Challenges for Automated Theorem Proving in Classical Higher-Order Logic

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St. Andrews, February 8, 2007

Overall Scientific Interest

Mathematics Assistance Systems

Mathematics Assistance Systems

- Computing

Overall Scientific Interest

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- Computing
- Proving

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- Exploring/Inventing

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Overall Scientific Interest

OMEGA GROUP

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- ...

Overall Scientific Interest

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- Computing
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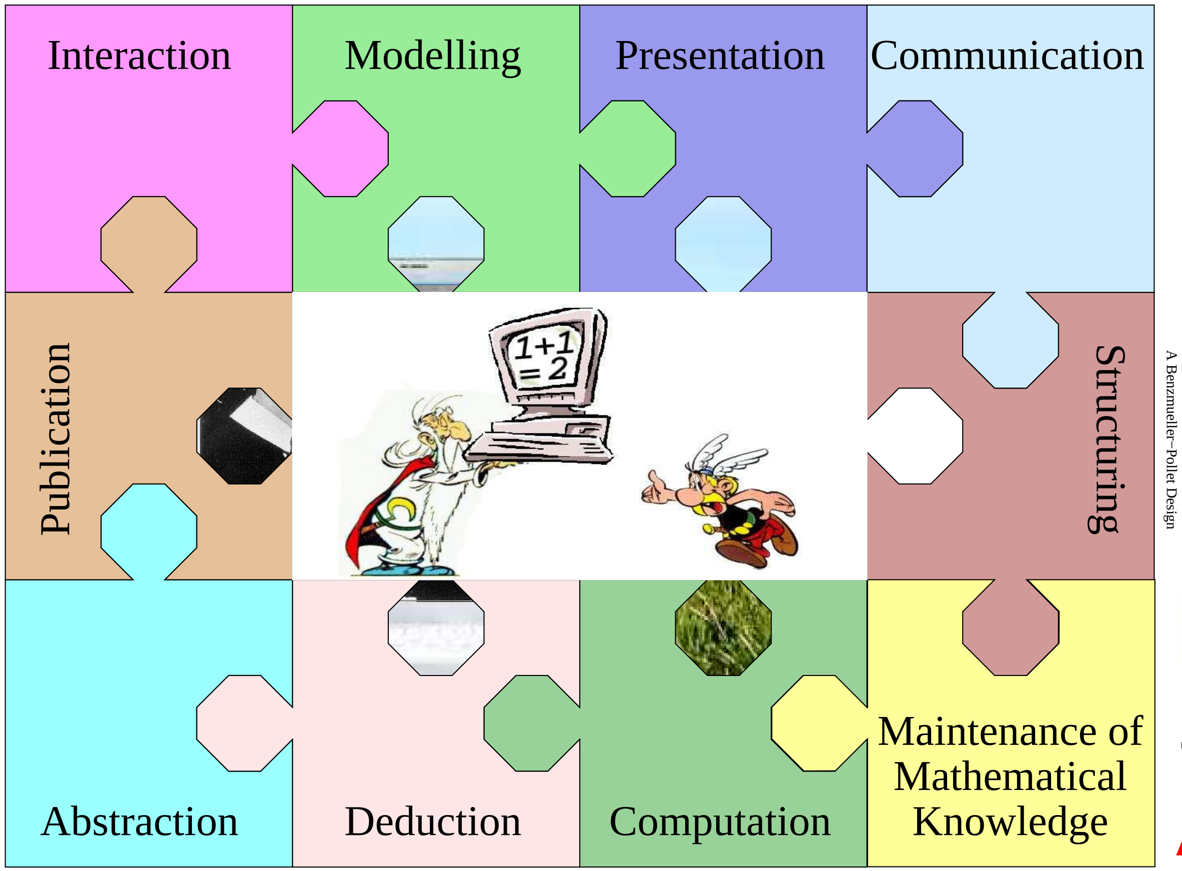
- . . .
- Architecture & HCI & . . .

Overall Scientific Interest

Mathematics Assistance Systems



. . .

Architecture & HCI & . . .

Overall Scientific Interest

Mathematics Assistance Systems

- Computing
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- ...
- (Architecture & HCI & ...)

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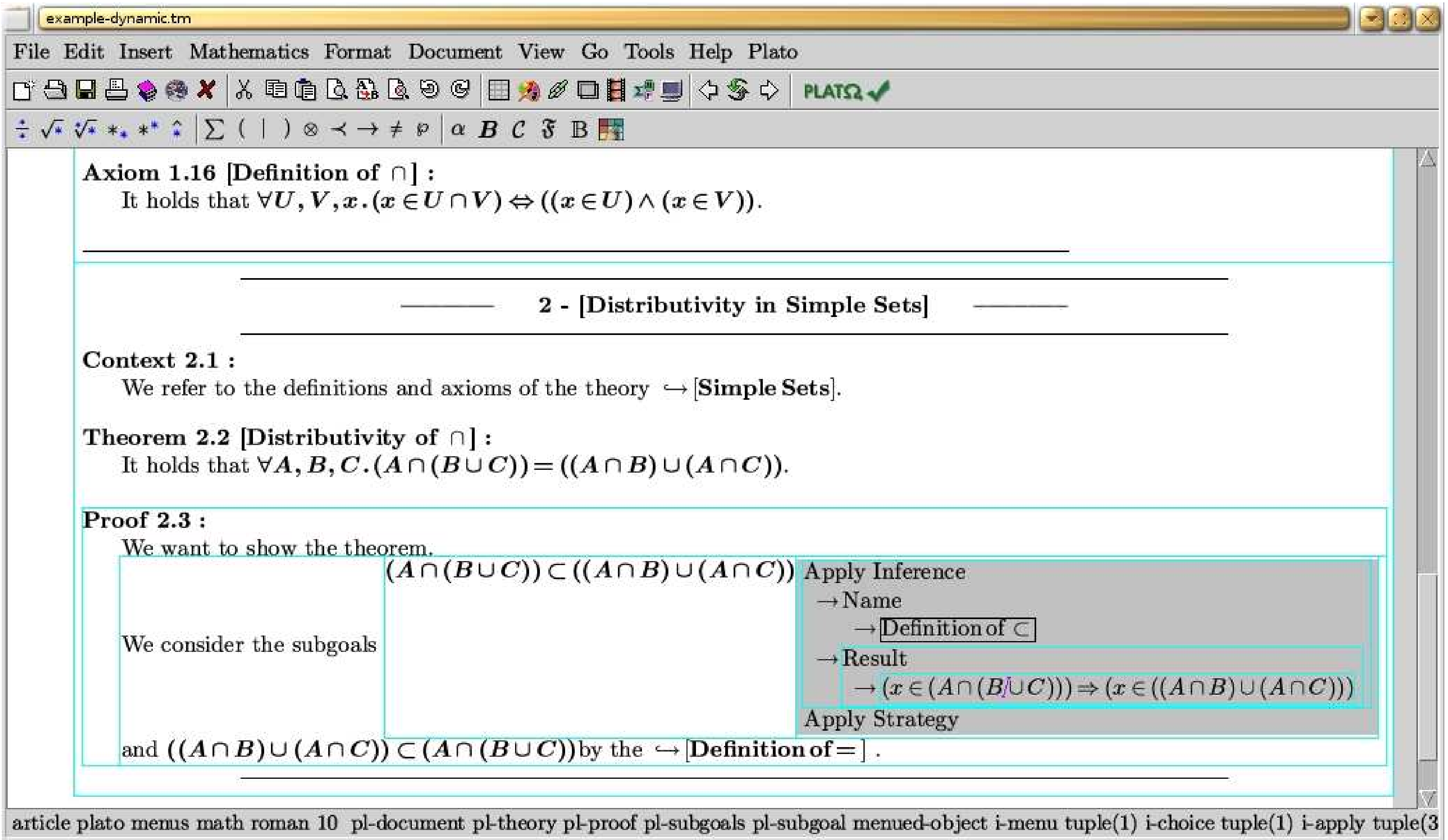

Overall Scientific Interest

Applications/Specialisations of Mathematics Assistance Systems

HW/SW-Verification Mathematics . . .

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HW/SW-Verification	Mathematics	...
E-Learning in HW/SW-Verif.	E-Learning in Maths	...

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Applications/Specialisations of Mathematics Assistance Systems

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Observation

Many proof assistants are based on higher-order logic

Overall Scientific Interest

Applications/Specialisations of Mathematics Assistance Systems

HW/SW-Verification	Mathematics	...
E-Learning in HW/SW-Verif.	E-Learning in Maths	...

Observation

Many proof assistants are based on higher-order logic

Motivation for

Automation of HOL (research is decades behind)

Own Research in HOL

Automated Theorem Proving

Own Research in HOL

Automated Theorem Proving

- Extensional Resolution, Equality Reasoning

Own Research in HOL

Semantics

Automated Theorem Proving

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- Model Classes (different extensionality properties)

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- Model Classes (different extensionality properties)
- Abstract Consistency Proof Method

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- Model Classes (different extensionality properties)
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- Test Problems

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- Combination with FO-ATP

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- LEO-II Project

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Proof Theory

Automated Theorem Proving

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- Model Classes (different extensionality properties)
- Abstract Consistency Proof Method
- Test Problems

Proof Theory

- Cut-simulation

Automated Theorem Proving

- Extensional Resolution, Equality Reasoning
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Own Research in HOL

Semantics

- Model Classes (different extensionality properties)
- Abstract Consistency Proof Method
- Test Problems

Proof Theory

- Cut-simulation (points to challenges for ATP)

Automated Theorem Proving

- Extensional Resolution, Equality Reasoning
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Syntax

Syntax

HOL-Syntax: Simple Typed λ-Calculus

	o	(truth values)
Simple Types $\mathcal{T}$:	ι	(individuals)
	$(\alpha \longrightarrow \beta)$	(functions from α to β)

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	o	(truth values)
Simple Types $\mathcal{T}$:	ι	(individuals)
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Typed Terms:

X_α	Variables ($\mathcal{V}$)
c_α	Constants & Parameters (Σ & $\mathcal{P}$)
$(\mathbf{F}_{\alpha\longrightarrow\beta}\, \mathbf{B}_\alpha)_\beta$	Application
$(\lambda Y_\alpha\, \mathbf{A}_\beta)_{\alpha\longrightarrow\beta}$	λ-abstraction

HOL-Syntax: Simple Typed λ-Calculus

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Equality of Terms: α, β, η

HOL: Adding Logical Connectives

$\top_o$ – true

$\bot_o$ – false

$\neg_{o \to o}$ – negation

$\vee_{o \to o \to o}$ – disjunction

$\wedge_{o \to o \to o}$ – conjunction

$\Rightarrow_{o \to o \to o}$ – implication

$\Leftrightarrow_{o \to o \to o}$ – equivalence

$\forall X_\alpha \blacksquare \ldots$ – universal quantification over type α ($\forall$ types α)

$\exists X_\alpha \blacksquare \ldots$ – existential quantification over type α ($\forall$ types α)

$=_{\alpha \to \alpha \to o}$ – equality at type α ($\forall$ types α)

HOL: Adding Logical Connectives

$\neg_{o \to o}$ – negation

$\vee_{o \to o \to o}$ – disjunction

$\forall X_\alpha. \ldots$ – universal quantification over type α ($\forall$ types α)

HOL: Leibniz Equality

Impredicative definition of equality

$\mathbf{A}_\alpha \doteq \mathbf{B}_\alpha$ means $\forall \mathsf{P}_{\alpha \longrightarrow o}(\mathsf{P}\,\mathbf{A} \Rightarrow \mathsf{P}\,\mathbf{B})$

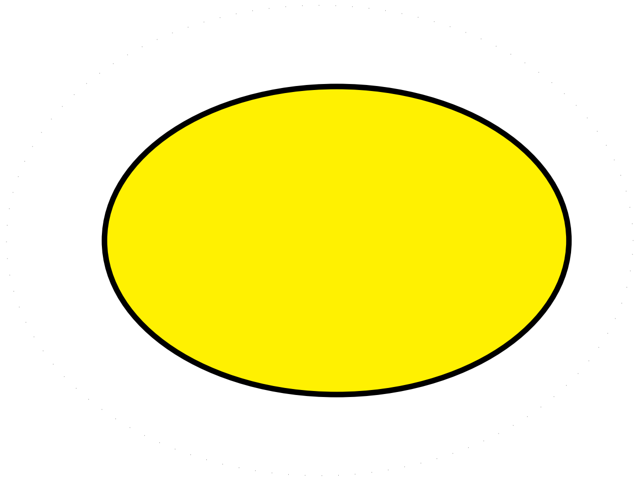
$\forall \mathsf{P}_{\alpha \longrightarrow o}(\neg \mathsf{P}\,\mathbf{A} \vee \mathsf{P}\,\mathbf{B})$

Semantics

Model Classes

(Extensionality)

Model Classes (Extensionality)



Standard Models $\mathfrak{ST}(\Sigma)$

- Idea of Standard Semantics:

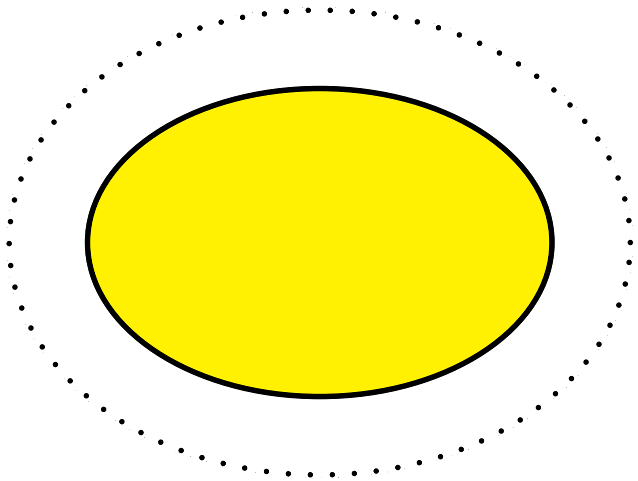
$$\iota \longrightarrow \mathcal{D}_\iota \qquad \text{(choose)}$$

$$o \longrightarrow \mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\} \qquad \text{(fixed)}$$

$$(\alpha \to \beta) \longrightarrow$$

$$\mathcal{D}_{\alpha\to\beta} = \mathcal{F}(\mathcal{D}_\alpha, \mathcal{D}_\beta) \qquad \text{(fixed)}$$

Model Classes (Extensionality)



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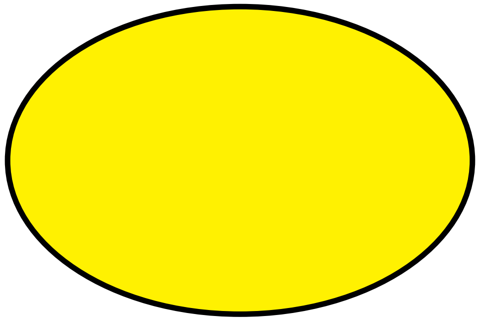
- Henkin's Generalization:

$$\mathcal{D}_{\alpha\to\beta} \subseteq \mathcal{F}(\mathcal{D}_\alpha, \mathcal{D}_\beta) \qquad \text{(choose)}$$

but elements are still functions!

[Henkin-50]

Model Classes (Extensionality)



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choose: $\mathcal{D}_\iota$
fixed: $\mathcal{D}_o, \mathcal{D}_{\alpha\to\beta}$, functions

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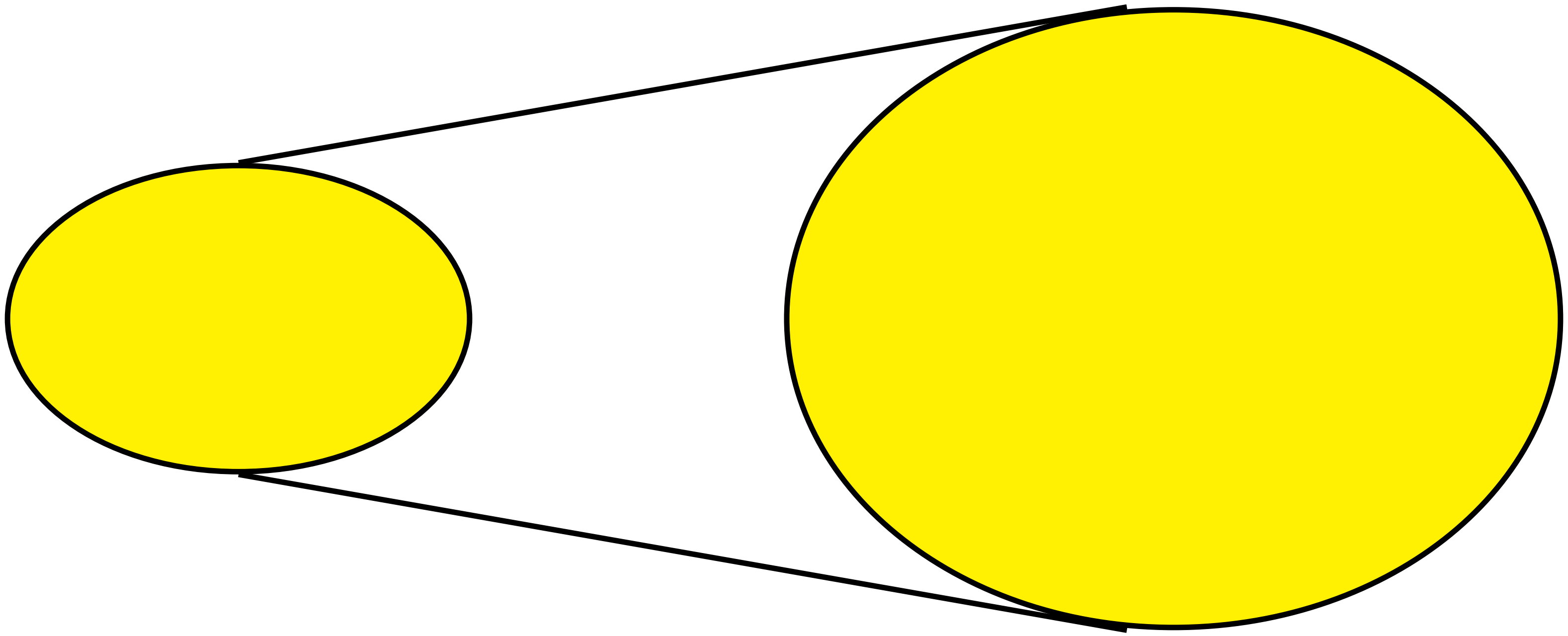

Standard Models $\mathfrak{S}\mathfrak{T}(\Sigma)$

Formulas valid in $\mathfrak{S}\mathfrak{T}(\Sigma)$

choose: $\mathcal{D}_\iota$
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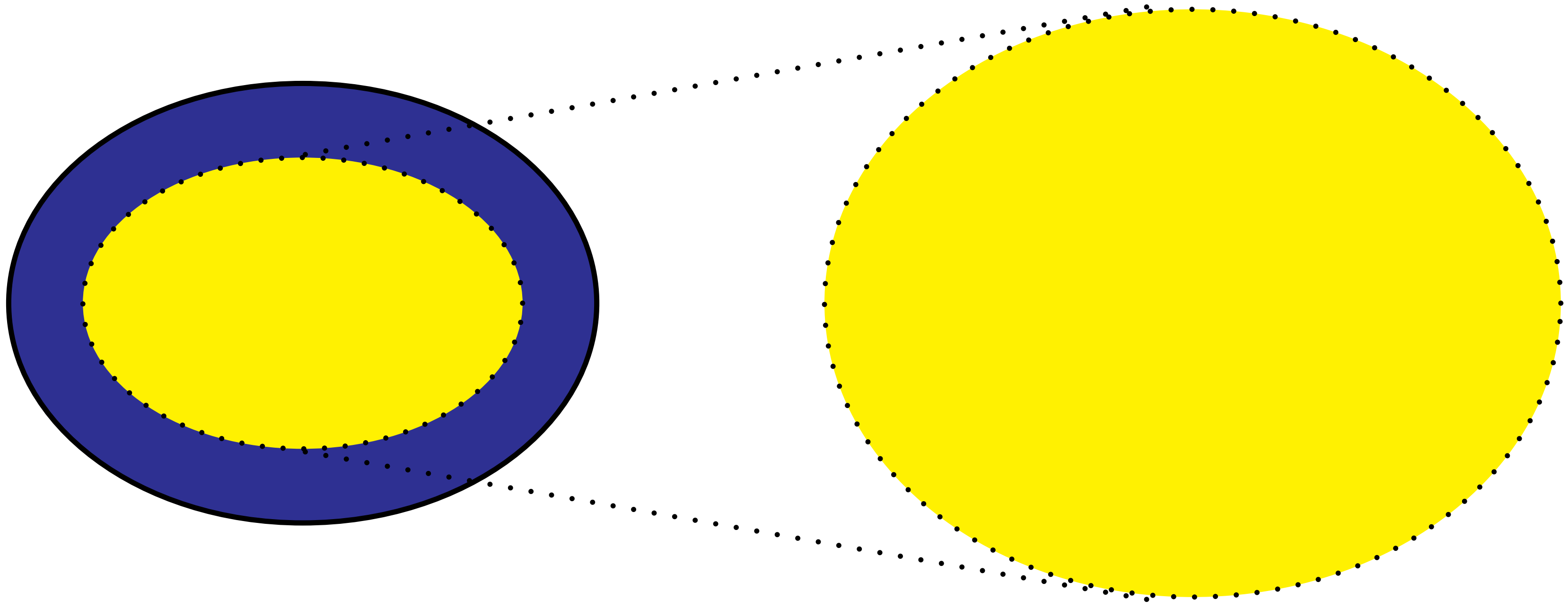

Henkin Models $\mathfrak{H}(\Sigma) = \mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma)$

Formulas valid in $\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma)$?

choose: $\mathcal{D}_\iota, \mathcal{D}_{\alpha\rightarrow\beta}$
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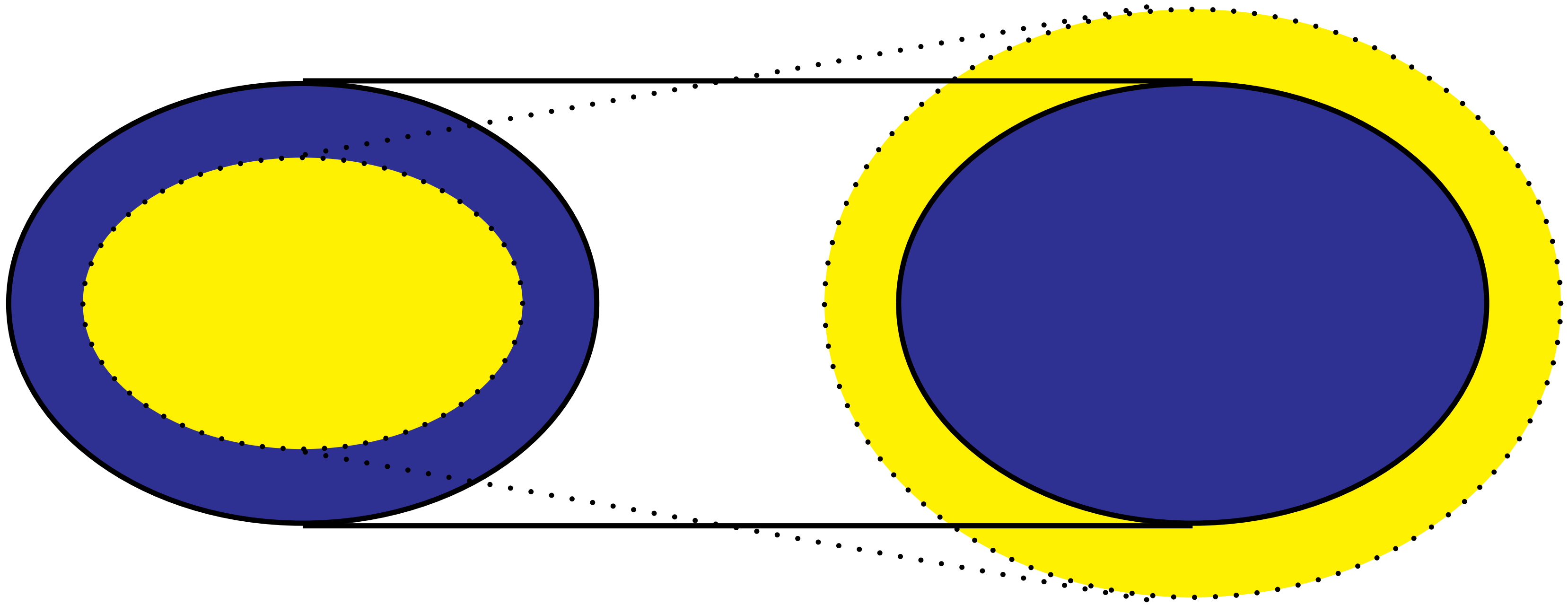
Model Classes (Extensionality)



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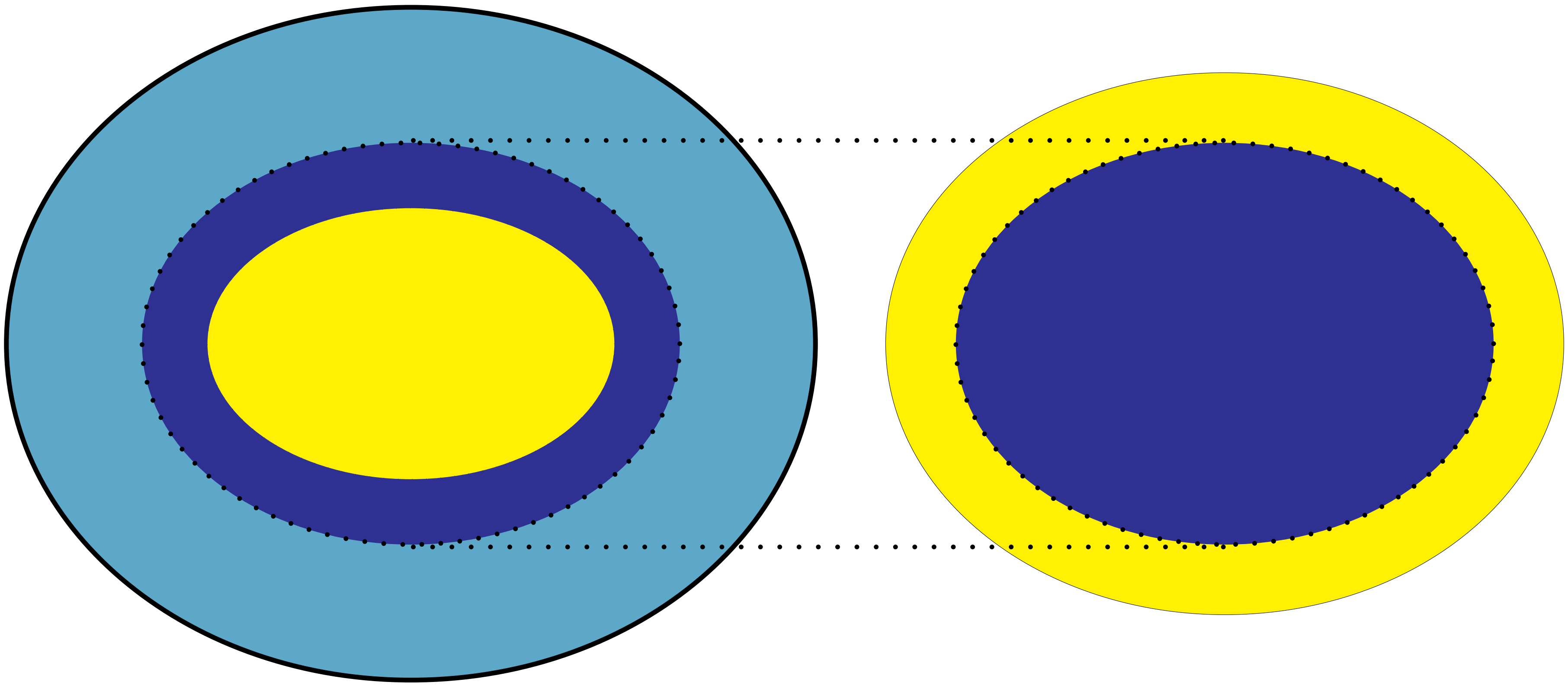


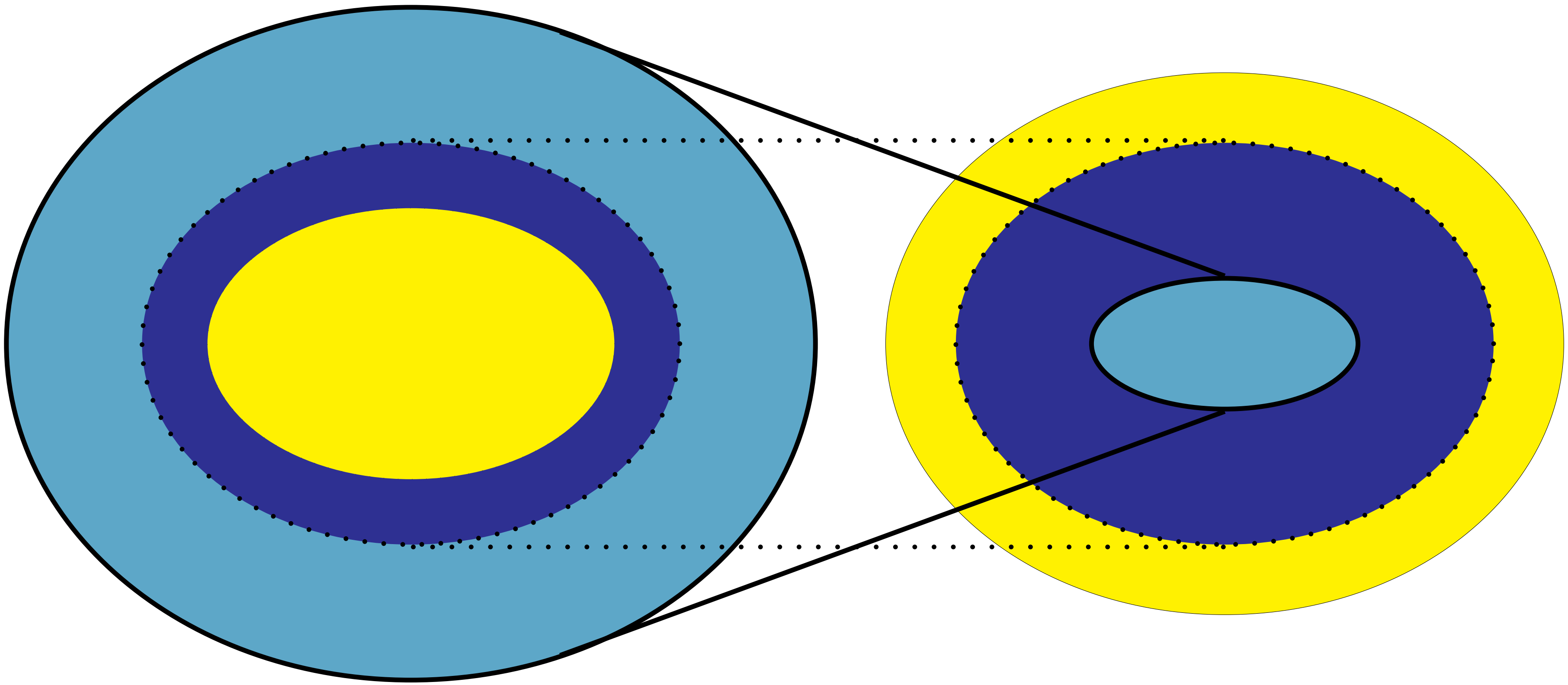
Non-Extensional Models $\mathfrak{M}_\beta(\Sigma)$

Formulas valid in $\mathfrak{M}_\beta(\Sigma)$?

choose: $\mathcal{D}_\iota, \mathcal{D}_{\alpha\rightarrow\beta}, \text{also non}-\text{functions}, \mathcal{D}_o$

fixed:

Model Classes (Extensionality)



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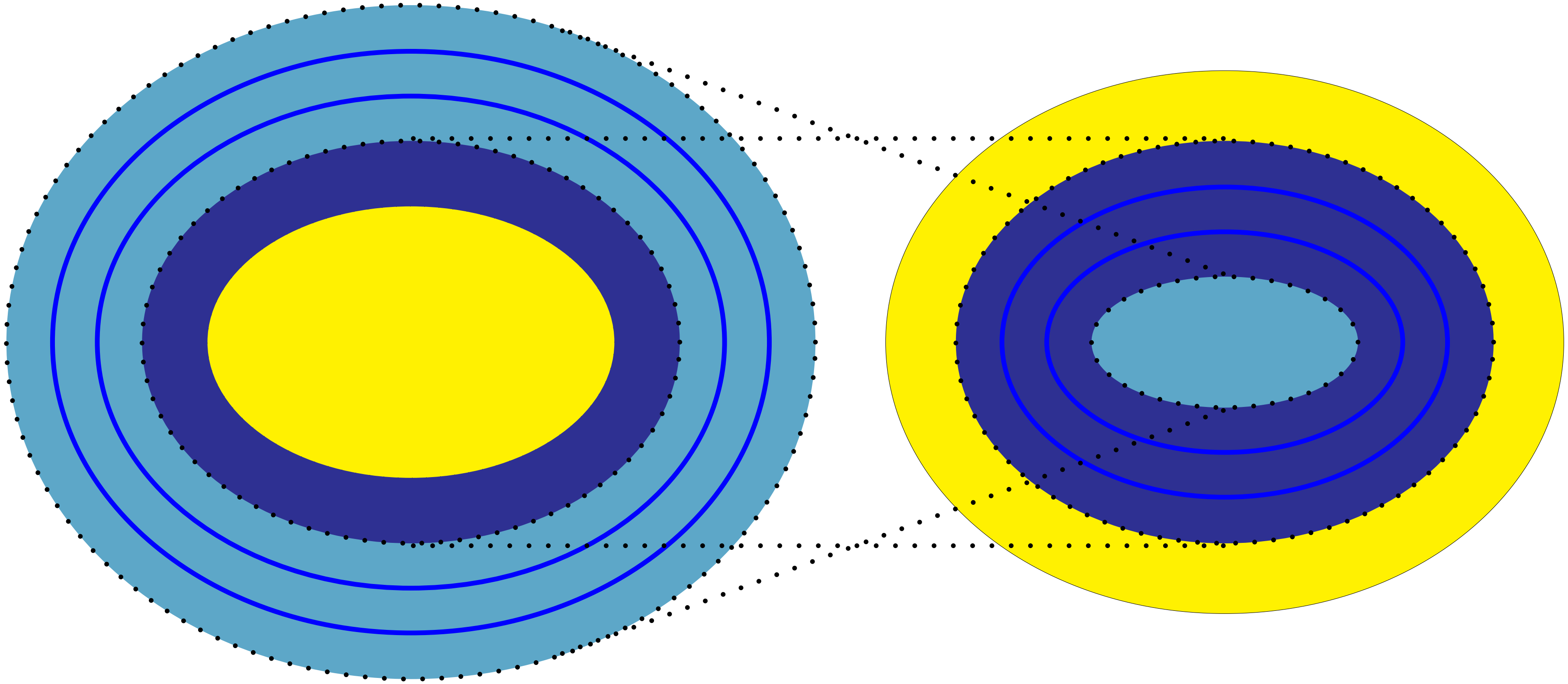

We additionally studied different model classes with 'varying degrees of extensionality'

Model Classes (Extensionality)

$\mathfrak{M}_{\beta}(\Sigma)$ non-extensional models

$\mathfrak{b}$: Boolean extensionality, $\mathcal{D}_{o} = \{\mathtt{T}, \mathtt{F}\}$

$\mathfrak{f}(= \eta + \xi)$: functional extensionality

η: η-functional

ξ: ξ-functionality

$\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma)$ Henkin models

full

$\mathfrak{ST}(\Sigma)$ standard models

Model Classes (Extensionality)

$$\begin{tikzcd}
\mathfrak{M}_{\beta}(\Sigma) \arrow[dr, "\mathfrak{b}"] \arrow[dd, dotted, "{\mathfrak{b}, \mathfrak{f}(= \eta + \xi)}"] & \\
& \mathfrak{M}_{\beta\mathfrak{b}}(\Sigma) \arrow[dl, "\mathfrak{f}"] \\
\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma) \arrow[d, dotted, "\text{full}"] & \\
\mathfrak{S}\mathfrak{T}(\Sigma) &
\end{tikzcd}$$

non-extensional models

Henkin models

standard models

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non-extensional models

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non-extensional models

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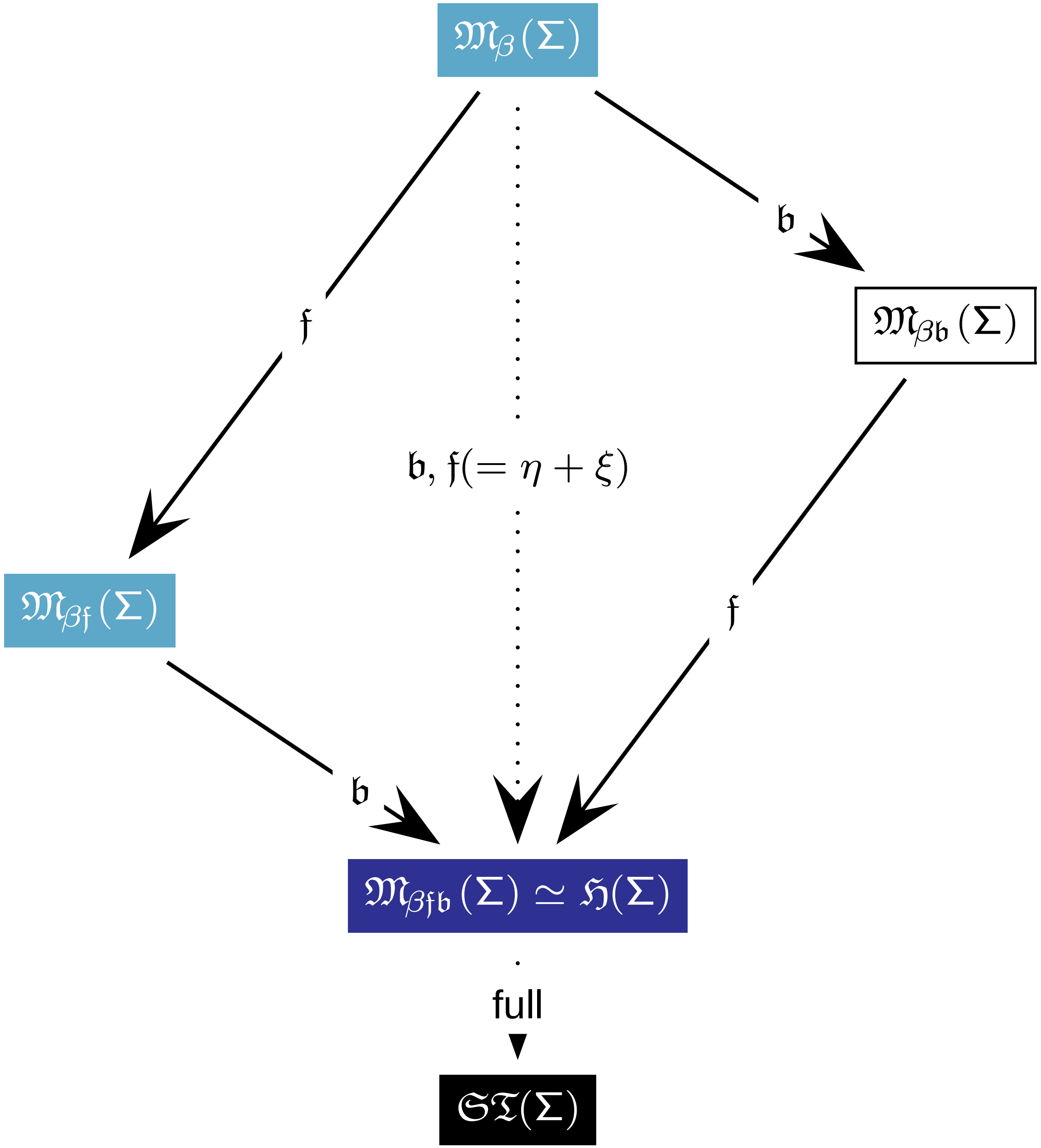

non-extensional models

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Model Classes (Extensionality)

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\begin{tikzcd}
 & \mathfrak{M}_{\beta}(\Sigma) \arrow[dl, "\xi"'] \arrow[ddl, dotted, "\mathfrak{f}"] \arrow[dr, "\mathfrak{b}"] & \\
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non-extensional models

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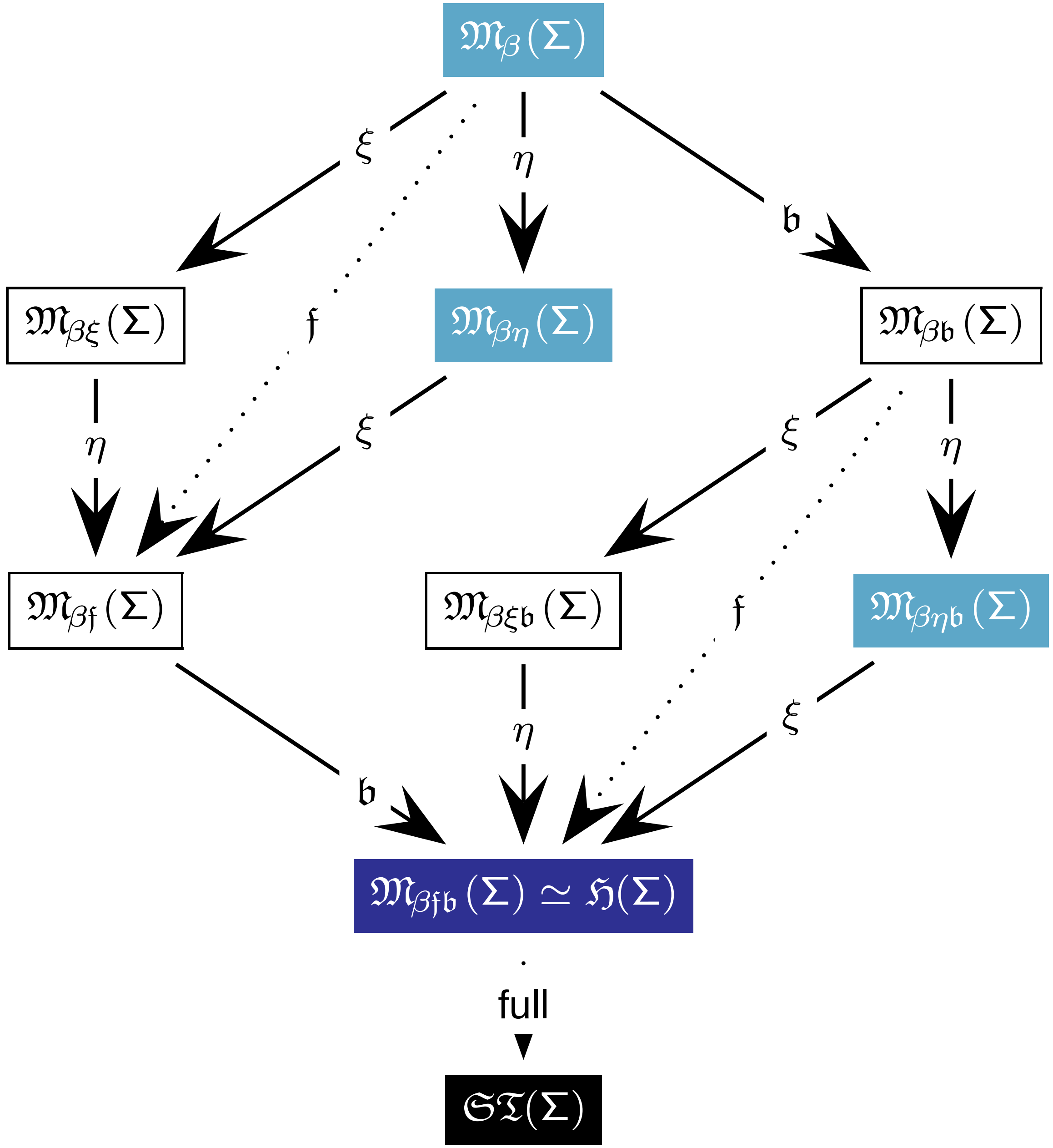

non-extensional models

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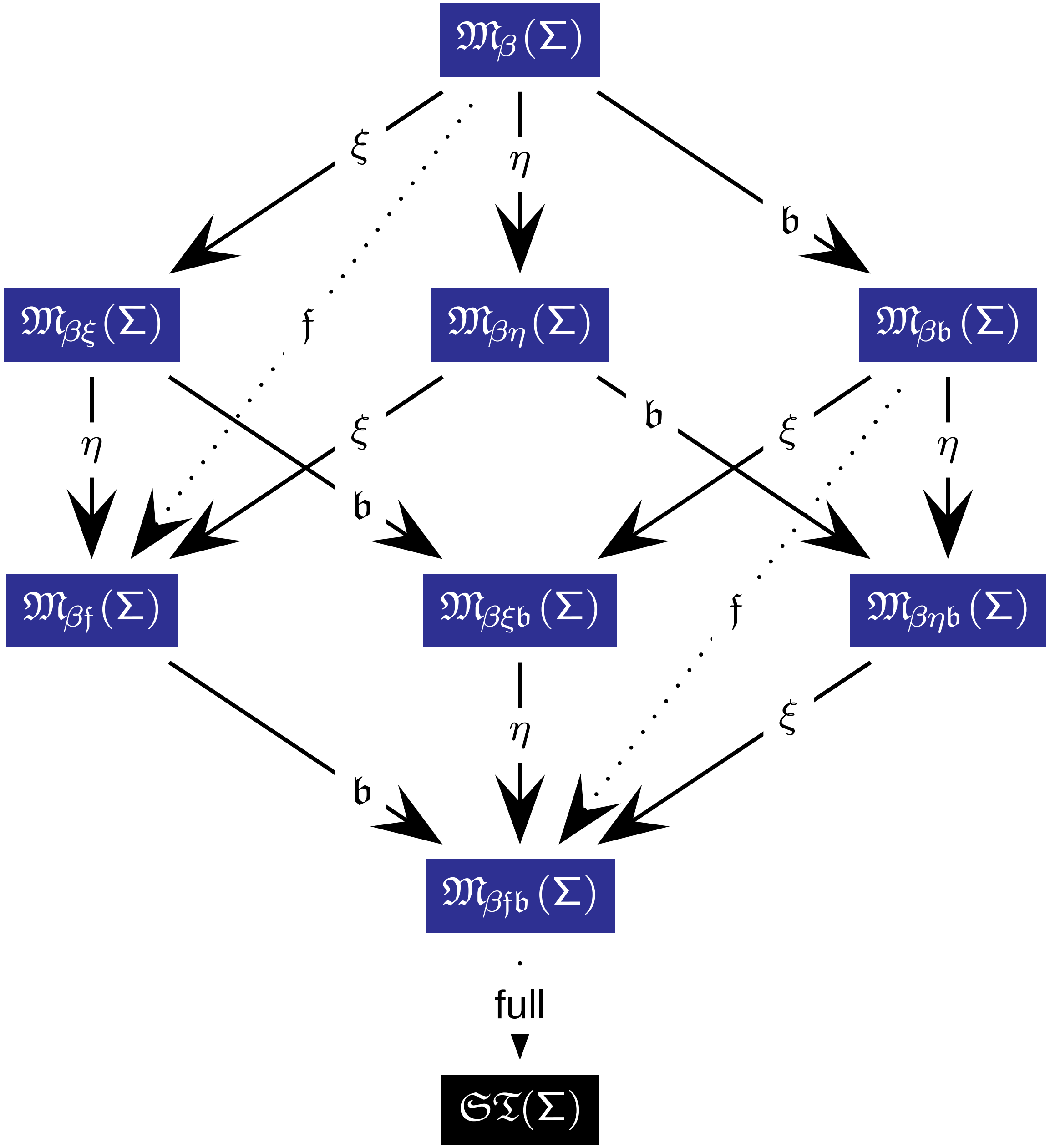


non-extensional models

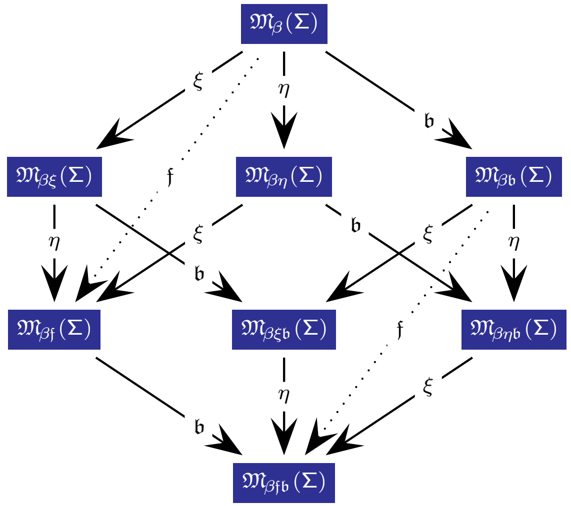
Henkin models

standard models

Semantics - Calculi - Abstract Consistency

Semantics:

Model Classes (Extensionality)



JSL(2004)69(4):1027-1088

Semantics - Calculi - Abstract Consistency

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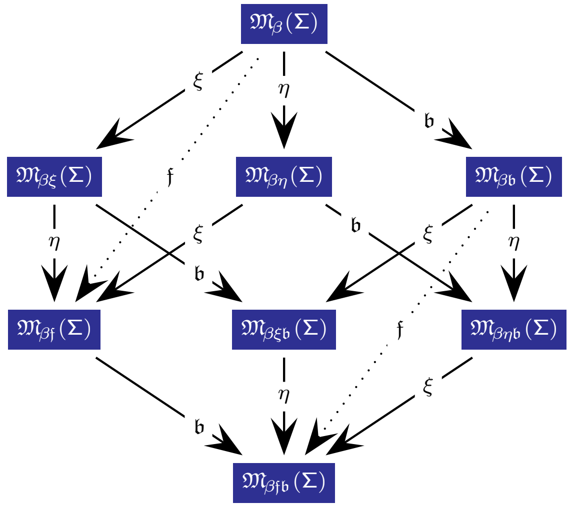
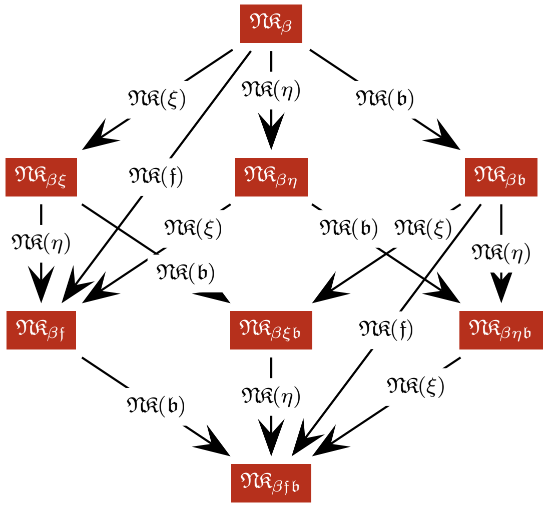


Reference Calculi:
ND (and others)



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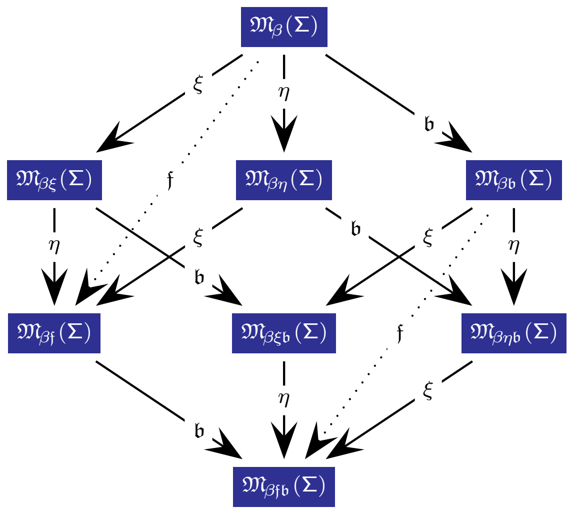

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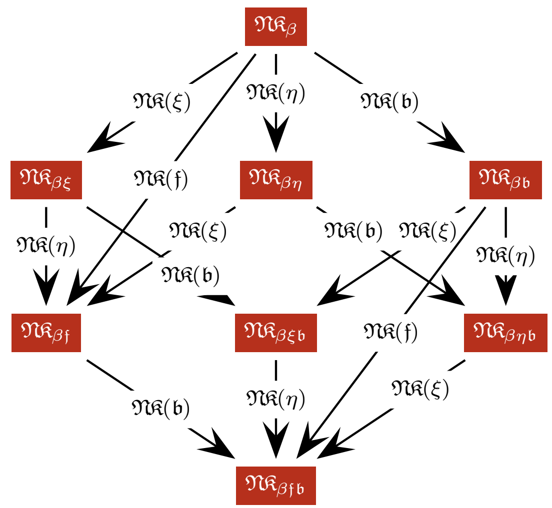

Abstract Consistency / Unifying Principle:
Extensions of Smullyan-63 and Andrews-71

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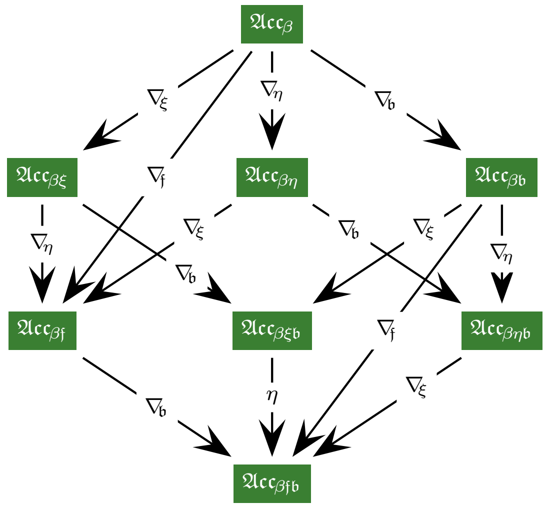
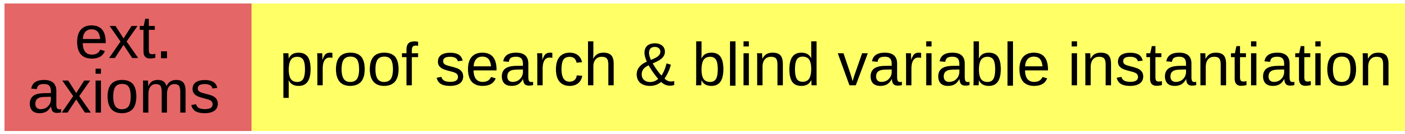


Extensional Resolution

Extensional HO Resolution $\mathcal{ER}$

- [Andrews-71]



- [Huet-73/75]

- [Benzmüller-99]

Cut-simulation

Sequent Calculi for HOL

We work with a one-sided sequent calculus:

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- examples for two-sided rules:

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$$\frac{}{\Gamma \Longrightarrow \Delta,\mathbf{A} \vee \mathbf{B}}\,\mathcal{G}(\vee_{Intro})$$

$\Delta,\mathbf{C}$ stands for $\Delta \cup \{\mathbf{C}\}$

Sequent Calculi for HOL

We work with a one-sided sequent calculus:

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$$\frac{}{\neg(\Gamma) \cup \mathbf{\Delta},\mathbf{A} \vee \mathbf{B}}\,\mathcal{G}(\vee_{+})$$

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$$\frac{\Gamma \Longrightarrow \mathbf{\Delta},\mathbf{A},\mathbf{B}}{\Gamma \Longrightarrow \mathbf{\Delta},\mathbf{A} \lor \mathbf{B}}\ \mathcal{G}(\lor_{Intro})$$

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Sequent Calculi for HOL

$$\frac{\mathbf{A} \text{ atomic (and } \beta\text{-normal)}}{\Delta,\neg\mathbf{A},\mathbf{A}}\ \mathcal{G}(init) \qquad \frac{\Delta,\mathbf{A}}{\Delta,\neg\neg\mathbf{A}}\ \mathcal{G}(\neg)$$

$$\frac{\Delta,\neg\mathbf{A} \quad \Delta,\neg\mathbf{B}}{\Delta,\neg(\mathbf{A}\vee\mathbf{B})}\ \mathcal{G}(\vee_-) \qquad \frac{\Delta,\mathbf{A},\mathbf{B}}{\Delta,(\mathbf{A}\vee\mathbf{B})}\ \mathcal{G}(\vee_+)$$

$$\frac{\Delta,\neg(\mathbf{A}\mathbf{C}){\downarrow_\beta} \quad \mathbf{C}\in \mathit{cwff}_\alpha(\Sigma)}{\Delta,\neg\forall X_\alpha \mathbf{A}}\ \mathcal{G}(\forall_-^{\mathbf{C}}) \qquad \frac{\Delta,(\mathbf{A}\mathsf{c}){\downarrow_\beta} \quad \mathsf{c}_\alpha\in\Sigma \text{ new}}{\Delta,\forall X_\alpha \mathbf{A}}\ \mathcal{G}(\forall_+^{c})$$

$\Delta,\mathbf{A}$ stands for $\Delta\cup\{\mathbf{A}\}$

Sequent Calculi for HOL: $\mathcal{G}_\beta$

The sequent calculus $\mathcal{G}_\beta$ is defined by the rules

$$\mathcal{G}(init), \mathcal{G}(\neg), \mathcal{G}(\vee_-), \mathcal{G}(\vee_+), \mathcal{G}(\forall_-^{\mathbf{C}}), \mathcal{G}(\forall_+^c)$$

- is sound for the eight model classes $\mathfrak{M}_*$
- is complete for the model class $\mathfrak{M}_\beta(\Sigma)$

- suitable for automation? $\longrightarrow$ Analysis of admissibility of cut:

$$\frac{\Delta, \mathbf{C} \quad \Delta, \neg\mathbf{C}}{\Delta}\,\mathcal{G}(cut)$$

- $\mathcal{G}_\beta$ is indeed cut-free

Cut-simulation with Leibnizequations

Leibniz-equations $\mathbf{M} \doteq^{\alpha} \mathbf{N}$ $(:= \forall \mathrm{P}_{o\alpha} \centerdot \neg \mathrm{P}\mathbf{M} \vee \mathrm{P}\mathbf{N})$ support cut-simulation in $\mathcal{G}_{\beta}$ in only 3 steps.

Proof:

$$\dfrac{\dfrac{\dfrac{\Delta, \mathbf{C}}{\Delta, \neg\neg\mathbf{C}}\ \mathcal{G}(\neg) \qquad \Delta, \neg\mathbf{C}}{\Delta, \neg(\neg\mathbf{C} \vee \mathbf{C})}\ \mathcal{G}(\vee_{-})}{\Delta' := \Delta, \neg\forall \mathrm{P}_{o\alpha} \centerdot \neg \mathrm{P}\mathbf{M} \vee \mathrm{P}\mathbf{N}}\ \mathcal{G}(\forall_{-}^{\lambda X_{\alpha} \centerdot \mathbf{C}})$$

Cut-simulation with Extensionality Axioms

The Boolean extensionality axiom $\mathcal{B}_o$ is:

$$\forall A_o. \forall B_o. (A \Leftrightarrow B) \Rightarrow A \doteq^o B$$

The infinitely many functional extensionality axioms $\mathcal{F}_{\alpha\beta}$ are:

$$\forall F_{\alpha \to \beta}. \forall G_{\alpha \to \beta}. (\forall X_\alpha. FX \doteq^\beta GX) \Rightarrow F \doteq^{\alpha \to \beta} G$$

Cut-simulation with Extensionality Axioms

> The functional extensionality axioms support effective cut-simulation in $\mathcal{G}_\beta$ in 11-steps.

Proof:

$$
\dfrac{
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\dfrac{\begin{array}{c}\text{3 steps; easy}\\ \vdots \\ \Delta, \mathsf{fa} \doteq^{\beta} \mathsf{fa}\end{array}}{\Delta, (\forall X_\alpha \mathbin{.} \mathsf{f}X \doteq^{\beta} \mathsf{f}X)}\ \mathcal{G}(\forall_{+}^{a_\alpha})
}{\Delta, \neg\neg\forall X_\alpha \mathbin{.} \mathsf{f}X \doteq^{\beta} \mathsf{f}X}\ \mathcal{G}(\neg)
\qquad
\begin{array}{c}\Delta, \mathbf{C} \quad \Delta, \neg\mathbf{C}\\ \vdots\ \text{3 steps; see before}\\ \Delta, \neg(\mathsf{f} \doteq^{\alpha\to\beta} \mathsf{f})\end{array}
}{\Delta, \neg(\neg(\forall X_\alpha \mathbin{.} \mathsf{f}X \doteq^{\beta} \mathsf{f}X) \vee \mathsf{f} \doteq^{\alpha\to\beta} \mathsf{f})}\ \mathcal{G}(\vee_{-})
}{\Delta, \neg\mathcal{F}_{\alpha\beta}}\ 2 \times \mathcal{G}(\forall_{-}^{f})
$$

Cut-simulation with Extensionality Axioms

It also works with Boolean extensionality axiom – in 14 steps.

Proof:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{7 steps; easy}\\ \vdots \\ \Delta, a \Leftrightarrow a\end{array}}{\Delta, \neg\neg(a \Leftrightarrow a)}\,\mathcal{G}(\neg) \qquad \begin{array}{c}\Delta, \mathbf{C} \quad \Delta, \neg\mathbf{C}\\ \vdots\ \text{3 steps; see before}\\ \Delta, \neg(a \doteq^{o} a)\end{array}}{\Delta, \neg(\neg(a \Leftrightarrow a) \vee a \doteq^{o} a)}\,\mathcal{G}(\vee_{-})}{\Delta, \neg\mathcal{B}_{o}}\,2 \times \mathcal{G}(\forall^{a}_{-})$$

Cut-simulation with other Axioms

- Reflexivity definition of equality (Andrews) 4 steps

$$\lambda X_\alpha \centerdot \lambda Y_\alpha \centerdot \forall Q_{\alpha\to\alpha\to o} \centerdot (\forall Z_\alpha \centerdot (Q\ Z\ Z)) \Rightarrow (Q\ X\ Y)$$

Cut-simulation with other Axioms

- Reflexivity definition of equality (Andrews) 4 steps
- Instances of Comprehension axioms 16 steps

$$\exists P_{\iota \to o} \centerdot \forall X_\iota \centerdot P X \Leftrightarrow X \doteq^\iota X$$

Cut-simulation with other Axioms

- Reflexivity definition of equality (Andrews) 4 steps
- Instances of Comprehension axioms 16 steps
- Axiom of Induction 18 steps

$$\forall P_{\iota \to o} \centerdot P0 \wedge (\forall X_{\iota} \centerdot PX \Rightarrow P(sX)) \Rightarrow \forall X_{\iota} \centerdot PX$$

Cut-simulation with other Axioms

- Reflexivity definition of equality (Andrews) 4 steps
- Instances of Comprehension axioms 16 steps
- Axiom of Induction 18 steps
- Axiom of Choice 7 steps

$$\exists I_{(\alpha\to o)\to\alpha} \centerdot \forall Q_{\alpha\to o} \centerdot \exists X_{\alpha} \centerdot QX \Rightarrow Q(IQ)$$

Cut-simulation with other Axioms

- Reflexivity definition of equality (Andrews) 4 steps
- Instances of Comprehension axioms 16 steps
- Axiom of Induction 18 steps
- Axiom of Choice 7 steps
- Axiom of Description 25 steps

$$\exists I_{(\alpha \to o) \to \alpha} . \forall Q_{\alpha \to o} . (\exists_1 Y_\alpha . QY) \Rightarrow Q(IQ)$$

Cut-simulation with other Axioms

- Reflexivity definition of equality (Andrews) 4 steps
- Instances of Comprehension axioms 16 steps
- Axiom of Induction 18 steps
- Axiom of Choice 7 steps
- Axiom of Description 25 steps
- Axiom of Excluded Middle 3 steps

$$\forall Q_o . Q \vee \neg Q$$

Cut-simulation with other Axioms

Reflexivity definition of equality (Andrews)	4 steps
Instances of Comprehension axioms	16 steps
Axiom of Induction	18 steps
Axiom of Choice	7 steps
Axiom of Description	25 steps
Axiom of Excluded Middle	3 steps
. . .	

Cut-simulation with other Axioms

Reflexivity definition of equality (Andrews)	4 steps
Instances of Comprehension axioms	16 steps
Axiom of Induction	18 steps
Axiom of Choice	7 steps
Axiom of Description	25 steps
Axiom of Excluded Middle	3 steps
...	

This motivates lots of further research on HOL automation:

How to avoid / treat cut-strong axioms and formulas?!?

$$\Gamma, \neg\forall P\, (P\, \mathbf{A}), \Delta$$

Conclusion

($\geq$) Two hearts are beating in my chest:

- Towards integrated mathematics assistance systems
  - ... by joining resources ...
- Foundations and automation (not only!) of HOL
  - semantics $\leftrightarrow$ proof theory $\leftrightarrow$ automation
  - automation still decades behind first-order ATP

Currently I am

- implementing LEO-II (new version of resolution prover LEO)
- want to integrate LEO-II with Isabelle/HOL (& others)
- involved in building up a HOTPTP and the THF syntax
- working towards a HOL prover competition

HOL Challenge: Impredicativity

Notion: (Impredicativity)

- quantification over sets and predicates
- support impredicative definitions and reflection

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HOL Challenge: Impredicativity

Notion: (Impredicativity)

- quantification over sets and predicates
- support impredicative definitions and reflection

Ex.: Automation already problematic for very simple quantifications over sets: $\exists P\,(P\ 1)$

- $P \longleftarrow \{x|\mathrm{true}\}$ $\qquad (\lambda X\ \mathbf{T}_o)$
- $P \longleftarrow \{x|x = 1\}$ $\qquad (\lambda X\ X = 1)$
- $P \longleftarrow \{x|x = 1 \vee x = 2\}$ $\qquad (\lambda X\ X = 1 \vee X = 2)$
- $P \longleftarrow \{x|x > 0\}$ $\qquad (\lambda X\ X > 0)$
- etc.

HOL Challenge: Impredicativity

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- support impredicative definitions and reflection

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- $P \longleftarrow \{x|x = 1\}$ $\qquad (\lambda X\ X = 1)$
- $P \longleftarrow \{x|x = 1 \lor x = 2\}$ $\qquad (\lambda X\ X = 1 \lor X = 2)$
- $P \longleftarrow \{x|x > 0\}$ $\qquad (\lambda X\ X > 0)$
- etc.

- unification not powerful enough $\Longrightarrow$ guessing is state of the art
- problem not limited to HOL

Extensional Resolution

Ex.: Extensional HO Resolution $\mathcal{ER}$

$$\forall B_{\alpha\to o}, C_{\alpha\to o}, D_{\alpha\to o}. B \cup (C \cap D) = (B \cup C) \cap (B \cup D)$$

Negation and definition expansion with

$\cup = \lambda A_{\alpha\to o}, B_{\alpha\to o}, X_{\alpha}. (A\ X) \vee (B\ X)$ $\quad$ $\cap = \lambda A_{\alpha\to o}, B_{\alpha\to o}, X_{\alpha}. (A\ X) \wedge (B\ X)$

leads to:

$$C_1 : [\lambda X_{\alpha}. (b\ X) \vee ((c\ X) \wedge (d\ X)) \neq^{?} \lambda X_{\alpha}. ((b\ X) \vee (c\ X)) \wedge ((b\ X) \vee (d\ X)))]$$

Goal directed functional and Boolean extensionality treatment:

$$C_2 : [(b\ x) \vee ((c\ x) \wedge (d\ x)) \Leftrightarrow ((b\ x) \vee (c\ x)) \wedge ((b\ x) \vee (d\ x)))]^{F}$$

Clause normalization results then in a pure propositional, i.e. decidable, set of clauses. Only these clauses are still in the search space of LEO(in total there are 33 clauses generated and LEO finds the proof on a 2,5GHz PC in 820ms).

Similar proof in case of embedded propositions:

$$\forall P_{(\alpha\to o)\to o}, B_{\alpha\to o}, C_{\alpha\to o}, D_{\alpha\to o}. P(B \cup (C \cap D)) \Rightarrow P((B \cup C) \cap (B \cup D))$$

Ex.: Extensional HO Resolution $\mathcal{ER}$

$$\forall P_{o \to o}.(P\ a_o) \wedge (P\ b_o) \Rightarrow (P\ (a_o \wedge b_o))$$

Negation and clause normalization

$$\mathcal{C}_1 : [p\ a]^T \quad \mathcal{C}_2 : [p\ b]^T \quad \mathcal{C}_3 : [p\ (a \wedge b)]^F$$

Resolution between $\mathcal{C}_1$ and $\mathcal{C}_3$ and between $\mathcal{C}_2$ and $\mathcal{C}_3$

$$\mathcal{C}_4 : [p\ a \neq^? p\ (a \wedge b)] \quad \mathcal{C}_5 : [p\ b \neq^? p\ (a \wedge b)]$$

Decomposition

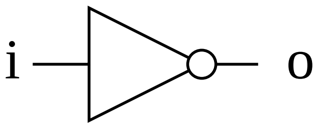
$$\mathcal{C}_6 : [a \neq^? (a \wedge b)] \quad \mathcal{C}_7 : [b \neq^? (a \wedge b)]$$

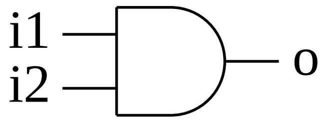
Goal directed extensionality treatment and clause normalisation:

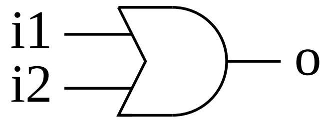
- from $\mathcal{C}_6$ $\quad \mathcal{C}_8 : [a]^F \vee [b]^F \quad \mathcal{C}_9 : [a]^T \vee [b]^T \quad \mathcal{C}_{10} : [a]^T$
- from $\mathcal{C}_7$ $\quad \mathcal{C}_{11} : [a]^F \vee [b]^F \quad \mathcal{C}_{12} : [a]^T \vee [b]^T \quad \mathcal{C}_{13} : [b]^T$

HOL Application: Hardware Verification

- Some Basic Devices



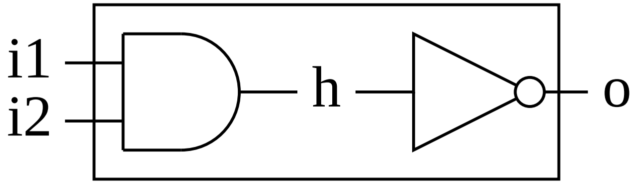




$\mathsf{NOT}(i, o) =$ $(o = \neg i)$	$\mathsf{AND}(i_1, i_2, o) =$ $(o = (i_1 \wedge i_2))$	$\mathsf{OR}(i_1, i_2, o) =$ $(o = (i_1 \vee i_2))$
$\mathsf{NOT}'(i, o) =$ $(\forall t. o(t) = \neg i(t))$	$\mathsf{AND}'(i_1, i_2, o) =$ $(\forall t. o(t) = (i_1(t) \wedge i_2(t)))$	$\mathsf{OR}'(i_1, i_2, o) =$ $(\forall t. o(t) = (i_1(t) \vee i_2(t)))$

HOL Application: Hardware Verification

- Specification of NAND Device

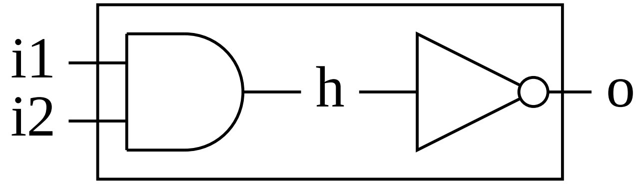


$$\mathsf{NAND{-}SPEC}(\mathsf{i}_1, \mathsf{i}_2, \mathsf{o}) =$$
$$(\mathsf{o} = \neg(\mathsf{i}_1 \wedge \mathsf{i}_2))$$

$$\mathsf{NAND{-}SPEC}'(\mathsf{i}_1, \mathsf{i}_2, \mathsf{o}) =$$
$$(\forall \mathsf{t}. \mathsf{o}(\mathsf{t}) = \neg(\mathsf{i}_1(\mathsf{t}) \wedge \mathsf{i}_2(\mathsf{t})))$$

HOL Application: Hardware Verification

- Implementation of NAND Device



$$\mathsf{NAND{-}IMP}(i_1, i_2, o) =$$
$$\exists h_o. \mathsf{AND}(i_1, i_2, h) \wedge \mathsf{NOT}(h, o)$$

$$\mathsf{NAND{-}IMP}'(i_1, i_2, o) =$$
$$\exists h_{\iota \to o}. \mathsf{AND}(i_1, i_2, h) \wedge \mathsf{NOT}(h, o)$$

HOL Application: Hardware Verification

- Implementation is correct

$$\mathsf{NAND{-}IMP}(\mathsf{i}_1, \mathsf{i}_2, \mathsf{o}) \Rightarrow \mathsf{NAND{-}SPEC}(\mathsf{i}_1, \mathsf{i}_2, \mathsf{o})$$
$$\mathsf{NAND{-}IMP}'(\mathsf{i}_1, \mathsf{i}_2, \mathsf{o}) \Rightarrow \mathsf{NAND{-}SPEC}'(\mathsf{i}_1, \mathsf{i}_2, \mathsf{o})$$

HOL Application: Hardware Verification

- Implementation is correct

  $\mathsf{NAND{-}IMP}(\mathsf{i}_1, \mathsf{i}_2, \mathsf{o}) \Rightarrow \mathsf{NAND{-}SPEC}(\mathsf{i}_1, \mathsf{i}_2, \mathsf{o})$

  $\mathsf{NAND{-}IMP}'(\mathsf{i}_1, \mathsf{i}_2, \mathsf{o}) \Rightarrow \mathsf{NAND{-}SPEC}'(\mathsf{i}_1, \mathsf{i}_2, \mathsf{o})$

- Definition expansion

HOL Application: Hardware Verification

- Implementation is correct

$$\mathsf{NAND{-}IMP}(i_1, i_2, o) \Rightarrow \mathsf{NAND{-}SPEC}(i_1, i_2, o)$$
$$\mathsf{NAND{-}IMP}'(i_1, i_2, o) \Rightarrow \mathsf{NAND{-}SPEC}'(i_1, i_2, o)$$

- Definition expansion

$$(o = \neg(i_1 \wedge i_2)) \Rightarrow (\exists h_o\text{.}\mathsf{AND}(i_1, i_2, h) \wedge \mathsf{NOT}(h, o))$$

HOL Application: Hardware Verification

- Implementation is correct

$$\mathsf{NAND{-}IMP}(i_1, i_2, o) \Rightarrow \mathsf{NAND{-}SPEC}(i_1, i_2, o)$$

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- Definition expansion

$$(o = \neg(i_1 \wedge i_2)) \Rightarrow (\exists h_o.\mathsf{AND}(i_1, i_2, h) \wedge \mathsf{NOT}(h, o))$$

$$(o = \neg(i_1 \wedge i_2)) \Rightarrow (\exists h_o.(h = (i_1 \wedge i_2)) \wedge (o = \neg h))$$

HOL Application: Hardware Verification

- Implementation is correct

$$\mathsf{NAND{-}IMP}(i_1, i_2, o) \Rightarrow \mathsf{NAND{-}SPEC}(i_1, i_2, o)$$
$$\mathsf{NAND{-}IMP}'(i_1, i_2, o) \Rightarrow \mathsf{NAND{-}SPEC}'(i_1, i_2, o)$$

- Definition expansion

$$(o = \neg(i_1 \land i_2)) \Rightarrow (\exists h_o.\, \mathsf{AND}(i_1, i_2, h) \land \mathsf{NOT}(h, o))$$
$$(o = \neg(i_1 \land i_2)) \Rightarrow (\exists h_o.\, (h = (i_1 \land i_2)) \land (o = \neg h))$$

$$(out = \neg(i_1 \land i_2)) \Rightarrow (\exists h_{\iota \rightarrow o}.\, \mathsf{AND}(i_1, i_2, h) \land \mathsf{NOT}(h, o))$$

HOL Application: Hardware Verification

- Implementation is correct

$$\mathsf{NAND{-}IMP}(i_1, i_2, o) \Rightarrow \mathsf{NAND{-}SPEC}(i_1, i_2, o)$$

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- Definition expansion

$$(o = \neg(i_1 \wedge i_2)) \Rightarrow (\exists h_o . \mathsf{AND}(i_1, i_2, h) \wedge \mathsf{NOT}(h, o))$$

$$(o = \neg(i_1 \wedge i_2)) \Rightarrow (\exists h_o . (h = (i_1 \wedge i_2)) \wedge (o = \neg h))$$

$$(out = \neg(i_1 \wedge i_2)) \Rightarrow (\exists h_{\iota \to o} . \mathsf{AND}(i_1, i_2, h) \wedge \mathsf{NOT}(h, o))$$

$$(out = \neg(i_1 \wedge i_2)) \Rightarrow$$
$$(\exists h_{\iota \to o} . (\forall t_i . (h(t) = (i_1(t) \wedge i_2(t)))) \wedge (\forall t_i . (o(t) = \neg h(t))))$$

HOL Application: Hardware Verification

- Implementation is correct

  $\mathsf{NAND{-}IMP}(i_1, i_2, o) \Rightarrow \mathsf{NAND{-}SPEC}(i_1, i_2, o)$

  $\mathsf{NAND{-}IMP}'(i_1, i_2, o) \Rightarrow \mathsf{NAND{-}SPEC}'(i_1, i_2, o)$

- Definition expansion

$$(o = \neg(i_1 \land i_2)) \Rightarrow (\exists h_o \centerdot \mathsf{AND}(i_1, i_2, h) \land \mathsf{NOT}(h, o))$$

$$(o = \neg(i_1 \land i_2)) \Rightarrow (\exists h_o \centerdot (h = (i_1 \land i_2)) \land (o = \neg h))$$

$$(out = \neg(i_1 \land i_2)) \Rightarrow (\exists h_{\iota \to o} \centerdot \mathsf{AND}(i_1, i_2, h) \land \mathsf{NOT}(h, o))$$

$$(out = \neg(i_1 \land i_2)) \Rightarrow$$
$$(\exists h_{\iota \to o} \centerdot (\forall t_i \centerdot (h(t) = (i_1(t) \land i_2(t)))) \land (\forall t_i \centerdot (o(t) = \neg h(t))))$$

- LEO's proof:

  time: 620ms, cl. gen.: 309, cl. fo-like: 68, proof length: 55 cl.

Extensionality Axioms as Clauses

- EXT-Func$^{\dot{=}}$: $\forall F_{\alpha\to\beta}\mathbf{.}\forall G_{\alpha\to\beta}(\forall X_{\beta}\mathbf{.}F\,X \doteq G\,X) \Rightarrow F \doteq G$

  Clauses:

  $\mathcal{C}_1 : [p_{\beta\to o}\ (F\ s_{\beta})]^T \vee [Q\ F]^F \vee [Q\ G]^T$

  $\mathcal{C}_2 : [p_{\beta\to o}\ (G\ s_{\beta})]^T \vee [Q\ F]^F \vee [Q\ G]^T$

- EXT-Bool$^{\dot{=}}$: $\forall A_o\mathbf{.}\forall B_o\mathbf{.}(A \Leftrightarrow B) \Leftrightarrow A \doteq^o B$

  Clauses:

  $\mathcal{C}_1 : [A]^F \vee [B]^F \vee [P\ A]^F \vee [P\ B]^T$

  $\mathcal{C}_2 : [A]^T \vee [B]^T \vee [P\ A]^F \vee [P\ B]^T,$

  $\mathcal{C}_3 : [A]^F \vee [B]^T \vee [p\ A]^T,$

  $\mathcal{C}_4 : [A]^F \vee [B]^T \vee [p\ B]^F,$

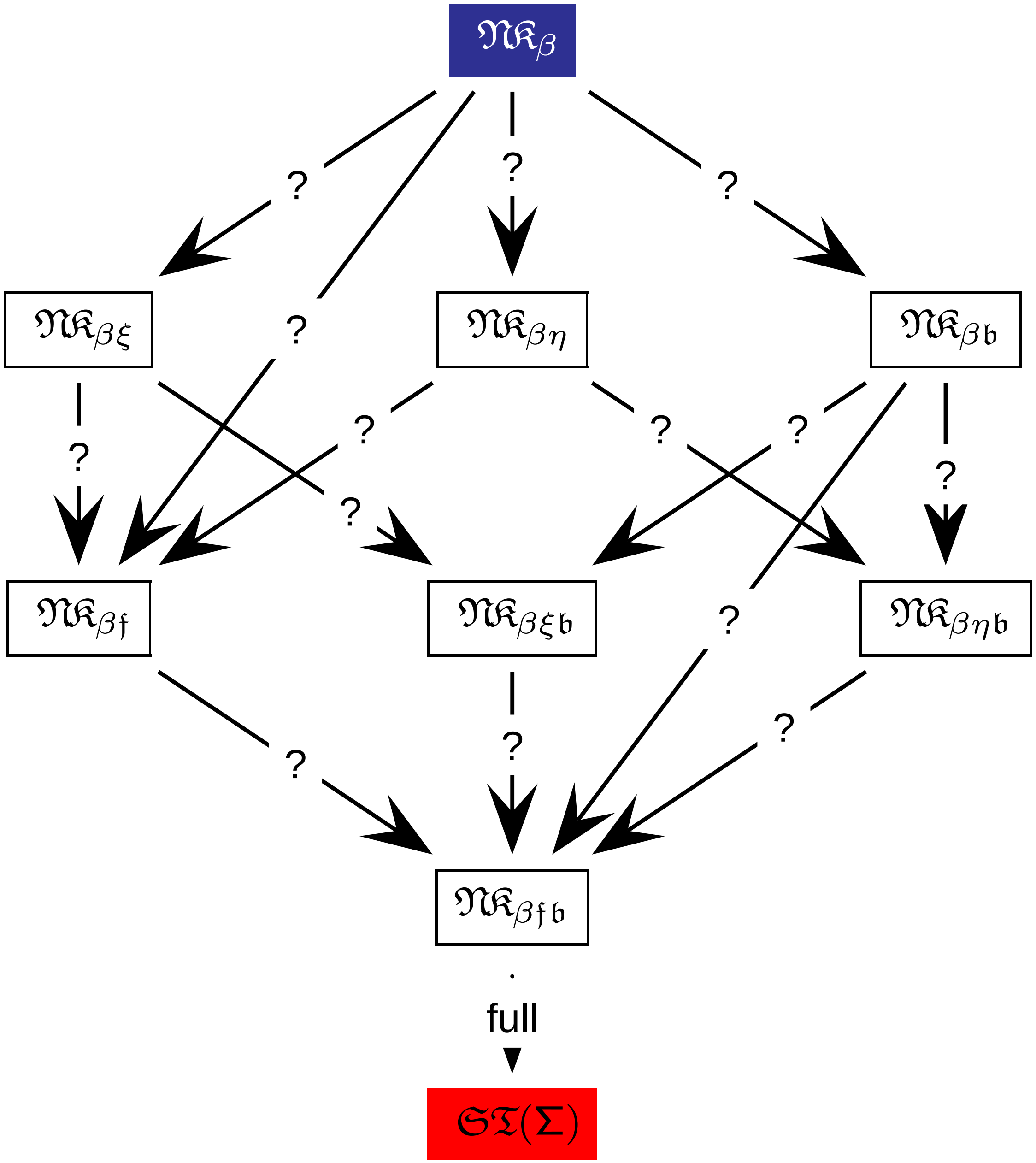
  $\mathcal{C}_5 : [A]^T \vee [B]^F \vee [p\ A]^T,$

  $\mathcal{C}_6 : [A]^T \vee [B]^F \vee [p\ B]^F$

Proof Theory

Calculi for HOL

ND Calculi for HOL



Base Calculus $\mathfrak{NK}_\beta$

—— $\mathfrak{NK}(Hyp)$ —— $\mathfrak{NK}(\beta)$

—— $\mathfrak{NK}(\neg I)$ —— $\mathfrak{NK}(\neg E)$

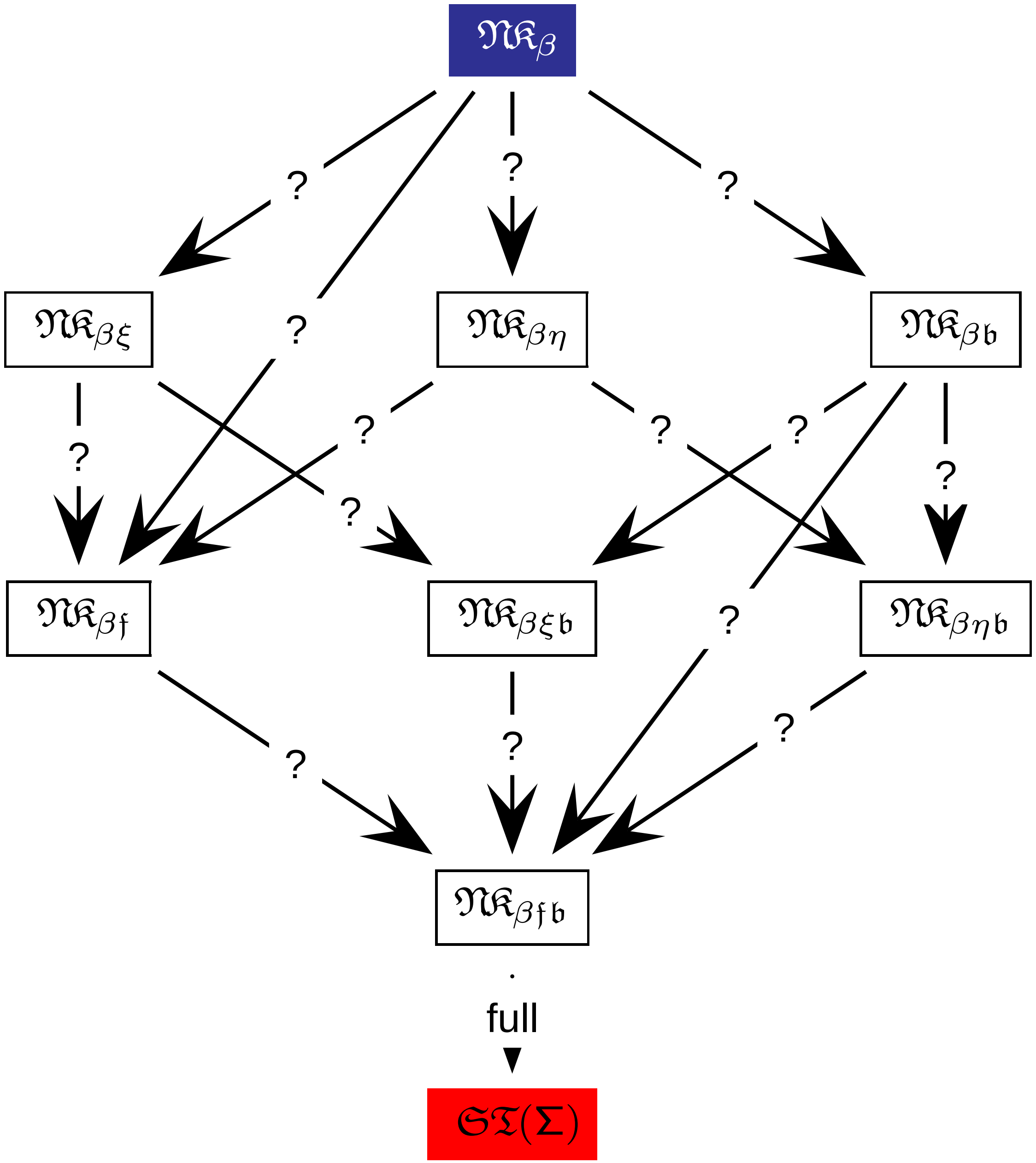
—— $\mathfrak{NK}(\vee I_L)$ —— $\mathfrak{NK}(\vee I_R)$

—— $\mathfrak{NK}(\vee E)$

—— $\mathfrak{NK}(\Pi I)^{\mathrm{w}}$

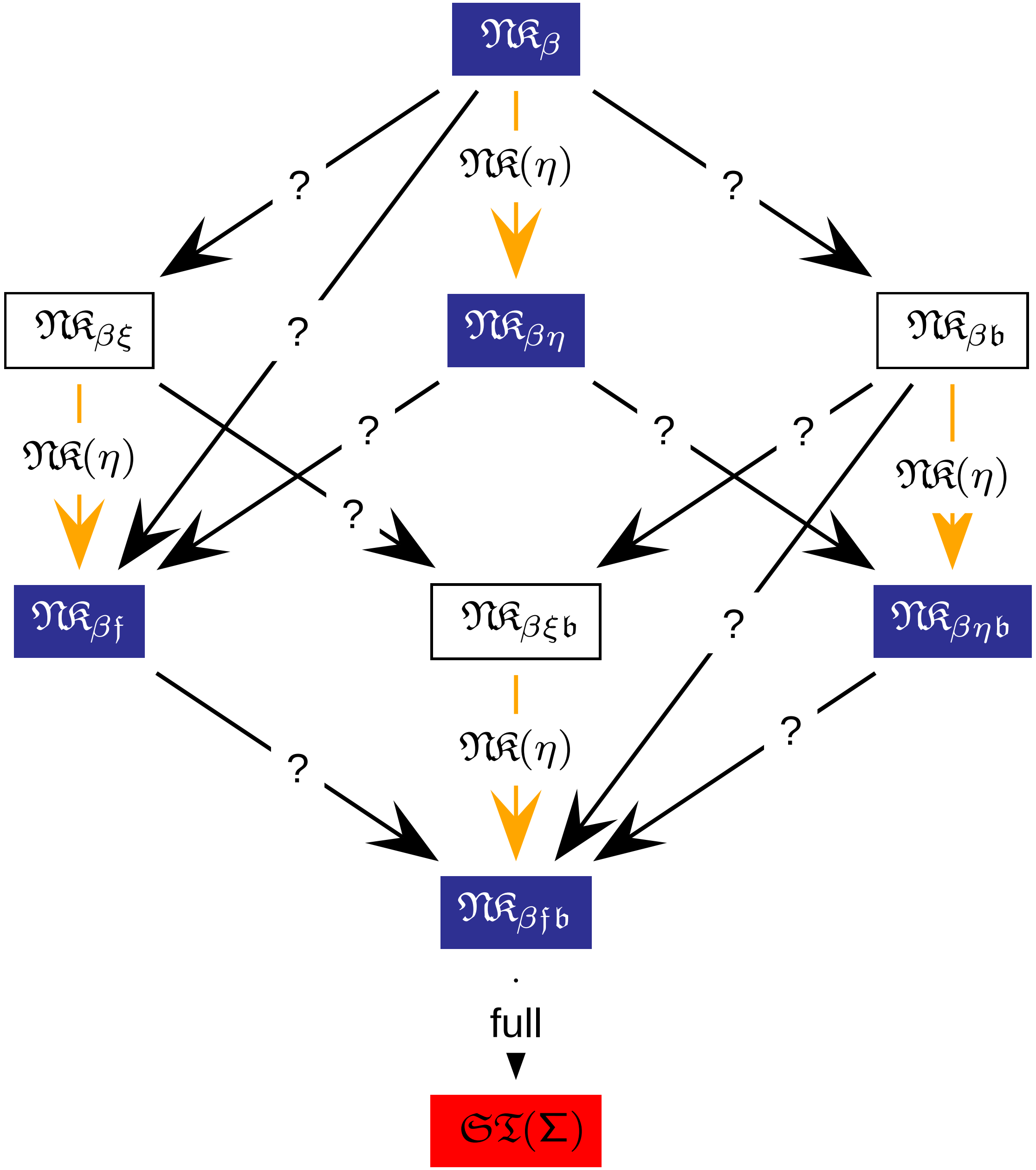
—— $\mathfrak{NK}(\Pi E)$ —— $\mathfrak{NK}(Contr)$

ND Calculi for HOL



Extensionality Rules

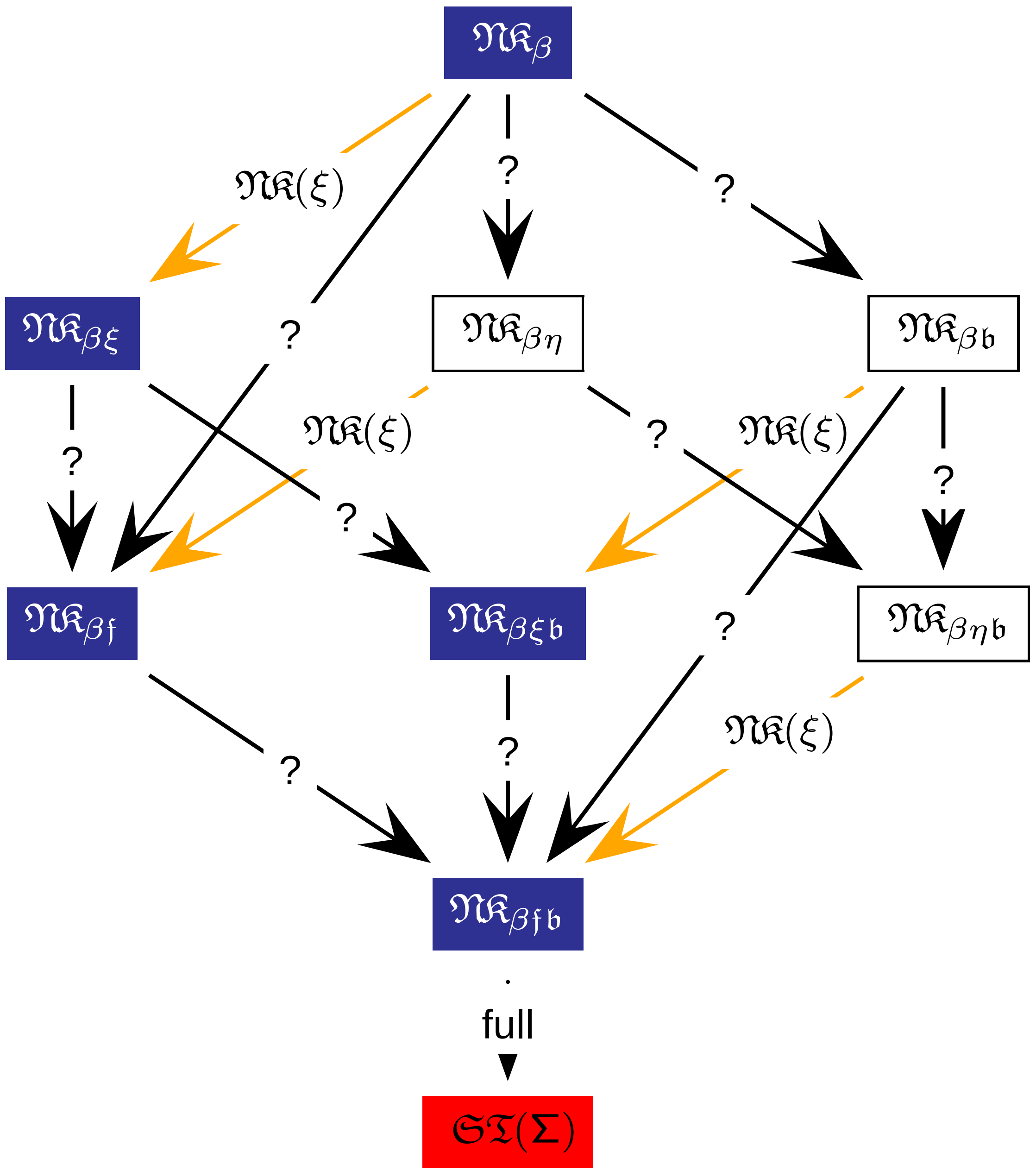
ND Calculi for HOL



Extensionality Rules

$$\frac{\mathbf{A} \overset{\beta\eta}{=} \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\ \mathfrak{NK}(\eta)$$

ND Calculi for HOL

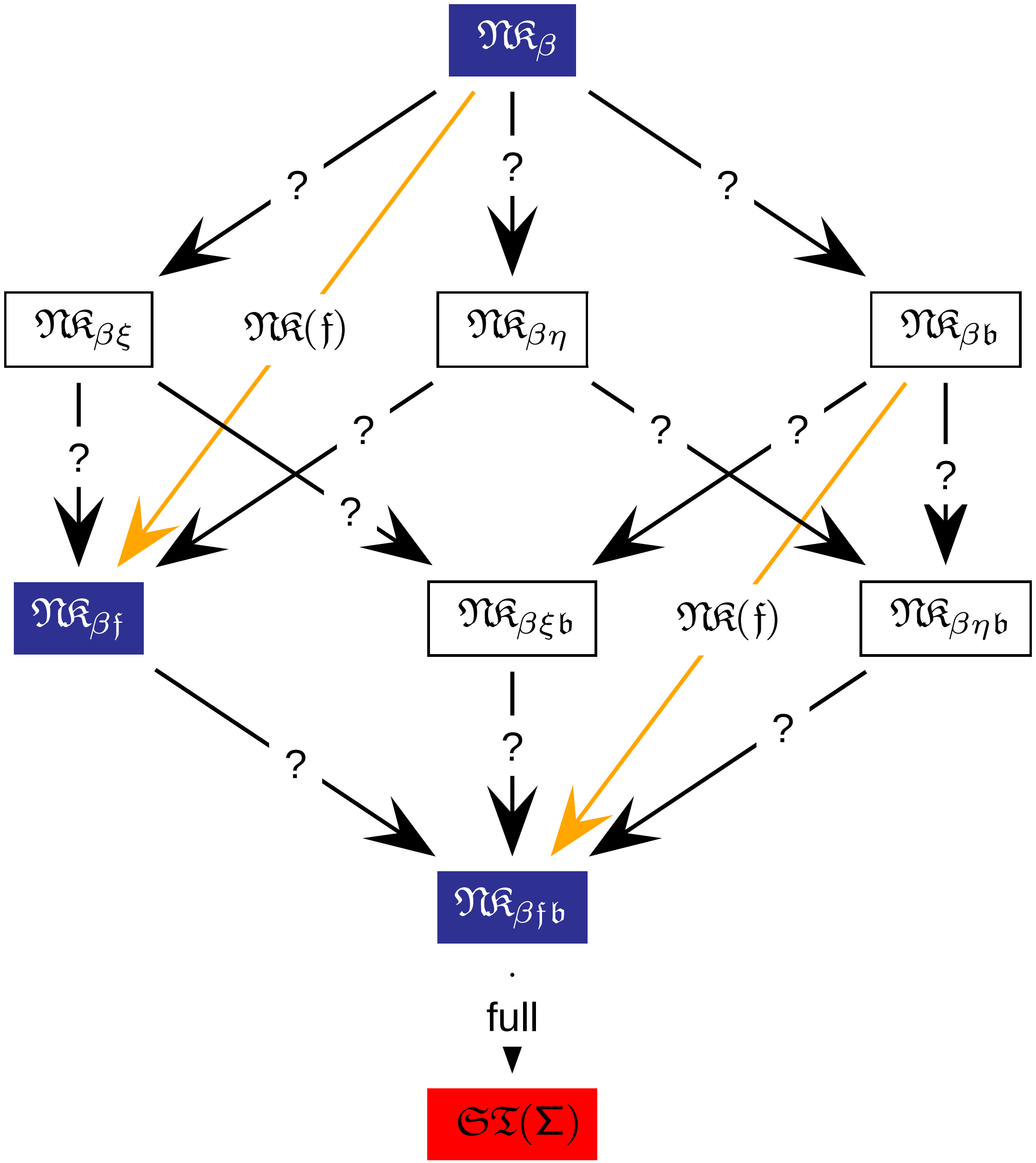


Extensionality Rules

$$\frac{\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\ \mathfrak{NK}(\eta)$$

$$\frac{\Phi \Vdash \forall x_\alpha . \mathbf{M} \doteq^{\beta} \mathbf{N}}{\Phi \Vdash (\lambda x_\alpha . \mathbf{M}) \doteq^{\beta\alpha} (\lambda x_\alpha . \mathbf{N})}\ \mathfrak{NK}(\xi)$$

ND Calculi for HOL



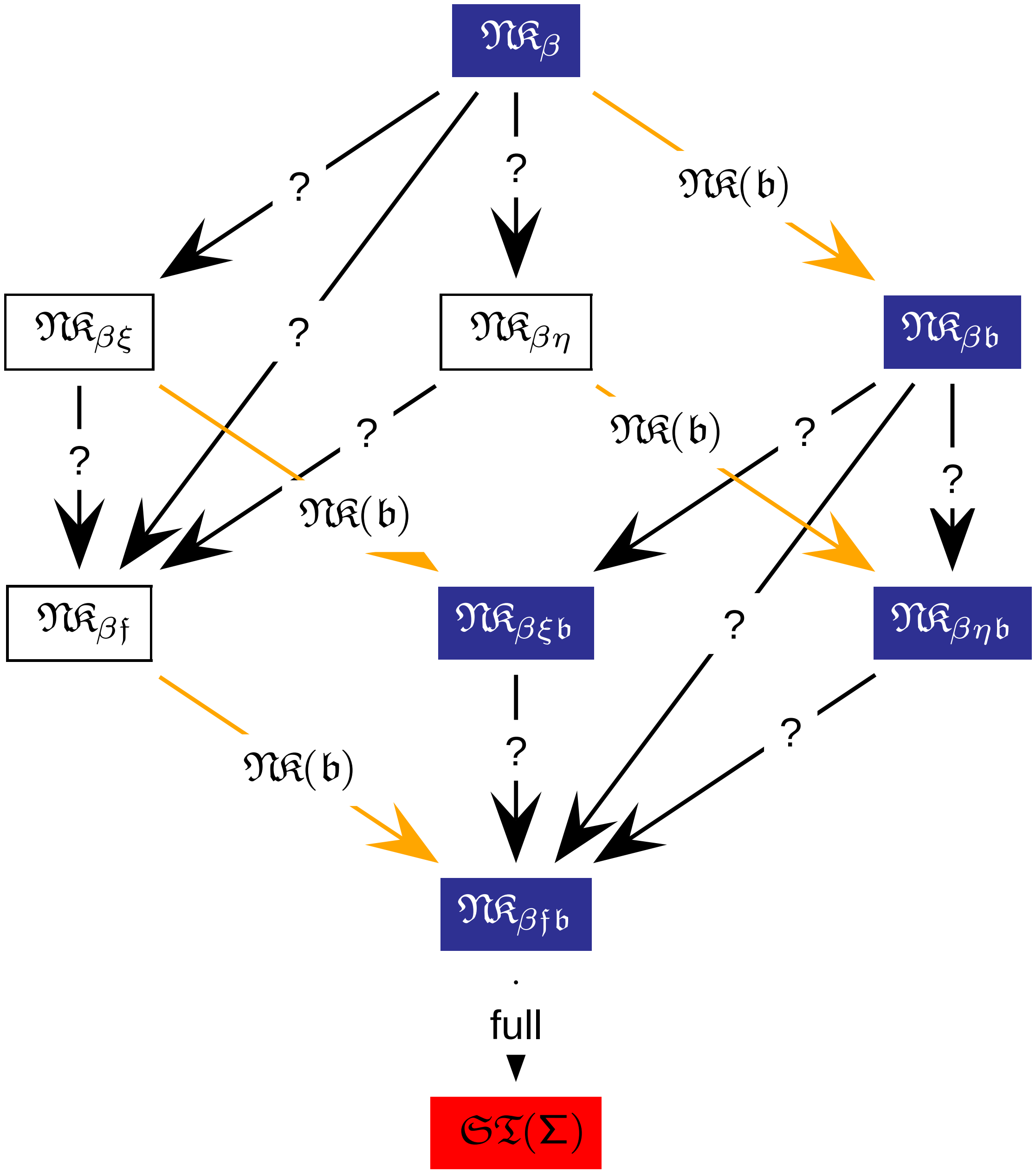
Extensionality Rules

$$\frac{\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\ \mathfrak{NK}(\eta)$$

$$\frac{\Phi \Vdash \forall \mathsf{x}_\alpha . \mathbf{M} \doteq^{\beta} \mathbf{N}}{\Phi \Vdash (\lambda \mathsf{x}_\alpha . \mathbf{M}) \doteq^{\beta\alpha} (\lambda \mathsf{x}_\alpha . \mathbf{N})}\ \mathfrak{NK}(\xi)$$

$$\frac{\Phi \Vdash \forall \mathsf{x}_\alpha . \mathbf{G}\mathsf{x} \doteq^{\beta} \mathbf{H}\mathsf{x}}{\Phi \Vdash \mathbf{G} \doteq^{\beta\alpha} \mathbf{H}}\ \mathfrak{NK}(\mathfrak{f})$$

ND Calculi for HOL



Extensionality Rules

$$\frac{\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\ \mathfrak{NK}(\eta)$$

$$\frac{\Phi \Vdash \forall X_\alpha . \mathbf{M} \doteq^{\beta} \mathbf{N}}{\Phi \Vdash (\lambda X_\alpha . \mathbf{M}) \doteq^{\beta\alpha} (\lambda X_\alpha . \mathbf{N})}\ \mathfrak{NK}(\xi)$$

$$\frac{\Phi \Vdash \forall X_\alpha . \mathbf{G}X \doteq^{\beta} \mathbf{H}X}{\Phi \Vdash \mathbf{G} \doteq^{\beta\alpha} \mathbf{H}}\ \mathfrak{NK}(\mathfrak{f})$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{B} \quad \Phi * \mathbf{B} \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \doteq^{o} \mathbf{B}}\ \mathfrak{NK}(\mathfrak{b})$$

Soundness and Completeness of $\mathfrak{NK}_*$

Thm.: Each calculus is sound wrt. the corresponding model class

Thm.: Each calculus complete wrt. the corresponding model class

For this we extended the

- abstract consistency proof method (unifying principle) of

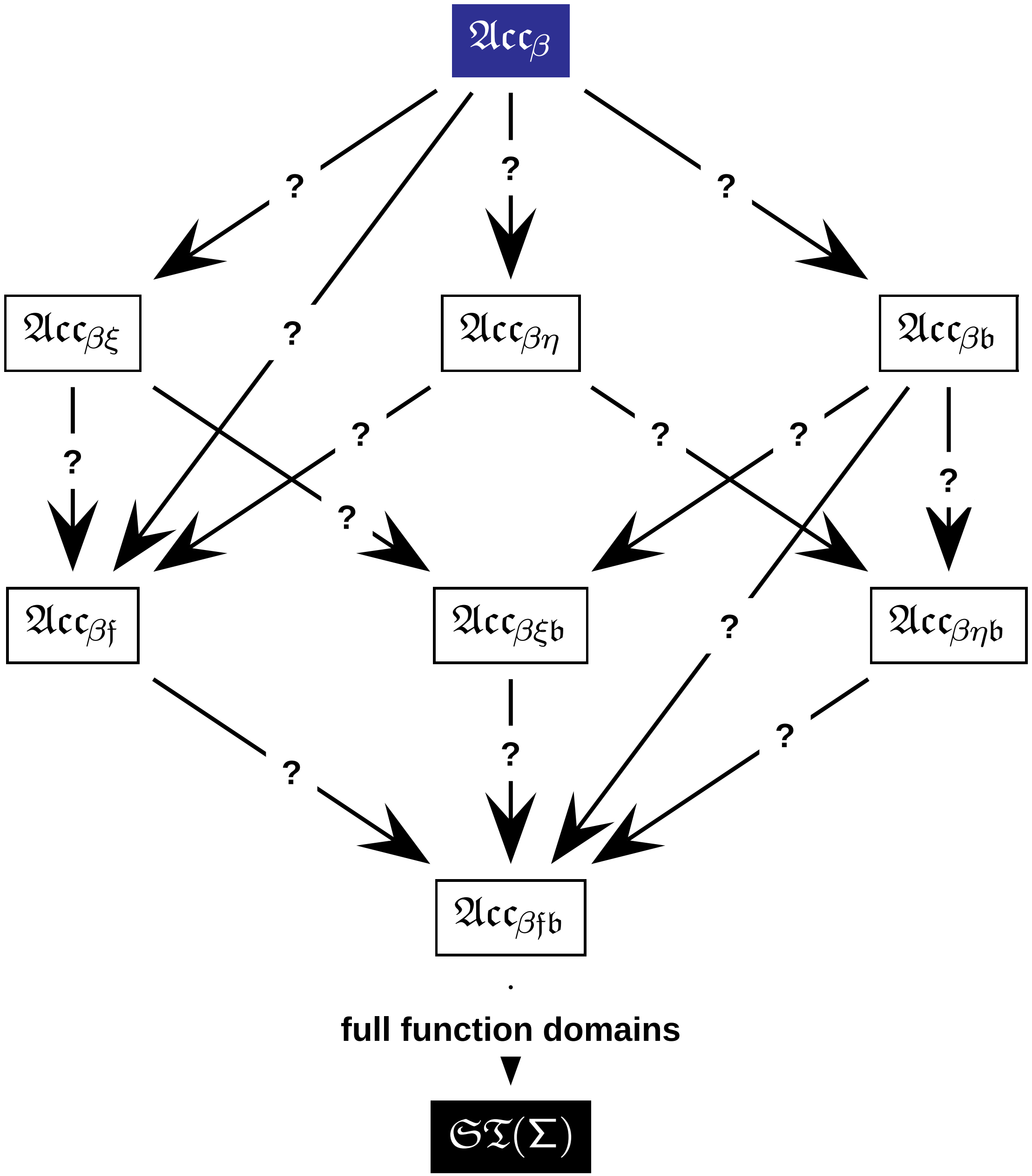
[Smullyan-63]

[Andrews-71]

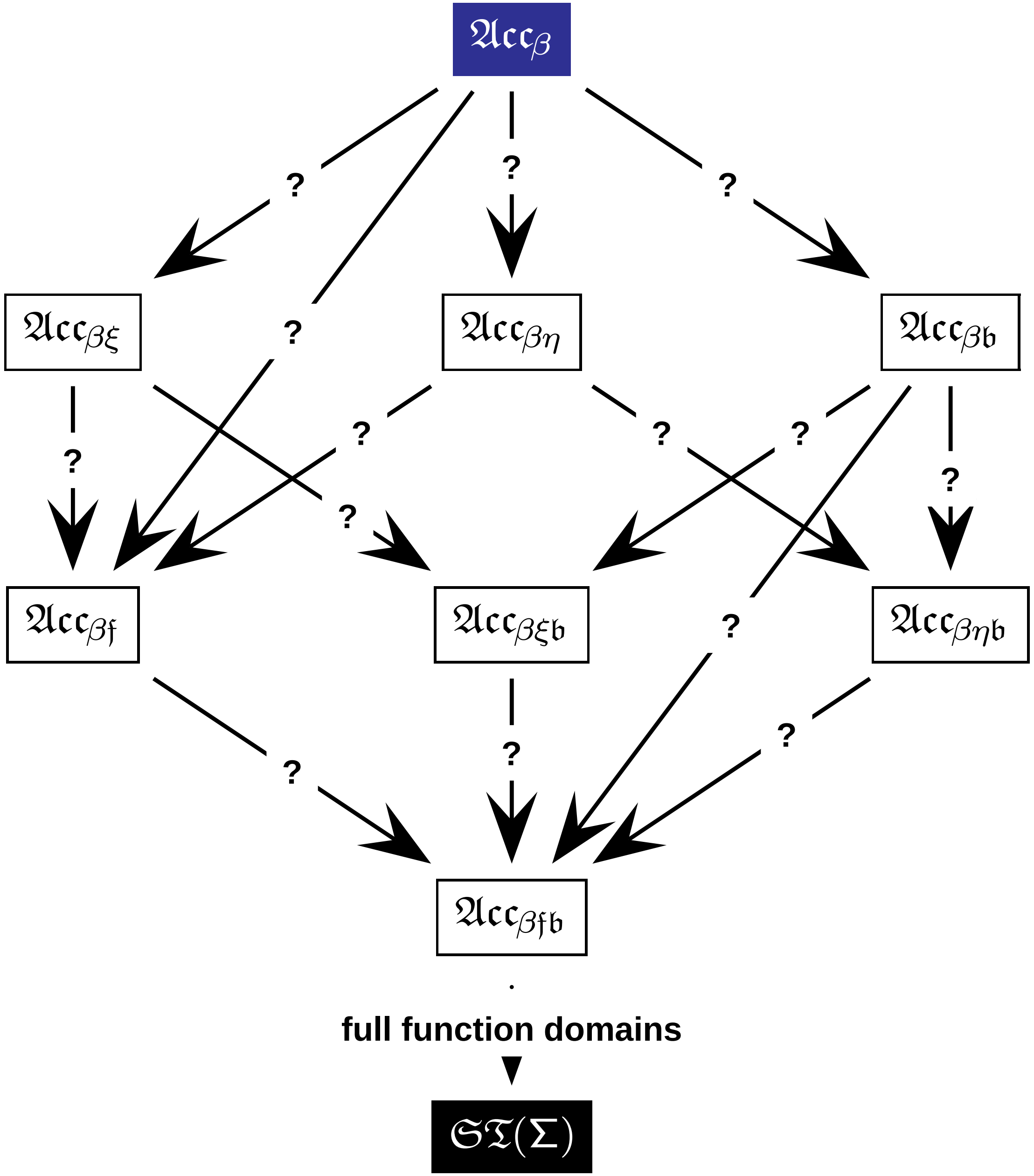
Semantics

Abstract Consistency Proof
Method

Abstract Consistency



Abstract Consistency



Properties for $\mathfrak{Acc}_\beta$: (Γ_Σ is class of sets of formulas; $\Phi \in \Gamma_\Sigma$)

∇_c If $\mathbf{A}$ is atomic, then $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$.

$\nabla_\neg$ If $\neg\neg\mathbf{A} \in \Phi$, then $\Phi,\mathbf{A} \in \Gamma_\Sigma$.

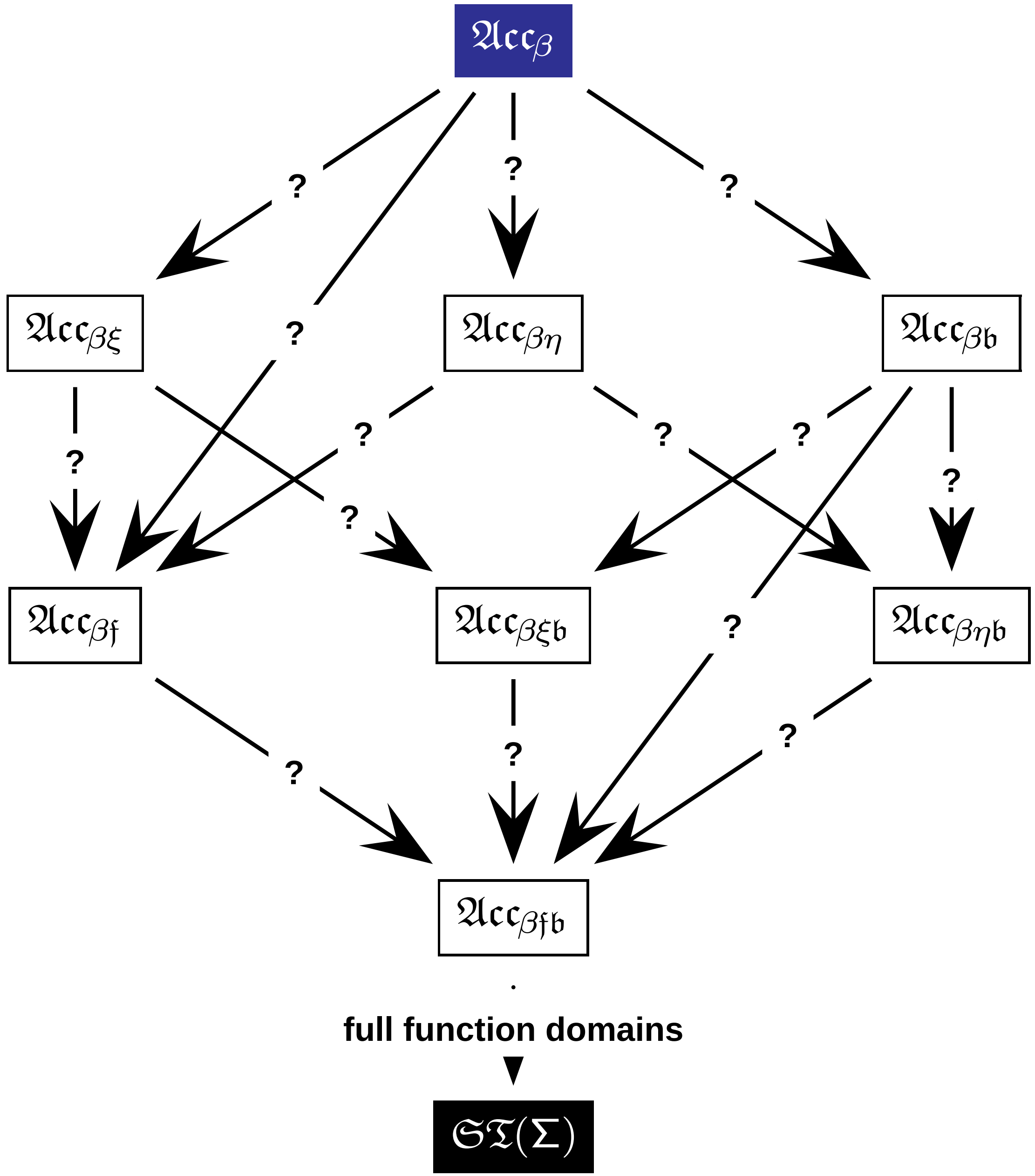
$\nabla_\vee$ If $\mathbf{A} \vee \mathbf{B} \in \Phi$, then $\Phi,\mathbf{A} \in \Gamma_\Sigma$ or $\Phi,\mathbf{B} \in \Gamma_\Sigma$.

$\nabla_\wedge$ If $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, then $\Phi,\neg\mathbf{A},\neg\mathbf{B} \in \Gamma_\Sigma$.

$\nabla_\forall$ If $\Pi^\alpha\mathbf{F} \in \Phi$, then $\Phi,\mathbf{F}\mathbf{W} \in \Gamma_\Sigma$ for each $\mathbf{W} \in cwff_\alpha(\Sigma)$.

$\nabla_\exists$ If $\neg\Pi^\alpha\mathbf{F} \in \Phi$, then $\Phi,\neg(\mathbf{F}w) \in \Gamma_\Sigma$ for any parameter $w_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

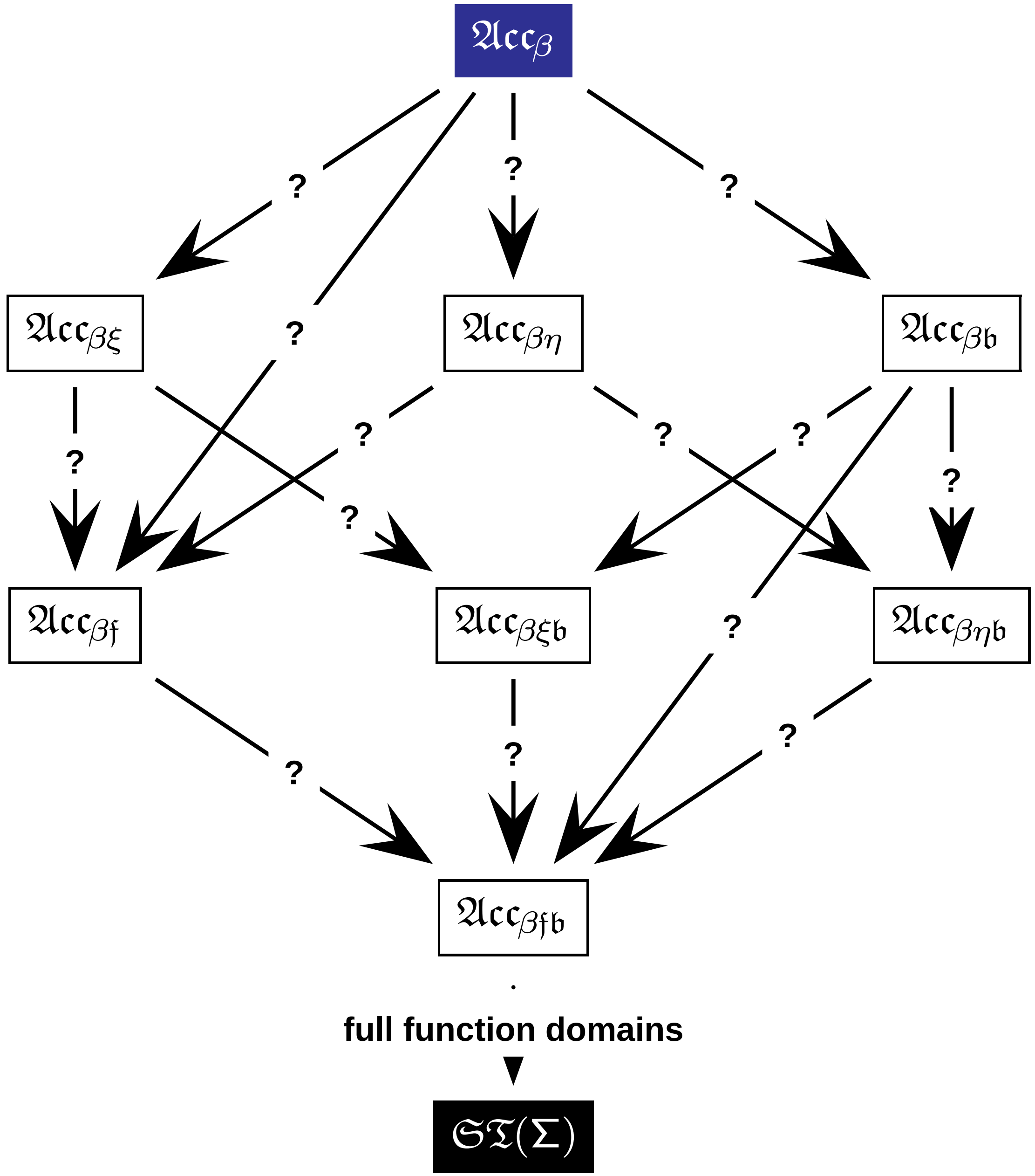
Abstract Consistency



Properties for $\mathfrak{Acc}_\beta$: (Γ_Σ is class of sets of formulas; $\Phi \in \Gamma_\Sigma$)

∇_c	If $\mathbf{A}$ is atomic, then $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$.
$\nabla_\neg$	If $\neg\neg\mathbf{A} \in \Phi$, then $\Phi,\mathbf{A} \in \Gamma_\Sigma$.
∇_β	If $\mathbf{A} =_\beta \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi,\mathbf{B} \in \Gamma_\Sigma$.
$\nabla_\vee$	If $\mathbf{A} \vee \mathbf{B} \in \Phi$, then $\Phi,\mathbf{A} \in \Gamma_\Sigma$ or $\Phi,\mathbf{B} \in \Gamma_\Sigma$.
$\nabla_\wedge$	If $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, then $\Phi,\neg\mathbf{A},\neg\mathbf{B} \in \Gamma_\Sigma$.
$\nabla_\forall$	If $\Pi^\alpha\mathbf{F} \in \Phi$, then $\Phi,\mathbf{F}\mathbf{W} \in \Gamma_\Sigma$ for each $\mathbf{W} \in cwff_\alpha(\Sigma)$.
$\nabla_\exists$	If $\neg\Pi^\alpha\mathbf{F} \in \Phi$, then $\Phi,\neg(\mathbf{F}w) \in \Gamma_\Sigma$ for any parameter $w_\alpha \in \Sigma_\alpha$ which does not occur in any sentence of Φ.

Abstract Consistency



Properties for $\mathfrak{Acc}_{\beta}$

∇_c ...

$\nabla_\neg$ ...

∇_β ...

$\nabla_\vee$ ...

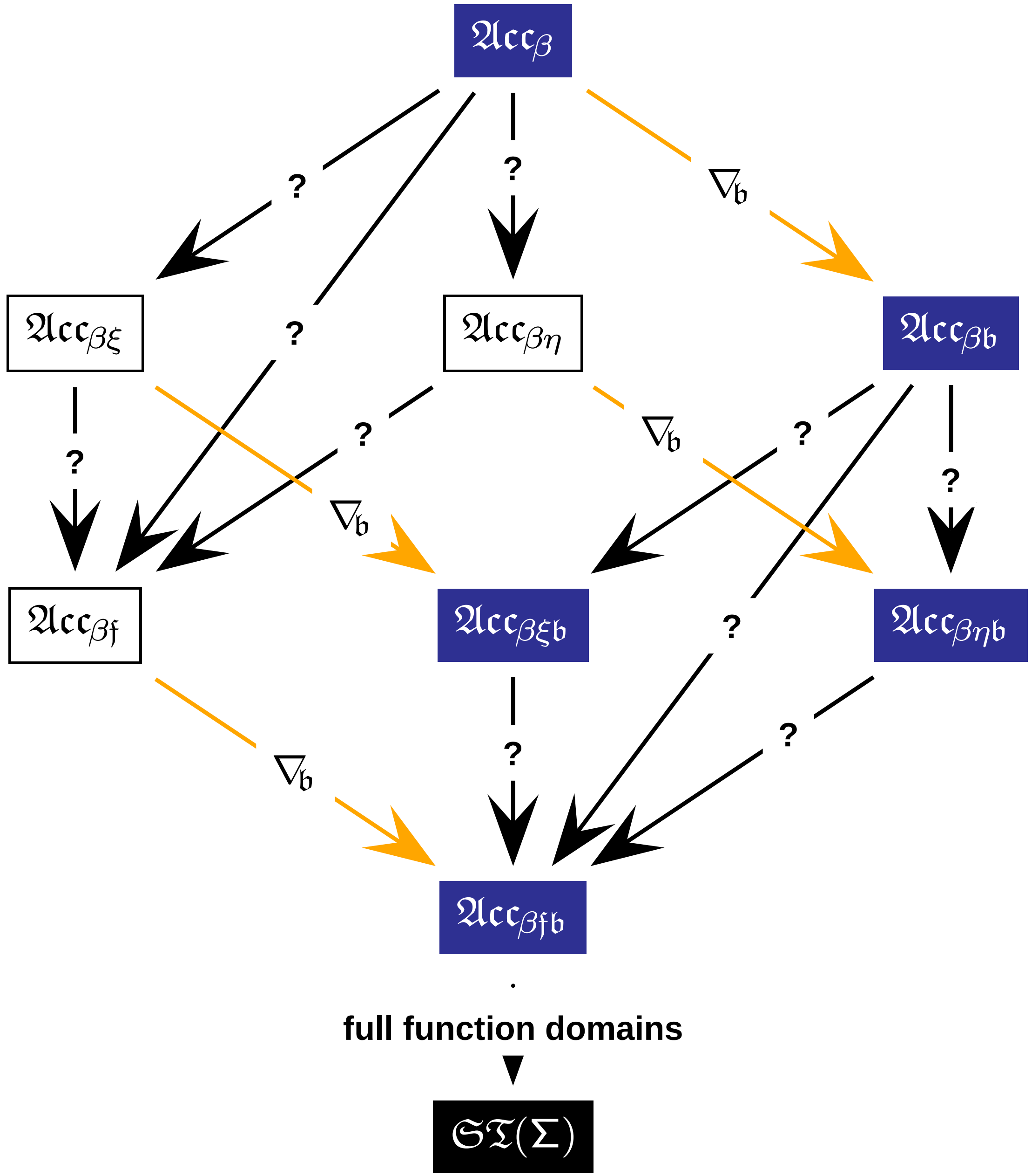
$\nabla_\wedge$ ...

$\nabla_\forall$ ...

$\nabla_\exists$ ...

Properties for Extensionality

Abstract Consistency



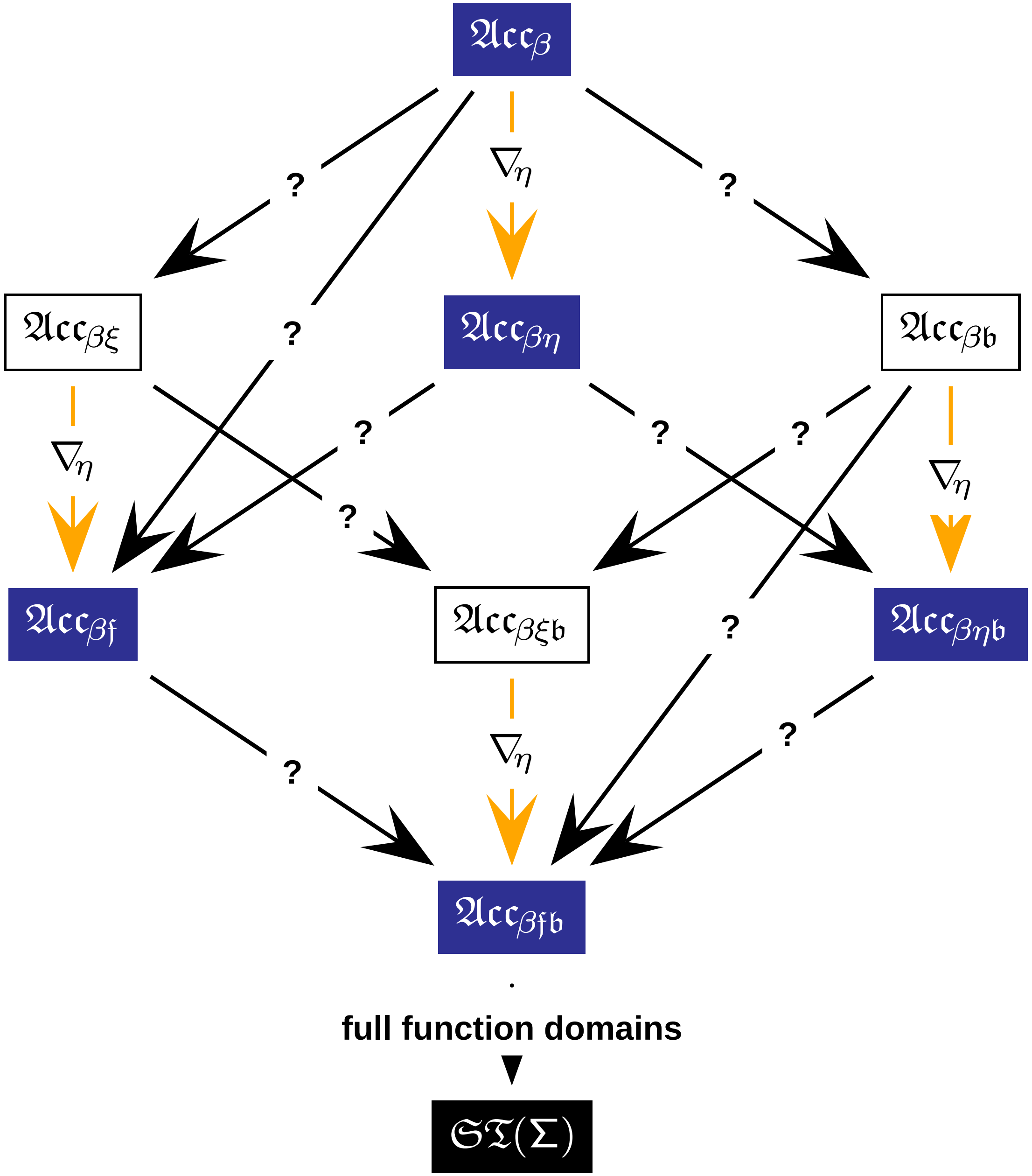
Properties for $\mathfrak{Acc}_\beta$

∇_c	...	$\nabla_\vee$	...
$\nabla_\neg$	...	$\nabla_\wedge$	...
∇_β	...	$\nabla_\forall$	...
		$\nabla_\exists$	...

Properties for Extensionality

$\nabla_\mathfrak{b}$ If $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$, then $\Phi, \mathbf{A}, \neg\mathbf{B} \in \Gamma_\Sigma$ or $\Phi, \neg\mathbf{A}, \mathbf{B} \in \Gamma_\Sigma$.

Abstract Consistency



Properties for $\mathfrak{Acc}_\beta$

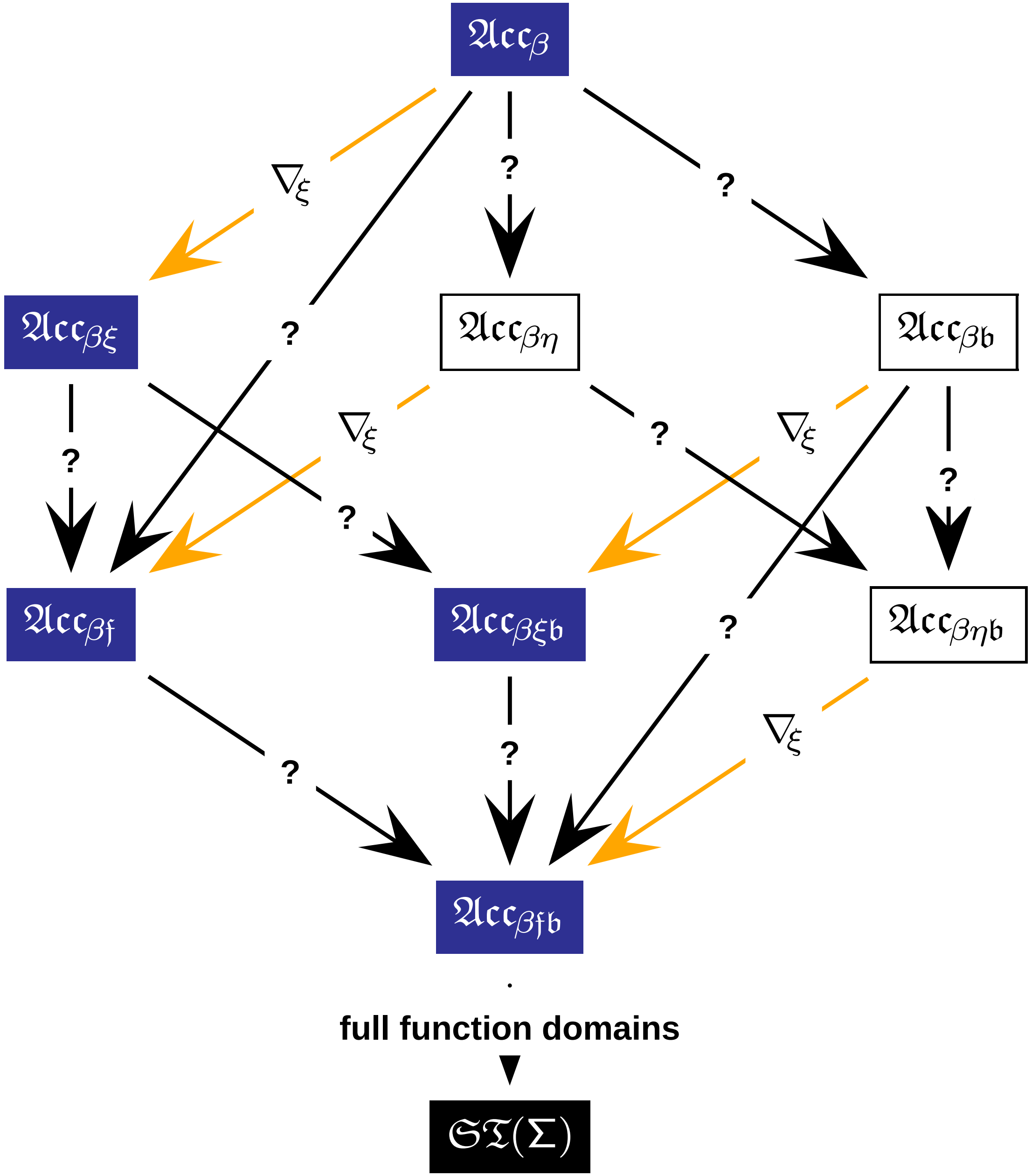
∇_c	...	$\nabla_\vee$	...
$\nabla_\neg$	...	$\nabla_\wedge$	...
∇_β	...	$\nabla_\forall$	...
		$\nabla_\exists$	...

Properties for Extensionality

$\nabla_\mathfrak{b}$ If $\neg(\mathbf{A} \doteq^\circ \mathbf{B}) \in \Phi$, then $\Phi,\mathbf{A},\neg\mathbf{B} \in \Gamma_\Sigma$ or $\Phi,\neg\mathbf{A},\mathbf{B} \in \Gamma_\Sigma$.

∇_η If $\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi,\mathbf{B} \in \Gamma_\Sigma$.

Abstract Consistency



Properties for $\mathfrak{Acc}_\beta$

∇_c	...	$\nabla_\vee$	...
$\nabla_\neg$	...	$\nabla_\wedge$	...
∇_β	...	$\nabla_\forall$	...
		$\nabla_\exists$	...

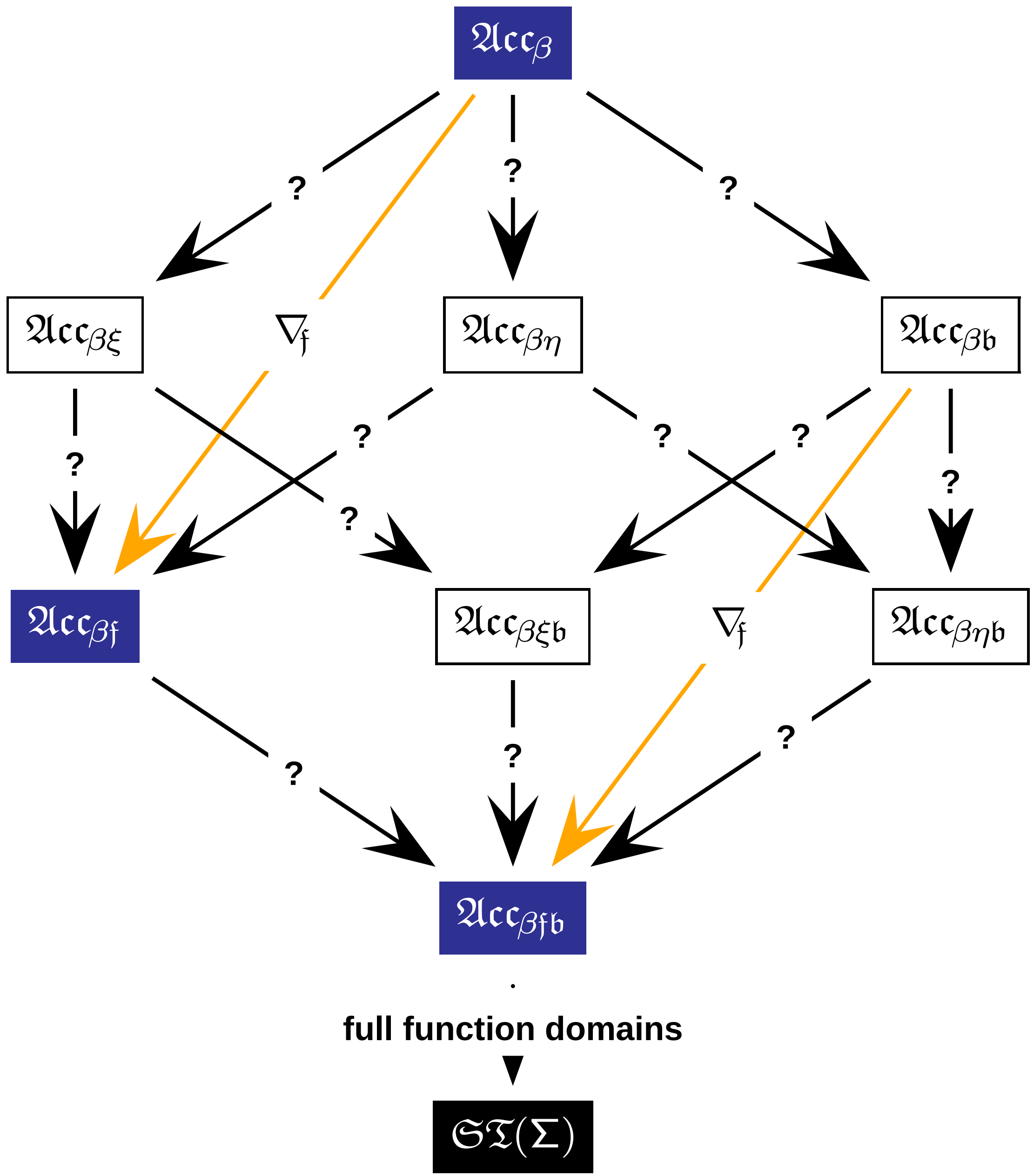
Properties for Extensionality

$\nabla_\mathfrak{b}$ If $\neg(\mathbf{A} \doteq^{o} \mathbf{B}) \in \Phi$, then $\Phi, \mathbf{A}, \neg\mathbf{B} \in \Gamma_\Sigma$ or $\Phi, \neg\mathbf{A}, \mathbf{B} \in \Gamma_\Sigma$.

∇_η If $\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi, \mathbf{B} \in \Gamma_\Sigma$.

∇_ξ If $\neg(\lambda X_\alpha.\mathbf{M} \doteq^{\alpha\to\beta} \lambda X_\alpha.\mathbf{N}) \in \Phi$, then $\Phi, \neg([w/X]\mathbf{M} \doteq^{\beta} [w/X]\mathbf{N}) \in \Gamma_\Sigma$ for any new $w_\alpha \in \Sigma_\alpha$.

Abstract Consistency



Properties for $\mathfrak{Acc}_\beta$

∇_c	...	$\nabla_\vee$	...
$\nabla_\neg$	...	$\nabla_\wedge$	...
∇_β	...	$\nabla_\forall$	...
		$\nabla_\exists$	...

Properties for Extensionality

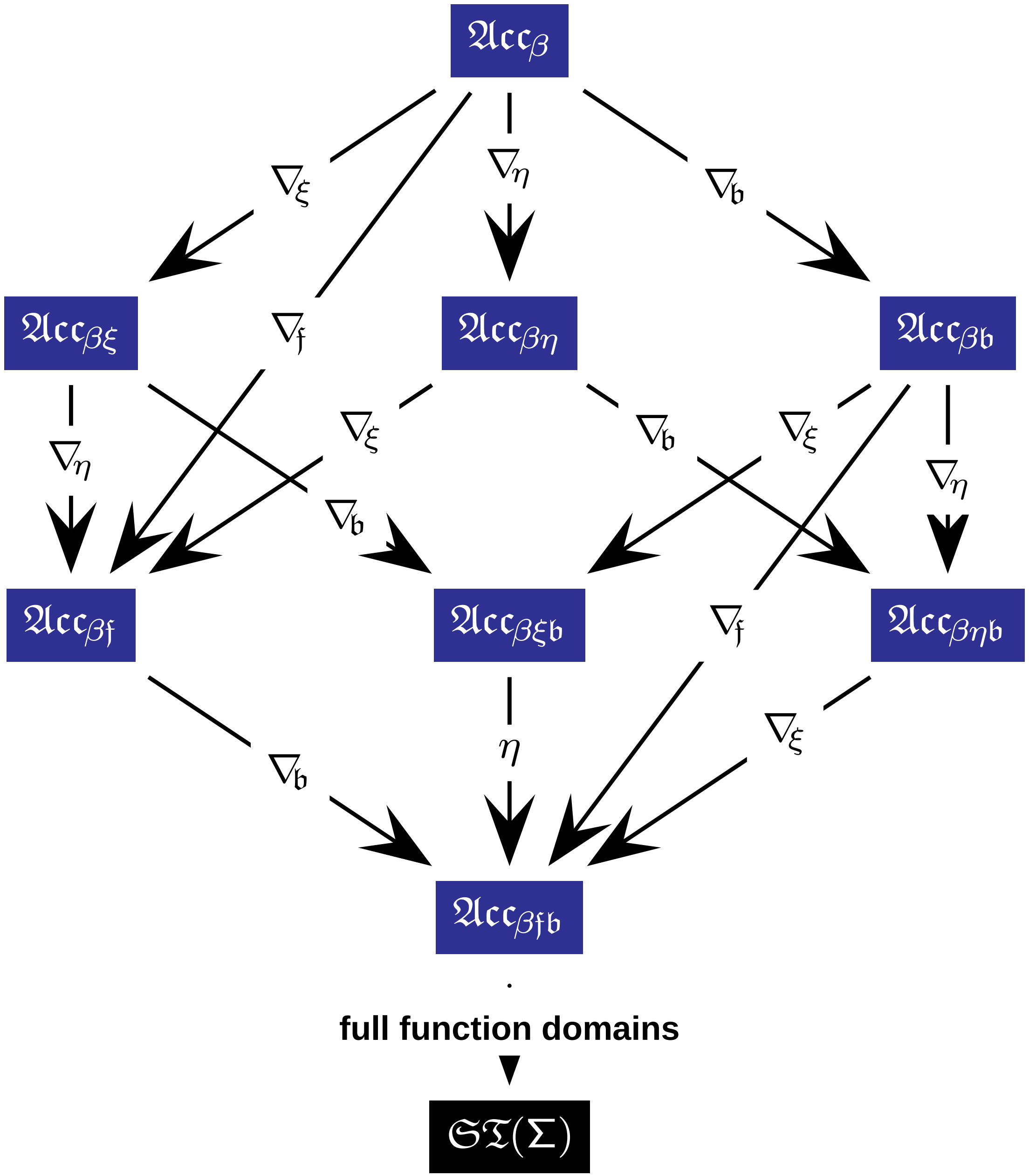
$\nabla_\mathfrak{b}$ If $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$, then $\Phi,\mathbf{A},\neg\mathbf{B} \in \Gamma_\Sigma$ or $\Phi,\neg\mathbf{A},\mathbf{B} \in \Gamma_\Sigma$.

∇_η If $\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi,\mathbf{B} \in \Gamma_\Sigma$.

∇_ξ If $\neg(\lambda X_\alpha.\mathbf{M} \doteq^{\alpha\to\beta} \lambda X_\alpha.\mathbf{N}) \in \Phi$, then $\Phi,\neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N}) \in \Gamma_\Sigma$ for any new $w_\alpha \in \Sigma_\alpha$.

$\nabla_\mathfrak{f}$ If $\neg(\mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H}) \in \Phi$, then $\Phi,\neg(\mathbf{G}w \doteq^\beta \mathbf{H}w) \in \Gamma_\Sigma$ for any new $w_\alpha \in \Sigma_\alpha$.

Abstract Consistency



Properties for $\mathfrak{Acc}_\beta$

∇_c	...	$\nabla_\vee$	...
$\nabla_\neg$	...	$\nabla_\wedge$	...
∇_β	...	$\nabla_\forall$	...
		$\nabla_\exists$	...

Properties for Extensionality

$\nabla_\mathfrak{b}$ If $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$, then $\Phi,\mathbf{A},\neg\mathbf{B} \in \Gamma_\Sigma$ or $\Phi,\neg\mathbf{A},\mathbf{B} \in \Gamma_\Sigma$.

∇_η If $\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B}$ and $\mathbf{A} \in \Phi$, then $\Phi,\mathbf{B} \in \Gamma_\Sigma$.

∇_ξ If $\neg(\lambda X_\alpha.\mathbf{M} \doteq^{\alpha\to\beta} \lambda X_\alpha.\mathbf{N}) \in \Phi$, then $\Phi,\neg([w/X]\mathbf{M} \doteq^\beta [w/X]\mathbf{N}) \in \Gamma_\Sigma$ for any new $w_\alpha \in \Sigma_\alpha$.

$\nabla_\mathfrak{f}$ If $\neg(\mathbf{G} \doteq^{\alpha\to\beta} \mathbf{H}) \in \Phi$, then $\Phi,\neg(\mathbf{G}w \doteq^\beta \mathbf{H}w) \in \Gamma_\Sigma$ for any new $w_\alpha \in \Sigma_\alpha$.

Abstract Consistency

Thm.: (Model Existence)
Saturated abstract consistency implies model existence

Appl.: (Completeness proofs by pure syntactical means)

$$\Gamma_\Sigma^G := \{\Phi | \Phi \text{ is } \mathcal{C}\text{-consistent}\} \text{ is a saturated } \mathfrak{Acc}_*$$