Formalising Category Theory by Automating Free Logic in Higher-Order Logic

D. Dormagen, I. Makarenko

Project of Computational Metaphysics 2016

C. Benzmüller, A. Steen, M. Wisniewski
Freie Universität Berlin, Institute of Computer Science, Germany

Freie Universität Berlin

Berlin Mathematical School

Introduction

Free logic extends classical logic by means to address non-existence of objects and to handle undefinedness in a suitable way. Reasoning about free logics can be realised, similar to other non-classical logics, by utilising semantical embeddings in classical higher-order logic [1]. Such an embedding has been employed for the axiomatisation of category theory, a theory, that depends on partiality and undefinedness and therefore significantly benefits from free logic in its formalisation. Parts of the category theory book *Categories, Allegories* by P. J. Freyd and A. Scedrov [4] have been formalised using an embedding of free logic in higher-order logic implemented in the mathematical proof assistant Isabelle/HOL [5]. The axiom system as presented in the book has been found to suffer from a constricted inconsistency [2], and hence an alternative axiom system provided by D. S. Scott [6] has been used in our formalisations.

Figure 1: Front page of *Categories, Allegories* [4]

Free Logic

Free logic is a logic free of any existence presumptions. While preserving the original quantification over a specific domain, terms may now denote undefined/non-existing objects. Free logic after Scott [7] distinguishes between a raw domain D and a particular subdomain E of D. D contains possibly non-existing objects while E holds only the existing entities. Free variables range over domain D and quantified variables only over domain E. Undefinedness is symbolized by a unique object $\star \in D \notin E$. A graphical illustration of this notion of free logic is presented in Figure 2.

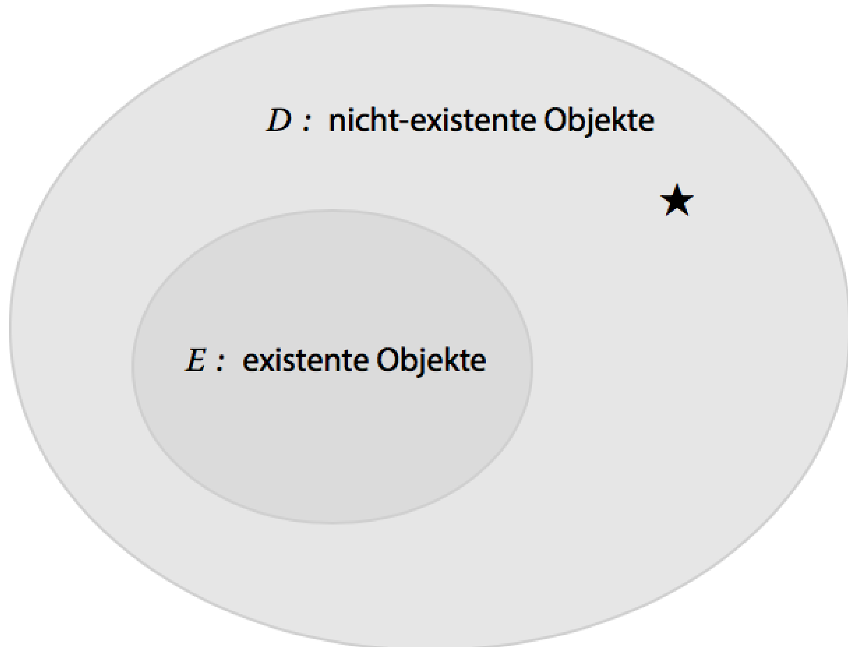

Figure 2: Graphical illustration of a domain and its subdomain

First Formalisations

Category theory after Freyd and Scedrov [4] introduces the source operator ($\Box X$), the target operator ($Y\Box$), as well as partial composition $X \cdot Y$ of two morphisms X and Y. Freyd and Scedrov assume Kleene equality ($=$) in most cases, and a directed Kleene equality ($\succeq$) in some special cases. Most of the formalised lemmas could be verified very easily by the theorem provers, for example, the lemma in Figure 3; only for some statements the provers failed.

1.13. $\Box(\Box x) = \Box x$ because
$\Box(\Box x) = \Box((\Box x)\Box) = (\Box x)\Box = \Box x$.
Similarly $(x\Box)\Box = x\Box$.

Figure 3: Lemma 1.13. of [4]: Some equations

1.18. Given categories **A** and **B**, a function
$F\colon \mathbf{A} \to \mathbf{B}$ is a FUNCTOR if

$$\Box x = y \quad \textit{implies} \quad \Box(Fx) = Fy\,,$$
$$x\Box = y \quad \textit{implies} \quad (Fx)\Box = Fy\,,$$
$$xy = z \quad \textit{implies} \quad (Fx)(Fy) = Fz\,.$$

Equivalently, F is a functor iff

$$F(\Box x) = \Box(Fx)\,,$$
$$F(x\Box) = (Fx)\Box\,,$$
$$F(xy) \succeq (Fx)(Fy)\,.$$

Figure 4: Lemma 1.18. of [4]: Equivalence of two functor definitions

The lemma shown in Figure 4, the equivalence of two functor definitions, could not be proved when formalized in a naive way. It is easy to show, that the first definition follows from the second one, but when assuming the first one and trying to prove the second one, the provers failed. Especially the third line of the second definition, the one with the directed equality, is not provable from the first definition. However, there exist some additional conditions, which can be added to make it derivable. However, it is unclear if Freyd and Scedrov did intend such conditions. Another issue to reconsider is the use of *if* and *iff* in the definitions. It has to be clarified if there is an intentional purpose for this distinction.

Related Work

Automating Free Logic in Isabelle/HOL [3] and *Axiomatising Category Theory in Free Logic* [2] by C. Benzmüller and D. Scott.

Results and Conclusion

The first chapter of *Categories, Allegories* has been formalised. There are many chapters to go, but with the help of a powerful proof assistant and an appropriate embedding techniques a complete formalisation of the book may be in reach.

References

[1] Christoph Benzmueller. A top-down approach to combining logics. In Joaquim Filipe and Ana Fred, editors, *Proc. of the 5th International Conference on Agents and Artificial Intelligence (ICAART)*, volume 1, pages 346–351, Barcelona, Spain, 2013. SCITEPRESS – Science and Technology Publications, Lda.

[2] Christoph Benzmueller and Dana Scott. Axiomatizing category theory in free logic, 2016.

[3] Christoph Benzmueller and Dana S. Scott. Automating free logic in Isabelle/HOL. In Gert-Martin Greuel, Thorsten Koch, Peter Paule, and Andrew Sommese, editors, *Mathematical Software – ICMS 2016, 5th International Congress, Proceedings*, volume 9725 of *LNCS*, pages 43–50, Berlin, 2016. Springer.

[4] Peter J. Freyd and Andre Scedrov. *Categories, Allegories*, volume 39 of *Mathematical Library*. North Holland, 1990.

[5] Tobias Nipkow, Lawrence C. Paulson, and Markus Wenzel. *Isabelle/HOL — A Proof Assistant for Higher-Order Logic*, volume 2283 of *LNCS*. Springer, 2002.

[6] Dana Scott. *Identity and Existence in Intuitionistic Logic*, pages 660–696. Springer Berlin Heidelberg, Berlin, Heidelberg, 1979.

[7] Dana S. Scott. *Existence and Description in Formal Logic*, pages 181–200. 1967.

Leibniz: *Calculemus!*

Computational Metaphysics is a interdisciplinary lecture course designed for advanced students of computer science, mathematics and philosophy. The main objective of the course is to teach the students how modern proof assistants based on expressive higher-order logic support the formal analysis of rational arguments in philosophy (and beyond). In our first course in Summer 2016 the focus has been on ontological arguments for the existence of God. However, some students picked formalisation projects also from other areas (including maths).
Computational Metaphysics was awarded the *Central Teaching Award 2015* of the FU Berlin.