Simple Type Theory
as Framework for
Combining Logics

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synonyms in this talk

Church’s Simple Type Theory
Classical Higher Order Logic (HOL)
simple types $\alpha, \beta ::= \nu | o | \alpha \rightarrow \beta$ (opt. further base types)

HOL defined by

\[
\begin{align*}
s, t & ::= \ p_\alpha \ | \ X_\alpha \\
& \quad | \ (\lambda X_\alpha \cdot s_\beta)_{\alpha \rightarrow \beta} \\
& \quad | \ (s_\alpha \rightarrow t_\alpha)_\beta \\
& \quad | \ (\neg o \rightarrow o \ s_0)_o \\
& \quad | \ (s_0 \lor o \rightarrow o \ t_0)_o \\
& \quad | \ (\Pi (\alpha \rightarrow o) \rightarrow o t_{\alpha \rightarrow o})_o \\
& \quad | \ (s_\alpha =_{\alpha \rightarrow o} t_\alpha)_o
\end{align*}
\]

HOL is well understood

- Origin (Church, J.Symb.Log., 1940)
Opinions about HOL:

- HOL is expressive

but ...

- HOL can not be effectively automated
- HOL is a classical logic and not easily compatible with
  - modal logics
  - intuitionistic logic
  - ...
- HOL can not fruitfully serve as a basis for combining logics
Opinions about HOL:

- HOL is expressive and we exploit this here

but...

- HOL can **not** be effectively automated (at least partly)
- HOL is a classical logic and **not** easily compatible with
  - (normal) modal logics
  - intuitionistic logic
  - ...

- HOL can **not** fruitfully serve as a basis for combining logics

...I will give theoretical and **practical** evidence
Quantified Multimodal Logics (QML) as HOL Fragments
(jww Larry Paulson)
Quantified Multimodal Logics (QML)

- QML defined by

\[ s, t \quad ::= \quad P \mid (k \, X^1 \ldots X^n) \]
\[ \mid \neg s \mid s \lor t \]
\[ \mid \Box_r s \]
\[ \mid \forall X. s \mid \forall P. s \]

- Kripke style semantics
  - different notions of models: \( QS5^- \)
    \( QS5^\pi \)
    \( QS5^\pi^+ \)

(Fitting, J.Symb.Log., 2005)
Quantified Multimodal Logics (QML)

- QML defined by

\[ s, t ::= P \mid (k \ X^1 \ldots X^n) \]
\[ \mid \neg s \mid s \lor t \]
\[ \mid \Box_r s \]
\[ \mid \forall X.s \mid \forall P.s \]

- Kripke style semantics
  - different notions of models: (Fitting, J.Symb.Log., 2005)
    \[ QS5_{\pi^-} \rightarrow QK_{\pi^-} \]
    \[ QS5_{\pi} \rightarrow QK_{\pi} \]
    \[ QS5_{\pi^+} \rightarrow QK_{\pi^+} \] (correspondence to Henkin models)
(Normal) QML as Fragment of HOL

— related, but significantly extending (Ohlbach, 1988/93) —

**Straightforward Encoding**

- base type $\nu$: non-empty set of possible worlds
- base type $\mu$: non-empty set of individuals

QML formulas $\longrightarrow$ HOL terms of type $\nu \rightarrow o$

**QML Operators as abbreviations for specific HOL terms**

- $\neg = \lambda \phi. \lambda W. W \phi$ $W$
- $\lor = \lambda \phi. \lambda \psi. \lambda W. W \phi W \lor \psi W$
- $\Box = \lambda R. \lambda \phi. \lambda W. \forall V. W R W V \lor \phi V$
- $\Pi^\mu = \lambda \tau. \lambda W. \forall X. W (\tau X)$ $W$ (quantif. over individuals)
- $\Pi^{\nu \rightarrow o} = \lambda \tau. \lambda W. \forall P. W (\tau P)$ $W$ (quantif. over propositions)
(Normal) QML as Fragment of HOL

— related, but significantly extending (Ohlbach, 1988/93) —

**Straightforward Encoding**

- base type $\nu$: non-empty set of possible worlds
- base type $\mu$: non-empty set of individuals

QML formulas $\longrightarrow$ HOL terms of type $\nu \to o$

**QML Operators as abbreviations for specific HOL terms**

\[
\neg = \lambda \phi_{\nu \to o} \cdot \lambda W_{\nu} \cdot \neg \phi W
\]

\[
\lor = \lambda \phi_{\nu \to o} \cdot \lambda \psi_{\nu \to o} \cdot \lambda W_{\nu} \cdot \phi W \lor \psi W
\]

\[
\Box = \lambda R_{\nu \to \nu} \cdot \lambda \phi_{\nu \to o} \cdot \lambda W_{\nu} \cdot \forall V_{\nu} \cdot \neg R W V \lor \phi V
\]

\[
\Pi^\mu = \lambda \tau_{\mu \to (\nu \to o)} \cdot \lambda W_{\nu} \cdot \forall X_{\mu} \cdot (\tau X) W
\]

\[
\Pi_{\nu \to o} = \lambda \tau_{(\nu \to o) \to (\nu \to o)} \cdot \lambda W_{\nu} \cdot \forall P_{\nu \to o} \cdot (\tau P) W
\]
(Normal) QML as Fragment of HOL

Encoding of validity

\[ \text{valid} = \lambda \phi \rightarrow \forall \phi \ W \]
Example: In all $r$-accessible worlds exists truth

Formulate problem in HOL using original QML syntax

$$\text{valid } \Box_r \exists P_{\rightarrow o \cdot} P$$

then automatically rewrite abbreviations

$$\Box_r \xrightarrow{\text{rewrite}} \ldots$$

$$\exists \xrightarrow{\text{rewrite}} \ldots$$

$$\text{valid } \xrightarrow{\text{rewrite}} \ldots$$

$$\beta \eta \downarrow \forall W_{\cdot \cdot} \forall Y_{\cdot \cdot} \neg r W Y \lor (\neg \forall P_{\rightarrow o \cdot} \neg (P Y))$$

and prove automatically (LEO-II, IsabelleP, TPS, Satallax, ... here the provers need to generate witness term $P = \lambda Y_{\cdot \cdot} \top$)
Example: In all $r$-accessible worlds exists truth

Formulate problem in HOL using original QML syntax

valid $\Box_r \exists P_{\rightarrow o} P$

then automatically rewrite abbreviations

\begin{align*}
\Box_r & \quad \xrightarrow{\text{rewrite}} \quad \ldots \\
\exists & \quad \xrightarrow{\text{rewrite}} \quad \ldots \\
\text{valid} & \quad \xrightarrow{\text{rewrite}} \quad \ldots \\
\beta\eta & \downarrow \quad \forall W_{\nu} \forall Y_{\nu} \neg r W Y \lor (\neg \forall P_{\rightarrow o} \neg (P Y))
\end{align*}

and prove automatically (LEO-II, IsabelleP, TPS, Satallax, ...) here the provers need to generate witness term $P = \lambda Y_{\nu}. T$
Example: In all $r$-accessible worlds exists truth

Formulate problem in HOL using original QML syntax

$$\text{valid } \Box_r \exists P_{\not\rightarrow o}. P$$

then automatically rewrite abbreviations

$$\begin{align*}
\Box_r & \quad \text{rewrite} \quad \ldots \\
\exists & \quad \text{rewrite} \quad \ldots \\
\text{valid} & \quad \text{rewrite} \quad \ldots \\
\beta\eta & \downarrow \\
& \quad \forall W_u. \forall Y_u. \neg r W Y \lor (\neg \forall P_{\not\rightarrow o}. \neg(P Y))
\end{align*}$$

and prove automatically (LEO-II, IsabelleP, TPS, Satallax, ... here the provers need to generate witness term $P = \lambda Y_u. \top$)
Soundness and Completeness

Theorem:

\[ \models_{QML}^{\mathcal{QK}_\pi} s \text{ if and only if } \models_{HOL}^{\text{Henkin}} \text{valid } s_{l \rightarrow o} \]

(Benzmüller Paulson, Techn. Report, 2009)
Further interesting Fragments of HOL

- Intuitionistic Logic
  (exploiting Gödel’s translation to S4)
  (BenzmüllerPaulson, Log.J.IGPL, 2010)

- Access Control Logics
  (exploiting a translation by Garg and Abadi)
  (Benzmüller, SEC, 2009)

- Region Connection Calculus — later in this talk

- ...
Reasoning about Combinations of Logics
Reasoning about Combinations of Logics: Correspondence

Correspondences between properties of accessibility relations

\[
\text{symmetric} \quad = \quad \lambda R. \forall S, T. R S T \Rightarrow R T S
\]
\[
\text{serial} \quad = \quad \lambda R. \forall S. \exists T. R S T
\]

and corresponding axioms

\[
\forall R. \text{symmetric } R \quad \iff \quad 0.0s \\
\quad \Rightarrow \quad \text{valid } \forall \phi. \phi \sqsubseteq \square_R \Diamond R \phi
\]

\[
\forall R. \text{serial } R \quad \iff \quad 0.0s \\
\quad \Rightarrow \quad \text{valid } \forall \phi. \square_R \phi \sqsubseteq \Diamond R \phi
\]

Such proofs can be automated with LEO-II in no-time!
Correspondences between properties of accessibility relations

\[
\text{symmetric} \quad = \quad \lambda R. \forall S, T. R S T \Rightarrow R T S
\]

\[
\text{serial} \quad = \quad \lambda R. \forall S. \exists T. R S T
\]

and corresponding axioms

\[
\forall R. \text{symmetric } R \quad \iff \quad 0.0_s \\
\Rightarrow \quad \text{valid } \forall \phi. \phi \supset \square R \Diamond R \phi
\]

\[
\forall R. \text{serial } R \quad \iff \quad 0.0_s \\
\Rightarrow \quad \text{valid } \forall \phi. \square R \phi \supset \Diamond R \phi
\]

Such proofs can be automated with LEO-II in no-time!
Reasoning about Combinations of Logics: Cube

(cf. John Halleck’s website)
Reasoning about Combinations of Logics: $M5 \leftrightarrow D4B$

\[\forall R.\]

\[
\begin{align*}
\text{valid } & \forall \phi. \square R \phi \supset \phi \\
\wedge & \quad \text{valid } \forall \phi. \Diamond R \phi \supset \square R \Diamond R \phi
\end{align*}
\]

\[= M5\]

\[\iff\]

\[
\begin{align*}
\text{valid } & \forall \phi. \square R \phi \supset \Diamond R \phi \\
\wedge & \quad \text{valid } \forall \phi. \square R \phi \supset \square R \square R \phi \\
\wedge & \quad \text{valid } \forall \phi. \phi \supset \square R \Diamond R \phi
\end{align*}
\]

\[= D4B\]
Reasoning about Combinations of Logics: $M5 \iff D4B$

\[
\forall R. \quad \begin{align*}
\text{valid } \forall \phi. \Box_R \phi & \supset \phi \\
\land \quad \text{valid } \forall \phi. \Diamond_R \phi & \supset \Box_R \Diamond_R \phi
\end{align*}
\begin{cases}
\text{=} M5
\end{cases}
\]

\[
\iff 
\begin{align*}
\text{serial } R \\
\land \quad \text{valid } \forall \phi. \Box_R \phi & \supset \Box_R \Box_R \phi \\
\land \quad \text{symmetric } R
\end{align*}
\begin{cases}
\text{=} D4B
\end{cases}
\]
Reasoning about Combinations of Logics: \( M5 \iff D4B \)

\[ \forall R. \]

\[
\begin{align*}
\text{reflexive } R \\
\land \\
\text{euclidean } R \\
\iff \\
\text{serial } R \\
\land \\
\text{transitive } R \\
\land \\
\text{symmetric } R
\end{align*}
\]

\[ = M5 \]

\[ = D4B \]
Reasoning about Combinations of Logics: $M5 \iff D4B$

$\forall R.$

- reflexive $R$
- euclidean $R$
- serial $R$
- transitive $R$
- symmetric $R$

$\iff$

$\{ \begin{align*}
\{ & = M5 \\
\{ & = D4B 
\end{align*} \}$

Proof with LEO-II in 0.1s
Reasoning *about* Combinations of Logics: Logics Cube

\[ S_5 = M_5 \iff M_{B5} \]
\[ \iff M_{4B5} \]
\[ \iff M_{45} \]
\[ \iff M_{4B} \]
\[ \iff D_{4B} \]
\[ \iff D_{4B5} \]
\[ \iff D_{B5} \]

\[ K_{B5} \iff K_{4B5} \]
\[ \iff K_{4B} \]
\[ M_5 \implies D_{45} \]
\[ D_{45} \implies M_5 \]
Reasoning about Combinations of Logics: Logics Cube

<table>
<thead>
<tr>
<th>S5 = M5</th>
<th>KB5</th>
<th>K4B5</th>
</tr>
</thead>
<tbody>
<tr>
<td>M5 ⇔ MB5</td>
<td>0.1s</td>
<td>K4B5 ⇔ 0.2s</td>
</tr>
<tr>
<td>M4B5 ⇔ M45</td>
<td>0.1s</td>
<td>0.1s</td>
</tr>
<tr>
<td>M45 ⇔ M4B</td>
<td>0.1s</td>
<td></td>
</tr>
<tr>
<td>D4B ⇔ D4B5</td>
<td>0.2s</td>
<td></td>
</tr>
<tr>
<td>DB5</td>
<td>0.1s</td>
<td></td>
</tr>
</tbody>
</table>

Proofs with LEO-II < 0.2s
Reasoning *about* Combinations of Logics: Logics Cube

\[
\begin{align*}
\text{S5} = \text{M5} & \iff \text{MB5} \\
& \iff \text{M4B5} \\
& \iff \text{M45} \\
& \iff \text{M4B} \\
& \iff \text{D4B} \\
& \iff \text{D4B5} \\
& \iff \text{DB5} \\
\text{KB5} & \iff \text{K4B5} \\
& \iff \text{K4B} \\
\text{M5} & \Rightarrow \text{D45} \\
\text{D45} & \not\Rightarrow \text{M5}
\end{align*}
\]

Countermodel 34.4s (IsabelleN)
(Segerberg, 1973) discusses a 2-dimensional logic providing two S5 modalities $\Box_a$ and $\Box_b$. He adds further axioms stating that these modalities are commutative and orthogonal. It actually turns out that orthogonality is already implied in this context.

reflexive $a$, transitive $a$, euclid. $a$,
reflexive $b$, transitive $b$, euclid. $b$,
valid $\forall \phi. \Box_a \Box_b \phi \iff \Box_b \Box_a \phi$

$\vdash_{HOL}$

valid $\forall \phi, \psi. \Box_a (\Box_a \phi \lor \Box_b \psi) \supset (\Box_a \phi \lor \Box_a \psi)$
\land
valid $\forall \phi, \psi. \Box_b (\Box_a \phi \lor \Box_b \psi) \supset (\Box_b \phi \lor \Box_b \psi)$
(Segerberg, 1973) discusses a 2-dimensional logic providing two S5 modalities $\Box_a$ and $\Box_b$. He adds further axioms stating that these modalities are commutative and orthogonal. It actually turns out that orthogonality is already implied in this context.

\[ \vdash_{HOL} \forall \phi. \Box_a \Box_b \phi \iff \Box_b \Box_a \phi \]

proof by LEO-II in 0.2s

valid $\forall \phi, \psi. \Box_a (\Box_a \phi \lor \Box_b \psi) \supset (\Box_a \phi \lor \Box_a \psi)$

$\land$

valid $\forall \phi, \psi. \Box_b (\Box_a \phi \lor \Box_b \psi) \supset (\Box_b \phi \lor \Box_b \psi)$
Reasoning within Combined Logics
Wise Men Puzzle

Once upon a time, a king wanted to find the wisest out of his three wisest men. He arranged them in a circle and told them that he would put a white or a black spot on their foreheads and that one of the three spots would certainly be white. The three wise men could see and hear each other but, of course, they could not see their faces reflected anywhere. The king, then, asked to each of them to find out the color of his own spot. After a while, the wisest correctly answered that his spot was white.
Wise Men Puzzle

(adopting (Baldoni, PhD, 1998))

- epistemic modalities:
  \( \Box_a, \Box_b, \Box_c \) : three wise men
  \( \Box_{\text{fool}} \) : common knowledge

- predicate constant:
  \( ws \) : 'has white spot'

Once upon a time, a king wanted to find the wisest out of his three wisest men. He arranged them in a circle and told them that he would put a white or a black spot on their foreheads and that one of the three spots would certainly be white. The three wise men could see and hear each other but, of course, they could not see their faces reflected anywhere. The king, then, asked to each of them to find out the color of his own spot. After a while, the wisest correctly answered that his spot was white.
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(adapting (Baldoni, PhD, 1998))

- common knowledge:
  at least one of the wise men has a white spot
  \[ \text{valid} \, \square \text{fool} (ws \, a) \lor (ws \, b) \lor (ws \, c) \]
  if \( X \) one has a white spot then \( Y \) can see this
  \[ (\text{valid} \, \square \text{fool} (ws \, X) \Rightarrow \square \gamma (ws \, X)) \]
  if \( X \) has not a white spot then \( Y \) can see this
  \[ \text{valid} \, \square \text{fool} \neg (ws \, X) \Rightarrow \square \gamma \neg (ws \, X) \]
  \[ X \neq Y \in \{ a, b, c \} \]
Reasoning within Combined Logics: Wise Men Puzzle

Wise Men Puzzle

Once upon a time, a king wanted to find the wisest out of his three wisest men. He arranged them in a circle and told them that he would put a white or a black spot on their foreheads and that one of the three spots would certainly be white. The three wise men could see and hear each other but, of course, they could not see their faces reflected anywhere. The king, then, asked each of them to find out the color of his own spot. After a while, the wisest correctly answered that his spot was white.

(adapting (Baldoni, PhD, 1998))

- if $X$ knows $\phi$ then $Y$ knows that he knows $\phi$

$$\text{valid } \forall \phi. \ (\Box X \phi \Rightarrow \Box Y \Box X \phi)$$

- if $X$ does not know $\phi$ then $Y$ does not know $\phi$

$$\text{valid } \forall \phi. \ (\neg \Box X \phi \Rightarrow \Box Y \neg \Box X \phi)$$

$$X \neq Y \in \{a, b, c\}$$

- axioms for common knowledge

$$\text{valid } \forall \phi. \ \Box_{\text{fool}} \phi \Rightarrow \phi$$

$$\text{valid } \forall \phi. \ \Box_{\text{fool}} \phi \Rightarrow \Box_{\text{fool}} \Box_{\text{fool}} \phi$$

$$\forall R. \text{valid } \forall \phi. \ \Box_{\text{fool}} \phi \Rightarrow \Box R \phi$$
Reasoning within Combined Logics: Wise Men Puzzle

Wise Men Puzzle

(adapting (Baldoni, PhD, 1998))

- a, b do not know that they have a white spot

\[ \text{valid} \not\vdash \square_a (ws \ a) \quad \text{valid} \not\vdash \square_b (ws \ b) \]

- prove that c does know he has a white spot:

\[ \ldots \vdash_{HOL} \text{valid} \ \square_c (ws \ c) \]
Wise Men Puzzle

(adapting (Baldoni, PhD, 1998))

- a, b do not know that they have a white spot
  \[
  \text{valid} \not\rightarrow \Box_a (ws\ a) \quad \text{valid} \not\rightarrow \Box_b (ws\ b)
  \]

- prove that c does know he has a white spot:
  \[
  \ldots \vdash^{HOL} \text{valid} \Box_c (ws\ c)
  \]

LEO-II can prove this result in 0.3s
Reasoning within Combined Logics: Epistemic & Spatial

Region Connection Calculus (RCC)

as fragment of HOL:

disconnected: \( dc = \lambda X_\tau.\lambda Y_\tau.\neg(c \ X \ Y) \)

part of: \( p = \lambda X_\tau.\lambda Y_\tau.\forall Z.((c \ Z \ X) \Rightarrow (c \ Z \ Y)) \)

identical with: \( eq = \lambda X_\tau.\lambda Y_\tau.((p \ X \ Y) \land (p \ Y \ X)) \)

overlaps: \( o = \lambda X_\tau.\lambda Y_\tau.\exists Z.((p \ Z \ X) \land (p \ Z \ Y)) \)

partially o: \( po = \lambda X_\tau.\lambda Y_\tau.((o \ X \ Y) \land \neg(p \ X \ Y) \land \neg(p \ Y \ X)) \)

ext. connected: \( ec = \lambda X_\tau.\lambda Y_\tau.((c \ X \ Y) \land \neg(o \ X \ Y)) \)

proper part: \( pp = \lambda X_\tau.\lambda Y_\tau.((p \ X \ Y) \land \neg(p \ Y \ X)) \)
	tangential pp: \( tpp = \lambda X_\tau.\lambda Y_\tau.((pp \ X \ Y) \land \exists Z.((ec \ Z \ X) \land (ec \ Z \ Y))) \)
nontang. pp: \( ntpp = \lambda X_\tau.\lambda Y_\tau.((pp \ X \ Y) \land \neg\exists Z.((ec \ Z \ X) \land (ec \ Z \ Y))) \)
Reasoning *within* Combined Logics: Epistemic & Spatial

A trivial problem for RCC:

Catalunya is a border region of Spain \( (tpp \text{ catalunya} \text{ span}) \),
Spain and France share a border \( (ec \text{ span} \text{ france}) \),
Paris is a region inside France \( (ntpp \text{ paris} \text{ france}) \)
\[ \vdash \text{HOL} \]

Catalunya and Paris are disconnected \( (dc \text{ catalunya} \text{ paris}) \)
\[ \land \]
Spain and Paris are disconnected \( (dc \text{ span} \text{ paris}) \)
Reasoning *within* Combined Logics: Epistemic & Spatial

A trivial problem for RCC:

- Catalunya is a border region of Spain
- Spain and France share a border
- Paris is a region inside France

\[ \text{(tpp catalunya spain),} \]
\[ \text{(ec spain france),} \]
\[ \text{(ntpp paris france)} \]

\[ \models_{\text{HOL}} 2.3s \]

- Catalunya and Paris are disconnected
- Spain and Paris are disconnected

\[ \text{(dc catalunya paris)} \]
\[ \land \]
\[ \text{(dc spain paris)} \]
valid $\forall \phi. \square_{\text{fool}} \phi \supset \square_{\text{bob}} \phi$,
valid $\square_{\text{fool}} (\lambda W. (ec \text{ span france}))$,
valid $\square_{\text{bob}} (\lambda W. (tpp \text{ catalunya span}))$,
valid $\square_{\text{bob}} (\lambda W. (ntpp \text{ paris france}))$
valid $\square_{\text{bob}} (\lambda W. ((dc \text{ catalunya paris}) \land (dc \text{ spain paris})))$
Reasoning *within* Combined Logics: Epistemic & Spatial

valid $\forall \phi. \Box_{\text{fool}} \phi \supset \Box_{\text{bob}} \phi$,
valid $\Box_{\text{fool}} (\lambda W. (ec \text{ spain france}))$,
valid $\Box_{\text{bob}} (\lambda W. (tpp \text{ catalunya spain}))$,
valid $\Box_{\text{bob}} (\lambda W. (ntpp \text{ paris france}))$

\(\vdash_{\text{HOL}}\) 20.4s

valid $\Box_{\text{bob}} (\lambda W. ((dc \text{ catalunya paris}) \land (dc \text{ spain paris})))$
valid $\forall \phi. \Box_{\text{fool}} \phi \supset \Box_{\text{bob}} \phi$,
valid $\Box_{\text{fool}} (\lambda W. (\text{ec spain france}))$,
valid $\Box_{\text{bob}} (\lambda W. (\text{tpp catalunya spain}))$,
valid $\Box_{\text{bob}} (\lambda W. (\text{ntpp paris france}))$

\[\vdash_{\text{HOL}} 20.4\text{s}\]
valid $\Box_{\text{bob}} (\lambda W. ((\text{dc catalunya paris}) \land (\text{dc spain paris})))$

\[\not\vdash_{\text{HOL}}\]
valid $\Box_{\text{fool}} (\lambda W. ((\text{dc catalunya paris}) \land (\text{dc spain paris})))$
Reasoning \textbf{within} Combined Logics: Epistemic & Spatial

\[ \forall \phi. \Box_{\text{fool}} \phi \supset \Box_{\text{bob}} \phi, \]
\[ \text{valid} \quad \Box_{\text{fool}} (\lambda W. (ec \ spain \ france)), \]
\[ \text{valid} \quad \Box_{\text{bob}} (\lambda W. (tpp catalunya spain)), \]
\[ \text{valid} \quad \Box_{\text{bob}} (\lambda W. (ntpp paris france)) \]
\[ \vdash_{\text{HOL}} 20.4s \quad \text{valid} \quad \Box_{\text{bob}} (\lambda W. ((dc catalunya paris) \land (dc spain paris))) \]
\[ \nabla_{\text{HOL}} 39.7s \quad \text{valid} \quad \Box_{\text{fool}} (\lambda W. ((dc catalunya paris) \land (dc spain paris))) \]
Reasoning *within* Combined Logics: Epistemic & Spatial

\[
\begin{align*}
\text{valid } & \forall \phi. \square_{\text{fool}} \phi \supset \square_{\text{bob}} \phi, \\
\text{valid } & \square_{\text{fool}} (\lambda \mathcal{W}. (ec \text{ spain france})) ,
\end{align*}
\]

\[\vdash_{HOL} 20.4s \quad \exists H_{\text{bob}} (\lambda \mathcal{W}. ((dc \text{ catalunya paris}) \land (dc \text{ spain paris})))\]

\[\forall H_{\text{bob}} (\lambda \mathcal{W}. ((dc \text{ catalunya paris}) \land (dc \text{ spain paris})))\]


Key idea is “Lifting” of RCC propositions to modal predicates:

\[
\begin{align*}
\underbrace{(tpp \text{ catalunya spain})}_{\text{type } \sigma} & \quad \rightarrow \quad \underbrace{(\lambda \mathcal{W}. (tpp \text{ catalunya spain}))}_{\text{type } \iota \rightarrow \sigma}
\end{align*}
\]
LEO-II
(EPRSC grant EP/D070511/1 at Cambridge University)

Thanks to Larry Paulson
LEO-II employs FO-ATPs: E, Spass, Vampire

http://www.ags.uni-sb.de/~leo
Conclusion

- HOL is suited as framework for combining logics
- automation of object-/meta-level reasoning — scalability?
- embeddings can possibly be fully verified in Isabelle/HOL? (consistency of QML embedding: 3.8s – IsabelleN)
- current work: application to ontology reasoning (SUMO)

You can use this framework right away! Try it!

- new TPTP infrastructure for automated HOL reasoning (jww Geoff Sutcliffe / EU grant THFTPTP)
  - standardized input / output language (THF)
  - problem library: 3000 problems
  - yearly CASC competitions

- provers and examples are online; demo: http://tptp.org

Wise Men Puzzle:

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Reasoning about Logics and Combined Logics

Reasoning within Combined Logics