Progress Report on Leo-II, an Automatic Theorem Prover for Higher-Order Logic

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LEO-II

An Effective Higher-Order Theorem Prover

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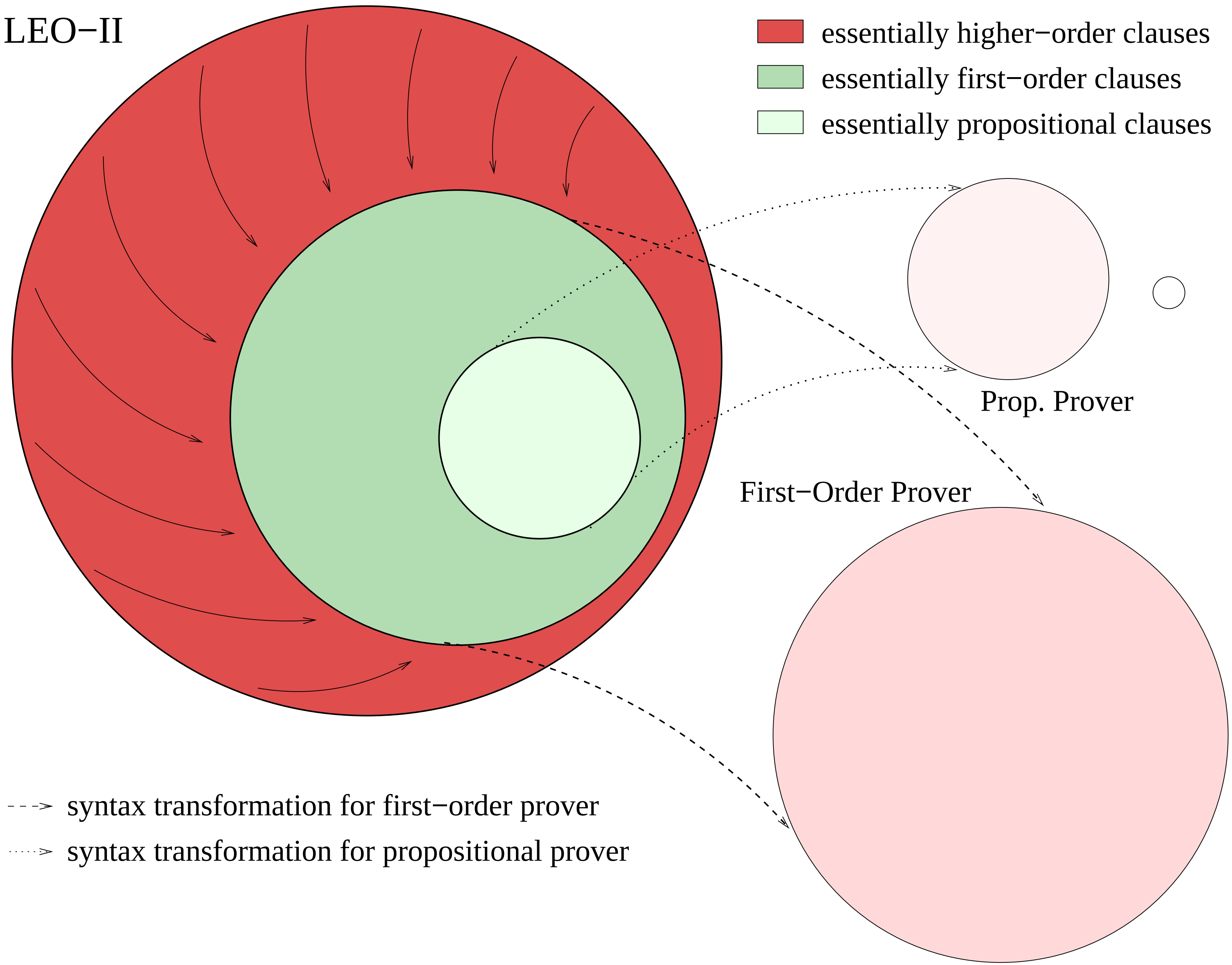
What is LEO-II

LEO-II
An Effective Higher-Order Theorem Prover

- Automatic theorem prover / successor of LEO
  - resolution based higher order theorem prover
  - standalone; implemented in Objective CAML (12K LoC)
  - cooperation with specialist provers, e.g. FO ATPs
  - term sharing and term indexing
  - experimentation with prover architecture(s)
- Interactive proof assistant
- Problem representation language: TPTP THF Syntax

Cooperation with Other Provers



First Experiments with LEO-II

LEO-II An Effective Higher-Order Theorem Prover

[1]: Intel(R) Pentium(R) 4 CPU 2.80GHz, 1GB, Linux, CPULimit 600s

[2]: Intel(R) Xeon(TM) 4 CPU 2.40GHz, 4GB, Linux, CPULimit 120s

[3]: Intel(R) Pentium(R) 1 CPU 1.60GHz, 1GB, Linux, CPULimit 60s

Problem	**Vampire 9.0**[1]	**LEO/Vamp.**[2]	**LEO-II/E**[3]
SET014+4	114.5	2.60	0.300
SET017+1	1.0	5.05	0.059
SET066+1	–	3.73	0.029
SET067+1	4.6	0.10	0.040
SET076+1	51.3	0.97	0.031
SET086+1	0.1	0.01	0.028
SET096+1	5.9	7.29	0.033
SET143+3	0.1	0.31	0.034
SET171+3	68.6	0.38	0.030
SET580+3	0.0	0.23	0.078
SET601+3	1.6	1.18	0.089
SET606+3	0.1	0.27	0.033
SET607+3	1.2	0.26	0.036
SET609+3	145.2	0.49	0.039
SET611+3	0.3	4.00	0.125
SET612+3	111.9	0.46	0.030
SET614+3	3.7	0.41	0.060
SET615+3	103.9	0.47	0.035
SET623+3	–	2.27	0.282
SET624+3	3.8	3.29	0.047
SET630+3	0.1	0.05	0.025
...	...	...	...

First Experiments with LEO-II

Problem	Vampire 9.0[1]	LEO/Vamp.[2]	LEO-II/E[3]
...	...	...	...
SET640+3	1.1	0.01	0.033
SET646+3	84.4	0.01	0.032
SET647+3	98.2	0.12	0.037
SET648+3	98.2	0.12	0.037
SET649+3	117.5	0.25	0.037
SET651+3	117.5	0.09	0.029
SET657+3	146.6	0.01	0.028
SET669+3	83.1	0.20	0.041
SET670+3	–	0.14	0.067
SET671+3	214.9	0.47	0.038
SET672+3	–	0.23	0.034
SET673+3	217.1	0.47	0.042
SET680+3	146.3	2.38	0.035
SET683+3	0.3	0.27	0.053
SET684+3	–	3.39	0.039
SET716+4	–	0.40	0.033
SET724+4	–	1.91	0.032
SET741+4	–	3.70	0.042
SET747+4	–	1.18	0.028
SET752+4	–	516.00	0.086
SET753+4	–	1.64	0.037
SET764+4	0.1	0.01	0.032
SET770+4	145.0	–	–

Average time (success) LEO-II = 0.048

(Normal) Multimodal Logic in HOL

- FOL encodings of modal logic well investigated
- HOL encodings of modal logic
  - Harrison's HOL-light primer
  - Hardt and Smolka, 2006
  - ...
  - here we pick-up, extend and explore an idea of Chad Brown; see talk in April 2005 at Loria Nancy, France

`http://mathgate.info/cebrown/papers/hybrid-hol.pdf`

(Normal) Multimodal Logic in HOL

LEO-II An Effective Higher-Order Theorem Prover

- base type ι: set of possible worlds
  certain terms of type $\iota \to \mathsf{o}$: multimodal logic formulas
- multimodal logic operators:

$$\neg_{(\iota\to o)\to(\iota\to o)} = \lambda A_{\iota\to o}.\, \lambda X_{\iota}.\, \neg A\, X$$

$$\vee_{(\iota\to o)\to(\iota\to o)\to(\iota\to o)} = \lambda A_{\iota\to o}.\, \lambda B_{\iota\to o}.\, \lambda X_{\iota}.\, A\, X \vee B\, X$$

$$\Box_{R\,(\iota\to\iota\to o)\to(\iota\to o)\to(\iota\to o)} = \lambda R_{\iota\to\iota\to o}.\, \lambda A_{\iota\to o}.\, \lambda X_{\iota}.\, \forall Y_{\iota}.\, R\, X\, Y \Rightarrow A\, Y$$

- multimodal logic propositions:
  - each constant $p_{\iota\to o} \in \Sigma$ is an atomic proposition
  - if φ and ψ are propositions,
    then so are $\neg\, \varphi$, $\varphi \vee \psi$ and $\Box_R\, \varphi$
- $\Rightarrow$, $\Leftrightarrow$, $\Diamond_r$, etc. defined as usual

(Normal) Multimodal Logic in HOL

LEO-II
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- We can also encode the notions of validity, satisfiability, etc.

$$\begin{aligned}
\texttt{valid} &= \lambda A_{\iota\rightarrow o}.\, \forall W_\iota.\, A\, W \\
\texttt{satisfiable} &= \lambda A_{\iota\rightarrow o}.\, \exists W_\iota.\, A\, W \\
\texttt{countersatisfiable} &= \lambda A_{\iota\rightarrow o}.\, \exists W_\iota.\, \neg A\, W \\
\texttt{invalid} &= \lambda A_{\iota\rightarrow o}.\, \forall W_\iota.\, \neg A\, W
\end{aligned}$$

Automation in LEO-II

LEO-II An Effective Higher-Order Theorem Prover

problem	LEO-II+E (sec)
valid($\Box_r \top$)	0.025
valid($\Box_r a \Rightarrow \Box_r a$)	0.026
valid($\Box_r a \Rightarrow \Box_s a$)	–
valid($\Box_s (\Box_r a \Rightarrow \Box_r a)$)	0.026
valid($\Box_r (a \land b) \Leftrightarrow (\Box_r a \land \Box_r b)$)	0.044
valid($\Diamond_r (a \Rightarrow b) \Rightarrow \Box_r a \Rightarrow \Diamond_r b$)	0.030
valid($\neg \Diamond_r a \Rightarrow \Box_r (a \Rightarrow b)$)	0.029
valid($\Box_r b \Rightarrow \Box_r (a \Rightarrow b)$)	0.026
valid($(\Diamond_r a \Rightarrow \Box_r b) \Rightarrow \Box_r (a \Rightarrow b)$)	0.027
valid($(\Diamond_r a \Rightarrow \Box_r b) \Rightarrow (\Box_r a \Rightarrow \Box_r b)$)	0.029
valid($(\Diamond_r a \Rightarrow \Box_r b) \Rightarrow (\Diamond_r a \Rightarrow \Diamond_r b)$)	0.030

Automation in LEO-II

Can we also automate reasoning about (normal) multimodal logics?

Example:

$$S4 = K + T + 4$$

with

T	$\Box_R A \Rightarrow A$
4	$\Box_R A \Rightarrow \Box_R \Box_R A$

Proving Properties of K

Essential properties of K are the necessitation rule N and the distribution axiom D:

N *If* A *is a theorem of* K, *then so is* $\Box_R A$

0.027sec $\forall R. \forall A. \mathtt{valid}(A) \Rightarrow \mathtt{valid}(\Box_R A)$

D $\Box_R (A \Rightarrow B) \Rightarrow (\Box_R A \Rightarrow \Box_R B)$

0.029sec $\forall R. \forall A. \forall B. \mathtt{valid}(\Box_R (A \Rightarrow B) \Rightarrow (\Box_R A \Rightarrow \Box_R B))$

Exploring Modal Logics in LEO-II+E

LEO-II
An Effective Higher-Order Theorem Prover

Is axiom T is valid in **K**?

$$\forall R. \forall A. \mathtt{valid}(\Box_R A \Rightarrow A)$$ no proof

Is there a relation R such that for all A axiom T is valid in **K**?

$$\exists R. \forall A. \mathtt{valid}(\Box_R A \Rightarrow A)$$ 0.539 sec

R ⟵ equality

Is axiom T indeed equivalent to reflexivity of R in **K**?

$$\forall R. (\forall A. \mathtt{valid}(\Box_R A \Rightarrow A) \Leftrightarrow \mathtt{refl}(R))$$ 0.039 sec

Exploring Modal Logics in LEO-II+E

LEO-II An Effective Higher-Order Theorem Prover

Is axiom 4 valid in **K**?

$$\forall R.\, \forall A.\, \mathtt{valid}(\Box_R A \Rightarrow \Box_R \Box_R A)$$ no proof

Is there a relation R such that for all A axiom 4 is valid in **K**?

$$\exists R.\, \forall A.\, \mathtt{valid}(\Box_R A \Rightarrow \Box_R \Box_R A)$$ 0.057 sec

$R \longleftarrow$ inequality

Is axiom 4 equivalent to transitivity of R in **K**?

$$\forall R.\, (\forall A.\, \mathtt{valid}(\Box_R A \Rightarrow \Box_R \Box_R A)) \Leftrightarrow \mathtt{trans}(R)$$ 0.195 sec

Exploring Modal Logics in LEO-II+E

Are T and 4 equivalent to reflexivity and transitivity of R in **K**?

$$\forall R.\, (\forall A.\, \mathtt{valid}(\Box_R A \Rightarrow A) \wedge \mathtt{valid}(\Box_R A \Rightarrow \Box_R \Box_R A))$$
$$\Leftrightarrow (\mathtt{refl}(R) \wedge \mathtt{trans}(R))$$

2.262 sec

LEO-II passes 70 clauses / E generates 21769 clauses

Better:

$\Rightarrow$ in 0.045 seconds

$\Leftarrow$ in 0.048 seconds

Summary

- Did not talk much about LEO-II $\longrightarrow$ come to poster!
- LEO-II appears to be suited for:

  reasoning within and about (normal) multimodal logics

- We have already extended the encoding to
  - normal first order quantified multimodal logics
  - normal higher order quantified multimodal logics
- Many future work directions, including

  LEO-II as a framework for exploring
  (normal) propositional and quantified multimodal logics

(Normal) FO Quantif. Multimodal Logic

LEO-II
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1. Λ_1^{mm}-terms are defined as (base type $\mu \neq \iota$):
   - Each constant $c_\mu \in \Sigma$ and variable $X_\mu \in \Sigma$ is a Λ_1^{mm}-term.
   - If $t^1_\mu, \ldots, t^n_\mu$ are Λ_1^{mm}-terms and $f_{\mu\rightarrow\ldots\rightarrow\mu\rightarrow\mu} \in \Sigma$ is an n-ary (curried) function symbol, then $(f\, t^1 \ldots t^n)_\mu$ is a Λ_1^{mm}-term.
2. The modal operators $\neg$, $\vee$, $\Box_r$ are defined as before.
3. $\forall X_\mu \centerdot \varphi_{\iota\rightarrow o}$ defined as $\lambda w_\iota \centerdot \forall X_\mu \centerdot \varphi\, w$
4. Λ_1^{mm}-propositions are defined by:
   - If $t^1_\mu, \ldots, t^n_\mu$ are Λ_1^{mm}-terms and let $p_{\mu\rightarrow\ldots\rightarrow\mu\rightarrow(\iota\rightarrow o)} \in \Sigma$, then $(p\, t^1 \ldots t^n)_{\iota\rightarrow o}$ is an atomic Λ_1^{mm}-proposition.
   - If φ and ψ be Λ_1^{mm}-propositions, then so are $\neg\, \varphi$, $\varphi \vee \psi$ and $\Box_r\, \varphi$, where $\neg$, $\vee$, $\Box_r$ are defined as above.
   - If $X_\mu \in \Sigma$ is a variable of type μ and $\varphi_{\iota\rightarrow o}$ is a Λ_1^{mm}-proposition, then $\forall X \centerdot \varphi$ is a Λ_1^{mm}-proposition.