Quantified Conditional Logics are Fragments of HOL

Christoph Benzmüller and Valerio Genovese

Freie Universität Berlin and University of Luxembourg

The International Conference on
Non-classical Modal and Predicate Logics
Guangzhou, China, 2011

My sincere apologies . . .

I was really looking forward to a break from . . .

. . . in Guangzhou!

Presentation Overview

Core questions of my current research:

1. Classical Higher Order Logic (HOL) as Universal Logic?
2. HOL Provers & Model Finders as Generic Reasoning Tools?
3. Integration of Specialist Reasoners (if available)?

Presentation Overview

Core questions of my current research:

1. Classical Higher Order Logic (HOL) as Universal Logic?
2. HOL Provers & Model Finders as Generic Reasoning Tools?
3. Integration of Specialist Reasoners (if available)?

In this talk:

- What is HOL?
- Examples of Fragments of HOL: Multimodal Logics & Others
- Quantified Conditional Logics are Natural Fragments of HOL
- Automation of Quantified Conditional Logics in HOL ATPs
- Conclusion

What is HOL?

(Classical Higher Order Logic/Church's Type Theory)

Expressivity	FOL	HOL	Example
Quantification over			
- Individuals	✓	✓	$\forall X.\, p(f(X))$
- Functions	–	✓	$\forall F.\, p(F(a))$
- Predicates/Sets/Rels	–	✓	$\forall P.\, P(f(a))$
Unnamed			
- Functions	–	✓	$(\lambda X.\, X)$
- Predicates/Sets/Rels	–	✓	$(\lambda X.\, X \neq a)$
Statements about			
- Functions	–	✓	$\mathit{continuous}(\lambda X.\, X)$
- Predicates/Sets/Rels	–	✓	$\mathit{reflexive}(=)$
Powerful abbreviations	–	✓	$\mathit{reflexive} = \lambda R.\, \lambda X.\, R(X, X)$

What is HOL? (Church's Type Theory, Alonzo Church, 1940)

Expressivity	FOL	HOL	Example
Quantification over			
- Individuals	✓	✓	$\forall X_{\iota}.\, p_{\iota\to o}(f_{\iota\to\iota}(X_{\iota}))$
- Functions	−	✓	$\forall F_{\iota\to\iota}.\, p_{\iota\to o}(F_{\iota\to o}(a_{\iota}))$
- Predicates/Sets/Rels	−	✓	$\forall P_{\iota\to o}.\, P_{\iota\to o}(f_{\iota\to\iota}(a_{\iota}))$
Unnamed			
- Functions	−	✓	$(\lambda X_{\iota}.\, X_{\iota})$
- Predicates/Sets/Rels	−	✓	$(\lambda X_{\iota\to\iota}.\, X_{\iota\to\iota} \neq_{\iota\to\iota\to p} a)_{\iota})$
Statements about			
- Functions	−	✓	$continuous_{(\iota\to\iota)\to o}(\lambda X_{\iota}.\, X_{\iota})$
- Predicates/Sets/Rels	−	✓	$reflexive_{(\iota\to\iota\to o)\to o}(=_{\iota\to\iota\to o})$
Powerful abbreviations	−	✓	$reflexive_{(\iota\to\iota\to o)\to o} = \lambda R_{(\iota\to\iota\to o)}.\, \lambda X_{\iota}.\, R$

Simple Types: Prevent Paradoxes and Inconsistencies

What is HOL? (Alonzo Church, 1940)

- Simple Types $\alpha ::= \iota \mid o \mid \alpha_1 \to \alpha_2$

- Simple Types $\alpha ::= \iota \mid o \mid \alpha_1 \rightarrow \alpha_2$

Individuals

Booleans (True and False)

Functions

What is HOL? (Alonzo Church, 1940)

- Simple Types $\alpha ::= \iota \mid \mu \mid o \mid \alpha_1 \to \alpha_2$

Possible worlds

Individuals

Booleans (True and False)

Functions

- Simple Types $\alpha ::= \iota \mid \mu \mid o \mid \alpha_1 \rightarrow \alpha_2$
- HOL Syntax

$$
\begin{aligned}
s, t \quad ::= \quad & p_\alpha \mid X_\alpha \\
& \mid (\lambda X_\alpha \text{.}\, s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta}\, t_\alpha)_\beta \\
& \mid (\neg_{o\rightarrow o}\, s_o)_o \mid (s_o \vee_{o\rightarrow o\rightarrow o} t_o)_o \mid (\forall X_\alpha \text{.}\, t_o)_o
\end{aligned}
$$

Constant Symbols

Variable Symbols

- Simple Types $\alpha ::= \iota \mid \mu \mid o \mid \alpha_1 \rightarrow \alpha_2$
- HOL Syntax

$$\begin{aligned} s,t \quad ::= \quad & p_\alpha \mid X_\alpha \\ & \mid (\lambda X_\alpha . s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta}\, t_\alpha)_\beta \\ & \mid (\neg_{o\rightarrow o}\, s_o)_o \mid (s_o \vee_{o\rightarrow o\rightarrow o} t_o)_o \mid (\forall X_\alpha . t_o)_o \end{aligned}$$

Constant Symbols
Variable Symbols
Abstraction
Application

What is HOL? (Alonzo Church, 1940)

- Simple Types $\alpha ::= \iota \mid \mu \mid o \mid \alpha_1 \rightarrow \alpha_2$
- HOL Syntax

$$
\begin{aligned}
s, t \quad ::= \quad & p_\alpha \mid X_\alpha \\
& \mid (\lambda X_\alpha . s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta}\, t_\alpha)_\beta \\
& \mid (\neg_{o\rightarrow o}\, s_o)_o \mid (s_o \vee_{o\rightarrow o\rightarrow o} t_o)_o \mid (\forall X_\alpha . t_o)_o
\end{aligned}
$$

Constant Symbols
Variable Symbols
Abstraction
Application
Logical Connectives

- Simple Types $\alpha ::= \iota \mid \mu \mid o \mid \alpha_1 \to \alpha_2$
- HOL Syntax

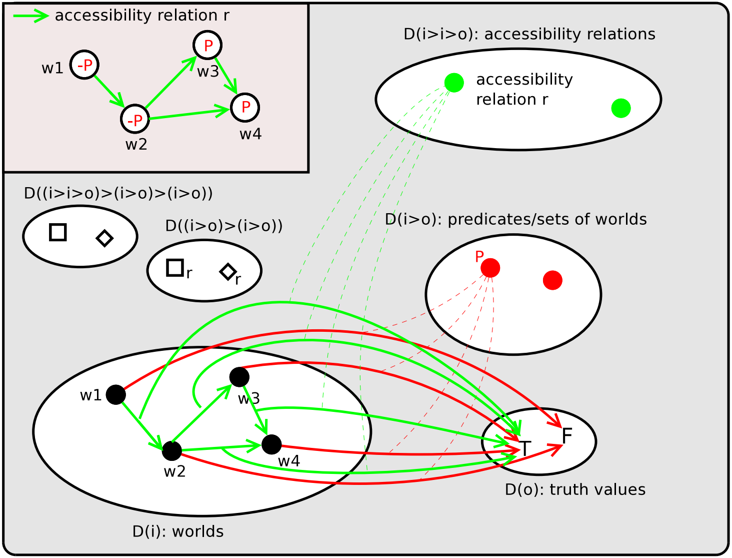
$$s, t ::= p_\alpha \mid X_\alpha \mid (\lambda X_\alpha.\, s_\beta)_{\alpha\to\beta} \mid (s_{\alpha\to\beta}\, t_\alpha)_\beta \mid (\neg_{o\to o}\, s_o)_o \mid (s_o \vee_{o\to o\to o} t_o)_o \mid \underbrace{(\forall X_\alpha.\, t_o)_o}_{(\Pi_{(\alpha\to o)\to o}\,(\lambda X_\alpha.\, t_o))_o}$$

- Simple Types $\alpha ::= \iota \mid \mu \mid o \mid \alpha_1 \rightarrow \alpha_2$
- HOL Syntax

$$\begin{aligned} s,t \quad ::= \quad & p_\alpha \mid X_\alpha \\ & \mid (\lambda X_\alpha . s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta}\, t_\alpha)_\beta \\ & \mid (\neg_{o\rightarrow o}\, s_o)_o \mid (s_o \vee_{o\rightarrow o\rightarrow o} t_o)_o \mid (\Pi_{(\alpha\rightarrow o)\rightarrow o}\, (\lambda X_\alpha . t_o))_o \end{aligned}$$

- HOL is (meanwhile) well understood
  - Origin [Church, J.Symb.Log., 1940]
  - Henkin-Semantics [Henkin, J.Symb.Log., 1950]
    [Andrews, J.Symb.Log., 1971, 1972]
  - Extens./Intens. [BenzmüllerEtAl., J.Symb.Log., 2004]
    [Muskens, J.Symb.Log., 2007]
- HOL with Henkin-Semantics: semi-decidable & compact (like FOL)

Examples of Fragments of HOL:
Multimodal Logics & Others


accessibility relation r
w1
-P
-P
w2
P
w3
P
w4
D(i>i>o): accessibility relations
accessibility
relation r
D((i>i>o)>(i>o)>(i>o))
D((i>o)>(i>o))
D(i>o): predicates/sets of worlds
P
w1
w3
w2
w4
T
F
D(o): truth values
D(i): worlds

- Syntax (MML): $s, t \quad ::= \quad P \mid \neg\, s \mid s \vee t \mid \Box_r\, s$

- Syntax MML
  - formulas s

  Kripke Semantics
  - worlds w
  - accessibility relations r

  explicit → transformation → First Order Logic

  e.g. work of Ohlbach

Not Needed!

- Syntax (MML): $s, t \quad ::= \quad P \mid \neg s \mid s \vee t \mid \Box_r s$

- HOL

Syntax MML		
- formulas s	$\longrightarrow$	terms $s_{\iota \to o}$
Kripke Semantics		
- worlds w	$\longrightarrow$	terms w_ι
- accessibility relations r	$\longrightarrow$	terms $r_{\iota \to \iota \to o}$

- Syntax (MML): $s, t \quad ::= \quad P \mid \neg\, s \mid s \vee t \mid \Box_r\, s$

- HOL

Syntax MML - formulas s	$\longrightarrow$	terms $s_{\iota\to o}$
Kripke Semantics - worlds w	$\longrightarrow$	terms w_ι
- accessibility relations r	$\longrightarrow$	terms $r_{\iota\to\iota\to o}$

- MML Syntax as Abbreviations of HOL-Terms

$$P = \lambda W_\iota.(P_{\iota\to o}\, W) = P_{\iota\to o}$$
$$\neg = \lambda S_{\iota\to o}.\lambda W_\iota.\neg(S\, W)$$
$$\vee = \lambda S_{\iota\to o}.\lambda T_{\iota\to o}.\lambda W_\iota.(S\, W) \vee (T\, W)$$
$$\Box = \lambda R_{\iota\to\iota\to o}.\lambda S_{\iota\to o}.\lambda W_\iota.\forall V_\iota.\neg(R\, W\, V) \vee (S\, V)$$

[BenzmüllerPaulson, Log.J.IGPL, 2010]

- Syntax (MML): $s, t \quad ::= \quad P \mid \neg\, s \mid s \vee t \mid \Box_r\, s$

- **HOL**

Syntax MML		
- formulas s	$\longrightarrow$	terms $s_{\iota\to o}$
Kripke Semantics		
- worlds w	$\longrightarrow$	terms w_ι
- accessibility relations r	$\longrightarrow$	terms $r_{\iota\to\iota\to o}$

- MML Syntax as Abbreviations of HOL-Terms

$$
\begin{aligned}
P &= \lambda W_\iota.(P_{\iota\to o}\, W) = P_{\iota\to o}\\
\neg &= \lambda S_{\iota\to o}.\lambda W_\iota.\neg(S\, W)\\
\vee &= \lambda S_{\iota\to o}.\lambda T_{\iota\to o}.\lambda W_\iota.(S\, W) \vee (T\, W)\\
\Box &= \lambda R_{\iota\to\iota\to o}.\lambda S_{\iota\to o}.\lambda W_\iota.\forall V_\iota.\neg(R\, W\, V) \vee (S\, V)\\
\forall^p, \forall^\mu &= \lambda Q_{\mu\to(\iota\to o)}.\lambda W_\iota.\forall X_\mu.(Q\, X\, W)
\end{aligned}
$$

[BenzmüllerPaulson, Logica Universalis, to appear]

- Syntax (MML): $s, t \quad ::= \quad P \mid \neg\, s \mid s \vee t \mid \Box_r\, s$

HOL

Syntax MML
- formulas s $\longrightarrow$ terms $s_{\iota\to o}$

Kripke Semantics
- worlds w $\longrightarrow$ terms w_ι
- accessibility relations r $\longrightarrow$ terms $r_{\iota\to\iota\to o}$

- MML Syntax as Abbreviations of HOL-Terms

$$
\begin{aligned}
P &= \lambda W_\iota . (P_{\iota\to o}\, W) = P_{\iota\to o}\\
\neg &= \lambda S_{\iota\to o} . \lambda W_\iota . \neg (S\, W)\\
\vee &= \lambda S_{\iota\to o} . \lambda T_{\iota\to o} . \lambda W_\iota . (S\, W) \vee (T\, W)\\
\Box &= \lambda R_{\iota\to\iota\to o} . \lambda S_{\iota\to o} . \lambda W_\iota . \forall V_\iota . \neg (R\, W\, V) \vee (S\, V)\\
\forall^p, \forall^\mu &= \lambda Q_{\mu\to(\iota\to o)} . \lambda W_\iota . \forall X_\mu . (Q\, X\, W)\\
s \Rightarrow t &= \lambda S_{\iota\to o} . \lambda T_{\iota\to o} . \lambda W_\iota . \forall V_\iota . \neg (f\, W\, S\, V) \vee (T\, V)
\end{aligned}
$$

[BenzmüllerGenovese, NCMPL, 2011]

- Validity

$$\text{valid} \quad = \quad \lambda \varphi_{\iota \to o} \centerdot \forall W_{\iota} \centerdot \varphi\, W$$

Also

- Satisfiability
- Countersatisfiability
- Unsatisfiability

As a consequence we have that . . .

Automation of (many) non-classical logics for free in classical HOL ATPs!

$$\models^{MML} \varphi \quad \text{iff} \quad \models^{HOL} \text{valid}\ \varphi_{\iota \to o}$$

- Propositional Multimodal Logics [BenzmüllerPaulson, Log.J.IGPL, 2010]
- Quantified Multimodal Logics

  [BenzmüllerPaulson, Logica Universalis, to appear (arXiv:0905.2435v1)]
- Propositional Conditional Logics

  [BenzmüllerGenoveseGabbayRispoli, submitted (arXiv:1106.3685v3)]
- Intuitionistic Logics: [BenzmüllerPaulson, Log.J.IGPL, 2010]
- Access Control Logics: [Benzmüller, IFIP SEC, 2009]
- Combinations of Logics: [Benzmüller, AMAI, 2011]

Work in progress

- Temporal Logics, Spatial Reasoning 'RCC', SUMO Ontology, OWL2 full, DOLCE Ontology, . . .

Quantified Conditional Logics are Fragments of HOL

This work extends

[BenzmüllerGenoveseGabbayRispoli, submitted (arXiv:1106.3685v3)]

[BenzmüllerPaulson, Logica Universalis, to appear (arXiv:0905.2435v1)]

Theory for (Reasoning with) Counterfactual Conditionals

If I had continued with competitive long-distance running in 1992,
I would have won the Olympic Games in 2000.

Problem: non-truth-functionality of counterfactual conditional statements

Solution (Stalnaker and Thomason)

- selection function semantics (a possible world semantics, extension of modal logics) [Stalnaker68]

$$\underbrace{\text{'If } A \text{ then } B\text{'}}_{(A \Rightarrow B)} \textit{ is true in world } w \quad \textit{iff} \quad B \textit{ is true for all } v \in f(w, A)$$

- idea: f selects worlds that are very *similar/close* to the actual world w
- many closely related theories: [Lewis73, Pollock76, Chellas75]

Quantified Conditional Logic – Syntax

$$\varphi, \psi ::= P \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi$$

Propositional Variables (PV) Individual Variables (IV) Constants (Sym)

$$\varphi, \psi ::= P \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi \mid \forall P.\varphi \mid \forall X.\varphi \mid k(X^1, \ldots, X^n)$$

Quantified Conditional Logic – Syntax

Propositional Variables (PV) Individual Variables (IV) Constants (Sym)

$$\varphi, \psi ::= P \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi \mid \forall P.\varphi \mid \forall X.\varphi \mid k(X^1, \ldots, X^n)$$

Quantified Conditional Logic – Syntax

Propositional Variables (PV)   Individual Variables (IV)   Constants (Sym)

$$\varphi, \psi ::= P \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi \mid \forall P.\varphi \mid \forall X.\varphi \mid k(X^1, \ldots, X^n)$$

Propositional Variables (PV)   Individual Variables (IV)   Constants (Sym)

$$\varphi, \psi ::= P \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi \mid \forall P.\varphi \mid \forall X.\varphi \mid k(X^1, \ldots, X^n)$$

Quantified Conditional Logic – Syntax

Propositional Variables (PV) Individual Variables (IV) Constants (Sym)

$$\varphi, \psi ::= P \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi \mid \forall P.\varphi \mid \forall X.\varphi \mid k(X^1, \ldots, X^n)$$

Logical Connectives and Quantifiers (others may be defined as usual)

Propositional Variables (PV) Individual Variables (IV) Constants (Sym)

$$\varphi, \psi ::= P \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi \mid \forall P.\varphi \mid \forall X.\varphi \mid k(X^1, \ldots, X^n)$$

Conditional (modal) operator

$$\varphi, \psi ::= P \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi \mid \forall P.\varphi \mid \forall X.\varphi \mid k(X^1, \ldots, X^n)$$

Interpretation

- is a structure $M = \langle S, f, D, Q, I \rangle$ with
  - S set of possible worlds
  - $f : S \times 2^S \mapsto 2^S$ is the selection function
  - D is a non-empty set of individuals (the first-order domain)
  - Q is a non-empty collection of subsets of S (the propositional domain)
  - I is a classical interpretation function where for each n-ary predicate symbol k, $I(k, w) \subseteq D^n$

Variable Assignment

- $g = \langle g^{iv}, g^{pv} \rangle$
  - $g^{iv} : IV \mapsto D$ maps individual variables to objects in D
  - $g^{pv} : PV \mapsto Q$ maps propositional variables to sets of worlds in Q

Quantified Conditional Logic – Semantics

$$\varphi, \psi ::= P \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi \mid \forall P.\varphi \mid \forall X.\varphi \mid k(X^1, \ldots, X^n)$$

Interpretation

- is a structure $M = \langle S, f, D, Q, I \rangle$ with
  - S set of possible worlds
  - $f : S \times 2^S \mapsto 2^S$ is the selection function
  - D is a non-empty set of individuals (the first-order domain)
  - Q is a non-empty collection of subsets of S (the propositional domain)
  - I is a classical interpretation function where for each n-ary predicate symbol k, $I(k, w) \subseteq D^n$

Variable Assignment

- $g = \langle g^{iv}, g^{pv} \rangle$
  - $g^{iv} : IV \mapsto D$ maps individual variables to objects in D
  - $g^{pv} : PV \mapsto Q$ maps propositional variables to sets of worlds in Q

Satisfiability $M, g, s \models \varphi$ defined as:

$M, g, s \models P$	iff	$s \in g(P)$
$M, g, s \models k(X^1, \ldots, X^n)$	iff	$s \in \langle g(X^1), \ldots, g(X^n) \rangle \in I(k, w)$
$M, g, s \models \neg\varphi$	iff	not $M, g, s \models \varphi$
$M, g, s \models \varphi \vee \psi$	iff	$M, g, s \models \varphi$ or $M, g, s \models \psi$
$M, g, s \models \varphi \Rightarrow \psi$	iff	$M, g, v \models \psi$ for all $v \in f(s, \overbrace{\{u \mid M, g, u \models \varphi\}}^{[\varphi]})$
$M, g, s \models \forall X.\varphi$	iff	$M, [d/X]g, s \models \varphi$ for all $d \in D$
$M, g, s \models \forall P.\varphi$	iff	$M, [p/P]g, s \models \varphi$ for all $p \in Q$

Validity

- $M \models \varphi$ iff for all worlds s and assignments g holds $M, g, s \models \varphi$
- $\models \varphi$ iff φ is valid in every model M

Satisfiability $M, g, s \models \varphi$ defined as:

$M, g, s \models P$	iff	$s \in g(P)$
$M, g, s \models k(X^1, \ldots, X^n)$	iff	$s \in \langle g(X^1), \ldots, g(X^n) \rangle \in I(k, w)$
$M, g, s \models \neg\varphi$	iff	not $M, g, s \models \varphi$
$M, g, s \models \varphi \vee \psi$	iff	$M, g, s \models \varphi$ or $M, g, s \models \psi$
$M, g, s \models \varphi \Rightarrow \psi$	iff	$M, g, v \models \psi$ for all $v \in f(s, \overbrace{\{u \mid M, g, u \models \varphi\}}^{[\varphi]})$
$M, g, s \models \forall X. \varphi$	iff	$M, [d/X]g, s \models \varphi$ for all $d \in D$
$M, g, s \models \forall P. \varphi$	iff	$M, [p/P]g, s \models \varphi$ for all $p \in Q$

Validity

- $M \models \varphi$ iff for all worlds s and assignments g holds $M, g, s \models \varphi$
- $\models \varphi$ iff φ is valid in every model M

Satisfiability $M, g, s \models \varphi$ defined as:

$$
\begin{array}{lll}
M, g, s \models P & \text{iff} & s \in g(P) \\
M, g, s \models k(X^1, \ldots, X^n) & \text{iff} & s \in \langle g(X^1), \ldots, g(X^n) \rangle \in I(k, w) \\
M, g, s \models \neg\varphi & \text{iff} & \text{not } M, g, s \models \varphi \\
M, g, s \models \varphi \vee \psi & \text{iff} & M, g, s \models \varphi \text{ or } M, g, s \models \psi \\
M, g, s \models \varphi \Rightarrow \psi & \text{iff} & M, g, v \models \psi \text{ for all } v \in f(s, \overbrace{\{u \mid M, g, u \models \varphi\}}^{[\varphi]}) \\
M, g, s \models \forall X. \varphi & \text{iff} & M, [d/X]g, s \models \varphi \text{ for all } d \in D \\
M, g, s \models \forall P. \varphi & \text{iff} & M, [p/P]g, s \models \varphi \text{ for all } p \in Q
\end{array}
$$

Validity

- $M \models \varphi$ iff for all worlds s and assignments g holds $M, g, s \models \varphi$
- $\models \varphi$ iff φ is valid in every model M

Satisfiability $M, g, s \models \varphi$ defined as:

$M, g, s \models P$ iff $s \in g(P)$

$M, g, s \models k(X^1, \ldots, X^n)$ iff $s \in \langle g(X^1), \ldots, g(X^n) \rangle \in I(k, w)$

$M, g, s \models \neg\varphi$ iff not $M, g, s \models \varphi$

$M, g, s \models \varphi \vee \psi$ iff $M, g, s \models \varphi$ or $M, g, s \models \psi$

$M, g, s \models \varphi \Rightarrow \psi$ iff $M, g, v \models \psi$ for all $v \in f(s, \overbrace{\{u \mid M, g, u \models \varphi\}}^{[\varphi]})$

$M, g, s \models \forall X.\varphi$ iff $M, [d/X]g, s \models \varphi$ for all $d \in D$

$M, g, s \models \forall P.\varphi$ iff $M, [p/P]g, s \models \varphi$ for all $p \in Q$

Validity

- $M \models \varphi$ iff for all worlds s and assignments g holds $M, g, s \models \varphi$
- $\models \varphi$ iff φ is valid in every model M

Satisfiability $M, g, s \models \varphi$ defined as:

$$
\begin{array}{lll}
M, g, s \models P & \text{iff} & s \in g(P) \\
M, g, s \models k(X^1, \ldots, X^n) & \text{iff} & s \in \langle g(X^1), \ldots, g(X^n) \rangle \in I(k, w) \\
M, g, s \models \neg\varphi & \text{iff} & \text{not } M, g, s \models \varphi \\
M, g, s \models \varphi \vee \psi & \text{iff} & M, g, s \models \varphi \text{ or } M, g, s \models \psi \\
M, g, s \models \varphi \Rightarrow \psi & \text{iff} & M, g, v \models \psi \text{ for all } v \in f(s, \overbrace{\{u \mid M, g, u \models \varphi\}}^{[\varphi]}) \\
M, g, s \models \forall X. \varphi & \text{iff} & M, [d/X]g, s \models \varphi \text{ for all } d \in D \\
M, g, s \models \forall P. \varphi & \text{iff} & M, [p/P]g, s \models \varphi \text{ for all } p \in Q
\end{array}
$$

Validity

- $M \models \varphi$ iff for all worlds s and assignments g holds $M, g, s \models \varphi$
- $\models \varphi$ iff φ is valid in every model M

Quantified Conditional Logic – Normality

Above semantics of $\Rightarrow$ enforces normality property:

if φ and φ' are equivalent, then they index the same formulas wrt. $\Rightarrow$

Above semantics of $\Rightarrow$ enforces normality property:

if φ and φ' are equivalent, then they index the same formulas wrt. $\Rightarrow$

The axiomatic counterpart of the normality condition given by rule (RCEA)

$$\frac{\varphi \leftrightarrow \varphi'}{(\varphi \Rightarrow \psi) \leftrightarrow (\varphi' \Rightarrow \psi)}\ (RCEA)$$

Above semantics forces also the following rules to hold:

$$\frac{(\varphi_1 \wedge \ldots \wedge \varphi_n) \leftrightarrow \psi}{(\varphi_0 \Rightarrow \varphi_1 \wedge \ldots \wedge \varphi_0 \Rightarrow \varphi_n) \rightarrow (\varphi_0 \Rightarrow \psi)}\ (RCK) \qquad \frac{\varphi \leftrightarrow \varphi'}{(\psi \Rightarrow \varphi) \leftrightarrow (\psi \Rightarrow \varphi')}\ (RCEC)$$

Above semantics of $\Rightarrow$ enforces normality property:

if φ and φ' are equivalent, then they index the same formulas wrt. $\Rightarrow$

The axiomatic counterpart of the normality condition given by rule (RCEA)

$$\frac{\varphi \leftrightarrow \varphi'}{(\varphi \Rightarrow \psi) \leftrightarrow (\varphi' \Rightarrow \psi)}\ (RCEA)$$

Above semantics forces also the following rules to hold:

$$\frac{(\varphi_1 \wedge \ldots \wedge \varphi_n) \leftrightarrow \psi}{(\varphi_0 \Rightarrow \varphi_1 \wedge \ldots \wedge \varphi_0 \Rightarrow \varphi_n) \rightarrow (\varphi_0 \Rightarrow \psi)}\ (RCK) \qquad \frac{\varphi \leftrightarrow \varphi'}{(\psi \Rightarrow \varphi) \leftrightarrow (\psi \Rightarrow \varphi')}\ (RCEC)$$

Logic CK: minimal logic closed under rules RCEA, RCEC and RCK.
In what follows only logic CK and its extensions are considered.

Kripke style semantics (higher-order) selection function!

$M, g, s \models P$	iff	$s \in g(P)$
$M, g, s \models k(X^1, \ldots, X^n)$	iff	$s \in \langle g(X^1), \ldots, g(X^n) \rangle \in I(k, w)$
$M, g, s \models \neg\varphi$	iff	not $M, g, s \models \varphi$
$M, g, s \models \varphi \vee \psi$	iff	$M, g, s \models \varphi$ or $M, g, s \models \psi$
$M, g, s \models \varphi \Rightarrow \psi$	iff	$M, g, v \models \psi$ for all $v \in f(s, \overbrace{\{u \mid M, g, u \models \varphi\}}^{[\varphi]})$
$M, g, s \models \forall X. \varphi$	iff	$M, [d/X]g, s \models \varphi$ for all $d \in D$
$M, g, s \models \forall P. \varphi$	iff	$M, [p/P]g, s \models \varphi$ for all $p \in Q$

Semantic embedding: ML $\longrightarrow$ HOL terms of type $\iota \rightarrow o$

$$
\begin{aligned}
P &= \lambda W_\iota . (P_{\iota\rightarrow o}\, W) = P_{\iota\rightarrow o} \\
k(X^1, \ldots, X^n) &= \lambda W_\iota . (k_{\mu^n\rightarrow(\iota\rightarrow o)}\, X^1_\mu \ldots X^n_\mu)\, W \\
\neg &= \lambda \varphi_{\iota\rightarrow o} . \lambda W_\iota . \neg(\varphi\, W) \\
\vee &= \lambda \varphi_{\iota\rightarrow o} . \lambda \psi_{\iota\rightarrow o} . \lambda W_\iota . (\varphi\, W) \vee (\psi\, W) \\
\Rightarrow &= \lambda \varphi_{\iota\rightarrow o} . \lambda \psi_{\iota\rightarrow o} . \lambda W_\iota . \forall V_\iota . \neg(f\, W\, \varphi\, V) \vee (\psi\, V) \\
\forall^\mu(\Pi^\mu) &= \lambda Q_{\mu\rightarrow(\iota\rightarrow o)} . \lambda W_\iota . \forall X_\mu . (Q\, X\, W) \\
\forall^p(\Pi^p) &= \lambda Q_{(\iota\rightarrow o)\rightarrow(\iota\rightarrow o)} . \lambda W_\iota . \forall P_{\iota\rightarrow o} . (Q\, P\, W)
\end{aligned}
$$

Kripke style semantics (higher-order) selection function!

$M, g, s \models P$ iff $s \in g(P)$

$M, g, s \models k(X^1, \ldots, X^n)$ iff $s \in \langle g(X^1), \ldots, g(X^n) \rangle \in I(k, w)$

$M, g, s \models \neg\varphi$ iff not $M, g, s \models \varphi$

$M, g, s \models \varphi \vee \psi$ iff $M, g, s \models \varphi$ or $M, g, s \models \psi$

$M, g, s \models \varphi \Rightarrow \psi$ iff $M, g, v \models \psi$ for all $v \in f(s, \overbrace{\{u \mid M, g, u \models \varphi\}}^{[\varphi]})$

$M, g, s \models \forall X . \varphi$ iff $M, [d/X]g, s \models \varphi$ for all $d \in D$

$M, g, s \models \forall P . \varphi$ iff $M, [p/P]g, s \models \varphi$ for all $p \in Q$

Semantic embedding: ML $\longrightarrow$ HOL terms of type $\iota \rightarrow o$

$$P = \lambda W_\iota . (P_{\iota \rightarrow o} W) = P_{\iota \rightarrow o}$$

$$k(X^1, \ldots, X^n) = \lambda W_\iota . (k_{\mu^n \rightarrow (\iota \rightarrow o)} X^1_\mu \ldots X^n_\mu) W$$

$$\neg = \lambda \varphi_{\iota \rightarrow o} . \lambda W_\iota . \neg(\varphi W)$$

$$\vee = \lambda \varphi_{\iota \rightarrow o} . \lambda \psi_{\iota \rightarrow o} . \lambda W_\iota . (\varphi W) \vee (\psi W)$$

$$\Rightarrow = \lambda \varphi_{\iota \rightarrow o} . \lambda \psi_{\iota \rightarrow o} . \lambda W_\iota . \forall V_\iota . \neg(f W \varphi V) \vee (\psi V)$$

$$\forall^\mu(\Pi^\mu) = \lambda Q_{\mu \rightarrow (\iota \rightarrow o)} . \lambda W_\iota . \forall X_\mu . (Q X W)$$

$$\forall^p(\Pi^p) = \lambda Q_{(\iota \rightarrow o) \rightarrow (\iota \rightarrow o)} . \lambda W_\iota . \forall P_{\iota \rightarrow o} . (Q P W)$$

Kripke style semantics (higher-order) selection function!

$M, g, s \models P$ iff $s \in g(P)$

$M, g, s \models k(X^1, \ldots, X^n)$ iff $s \in \langle g(X^1), \ldots, g(X^n) \rangle \in I(k, w)$

$M, g, s \models \neg\varphi$ iff not $M, g, s \models \varphi$

$M, g, s \models \varphi \vee \psi$ iff $M, g, s \models \varphi$ or $M, g, s \models \psi$

$M, g, s \models \varphi \Rightarrow \psi$ iff $M, g, v \models \psi$ for all $v \in f(s, \overbrace{\{u \mid M, g, u \models \varphi\}}^{[\varphi]})$

$M, g, s \models \forall X. \varphi$ iff $M, [d/X]g, s \models \varphi$ for all $d \in D$

$M, g, s \models \forall P. \varphi$ iff $M, [p/P]g, s \models \varphi$ for all $p \in Q$

Semantic embedding: ML $\longrightarrow$ HOL terms of type $\iota \rightarrow o$

$$P = \lambda W_\iota. (P_{\iota\rightarrow o}\, W) = P_{\iota\rightarrow o}$$

$$k(X^1, \ldots, X^n) = \lambda W_\iota. (k_{\mu^n\rightarrow(\iota\rightarrow o)}\, X^1_\mu \ldots X^n_\mu)\, W$$

$$\neg = \lambda\varphi_{\iota\rightarrow o}. \lambda W_\iota. \neg(\varphi\, W)$$

$$\vee = \lambda\varphi_{\iota\rightarrow o}. \lambda\psi_{\iota\rightarrow o}. \lambda W_\iota. (\varphi\, W) \vee (\psi\, W)$$

$$\Rightarrow = \lambda\varphi_{\iota\rightarrow o}. \lambda\psi_{\iota\rightarrow o}. \lambda W_\iota. \forall V_\iota. \neg(f\, W\, \varphi\, V) \vee (\psi\, V)$$

$$\forall^\mu(\Pi^\mu) = \lambda Q_{\mu\rightarrow(\iota\rightarrow o)}. \lambda W_\iota. \forall X_\mu. (Q\, X\, W)$$

$$\forall^p(\Pi^p) = \lambda Q_{(\iota\rightarrow o)\rightarrow(\iota\rightarrow o)}. \lambda W_\iota. \forall P_{\iota\rightarrow o}. (Q\, P\, W)$$

Kripke style semantics (higher-order) selection function!

$M, g, s \models P$	iff	$s \in g(P)$
$M, g, s \models k(X^1, \ldots, X^n)$	iff	$s \in \langle g(X^1), \ldots, g(X^n) \rangle \in I(k, w)$
$M, g, s \models \neg\varphi$	iff	not $M, g, s \models \varphi$
$M, g, s \models \varphi \vee \psi$	iff	$M, g, s \models \varphi$ or $M, g, s \models \psi$
$M, g, s \models \varphi \Rightarrow \psi$	iff	$M, g, v \models \psi$ for all $v \in f(s, \overbrace{\{u \mid M, g, u \models \varphi\}}^{[\varphi]})$
$M, g, s \models \forall X. \varphi$	iff	$M, [d/X]g, s \models \varphi$ for all $d \in D$
$M, g, s \models \forall P. \varphi$	iff	$M, [p/P]g, s \models \varphi$ for all $p \in Q$

Semantic embedding: ML $\longrightarrow$ HOL terms of type $\iota \to o$

$$P = \lambda W_\iota. (P_{\iota\to o}\, W) = P_{\iota\to o}$$
$$k(X^1, \ldots, X^n) = \lambda W_\iota. (k_{\mu^n\to(\iota\to o)}\, X^1_\mu \ldots X^n_\mu)\, W$$
$$\neg = \lambda\varphi_{\iota\to o}. \lambda W_\iota. \neg(\varphi\, W)$$
$$\vee = \lambda\varphi_{\iota\to o}. \lambda\psi_{\iota\to o}. \lambda W_\iota. (\varphi\, W) \vee (\psi\, W)$$
$$\Rightarrow = \lambda\varphi_{\iota\to o}. \lambda\psi_{\iota\to o}. \lambda W_\iota. \forall V_\iota. \neg(f\, W\, \varphi\, V) \vee (\psi\, V)$$
$$\forall^\mu(\Pi^\mu) = \lambda Q_{\mu\to(\iota\to o)}. \lambda W_\iota. \forall X_\mu. (Q\, X\, W)$$
$$\forall^p(\Pi^p) = \lambda Q_{(\iota\to o)\to(\iota\to o)}. \lambda W_\iota. \forall P_{\iota\to o}. (Q\, P\, W)$$

Quantified Conditional Logics as Fragments of HOL

Kripke style semantics (higher-order) selection function!

$M,g,s \models P$	iff	$s \in g(P)$
$M,g,s \models k(X^1,\dots,X^n)$	iff	$s \in \langle g(X^1),\dots,g(X^n)\rangle \in I(k,w)$
$M,g,s \models \neg\varphi$	iff	not $M,g,s \models \varphi$
$M,g,s \models \varphi \vee \psi$	iff	$M,g,s \models \varphi$ or $M,g,s \models \psi$
$M,g,s \models \varphi \Rightarrow \psi$	iff	$M,g,v \models \psi$ for all $v \in f(s,\overbrace{\{u \mid M,g,u \models \varphi\}}^{[\varphi]})$
$M,g,s \models \forall X.\varphi$	iff	$M,[d/X]g,s \models \varphi$ for all $d \in D$
$M,g,s \models \forall P.\varphi$	iff	$M,[p/P]g,s \models \varphi$ for all $p \in Q$

Semantic embedding: ML $\longrightarrow$ HOL terms of type $\iota \to o$

$$
\begin{aligned}
P &= \lambda W_\iota.(P_{\iota\to o}\,W) = P_{\iota\to o}\\
k(X^1,\dots,X^n) &= \lambda W_\iota.(k_{\mu^n\to(\iota\to o)}\,X^1_\mu \dots X^n_\mu)\,W\\
\neg &= \lambda\varphi_{\iota\to o}.\lambda W_\iota.\neg(\varphi\,W)\\
\vee &= \lambda\varphi_{\iota\to o}.\lambda\psi_{\iota\to o}.\lambda W_\iota.(\varphi\,W)\vee(\psi\,W)\\
\Rightarrow &= \lambda\varphi_{\iota\to o}.\lambda\psi_{\iota\to o}.\lambda W_\iota.\forall V_\iota.\neg(f\,W\,\varphi\,V)\vee(\psi\,V)\\
\forall^\mu(\Pi^\mu) &= \lambda Q_{\mu\to(\iota\to o)}.\lambda W_\iota.\forall X_\mu.(Q\,X\,W)\\
\forall^p(\Pi^p) &= \lambda Q_{(\iota\to o)\to(\iota\to o)}.\lambda W_\iota.\forall P_{\iota\to o}.(Q\,P\,W)
\end{aligned}
$$

Quantified Conditional Logics as Fragments of HOL

Kripke style semantics (higher-order) selection function!

$$
\begin{array}{lll}
M, g, s \models P & \text{iff} & s \in g(P) \\
M, g, s \models k(X^1, \ldots, X^n) & \text{iff} & s \in \langle g(X^1), \ldots, g(X^n) \rangle \in I(k, w) \\
M, g, s \models \neg\varphi & \text{iff} & \text{not } M, g, s \models \varphi \\
M, g, s \models \varphi \vee \psi & \text{iff} & M, g, s \models \varphi \text{ or } M, g, s \models \psi \\
M, g, s \models \varphi \Rightarrow \psi & \text{iff} & M, g, v \models \psi \text{ for all } v \in f(s, \overbrace{\{u \mid M, g, u \models \varphi\}}^{[\varphi]}) \\
M, g, s \models \forall X. \varphi & \text{iff} & M, [d/X]g, s \models \varphi \text{ for all } d \in D \\
M, g, s \models \forall P. \varphi & \text{iff} & M, [p/P]g, s \models \varphi \text{ for all } p \in Q
\end{array}
$$

Semantic embedding: ML $\longrightarrow$ HOL terms of type $\iota \to o$

$$
\begin{array}{rcl}
P & = & \lambda W_\iota. (P_{\iota\to o}\, W) = P_{\iota\to o} \\
k(X^1, \ldots, X^n) & = & \lambda W_\iota. (k_{\mu^n \to (\iota\to o)}\, X^1_\mu \ldots X^n_\mu)\, W \\
\neg & = & \lambda \varphi_{\iota\to o}. \lambda W_\iota. \neg(\varphi\, W) \\
\vee & = & \lambda \varphi_{\iota\to o}. \lambda \psi_{\iota\to o}. \lambda W_\iota. (\varphi\, W) \vee (\psi\, W) \\
\Rightarrow & = & \lambda \varphi_{\iota\to o}. \lambda \psi_{\iota\to o}. \lambda W_\iota. \forall V_\iota. \neg(f\, W\, \varphi\, V) \vee (\psi\, V) \\
\forall^\mu(\Pi^\mu) & = & \lambda Q_{\mu\to(\iota\to o)}. \lambda W_\iota. \forall X_\mu. (Q\, X\, W) \\
\forall^p(\Pi^p) & = & \lambda Q_{(\iota\to o)\to(\iota\to o)}. \lambda W_\iota. \forall P_{\iota\to o}. (Q\, P\, W)
\end{array}
$$

Quantified Conditional Logics as Fragments of HOL

Kripke style semantics (higher-order) selection function!

$M, g, s \models P$	iff	$s \in g(P)$
$M, g, s \models k(X^1, \ldots, X^n)$	iff	$s \in \langle g(X^1), \ldots, g(X^n) \rangle \in I(k, w)$
$M, g, s \models \neg\varphi$	iff	not $M, g, s \models \varphi$
$M, g, s \models \varphi \vee \psi$	iff	$M, g, s \models \varphi$ or $M, g, s \models \psi$
$M, g, s \models \varphi \Rightarrow \psi$	iff	$M, g, v \models \psi$ for all $v \in f(s, \overbrace{\{u \mid M, g, u \models \varphi\}}^{[\varphi]})$
$M, g, s \models \forall X. \varphi$	iff	$M, [d/X]g, s \models \varphi$ for all $d \in D$
$M, g, s \models \forall P. \varphi$	iff	$M, [p/P]g, s \models \varphi$ for all $p \in Q$

Semantic embedding: ML $\longrightarrow$ HOL terms of type $\iota \rightarrow o$

$$
\begin{aligned}
P &= \lambda W_\iota . (P_{\iota \rightarrow o}\, W) = P_{\iota \rightarrow o} \\
k(X^1, \ldots, X^n) &= \lambda W_\iota . (k_{\mu^n \rightarrow (\iota \rightarrow o)}\, X^1_\mu \ldots X^n_\mu)\, W \\
\neg &= \lambda \varphi_{\iota \rightarrow o} . \lambda W_\iota . \neg(\varphi\, W) \\
\vee &= \lambda \varphi_{\iota \rightarrow o} . \lambda \psi_{\iota \rightarrow o} . \lambda W_\iota . (\varphi\, W) \vee (\psi\, W) \\
\Rightarrow &= \lambda \varphi_{\iota \rightarrow o} . \lambda \psi_{\iota \rightarrow o} . \lambda W_\iota . \forall V_\iota . \neg(f\, W\, \varphi\, V) \vee (\psi\, V) \\
\forall^\mu(\Pi^\mu) &= \lambda Q_{\mu \rightarrow (\iota \rightarrow o)} . \lambda W_\iota . \forall X_\mu . (Q\, X\, W) \\
\forall^p(\Pi^p) &= \lambda Q_{(\iota \rightarrow o) \rightarrow (\iota \rightarrow o)} . \lambda W_\iota . \forall P_{\iota \rightarrow o} . (Q\, P\, W)
\end{aligned}
$$

Validity defined as before

$$\text{valid} \quad = \quad \lambda\varphi_{\iota\to o}.\forall W_\iota.\varphi\, W$$

Soundness and Completeness Theorem

$$\models^{QCL} \varphi \quad \text{iff} \quad \models^{HOL} \text{valid}\, \varphi_{\iota\to o}$$

Proof Idea:

Explicate and analyze the relation between selection functions semantics and corresponding Henkin models; see paper for details.

For Propositional Conditional Logics see

[BenzmüllerGenoveseGabbayRispoli, submitted (arXiv:1106.3685v3)]

For Quantified Multimodal Logics see

[BenzmüllerPaulson, Logica Universalis, to appear (arXiv:0905.2435v1)]

Automation of Quantified Conditional Logics
in HOL ATPs

Proof of the Barcan formula (confirms constant domain)

$$(\forall X.\varphi \Rightarrow \psi(X)) \rightarrow (\varphi \Rightarrow \forall X.\psi(X))$$

Proof of the Barcan formula (confirms constant domain)

$$(\forall X.\varphi \Rightarrow \psi(X)) \rightarrow (\varphi \Rightarrow \forall X.\psi(X))$$

$$\text{valid}\,(\forall X.\varphi \Rightarrow (\psi\, X)) \rightarrow (\varphi \Rightarrow \forall X.(\psi\, X))$$

Proof of the Barcan formula (confirms constant domain)

$$(\forall X.\varphi \Rightarrow \psi(X)) \rightarrow (\varphi \Rightarrow \forall X.\psi(X))$$

$$\text{valid}\, \neg(\Pi^{\mu} \lambda X.\, \varphi \Rightarrow (\psi\, X)) \vee (\varphi \Rightarrow \Pi^{\mu} \lambda X.\, (\psi\, X))$$

Proof of the Barcan formula (confirms constant domain)

$$(\forall X.\varphi \Rightarrow \psi(X)) \rightarrow (\varphi \Rightarrow \forall X.\psi(X))$$

$$\forall W_\iota\mathbf{.}(\neg(\Pi^\mu \lambda X\mathbf{.}\varphi \Rightarrow (\psi\, X)) \vee (\varphi \Rightarrow \Pi^\mu \lambda X\mathbf{.}(\psi\, X)))W$$

Proof of the Barcan formula (confirms constant domain)

$$(\forall X.\varphi \Rightarrow \psi(X)) \rightarrow (\varphi \Rightarrow \forall X.\psi(X))$$

$$\forall W_\iota\text{.}(\lambda V_\iota\text{.}((\neg(\Pi^\mu \lambda X\text{.}\varphi \Rightarrow (\psi\, X))\, V) \vee ((\varphi \Rightarrow \Pi^\mu \lambda X\text{.}(\psi\, X))\, V)))\, W$$

Proof of the Barcan formula (confirms constant domain)

$$(\forall X.\varphi \Rightarrow \psi(X)) \rightarrow (\varphi \Rightarrow \forall X.\psi(X))$$

$$\forall W_\iota.(\neg(\Pi^\mu \lambda X.\varphi \Rightarrow (\psi\, X))\, W \vee (\varphi \Rightarrow \Pi^\mu \lambda X.(\psi\, X))\, W)$$

Proof of the Barcan formula (confirms constant domain)

$$(\forall X.\varphi \Rightarrow \psi(X)) \rightarrow (\varphi \Rightarrow \forall X.\psi(X))$$

$$\dots$$

by LEO-II or Satallax in 0.01 seconds

Proof of the Barcan formula (confirms constant domain)

$$(\forall X.\varphi \Rightarrow \psi(X)) \rightarrow (\varphi \Rightarrow \forall X.\psi(X))$$

$$\ldots$$

by LEO-II or Satallax in 0.01 seconds

Proof of the Converse Barcan formula

$$(\varphi \Rightarrow \forall X.\psi(x)) \rightarrow (\forall X.\varphi \Rightarrow \psi(x))$$

by LEO-II or Satallax in 0.01 seconds

You may easily try it yourself!

All you need to do

- reuse the THF encoding of Quantified Multimodal Logics in HOL from the attachment of our paper
- formulate your conjecture in THF Syntax and add it to the above file
- go to `www.tptp.org` ⟶ SystemOnTPTP
  - upload your file
  - choose a THF reasoners: LEO-II, Satallax, TPS, Isabelle, Refute, Nitpick
  - run the selected system
  - some systems even provide detailed proof output

Question to audience:

- Is there any direct ATP for Quantified Conditional Logics which we can employ for comparative evaluation of our approach?

Conclusion

Embedding of Quantified Conditional Logics in HOL

- is more challenging than others because of selection function semantics (I do not know of a translation into FOL)
- is nevertheless quite straightforward
- provides evidence for potential of HOL (ATPs) as universal logic (reasoners)

Further work includes

- experiments: scalability of proof automation
- combinations with other logics
- further logics