Extensional Higher-Order Paramodulation and RUE-Resolution

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Outline

- Motivation
- Higher-Order Logic (with/without primitive Equality)
- **From**
  - Extensional Higher-Order Resolution $\mathcal{ER}$

  To
  - Extensional Higher-Order Paramodulation $\mathcal{EP}$
  - Extensional Higher-Order RUE-Resolution $\mathcal{ERUE}$
- Completeness: Abstract Consistency & Model-Existence
- Conclusion

Motivation

- Automated Higher-Order Theorem Proving [Andrews71, JensenPietrowski72, Huet72, Wolfram93, Kohlhase94, Kohlhase95]

  $\Rightarrow$ **equality/extensionality treatment not sufficiently solved**

- HO Termrewriting/Narrowing [NipkowPrehofer98, Prehofer98, NipkowMayr98, . . .]

  $\Rightarrow$ **does not address Henkin complete HO-ATP**

- HO E-Unification [Snyder90, Qian93, QianWang96]

  $\Rightarrow$ **restricted to FO theories; no full extensionality**

- Extensional HO-Resolution [BenzmuellerKohlhase98]

  $\Rightarrow$ **only for defined equality**

This talk addresses **Henkin complete HO-ATP with/without primitive equality**

Classical Type Theory ($\frac{\text{HOL}}{\Lambda^{\rightarrow}}$): Syntax

- **Types:** (i) $\{i, o\} \in \mathcal{T}$ (ii) $\alpha, \beta \in \mathcal{T} \quad \rightsquigarrow \quad \alpha \rightarrow \beta \in \mathcal{T}$
- **Terms $\Lambda^{\rightarrow}$:**
  - (i) (infinite) sets of variables of type α: $V_\alpha \subseteq \Lambda^{\rightarrow}$ **(Notation X_α)**
  - (ii) Constants of type α: $C_\alpha \subseteq \Lambda^{\rightarrow}$ **(Notation d_α)**
    Required: $\neg \in C_{o\rightarrow o}, \vee \in C_{o\rightarrow(o\rightarrow o)}, \Pi \in C_{(\alpha\rightarrow o)\rightarrow o}$
  - (iii) Application: $\mathbf{A}_{\alpha\rightarrow\beta}, \mathbf{B}_\alpha \in \Lambda^{\rightarrow} \quad \rightsquigarrow \quad (\mathbf{A}\ \mathbf{B})_\beta \in \Lambda^{\rightarrow}$
  - (iii) Abstraction: $X_\alpha \in V_\alpha, \mathbf{A}_\beta \in \Lambda^{\rightarrow} \quad \rightsquigarrow \quad (\lambda X . \mathbf{A})_{\alpha\rightarrow\beta} \in \Lambda^{\rightarrow}$
- **Normalforms** (e.g. $\beta\eta$-normalform / $\beta\eta$-headnormalform):
  - (i) Abstraction from bound variables: $\lambda X_\gamma . \mathbf{A} \quad \longleftrightarrow^\alpha \quad \lambda Y_\gamma . \mathbf{A}[Y/X]$
  - (ii) λ-Conversion: $(\lambda X_\gamma . \mathbf{A})\ \mathbf{B}_\gamma \quad \longrightarrow^\beta \quad \mathbf{A}[\mathbf{B}/X]$
    (if X not free in $\mathbf{A}$) $\lambda X . \mathbf{A}\ X \quad \longrightarrow^\eta \quad \mathbf{A}$

Classical Type Theory ($\frac{\text{HOL}}{\Lambda^{\rightarrow}}$): Semantics

Standardsemantics	Choose	Required
Semantical Domains	D_ι	$D_o = \{\bot, \top\}$, $D_{\alpha\rightarrow\beta} = \mathcal{F}(D_\alpha, D_\beta)$
Interpretation of Const.	$I : (I_\alpha : C_\alpha \longrightarrow D_\alpha)_{\alpha\in\mathcal{T}}$	$I(\neg), I(\vee)$, $I(\Pi)$ **as intended**
Variable Assignment	$\varphi : (\varphi_\alpha : V_\alpha \longrightarrow D_\alpha)_{\alpha\in\mathcal{T}}$	
Interpretation of terms	$I_\varphi(X) = \varphi(X), \quad I_\varphi(c) = I(c), \quad I_\varphi(\mathbf{A}\ \mathbf{B}) = I_\varphi(A)@I_\varphi(B)$,	
$I_\varphi : \Lambda^{\rightarrow} \longrightarrow D$ **def. by**	$I_\varphi(\lambda X_\alpha.\mathbf{B}_\beta) = f \in D_{\alpha\rightarrow\beta}$, **such that** $\forall a : f@a = I_{\varphi[a/X]}(\mathbf{B})$	

Model: $\mathcal{M} = (\mathcal{D} : \{D_\alpha\}, \mathcal{I} : \{I_\alpha\})$; **satisfisfiability and validity defined as usual**

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Henkin semantics	**Choose**	**Required**
Semantical Domains	D_ι, $D_{\alpha\rightarrow\beta} \subseteq \mathcal{F}(D_\alpha, D_\beta)$	$D_o = \{\bot, \top\}$, **Totality of** I_φ
Interpretation of Const.	**as above**	**as above**
Variable Assignment	**as above**	**as above**
Interpretation of terms	**as above**	

Model: $\mathcal{M} = (\mathcal{D} : \{D_\alpha\}, \mathcal{I} : \{I_\alpha\})$; **satisfisfiability and validity defined as usual**

Remarks on Classical Type Theory ($\frac{\text{HOL}}{\Lambda^{\rightarrow}}$)

- **[Gödel 1931]** Standardsemantics **does not allow** complete calculi
- **[Henkin 1950]** Henkin semantics **does allow** complete calculi
- **Comprehension Principles** are built-in $(\exists F_{\alpha\rightarrow\beta} \centerdot \forall X_{\alpha} \centerdot (F\ X) = \mathbf{A}_{\beta})$

  $\rightsquigarrow \lambda X_{\alpha} \centerdot \mathbf{A}$
- **Axiom of choice** and **Descriptionoperator** ι are optional
- **Equality** is built-in (Leibniz Equality denotes a functional congruence)

$$\boxed{\doteq^{\alpha} \;:= \lambda X_{\alpha} \centerdot \lambda Y_{\alpha} \centerdot \forall P_{\alpha\rightarrow o} \centerdot PX \Rightarrow PY}$$

  but **infinetely many extensionality axioms** are required

Nasty Extensionality Axioms

$$\mathbf{EXT}^{\doteq}_{\alpha\to\beta} := \forall F_{\alpha\to\beta} \centerdot \forall G_{\alpha\to\beta} (\forall X_{\beta} \centerdot F\ X \doteq G\ X) \Rightarrow F \doteq G \quad \overset{\text{CNF}}{\rightsquigarrow}$$

$$\mathcal{C}_1 : [p_{\beta\to o}\ (F\ s_{\beta})]^T \vee [\mathbf{Q}\ F]^F \vee [\mathbf{Q}\ G]^T,\ \mathcal{C}_2 : [p_{\beta\to o}\ (G\ s_{\beta})]^T \vee [\mathbf{Q}\ F]^F \vee [\mathbf{Q}\ G]^T$$

$$\mathbf{EXT}^{\doteq}_{o} := \forall A_o \centerdot \forall B_o \centerdot (A \Leftrightarrow B) \Leftrightarrow A \doteq^o B \quad \overset{\text{CNF}}{\rightsquigarrow}$$

$$\mathcal{C}_1 : [\mathbf{A}]^F \vee [\mathbf{B}]^F \vee [\mathbf{P}\ A]^F \vee [\mathbf{P}\ B]^T,\ \mathcal{C}_2 : [\mathbf{A}]^T \vee [\mathbf{B}]^T \vee [\mathbf{P}\ A]^F \vee [\mathbf{P}\ B]^T,\ \mathcal{C}_3 : [\mathbf{A}]^F \vee [\mathbf{B}]^T \vee [p\ A]^T,\ \mathcal{C}_4 : [\mathbf{A}]^F \vee [\mathbf{B}]^T \vee [p\ B]^F,\ \mathcal{C}_5 : [\mathbf{A}]^T \vee [\mathbf{B}]^F \vee [p\ A]^T,\ \mathcal{C}_6 : [\mathbf{A}]^T \vee [\mathbf{B}]^F \vee [p\ B]^F$$

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$\mathbf{EXT}^{=}_{\alpha\to\beta} \quad \overset{\text{CNF}}{\rightsquigarrow} \quad \mathcal{C}_1 : [\mathbf{F}_{\alpha\to\beta}\ X_\alpha = \mathbf{G}_{\alpha\to\beta}\ X_\alpha]^F \vee [\mathbf{F}_{\alpha\to\beta} = \mathbf{G}_{\alpha\to\beta}]^T$

$\mathbf{EXT}^{=}_{o} \quad \overset{\text{CNF}}{\rightsquigarrow} \quad \mathcal{C}_1 : [\mathbf{A} = \mathbf{B}]^F \vee [\mathbf{A}]^F \vee [\mathbf{B}]^T, \mathcal{C}_2 : [\mathbf{A} = \mathbf{B}]^F \vee [\mathbf{A}]^T \vee [\mathbf{B}]^F,$
$\mathcal{C}_3 : [\mathbf{A} = \mathbf{B}]^T \vee [\mathbf{A}]^F \vee [\mathbf{B}]^F, \mathcal{C}_4 : [\mathbf{A} = \mathbf{B}]^T \vee [\mathbf{A}]^T \vee [\mathbf{B}]^T$

$\rightsquigarrow$ avoid the extensionality axioms

Extensional HO Resolution: $\mathcal{ER}$

Constrained Resolution [Huet72]

$$\frac{\mathcal{D} \quad \mathcal{C} \in \mathcal{CNF}(\mathcal{D})}{\mathcal{C}}\ \textbf{Cnf} \qquad \frac{}{}\ \textbf{PrimSubst} \qquad \frac{[P\ a\ b]^\alpha \vee \mathcal{C}}{\begin{array}{c}[\neg(P'\ a\ b)]^\alpha \vee \mathcal{C}\\ [(P'\ a\ b) \vee (P''\ a\ b)]^\alpha \vee \mathcal{C}\\ [\Pi^\gamma(P'\ a\ b)]^\alpha \vee \mathcal{C}\end{array}}$$

$$\frac{[\mathbf{A}]^\alpha \vee \mathcal{C} \quad [\mathbf{B}]^\beta \vee \mathcal{D} \quad \alpha,\beta \in \{T,F\}, \alpha \neq \beta}{\mathcal{C} \vee \mathcal{D} \vee [\mathbf{A} \neq^? \mathbf{B}]}\ \textbf{Res} \qquad \frac{[\mathbf{A}]^\alpha \vee [\mathbf{B}]^\alpha \vee \mathcal{C} \quad \alpha \in \{T,F\}}{[\mathbf{A}]^\alpha \vee \mathcal{C} \vee [\mathbf{A} \neq^? \mathbf{B}]}\ \textbf{Fac}$$

+ **HO-(pre)-unification** **+**

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$$\frac{[\mathbf{A}]^{\alpha} \lor \mathcal{C} \quad [\mathbf{B}]^{\beta} \lor \mathcal{D} \quad \alpha, \beta \in \{T, F\}, \alpha \neq \beta}{\mathcal{C} \lor \mathcal{D} \lor [\mathbf{A} = \mathbf{B}]^{F}}\ \textbf{Res} \qquad \frac{[\mathbf{A}]^{\alpha} \lor [\mathbf{B}]^{\alpha} \lor \mathcal{C} \quad \alpha \in \{T, F\}}{[\mathbf{A}]^{\alpha} \lor \mathcal{C} \lor [\mathbf{A} = \mathbf{B}]^{F}}\ \textbf{Fac}$$

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$$\frac{[\mathbf{A}]^{\alpha} \vee \mathcal{C} \quad [\mathbf{B}]^{\beta} \vee \mathcal{D} \quad \alpha, \beta \in \{T, F\}, \alpha \neq \beta}{\mathcal{C} \vee \mathcal{D} \vee [\mathbf{A} = \mathbf{B}]^{F}}\ \textbf{Res} \qquad \frac{[\mathbf{A}]^{\alpha} \vee [\mathbf{B}]^{\alpha} \vee \mathcal{C} \quad \alpha \in \{T, F\}}{[\mathbf{A}]^{\alpha} \vee \mathcal{C} \vee [\mathbf{A} = \mathbf{B}]^{F}}\ \textbf{Fac}$$

+ **HO-(pre)-unification** **+**

Integration of Unification and Theorem Proving (mutual recursive calls)

$$\frac{\mathbf{C} \vee [\mathbf{M}_{o} = \mathbf{N}_{o}]^{F}}{\mathbf{C} \vee [\mathbf{M}_{o} \Leftrightarrow \mathbf{N}_{o}]^{F}}\ \textbf{Equiv} \qquad \frac{\mathbf{C} \vee [\mathbf{M}_{\alpha} = \mathbf{N}_{\alpha}]^{F}}{\mathbf{C} \vee [\forall P_{\alpha \to o} \blacksquare P\ \mathbf{M} \Rightarrow P\ \mathbf{N}]^{F}}\ \textbf{Leib}$$

Extensional HO (Pre-)Unifikation

Employ usual rules of [GallierSnyder89]

Triv, Dec, FlexRigid, Solve, FlexFlex, Func $\dfrac{\mathcal{C} \lor [\mathbf{A}_{\alpha\to\beta} = \mathbf{B}_{\alpha\to\beta}]^F}{\mathcal{C} \lor [\mathbf{A}\ s_\alpha = \mathbf{B}\ s_\alpha]^F}$

Recursive calls to ER: realises general HO-E-unification

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Recursive calls to ER: realises general HO-E-unification

Example1:

$$\dfrac{[(\lambda X_\iota.\, \text{red}\ X \Rightarrow \text{red}\ X) = (\lambda X_\iota.\, \text{blue}\ X \vee \neg\text{blue}\ X)]^F}{\dfrac{[(\text{red}\ s \Rightarrow \text{red}\ s) = (\text{blue}\ s \vee \neg\text{blue}\ s)]^F}{[(\text{red}\ s \Rightarrow \text{red}\ s) \equiv (\text{blue}\ s \vee \neg\text{blue}\ s)]^F}\ \text{Equiv}}\ \text{Func}$$

Example2: Given $[P\ (f\ X)]^F \vee [P\ (g\ X)]^T$ (i.e. $\forall X_\iota.\, f\ X \doteq g\ X$)

Then $[f = g]^F$ is unifiable

Example in $\mathcal{ER}$

Example: Let a_o, b_o, c_o be propositions, then $\boxed{\forall \mathbf{op}_{o\to o}\text{.} (\mathbf{op}\ a) \wedge (\mathbf{op}\ b) \Rightarrow \mathbf{op}\ (a \wedge b)}$

$$\overset{\text{CNF}}{\rightsquigarrow} \quad [\text{op}\ a]^T \quad [\text{op}\ b]^T \quad [\text{op}\ (a \wedge b)]^F$$

Proof: **Difference-Reduction & recursive calls to the Theorem Prover**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\text{op}\ (a\wedge b)]^F \quad [\text{op}\ a]^T}{[(a\wedge b) = a]^F}\text{Res,Dec}}{[(a\wedge b) \equiv a]^F}\text{Equiv}}{[a]^F \vee [b]^F \qquad [a]^T}\text{Cnf}}{[b]^F}\text{Res,Triv} \qquad \dfrac{\dfrac{\dfrac{[\text{op}\ (a\wedge b)]^F \quad [\text{op}\ b]^T}{[(a\wedge b) = b]^F}\text{Res,Dec}}{[(a\wedge b) \equiv b]^F}\text{Equiv}}{[b]^T}\text{Cnf}}{\square}\text{Res,Triv}$$

Other examples: $(X \cap Y) \cup (X \backslash Y) = X, \quad \wp(\emptyset) = \{\emptyset\}, \quad \ldots$ (LEO $<$ 1 second)

Adding Primitive Equality

Motivation: Leibniz equality introduces many flexible literals

$\rightsquigarrow$ employ primitive equality instead

Question:

We will now introduce a primitive equality treatment — do we still have to care about defined equality?

Yes

Adding Primitive Equality

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$\leadsto$ employ primitive equality instead

Question:

We will now introduce a primitive equality treatment — do we still have to care about defined equality?

Yes

Other valid definitions of equality: (apart from Leibniz equality)

Reflexivity Definition: $\dot{\ddot{=}} := \lambda X_\bullet \lambda Y_\bullet \forall Q_\bullet (\forall Z_\bullet (Q\ Z\ Z)) \Rightarrow (Q\ X\ Y)$

Modified Leibniz Equality: $\dot{\dot{\ddot{=}}} := \lambda X_\bullet \lambda Y_\bullet \forall P_\bullet ((\mathbf{a} \vee \neg\, \mathbf{a}) \wedge P\ X) \Rightarrow ((\mathbf{b} \Rightarrow \mathbf{b}) \wedge P\ Y)$

Modified Reflexivity Definition: $\ldots$

Consequence:

In order to obtain a Henkin complete calculus with primitive equality we have to provide an appropriate treatment of defined <u>and</u> primitive equality

Extensional HO Paramodulation: $\mathcal{EP}$

$$\frac{[\mathbf{A}[\mathbf{T}_\beta]]^\alpha \vee C \quad [\mathbf{L} =^\beta \mathbf{R}]^T \vee D}{[\mathbf{A}[\mathbf{R}]]^\alpha \vee C \vee D \vee [\mathbf{T} =^\beta \mathbf{L}]^F}\ \textbf{Para} \qquad \boxed{\text{or}} \qquad \frac{[\mathbf{A}]^\alpha \vee C \quad [\mathbf{L} =^\beta \mathbf{R}]^T \vee D}{[P_{\alpha\to o}\ \mathbf{R}]^\alpha \vee C \vee D \vee [\mathbf{A} =^o P_{\beta\to o}\ \mathbf{L}]^F}\ \textbf{Para'}$$

- **negative equation literals are still handled as unification constraints**
- **not needed:**
  - **Reflexivity Rule, terms like $[(f X) = (f a)]^F$ are tackled by UNI**
  - **Resolution/Factorisation on unification constraints**
  - **Paramodulation into unification constraints**

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$$\frac{[p\ (f\ (f\ a))]^T \quad [f = g]^T}{[p\ (f\ (g\ a))]^T} \text{ Para,Uni}$$

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$$\cfrac{\cfrac{[p\ (f\ (f\ a))]^T \quad [f = g]^T}{[P\ g]^T \vee [P\ f = (p\ (f\ (f\ a)))]^F} \text{ Para'}}{\begin{array}{ll} [p\ (f\ (f\ a))]^T & with\ [\lambda X_\blacksquare (p\ (f\ (f\ a)))/P] \\ [p\ (f\ (g\ a))]^T & with\ [\lambda X_\blacksquare (p\ (f\ (X\ a)))/P] \\ [p\ (g\ (f\ a))]^T & with\ [\lambda X_\blacksquare (p\ (X\ (f\ a)))/P] \\ [p\ (g\ (g\ a))]^T & with\ [\lambda X_\blacksquare (p\ (X\ (X\ a)))/P] \end{array}} \text{ UNI}$$

Contradict. Positive Primitive Equations

Note: some of the semantical domains do always contain **fix-point free functions**!

$$(\lambda X_o \mathbin{.} \neg X) \in \mathcal{D}_{o \to o} \qquad \overbrace{(\lambda P_{\iota \to o} \mathbin{.} \lambda Y_\iota \mathbin{.} \neg (P\, Y))}^{\text{set complement}} \in \mathcal{D}_{(\iota \to o) \to (\iota \to o)} \qquad \ldots$$

Problem: (single) positive primitive equations literals may be **contradictory but not refutable**

$$[\mathbf{A}_o = \neg \mathbf{A}_o]^T \qquad [\overbrace{(\lambda X \mathbin{.} \textsf{male}\ X) = (\lambda X \mathbin{.} \neg(\textsf{male}\ X))}^{\{X | \textsf{male}\ X\} = \overline{\{X | \textsf{male}\ X\}}}]^T$$

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Solution: extensionality axioms for primitive equality or **new extensionality rules**

$$\frac{\mathcal{C} \vee [\mathbf{M}_o = \mathbf{N}_o]^T}{\mathcal{C} \vee [\mathbf{M}_o \Leftrightarrow \mathbf{N}_o]^T}\ \text{Equiv'} \qquad \frac{\mathcal{C} \vee [\mathbf{M}_{\alpha \to \beta} = \mathbf{N}_{\alpha \to \beta}]^T \quad X \text{ new}}{\mathcal{C} \vee [\mathbf{M}\, X = \mathbf{N}\, X]^T}\ \text{Func'}$$

Thm: $\mathcal{EP} := \mathcal{ER} \cup \{$Para,Equiv',Func'$\}$ is a Henkin complete calculus with primitive equality (yet proven only when FlexFlex-rule is available)

Examples in $\mathcal{EP}$

Example1: $(a \cap b = d \cap c) \wedge (\textbf{empty}\ (a \cap b) \cap e) \Rightarrow (\textbf{empty}\ (d \cap c) \cap e)$

$$\dfrac{\dfrac{[\textbf{empty}\ (a \cap b)\ \cap e]^T \quad [a \cap b = d \cap c]^T}{[\textbf{empty}\ (d \cap c)\ \cap e]^T}\ \textbf{Para} \qquad [\textbf{empty}\ (d \cap c) \cap e]^F}{\Box}\ \textbf{Res,Triv}$$

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Example2: $(a \cap b = d \cap c) \wedge (\textbf{empty}\ (a \cap e) \cap b) \Rightarrow (\textbf{empty}\ (d \cap e) \cap c)$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\textbf{empty}\ (a \cap e) \cap b]^T \quad [\textbf{empty}\ (d \cap e) \cap c]^F}{[\textbf{empty}\ (a \cap e) \cap b = \textbf{empty}\ (d \cap e) \cap c]^F}\ \textbf{Res}}{[(a\ s \wedge e\ s) \wedge b\ s \equiv (d\ s \wedge e\ s) \wedge c\ s]^F}\ \textbf{Dec,Func,Equiv}}{\vdots}\ \textbf{Cnf} \qquad \dfrac{\dfrac{[a \cap b = d \cap c]^T}{[a\ X \wedge b\ X \equiv d\ X \wedge e\ X]^T}\ \textbf{Func,Equiv}}{\vdots}\ \textbf{Cnf}}{\Box}$$

Definition: $\cap := \lambda M_{\alpha\to o} \centerdot \lambda N_{\alpha\to o} \centerdot \lambda X_{\alpha} \centerdot M\ X \wedge N\ X$

Extensional HO RUE-Resolution: $\mathcal{ERUE}$

Motivation: Development of a pure Difference-Reduction Approach

Idea: Avoid paramodulation and instead allow to resolve on unification constraints

$$\cfrac{\cfrac{[P\ a]^T \vee [P\ b]^F \qquad [a = b]^F}{[P\ b]^F \vee [P\ a = (a = b)]^F}\ \textbf{Res'}}{\cfrac{[b = b]^F}{\Box}\ \textbf{Triv}}\ \textbf{Uni,Subst} : \{P \leftarrow \lambda X_{\bullet} X = b\}$$

At least theoretically not needed:

$$\frac{\mathcal{C} \vee [\mathbf{L} = \mathbf{R}]^F}{\mathcal{C} \vee [\mathbf{R} = \mathbf{L}]^F}\ \textbf{Sym}$$

Thm: $\mathcal{ERUE} := \mathcal{ER} \setminus \{$**Res**$\} \cup \{$**Res'**,**Equiv'**,**Func'**$\}$ is a Henkin complete calculus with primitive equality (yet proven only when FlexFlex-rule is available)

Examples in $\mathcal{ERUE}$

Example1: $(a \cap b = d) \wedge (\textbf{empty}\ (a \cap b) \cap c) \Rightarrow (\textbf{empty}\ d \cap c)$

$$\dfrac{\dfrac{\dfrac{[\textbf{empty}\ (a \cap b) \cap e]^T \quad [\textbf{empty}\ (d \cap c) \cap e]^F}{[(a \cap b) \cap e = (d \cap c) \cap e]^F}\ \textbf{Res,Dec,Triv}}{[a \cap b = d \cap c]^F}\ \textbf{Dec,Triv} \qquad [a \cap b = d \cap c]^T}{\square}\ \textbf{Res,Triv}$$

Example2: . . . as we have seen before . . .

RUE-aspects:

- avoid subterm-replacement
- try to reduce the differences between the resolution literals
- compute disagreement set (i.e., the clashing pairs within the unification attempt)
- disagreement set represented as negative equations (unification constraints)

Completeness Proofs

Completeness of $\mathcal{ER}|\mathcal{EP}|\mathcal{ERUE}$

The calculi $\mathcal{ER}, \mathcal{EP}, \mathcal{ERUE}$ are complete with respect to Henkin semantics.

Proof

Let Γ_Σ be the set of Σ-sentences which cannot be refuted by calculus $\mathcal{ER}|\mathcal{EP}|\mathcal{ERUE}$ ($\Gamma_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) | \textbf{Cnf}(\Phi) \nvdash_{\mathcal{ER}|\mathcal{EP}|\mathcal{ERUE}} \Box\}$), then we show that Γ_Σ is a saturated abstract consistency class for Henkin models (with primitive equality). This entails completeness of $\mathcal{ER}|\mathcal{EP}|\mathcal{ERUE}$ by the model existence theorem.

Theorem (Model Existence) **For a given abstract consistency class Γ_Σ for Henkin models (with primitive equality) and a set $H \in \Gamma_\Sigma$ there exists a Henkin model $\mathcal{M}$ for H.**

Abstract Consistency Classes (Acc)

Let Γ_Σ be a class of sets of Σ-sentences. Γ_Σ is an Acc, if for all $\Phi \in \Gamma_\Sigma$ and all propositions $\mathbf{A}, \mathbf{B}$*cwff*(Σ):

Saturated	$\mathbf{A} \in \Phi$ **or** $\neg\mathbf{A} \in \Phi$
∇_c	**If $\mathbf{A}$ is atomic, then $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$.**
$\nabla_\neg$	**If $\neg\neg\mathbf{A} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$.**
∇_β	**If $\mathbf{A} \in \Phi$ and $\mathbf{B}$ is the β-normal form of $\mathbf{A}$, then $\mathbf{B} * \Phi \in \Gamma_\Sigma$.**
$\nabla_\mathfrak{f}$	**If $\mathbf{A} \in \Phi$ and $\mathbf{B}$ is the $\beta\eta$-normal form of $\mathbf{A}$, then $\mathbf{B} * \Phi \in \Gamma_\Sigma$.**
$\nabla_\vee$	**If $\mathbf{A} \vee \mathbf{B} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$ or $\Phi * \mathbf{B} \in \Gamma_\Sigma$.**
$\nabla_\wedge$	**If $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, then $\Phi \cup \{\neg\mathbf{A}, \neg\mathbf{B}\} \in \Gamma_\Sigma$.**
$\nabla_\forall$	**If $\Pi^\alpha\mathbf{F} \in \Phi$, then $\Phi * \mathbf{F}\mathbf{W} \in \Gamma_\Sigma$ for each $\mathbf{W} \in$ *cwff*$_\alpha(\Sigma)$.**
$\nabla_\exists$	**If $\neg\Pi^\alpha\mathbf{F} \in \Phi$, then $\Phi * \neg(\mathbf{F}w) \in \Gamma_\Sigma$ for any constant $w \in \Sigma_\alpha$, which does not occur in Φ.**
$\nabla_\mathfrak{b}$	**If $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$, then $\Phi \cup \{\mathbf{A}, \neg\mathbf{B}\} \in \Gamma_\Sigma$ or $\Phi \cup \{\neg\mathbf{A}, \mathbf{B}\} \in \Gamma_\Sigma$.**
$\nabla_\mathfrak{q}$	**If $\neg(\mathbf{F} \doteq^{\alpha\rightarrow\beta} \mathbf{G}) \in \Phi$, then $\Phi * \neg(\mathbf{F}w \doteq^\beta \mathbf{G}w) \in \Gamma_\Sigma$ for any new constant $w \in \Sigma_\alpha$**
$\nabla_\mathfrak{e}^r$	$\neg(\mathbf{A} =^\alpha \mathbf{A}) \notin \Phi$
$\nabla_\mathfrak{e}^s$	**if $\mathbf{F}[\mathbf{A}]_p \in \Phi$ and $\mathbf{A} = \mathbf{B} \in \Phi$, then $\Phi * \mathbf{F}[\mathbf{B}]_p \in \Gamma_\Sigma$**

Conclusion

- First **Henkin complete** refutation approaches for classical Type Theory (with/without primitive equality) that **avoid additional extensionality axioms** in the search space
  - Extensional HO Resolution $\mathcal{ER}$
  - Extensional HO Paramodulation $\mathcal{EP}$
  - Extensional HO RUE-Resolution $\mathcal{ERUE}$
- General approaches to **extensional HO E-Unification**
- For Completeness Proofs: **Adaption of Smullyan's / Andrews' Unifying Principle to Henkin Semantics** (for HOL with/without primitive equality)
- Further work:
  - Turn theoretical approaches into practical ones (restrictions, heuristics)
  - Investigate/prove admissibility of FlexFlex-rule
  - Case studies