Cut-free Calculi for Challenge Logics in a Lazy Way

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Cut-elimination in classical higher-order logic (HOL)

- History: Takeuti, . . . , Andrews
- Here: Own cut-free one-sided sequent calculus

Many non-classical logics are fragments of HOL

- Here: Quantified conditional logics

Cut-elimination for free

Beware of cut-simulation

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Classical Higher-Order Logic (HOL)

Syntax

$$s, t ::= c_\alpha \mid X_\alpha \mid (\lambda X_\alpha s_\beta)_{\alpha\to\beta} \mid (s_{\alpha\to\beta}\, t_\alpha)_\beta \mid$$
$$(\neg_{o\to o}\, s_o)_o \mid (s_o \vee_{o\to o\to o} t_o)_o \mid (\Pi_{(\alpha\to o)\to o}\, s_{\alpha\to o})_o$$

(Note: Binder notation $\forall X_\alpha t_o$ as syntactic sugar for $\Pi_{(\alpha\to o)\to o}\lambda X_\alpha t_o$)

HOL with Henkin semantics is (meanwhile) well understood

- Origin [Church,JSymbLog,1940]
- Henkin semantics [Henkin,JSymb.Log,1950]
  [Andrews, JSymbLog,1971,1972]
- Extens./Intens. [BenzmüllerEtAl,JSymbLog,2004]
  [Muskens,JSymbLog,2007]

Sound and complete provers do exists

Cut-free Calculi for HOL

- Takeuti (1953): defined GLC (generalized logical calculus) by extending Gentzen's LK; conjectured cut-elimination for GLC
- Schütte (1960): simplified verion GLC; gave a semantic characterization Takeuti's conjecture.
- Tait (1966): proved Schütte's conjecture.
- Takahashi (1967), Prawitz (1968): proved higher-order versions of the conjecture.
- Girard (1971): Takeuti's conjecture as a consequence of strong normalization for System F.
- Andrews (1971): Completeness of resolution in elementary type theory with abstract consistency technique.
- Takeuti (1975): Henkin complete cut-free sequent calculus with extensionality.
- Brown (2004), Benzmüller et al. (2004, 2009), and Brown and Smolka (2010): Various complete cut-free calculi with/without extensionality, use of abstract consistency technique

Cut-free calculi for HOL

One-sided sequent calculus $\mathcal{G}_{\beta\mathfrak{fb}}$ [BenzmüllerBrownKohlhase, LMCS, 2009]
(Δ: finite sets of β-normal closed formulas, $\Delta * \mathbf{A}$ stands for $\Delta \cup \{\mathbf{A}\}$,
$cwff_\alpha$: set of closed terms of type α, $\doteq$ abbreviates Leibniz equality):

Base Rules

$$\frac{\mathbf{A} \text{ atomic (\& } \beta\text{-normal)}}{\Delta * \mathbf{A} * \neg\mathbf{A}}\,\mathcal{G}(init) \qquad \frac{\Delta * \mathbf{A}}{\Delta * \neg\neg\mathbf{A}}\,\mathcal{G}(\neg) \qquad \frac{\Delta * \neg\mathbf{A} \quad \Delta * \neg\mathbf{B}}{\Delta * \neg(\mathbf{A} \vee \mathbf{B})}\,\mathcal{G}(\vee_-)$$

$$\frac{\Delta * \mathbf{A} * \mathbf{B}}{\Delta * (\mathbf{A} \vee \mathbf{B})}\,\mathcal{G}(\vee_+) \qquad \frac{\Delta * \neg\,(\mathbf{AC}){\downarrow_\beta} \quad \mathbf{C} \in cwff_\alpha}{\Delta * \neg\Pi^\alpha\mathbf{A}}\,\mathcal{G}(\Pi^{\mathbf{C}}_-) \qquad \frac{\Delta * (\mathbf{A}c){\downarrow_\beta} \quad c_\alpha\, new}{\Delta * \Pi^\alpha\mathbf{A}}\,\mathcal{G}(\Pi^c_+)$$

Full Extensionality

$$\frac{\Delta * (\forall X_\alpha . \mathbf{A}X \doteq^\beta \mathbf{B}X){\downarrow_\beta}}{\Delta * (\mathbf{A} \doteq^{\alpha\to\beta} \mathbf{B})}\,\mathcal{G}(\mathfrak{f}) \qquad \frac{\Delta * \neg\mathbf{A} * \mathbf{B} \quad \Delta * \neg\mathbf{B} * \mathbf{A}}{\Delta * (\mathbf{A} \doteq^o \mathbf{B})}\,\mathcal{G}(\mathfrak{b})$$

Initial. and Decomp. of Leibniz Equality

$$\frac{\Delta * (\mathbf{A} \doteq^o \mathbf{B}) \quad \mathbf{A},\mathbf{B} \text{ atomic}}{\Delta * \neg\mathbf{A} * \mathbf{B}}\,\mathcal{G}(Init^{\doteq})$$

$$\frac{\Delta * (\mathbf{A}^1 \doteq^{\alpha_1} \mathbf{B}^1) \;\cdots\; \Delta * (\mathbf{A}^n \doteq^{\alpha_n} \mathbf{B}^n) \quad n \geq 1, \beta \in \{o, \iota\}, h_{\overline{\alpha^n}\to\beta} \in \Sigma}{\Delta * (h\overline{\mathbf{A}^n} \doteq^\beta h\overline{\mathbf{B}^n})}\,\mathcal{G}(d)$$

Soundness and Completeness for HOL

Theorem — [BenzmüllerBrownKohlhase, LMCS, 2009] **1**

$$\models^{HOL} \mathbf{A}_o \quad \textit{iff} \quad \vdash^{\mathcal{G}_{\beta\mathfrak{fb}}}_{\textit{cut-free}} \mathbf{A}_o$$

resp.

$$\mathbf{Ax} \models^{HOL} \mathbf{A}_o \quad \textit{iff} \quad \mathbf{Ax} \vdash^{\mathcal{G}_{\beta\mathfrak{fb}}}_{\textit{cut-free}} \mathbf{A}_o$$

Calculus $\mathcal{G}_{\beta\mathfrak{fb}}$

- cut-free, sound and complete for HOL with Henkin semantics
- base types ι and o considered
- works also for more than two base types

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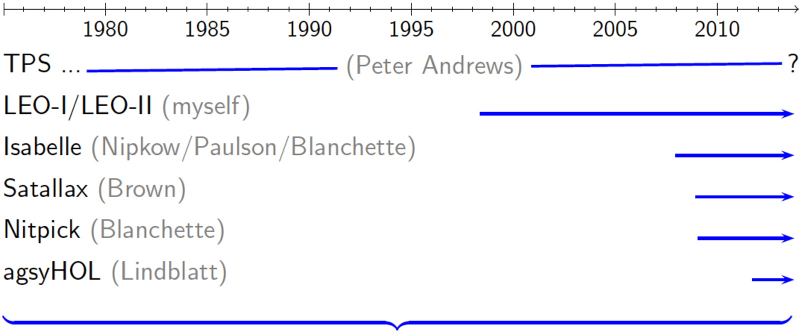
ATPs for HOL



- all accept TPTP THF Syntax [SutcliffeBenzmüller, J.Form.Reas, 2009]
- can be called remotely via SystemOnTPTP at Miami
- they significantly gained in strength over the last years
- they can be bundled into a combined prover HOL-P

Exploit HOL with Henkin semantics as metalogic
Automate other logics (& combinations) via semantic embeddings
— HOL-P becomes a Universal Reasoner —

Quantified non-classical logics as fragments of HOL

- Non-classical logics often come with a Kripke style semantics
- use HOL as meta-logic to encode these Kripke structures
- embedded logics become (lifted) predicates on worlds/states
- elegant and transparent encodings by exploiting λ-abstraction
- soundness and completeness wrt. Henkin semantics for HOL
- cut-elimination for free
- automation for free with HOL ATPs

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Automation with HOL ATPs

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Holy Logic: Computer Scientists 'Prove' God Exists

By David Knight

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Austrian mathematician Kurt Gödel kept his proof of God's existence a secret for decades. Now two scientists say they have proven it mathematically using a computer.

Two scientists have formalized a theorem regarding the existence of God penned by mathematician Kurt Gödel. But the God angle is somewhat of a red herring -- the real step forward is the example it sets of how computers can make scientific progress simpler.

Gödel's ontological argument: Second-Order Modal Logic

Logic studied in this talk: Quantified Conditional Logics (QCL)

– see also [Benzmüller, IJCAI, 2013] –

Quantified Conditional Logics (QCL)

Syntax

$$\varphi, \psi ::= \quad P \mid k(X^1, \ldots, X^n) \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi \mid$$
$$\forall^{co} X \varphi \mid \forall^{va} X \varphi \mid \forall P \varphi$$

Kripke style semantics

$$M, g, s \models \varphi \vee \psi \text{ iff } M, g, s \models \varphi \text{ or } M, g, s \models \psi$$

$$\ldots\ldots$$

$$M, g, s \models \varphi \Rightarrow \psi \text{ iff } M, g, t \models \psi \text{ for all } t \in S \text{ such that}$$
$$t \in f(s, [\varphi]) \text{ where } [\varphi] = \{u \mid M, g, u \models \varphi\}$$

Very expressive (e.g.: $\Box\varphi := \varphi \Rightarrow \varphi$), many applications

Cut-elimination: for some prop. QCLs [PattinsonSchröder, LMCS, 2011]
Provers: for some prop. QCLs [OlivettiPozzato, JANCL, 2008]

QCLs are fragments of HOL

QCL formulas φ are identified with (lifted) HOL terms φ_τ where $\tau := \iota \rightarrow o$

Semantic embedding exploits Kripke style semantics

$$
\begin{aligned}
\neg &= \lambda A_\tau \lambda X_\iota \neg(A\,X) \\
\vee &= \lambda A_\tau \lambda B_\tau \lambda X_\iota (A\,X \vee B\,X) \\
\Rightarrow &= \lambda A_\tau \lambda B_\tau \lambda X_\iota \forall V_\iota (f\,X\,A\,V \rightarrow B\,V) \\
\Pi^{co} &= \lambda Q_{u\rightarrow\tau} \lambda V_\iota \forall X_u (Q\,X\,V) \\
\Pi^{va} &= \lambda Q_{u\rightarrow\tau} \lambda V_\iota \forall X_u (eiw\,V\,X \rightarrow Q\,X\,V) \\
\Pi &= \lambda R_{\tau\rightarrow\tau} \lambda V_\iota \forall P_\tau (R\,P\,V)
\end{aligned}
$$

Meta-notion of validity defined as: $\text{valid} = \lambda A_\tau \forall S_\iota (A\,S)$

Varying domains are non-empty: $\forall W_\iota \exists X_u (eiw\,W\,X)$

Assume that set Ax contains all the above HOL axioms.

QCLs are fragments of HOL

Theorem — [Benzmüller, IJCAI, 2013] **2**

$$\models^{QCL} \varphi \quad \textit{iff} \quad \mathbf{Ax} \models^{HOL} \textit{vld}\, \varphi_\tau$$

QCLs are fragments of HOL

ID	Axiom Condition	$A \Rightarrow A$ $f(w, [A]) \subseteq [A]$
MP	Axiom Condition	$(A \Rightarrow B) \rightarrow (A \rightarrow B)$ $w \in [A] \rightarrow w \in f(w, [A])$
CS	Axiom Condition	$(A \wedge B) \rightarrow (A \Rightarrow B)$ $w \in [A] \rightarrow f(w, [A]) \subseteq \{w\}$
CEM	Axiom Condition	$(A \Rightarrow B) \vee (A \Rightarrow \neg B)$ $\lvert f(w, [A]) \rvert \leq 1$
AC	Axiom Condition	$(A \Rightarrow B) \wedge (A \Rightarrow C) \rightarrow (A \wedge C \Rightarrow B)$ $f(w, [A]) \subseteq [B] \rightarrow f(w, [A \wedge B]) \subseteq f(w, [A])$
RT	Axiom Condition	$(A \wedge B \Rightarrow C) \rightarrow ((A \Rightarrow B) \rightarrow (A \Rightarrow C))$ $f(w, [A]) \subseteq [B] \rightarrow f(w, [A]) \subseteq f(w, [A \wedge B])$
CV	Axiom Condition	$(A \Rightarrow B) \wedge \neg(A \Rightarrow \neg C) \rightarrow (A \wedge C \Rightarrow B)$ $(f(w, [A]) \subseteq [B]$ and $f(w, [A]) \cap [C] \neq \emptyset) \rightarrow f(w, [A \wedge C]) \subseteq [B]$
CA	Axiom Condition	$(A \Rightarrow B) \wedge (C \Rightarrow B) \rightarrow (A \vee C \Rightarrow B)$ $f(w, [A \vee B]) \subseteq f(w, [A]) \cup f(w, [B])$

QCLs are fragments of HOL

For obtaining cut-free calculus for QCL logic ID simply add

$$\text{valid } \Pi \lambda A. A \Rightarrow A$$

or

$$(\forall A, W.(f\ W\ A) \subseteq A)$$

to the set of axioms **AX**.

We have:

$$\models^{QCL(ID)} \varphi \quad \text{iff} \quad \mathbf{Ax} \cup \{ID\} \models^{\text{HOL}} \text{vld}\, \varphi_\tau$$

Cut-elimination for QCL

Theorem — Combining Theorems 1 and 2 **3**

$$\models^{QCL(*)} \varphi \quad \textit{iff} \quad \mathbf{AX} \cup \{*\} \vdash^{\mathcal{G}_{\beta\mathfrak{fb}}}_{\textit{cut-free}} \textit{vld}\, \varphi_\tau$$

$* \in \{ID, MP, CS, \ldots\}$

A „lean" QCL Theorem Prover

```
%---- file: Axioms.ax ----------------------------------
%--- type mu for individuals
thf(mu,type,(mu:$tType)).
%--- reserved constant for selection function f
thf(f,type,(f:$i>($i>$o)>$i>$o)).
%--- `exists in world' predicate for varying domains;
%--- for each v we get a non-empty subdomain eiw@v
thf(eiw,type,(eiw:$i>mu>$o)).
thf(nonempty,axiom,(![V:$i]:?[X:mu]:(eiw@V@X))).
%--- negation, disjunction, material implication
thf(not,type,(not:($i>$o)>$i>$o)).
thf(or,type,(or:($i>$o)>($i>$o)>$i>$o)).
thf(impl,type,(impl:($i>$o)>($i>$o)>$i>$o)).
thf(not_def,definition,(not = (^[A:$i>$o,X:$i]:~(A@X)))).
thf(or_def,definition,(or
 = (^[A:$i>$o,B:$i>$o,X:$i]:((A@X)|(B@X))))).
thf(impl_def,definition,(impl
 = (^[A:$i>$o,B:$i>$o,X:$i]:((A@X)=>(B@X))))).
%--- conditionality
thf(cond,type,(cond:($i>$o)>($i>$o)>$i>$o)).
thf(cond_def,definition,(cond
 = (^[A:$i>$o,B:$i>$o,X:$i]:![W:$i]:((f@X@A@W)=>(B@W))))).
%--- quantification (constant dom., varying dom., prop.)
thf(all_co,type,(all_co: (mu>$i>$o)>$i>$o)).
thf(all_va,type,(all_va:(mu>$i>$o)>$i>$o)).
thf(all,type,(all:(($i>$o)>$i>$o)>$i>$o)).
thf(all_co_def,definition,(all_co
 = (^[A:mu>$i>$o,W:$i]:![X:mu]:(A@X@W)))).
thf(all_va_def,definition,(all_va
 = (^[A:mu>$i>$o,W:$i]:![X:mu]:((eiw@W@X)=>(A@X@W))))).
thf(all_def,definition,(all
 = (^[A:($i>$o)>$i>$o,W:$i]:![P:$i>$o]:(A@P@W)))).
%--- box operator based on conditionality
thf(box,type,(box:($i>$o)>$i>$o)).
thf(box_def,definition,(box
 = (^[A:$i>$o]:(cond@(not@A)@A)))).
%--- notion of validity of a conditional logic formula
thf(vld,type,(vld:($i>$o)>$o)).
thf(vld_def,definition,(vld
 = (^[A:$i>$o]:![S:$i]:(A@S)))).
%---- end file: Axioms.ax ------------------------------
```

Beware of Cut-Simulation

- In impredicative logics cut-elimination maybe be worthless ...

  ... why? ...

- ... since the cut-rule can (eventually) be effectively simulated.

Cut-Simulation with Excluded Middle

$$\dfrac{\dfrac{\Delta * \mathbf{C}}{\Delta * \neg\neg\mathbf{C}}\ \mathcal{G}(\neg) \qquad \Delta * \neg\mathbf{C}}{\dfrac{\Delta * \neg(\neg\mathbf{C} \vee \mathbf{C})}{\Delta * \neg\Pi(\lambda P_o . \neg P \vee P)}\ \mathcal{G}(\Pi^{\mathbf{C}}_{-})}\ \mathcal{G}(\vee_{-})$$

Cut-Simulation with Leibniz Equations

$$\dfrac{\dfrac{\dfrac{\Delta * \mathbf{C}}{\Delta * \neg\neg\mathbf{C}}\;\mathcal{G}(\neg) \qquad \Delta * \neg\mathbf{C}}{\Delta * \neg(\neg\mathbf{C} \vee \mathbf{C})}\;\mathcal{G}(\vee_{-})}{\Delta * \neg\Pi(\lambda P_{\alpha\to o}.\neg P\mathbf{M} \vee P\mathbf{N})}\;\mathcal{G}(\Pi_{-}^{\lambda X.\mathbf{C}})$$

Cut-Simulation with Prominent Axioms

▸ Axiom of excluded middle	3 steps
▸ Leibniz equations (axioms/hypotheses)	3 steps
▸ Axiom of functional extensionality	11 steps
▸ Axiom of Boolean extensionality	14 steps
▸ Reflexivity definition of equality (Andrews)	4 steps
▸ Instances of the comprehension axioms	16 steps
▸ Axiom of Induction	18 steps
▸ Axioms of choice	7 steps
▸ Axiom of description	25 steps

Cut-free calculi for HOL

One-sided sequent calculus $\mathcal{G}_{\beta\mathfrak{fb}}$ [BenzmüllerBrownKohlhase, LMCS, 2009]
(Δ and Δ': finite sets of β-normal closed formulas, $\Delta * \mathbf{A}$: $\Delta \cup \{\mathbf{A}\}$,
$cwff_\alpha$: set of closed terms of type α, $\doteq$ is Leibniz equality):

Base Rules

$$\frac{\mathbf{A} \text{ atomic (\& } \beta\text{-normal)}}{\Delta * \mathbf{A} * \neg\mathbf{A}}\,\mathcal{G}(init) \qquad \frac{\Delta * \mathbf{A}}{\Delta * \neg\neg\mathbf{A}}\,\mathcal{G}(\neg) \qquad \frac{\Delta * \neg\mathbf{A} \quad \Delta * \neg\mathbf{B}}{\Delta * \neg(\mathbf{A} \vee \mathbf{B})}\,\mathcal{G}(\vee_-)$$

$$\frac{\Delta * \mathbf{A} * \mathbf{B}}{\Delta * (\mathbf{A} \vee \mathbf{B})}\,\mathcal{G}(\vee_+) \qquad \frac{\Delta * \neg\,(\mathbf{AC})\big\downarrow_\beta \quad \mathbf{C} \in cwff_\alpha}{\Delta * \neg\Pi^\alpha\mathbf{A}}\,\mathcal{G}(\Pi^{\mathbf{C}}_-) \qquad \frac{\Delta * (\mathbf{A}c)\big\downarrow_\beta \quad c_\alpha\, new}{\Delta * \Pi^\alpha\mathbf{A}}\,\mathcal{G}(\Pi^c_+)$$

Full Extensionality

$$\frac{\Delta * (\forall X_\alpha.\mathbf{A}X \doteq^\beta \mathbf{B}X)\big\downarrow_\beta}{\Delta * (\mathbf{A} \doteq^{\alpha\to\beta} \mathbf{B})}\,\mathcal{G}(\mathfrak{f}) \qquad \frac{\Delta * \neg\mathbf{A} * \mathbf{B} \quad \Delta * \neg\mathbf{B} * \mathbf{A}}{\Delta * (\mathbf{A} \doteq^o \mathbf{B})}\,\mathcal{G}(\mathfrak{b})$$

Initial. and Decomp. of Leibniz Equality

$$\frac{\Delta * (\mathbf{A} \doteq^o \mathbf{B}) \quad \mathbf{A},\mathbf{B} \text{ atomic}}{\Delta * \neg\mathbf{A} * \mathbf{B}}\,\mathcal{G}(Init^{\doteq})$$

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How about these Axioms?

$$
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\end{aligned}
$$

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How about the additional Axioms of QCL?

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Homework: Study cut-simulation for these axioms!

Cut-Simulation with ID

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{\Delta * fM(\lambda x. \neg C \vee C)N}{\Delta * \neg\neg fM(\lambda x. \neg C \vee C)N}\ \mathcal{G}(\neg)
        \qquad
        \dfrac{\dfrac{\Delta * \mathbf{C}}{\Delta * \neg\neg\mathbf{C}}\ \mathcal{G}(\neg) \qquad \Delta * \neg\mathbf{C}}{\Delta * \neg(\neg C \vee C)}\ \mathcal{G}(\vee_{-})
      }{\Delta * \neg(\neg fM(\lambda x. \neg C \vee C)N \vee (\neg C \vee C))}\ \mathcal{G}(\vee_{-})
    }{\Delta * \neg\Pi\lambda Y. (\neg fM(\lambda x. \neg C \vee C)Y \vee (\neg C \vee C))}\ \mathcal{G}(\Pi_{-}^{\mathbf{N}})
  }{\Delta * \neg\Pi\lambda A. \Pi\lambda Y. \neg fMAY \vee AY}\ \mathcal{G}(\Pi_{-}^{\lambda x. \neg C \vee C})
}{\Delta * \neg\Pi\lambda X. \Pi\lambda A. \Pi\lambda Y. \neg fXAY \vee AY}\ \mathcal{G}(\Pi_{-}^{\mathbf{M}})
$$

Conclusion

So, what is actually the point about cut-elimination?

- **Cut-elimination holds for HOL**
- **Many non-classical logics are just fragments of HOL**
- **Cut-elimination for free!**
- **Applies to many propositional and quantified logics**

- **However, in HOL it makes little sense to study cut-elimination and to neglect cut-simulation**