

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∀P.[◇□P → □P]

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for all $M, M | F$ $→$ □F

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exists $M, M | F$ $→$ ◊F

What about iterations?

◊□◊◊F

weak intuitions $→$ dozens of modal logics

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If something is possibly necessary, then it is necessary.

◇□(A ∨ ¬A) □(A ∨ ¬A)

logical necessity ~ validity  logical possibility ~ satisfiability

for all M, M |= F → □F exists M, M |= F → ◇F

What about iterations?

◇◇◇◇F

weak intuitions ⇒ dozens of modal logics

S5 is considered adequate

(But KB is sufficient!)
∀P.[◇□P → □P]

If something is possibly necessary, then it is necessary.

◇□(A ∨ ¬A) □(A ∨ ¬A)

logical necessity ∼ validity logical possibility ∼ satisfiability

for all M, M |= F → □F exists M, M |= F → ◇F

What about iterations?

◇□◇◇F

weak intuitions ⇒ dozens of modal logics

S5 is considered adequate

(But KB is sufficient!)
∀P. [◇□P → □P]

If something is possibly necessary, then it is necessary.

◇□(A ∨ ¬A)  □(A ∨ ¬A)

logical necessity ∼ validity  logical possibility ∼ satisfiability

for all M, M |= F  →  □F  exists M, M |= F  →  ◇F

What about iterations?

◇□◇◇F

weak intuitions ⇒ dozens of modal logics

S5 is considered adequate

(But KB is sufficient!)
∀\(P.[\Diamond\Box P \rightarrow \Box P]\)

If something is possibly necessary, then it is necessary.

\(\Diamond_c \Box_c (A \lor \neg A) \quad \Box_c (A \lor \neg A)\)

logical necessity \(\sim\) validity

\[\text{for all } M, M \models F \quad \rightarrow \quad \Box F\]

logical possibility \(\sim\) satisfiability

\[\text{exists } M, M \models F \quad \rightarrow \quad \Diamond F\]

**What about iterations?**

\(\Diamond \square \Diamond \Diamond F\)

weak intuitions \(\Rightarrow\) dozens of modal logics

**S5 is considered adequate**

(But KB is sufficient!)
∀P. [◇□P → □P]

If something is possibly necessary, then it is necessary.

◇c□c(A ∨ ¬A)  □c(A ∨ ¬A)

logical necessity ~ validity  logical possibility ~ satisfiability

for all M, M ⊨ F  →  □F  exists M, M ⊨ F  →  ◇F

What about iterations?

◇◇◇◇F

weak intuitions ⇒ dozens of modal logics

S5 is considered adequate

(But KB is sufficient!)
∀P. [P → □P]

Everything that is the case is so necessarily.

Follows from T2, T3 and D2.

There are no contingent “truths”.
Everything is determined.
There is no free will.

Many proposed solutions: Anderson, Fitting, Hájek, …
∀P. [P → □P]

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\forall P. [P \rightarrow \Box P]

Everything that is the case is so necessarily.

Follows from T2, T3 and D2.

There are no contingent “truths”.
Everything is determined.

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∀φ[\text{P}(\neg \phi) \leftrightarrow \neg \text{P}(\phi)]

Either a property is positive or its negation is (but never both)

Are the following properties positive or negative?

\begin{align*}
\lambda x. G(x) & \\
\lambda x. E(x) & \\
\lambda x. x = x & \\
\lambda x. \top & \\
\lambda x. \text{blue}(x) & \\
\lambda x. \text{white}(x) & \\
\lambda x. \text{human}(x) & 
\end{align*}

Solution:

“...positive in the moral aesthetic sense (independently of the accidental structure of the world). Only then the ax. true. . . . ”

- Gödel, 1970
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Part F: Conclusions
The \textit{(new)} insights we gained from experiments include:

- Logic K sufficient for T1, C and T2
- Logic S5 not needed for T3
- Logic KB sufficient for T3 (not well known)
- We found a simpler new proof of C
- Gödel’s axioms (without conjunct $\phi(x)$ in D2) are inconsistent
- Scott’s axioms are consistent
- For T1, only half of A1 (A1a) is needed
- For T2, the other half (A1b) is needed
Our novel contributions to the theorem proving community include

- Powerful infrastructure for reasoning with QML
- A new natural deduction calculus for higher-order modal logic
- Difficult new benchmarks problems for HOL provers
- Huge media attention
Conclusion

What have we achieved

- Verification of Gödel’s ontological argument with HOL provers
  - exact parameters known: constant domain quantification, Henkin Semantics
  - experiments with different parameters could be performed

- Gained some novel results and insights

- Major step towards Computer-assisted Theoretical Philosophy
  - see also Ed Zalta’s Computational Metaphysics project at Stanford University
  - see also John Rushby’s recent verification of Anselm’s proof in PVS
  - remember Leibniz’ dictum — *Calculemus!*

- Interesting bridge between CS, Philosophy and Theology

Ongoing and future work

- Formalize and verify literature on ontological arguments
  - …in particular the criticism and improvements to Gödel

- Own contributions — supported by theorem provers
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- Own contributions — supported by theorem provers
I'm sure that God would be impressed with your proof, if only he existed :-)

Larry

Die Philosophen können so schön staunen.

Sie packen Dinge in Begriffe (gucken dabei in die Luft) werfen die Begriffe dann in ihre Philosophiekiste, schütteln ganz dolle, und freuen sich, dass ganz genau rauskommt, was sie vorher reingetan haben. Und das geht sogar, wenn eine Maschine die Kiste schüttelt.

Unerstaunt
2017cp

60. Suchlauf

souveränsatt 09.09.2013

man kann auch auf andere Weise in diesem Zusammenhang methodisch vorgehen: bei einer längeren Autofahrt das Radio auf automatischen Suchlauf stellen. Nach zwei Tagen sieht man Gott

... find more on the internet ...
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TrifidNebula